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p-adic Galois representations and elliptic curves

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Notations

$\mathbb{N}$, the set of natural numbers.

$\mathbb{Z}$, the set of integers.

$\mathbb{Q}$, the set of rational numbers.

p, a prime number.

$\mathbb{Q}_p$, the p-adic completion of $\mathbb{Q}$.

$\mathbb{Z}_p$, the p-adic integers in $\mathbb{Q}_p$.

K, a p-adic field in the sense of Definition A.1.

$\overline{K}$, a fixed algebraic closure of K.

$\mathbb{C}_K = \widehat{\overline{K}}$, p-adic completion of $\overline{K}$.

K^{un}, maximal unramified extension of K inside $\overline{K}$.

k, residue field of K.

$K_0 = W(k)[1/p]$, completed maximal unramified extension of $\mathbb{Q}_p$ contained in K.

$\sigma : \mathrm{Gal}(K^{\mathrm{un}}/K) \to \mathrm{Gal}(K^{\mathrm{un}}/K)$, the absolute Frobenius $x \mapsto x^p$.

$G_K = \mathrm{Gal}(\overline{K}/K)$, the absolute Galois group of K.

Preface

The aim of this work is to understand the basic theory of p-adic Galois representations[1]and give a complete description of such representations arising from elliptic curves defined over $\mathbb{Q}_p$. Below is a brief discussion on the content of each chapter.

1. *Fontaine's period rings.* In this chapter we recall the basics of Fontaine's theory of period rings. This includes an introduction and explicit construction of period rings $B_{\mathrm{dR}}, B_{\mathrm{cris}}$ and B_{st}. We also discuss certain properties of these rings and construct some explicit elements which recurrently appear in all the theory that follows.

2. *Filtered (φ, N)-modules.* This chapter is dedicated to the study of the category of filtered (φ, N)-modules . The motivation for such objects would come later in Chapter 3. In this chapter we also introduce certain invariants attached to these modules, namely the Newton number and Hodge number. Based on this, we mention an admissibility criteria for these modules, which helps in classifying admissible modules in dimension 1 and 2 over $\mathbb{Q}_p$.

3. *p-adic Galois representations.* In the third chapter we recall some formalism and use it to explain the construction of various types of p-adic representations. Here we also give an important example of semistable representation using Tate's elliptic curves which would prove to be crucial in the classification done in chapter 4. In Section 3.5, after recalling some basic facts, we construct a p-adic pairing and verify the p-adic de Rham comparison theorem for two examples explicitly. The example of 1-dimensional non-split torus over a p-adic field is the content of Subsection 3.5.2 which is new in the sense that it is not part of any previous literature.

4. *An equivalence of categories.* The content of fourth chapter is on the equivalence of the category of semistable p-adic Galois representations and admissible filtered (φ, N)-modules. First we recollect the construction of a quasi inverse functor and then show its full faithfullness. Establishing the equivalence requires more work, so only a small part of the proof is shown in this chapter. In the end, using the classification done in Chapter 2 for admissible filtered (φ, N)-modules over $\mathbb{Q}_p$ in dimension 1 and 2, we give the associated representations.

5. *p-adic Galois representations from elliptic curves over $\mathbb{Q}_p$.* In this last chapter we classify all p-adic representations coming from elliptic curves over $\mathbb{Q}_p$. With the help of the article [Vol00], we describe a list of objects from the category of filtered (φ, N)-modules. After this, we consider all possible representations coming from elliptic curves and relate these to the corresponding objects on the list. For the converse, examples of elliptic curves in short Weierstrass form are given for some of the objects on the list.

A. *Hodge-Tate representations.* This is an appendix on Hodge-Tate representations which could be seen as a motivation for Fontaine's theory. In this chapter we collect certain definitions and results that will be used throughout in the text. It is meant to serve as a quick introduction and therefore all the proofs have been omitted.

[1]A p-adic representation of G_K is a representation $\rho : G_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V)$ of G_K on a finite dimensional $\mathbb{Q}_p$-vector space V such that ρ is linear and continuous. The category of such representations is denoted $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

B. *Kähler differentials.* In this appendix chapter we recall some basic facts about Kähler differentials and the definition of algebraic de Rham cohomology needed in our computations in Section 3.5.

A good reference for first three chapters is the book (under preparation) by Fontaine and Ouyang [FO08]. A quick introduction to the theory could also be found in the notes of Berger [Ber04]. For the fourth chapter as well as the appendix on Hodge-Tate representations, one could take a look at the online available notes of Brinon and Conrad [BC09]. Apart from these sources, wherever necessary, we mention appropriate references for the results used.

Please note that the none of these results are new and have been written based on my understanding of the literature. However, if you find any mistakes please contact me directly. Thank you for reading this article.

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Contents

Chapter 1

Fontaine's period rings

1.1 The functor R: perfection of a ring

1.1.1 The ring R(A)

Let A be a (commutative) ring with unity. Let p be a prime number such that char $A = p$. The *absolute Frobenius* map on A is the homomorphism,

$$\begin{aligned} \varphi : A &\longrightarrow A \\ a &\longmapsto a^p. \end{aligned}$$

If φ is an isomorphism then A is called *perfect*. Also A is reduced if and only if φ is injective where A is reduced if it has no nontrivial nilpotents.

Definition 1.1. Let A be a commutative ring with unity with char $A = p$, define

$$R(A) := \varprojlim_{n \in \mathbb{N}} A_n,$$

where $A_n = A$ and the transition map $\varphi : A_{n+1} \to A_n$ is given by $\varphi(x_{n+1}) = x_{n+1}^p$. In other words, the inverse system is given as

$$\cdots \to A_{n+2} \xrightarrow{x \mapsto x^p} A_{n+1} \xrightarrow{x \mapsto x^p} A_n \to \cdots .$$

An element $x \in R(A)$ is a sequence $x = (x_n)_{n \in \mathbb{N}}$ such that $x_n \in A$, $x_{n+1}^p = x_n$.

Proposition 1.2. *$R(A)$ is a perfect ring.*

Proof. Let $\varphi : R(A) \to R(A)$ be a homomorphism which sends $x \mapsto x^p$. We need to show that φ is bijective. For injection, we take $x = (x_n)_{n \in \mathbb{N}} \in R(A)$ such that $x^p = 0$. This means $x_n^p = 0$ for each $n \in \mathbb{N}$. But then $x_{n+1}^p = x_n = 0$ for each $n \geq 1$, therefore $x = 0$, proving that φ is injective. For $x = (x_n)_{n \in \mathbb{N}} \in R(A)$, we simply put $y = (x_{n+1})_{n \in \mathbb{N}}$. Obviously, $y^p = x$ which implies the surjection of φ. ∎

This construction is sometimes called *perfection* of a ring of characteristic p.

For any $n \in \mathbb{N}$, let us consider the projection map

$$\begin{aligned} \theta_n : R(A) &\longrightarrow A \\ (x_n)_{n \in \mathbb{N}} &\longmapsto x_n. \end{aligned}$$

If A is perfect then θ_n is an isomorphism for each n. If A is reduced then each θ_n is injective and $\theta_m(R(A)) = \cap_{n \geq m} \varphi^n(A)$. This is easy to check. Indeed, let $x \in R(A)$ i.e., $x = (x_n)_{n \in \mathbb{N}}$ such that $x_n \in A$ for each $n \in \mathbb{N}$ and $\theta_m(x) = x_m$. We know that $x_{n+1}^p = x_n$ for each n. So, $x_m \in \varphi^n(A)$ for each $n \geq m$. This gives $\theta_m(R(A)) \subset \cap_{n \geq m} \varphi^n(A)$. To prove $\cap_{n \geq m} \varphi^n(A) \subset \theta_m(R(A))$, we take

$y \in \cap_{n \geq m} \varphi^n(A)$. Then for each $n \geq m$ there exists an $x_n \in A$ such that $x_n^{p^n} = y$ and $x_{n+1}^p = x_n$. Consider the following p-power compatible sequence in A, $z_0 = y^{p^m}, z_1 = y^{p^{m-1}}, \ldots, z_{m-1} = y^p, z_m = y, z_{m+1} = x_{m+1}^{p^m}, z_{m+2} = x_{m+2}^{p^m}, \ldots$. Then clearly, $z = (z_n)_{n \in \mathbb{N}} \in R(A)$ such that $\theta_m(z) = y$. Hence $\theta_m(R(A)) = \cap_{n \geq m} \varphi^n(A)$. Taking A to be reduced makes sure that θ_n is injective for each $n \in \mathbb{N}$.

Remark 1.3. If A is a topological ring, then $R(A)$ can be given the topology of the inverse limit.

Proposition 1.4. *Let A be a ring, separated and complete for the p-adic topology, i.e., $A \xrightarrow{\sim} \varprojlim_n A/p^n A$. Consider $R(A/pA)$. There exists a bijection between $R(A/pA)$ and the set $S := \{(x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in A, (x^{(n+1)})^p = x^{(n)}\}$.*

Proof. Let $x = (x_n)_{n \in \mathbb{N}} \in R(A/pA)$ with $x_n \in A/pA$ and $x_{n+1}^p = x_n$. Let $\widehat{x}_n$ be a lifting of x_n to A. Then $\widehat{x}_{n+1}^p = \widehat{x}_n \mod pA$. We make an important observation that if $\alpha = \beta \mod p^m A$ then $\alpha^p = \beta^p \mod p^{m+1} A$. So for each $m, n \in \mathbb{N}$ we have $\widehat{x}_{n+m+1}^{p^{m+1}} = \widehat{x}_{n+m}^{p^m} \mod p^m A$. And therefore, $\widehat{x}_{n+m}^{p^m}$ converges in A as $m \to +\infty$. Let $x^{(n)} = \lim_{m \to +\infty} \widehat{x}_{n+m}^{p^m}$. Then clearly, $(x^{(n+1)})^p = x^{(n)}$ and $x^{(n)}$ is a lifting of x_n i.e., $x^{(n)} \mod pA = x_n$. This limit $x^{(n)}$ is independent of the choice of the liftings $\widehat{x}_n$. So, we have a map $R(A/pA) \to S$ where $(x_n)_{n \in \mathbb{N}} \mapsto (x^{(n)})_{n \in \mathbb{N}}$ from the construction above. While on the other hand $A \to A/pA$ reduction mod p gives a natural map $S \to R(A/pA)$ where $(x^{(n)})_{n \in \mathbb{N}} \mapsto (x^{(n)} \mod p)_{n \in \mathbb{N}}$. By the construction, it is immediate that these two maps are inverse to each other. ■

Remark 1.5. Any element $x \in R(A/pA)$ can be written in two different ways,

(i) $x = (x_n)_{n \in \mathbb{N}}$ such that $x_n \in A/pA$.

(ii) $x = (x^{(n)})_{n \in \mathbb{N}}$ such that $x^{(n)} \in A$.

Now, we want to see how addition and multiplication of elements should be defined in S in order to be compatible with the operations in $R(A/pA)$. Let $x = (x^{(n)})_{n \in \mathbb{N}}, y = (y^{(n)})_{n \in \mathbb{N}} \in R(A/pA)$, then $(xy)^{(n)} = (x^{(n)} y^{(n)})$ and $(x+y)^{(n)} = \lim_{m \to +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$. Also, if p is odd then $(-1)^p = -1$ in A, so $(-x^{(n)})_{n \in \mathbb{N}}$ is a p-power compatible sequence for any x. Hence, by definition of addition in S we see that $(-x)^{(n)} = -x^{(n)}$ for all $n \geq 0$ and all x when $p \neq 2$. If $p = 2$, this argument would not work but we observe that $(-x)^{(n)} = x^{(n)}$ for all $n \geq 0$ since $-x = x$ in such cases.

1.1.2 Properties of the ring R

In the last section we discussed perfection of A/pA for A a commutative ring with unity. In this section we let $A = \mathcal{O}_L$ where L is a subfield of $\overline{K}$ containing K_0.

Lemma 1.6. $\mathcal{O}_L/p\mathcal{O}_L = \mathcal{O}_{\widehat{L}}/p\mathcal{O}_{\widehat{L}}$ *where $\widehat{L}$ is the p-adic completion of the field L.*

Proof. Let $x \in \mathcal{O}_{\widehat{L}}$ but $x \notin p\mathcal{O}_{\widehat{L}}$ and let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in L such that converging to x. We know that for any $\epsilon \in \mathbb{R}_{>0}$ there exists $N_\epsilon \in \mathbb{N}$ such that $\upsilon_L(x_n - x) > \epsilon$ for every $n \geq N_\epsilon$. Let $\epsilon = \upsilon_L(p)$ then $\upsilon_L(x_n - x) > \upsilon_L(p)$ for every $n \geq N$ for some N large enough. But also, we must have $\upsilon_L(x) < \upsilon_L(p)$ and $\upsilon_L(x_n) < \upsilon_L(p)$ for all $n \in \mathbb{N}$. Hence $x_n = x \mod p$. ■

Using this lemma we can write $R(\mathcal{O}_L/p\mathcal{O}_L) = R(\mathcal{O}_{\widehat{L}}/p\mathcal{O}_{\widehat{L}}) = \{x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in \mathcal{O}_{\widehat{L}} \text{ and } (x^{(n+1)})^p = x^{(n)}\}$.

Definition 1.7. $R := R(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) = R(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$.

Remark 1.8. (i) An element $x \in R$ is a unit if and only if $x_0 \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ is a unit, so R is a local ring.

(ii) We have valuation $\upsilon = \upsilon_p$ on $\mathbb{C}_K$ normalized by $\upsilon(p) = 1$. This enables us to give a valuation of R by $\upsilon(x) := \upsilon_R(x) := \upsilon(x^{(0)})$ on R.

(iii) R is not noetherian.

Proposition 1.9. *The ring R is a complete valuation ring with the valuation given by υ. It is perfect of characteristic p. Its maximal ideal is $\mathfrak{m}_R = \{x \in R \mid \upsilon(x) > 0\}$ and the residue field is isomorphic to $\overline{k}$. The fraction field* Fr R *of R is a complete nonarchimedean perfect field of characteristic p.*

Proof. First of all we need to check that the valuation defined above is ultrametric. For this we notice that since $R \to \mathcal{O}_{\mathbb{C}_K}, x \to x^{(0)}$ is a surjective map and $\upsilon(x) = \upsilon_R(x) = \upsilon(x^{(0)})$, we have $\upsilon(R) = \mathbb{Q}_{\geq 0} \cup \{+\infty\}$. Now, $\upsilon(x) = +\infty$ if and only if $\upsilon(x^{(0)}) = +\infty$ if and only if $x^{(0)} = 0$ i.e., $x = 0$. To check the ultrametric inequatilty $\upsilon(x+y) \geq \min\{\upsilon(x), \upsilon(y)\}$ for every $x, y \in R$. Since $\upsilon(x) = \upsilon(x^{(0)})$ and $(x^{(n+1)})^p = x^{(n)}$ for every $n \geq 0$, we have that $\upsilon(x) = \upsilon(x^{(0)}) = p^n \upsilon(x^{(n)})$ for every $n \geq 0$. So, for any $x, y \in R$ there exists $n \in \mathbb{N}$ such that $\upsilon(x^{(n)}) < 1$ and $\upsilon(y^{(n)}) < 1$. Since $(x+y)^{(n)} = \lim_{m \to +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$, we know that $(x+y)^{(n)} = x^{(n)} + y^{(n)} \mod p$ i.e., $\upsilon((x+y)^{(n)}) = \upsilon(x^{(n)} + y^{(n)} + p\lambda)$ for some $\lambda \in \mathbb{C}_K$. From this we get $\upsilon((x+y)^{(n)}) \geq \min\{\upsilon(x^{(n)}), \upsilon(y^{(n)}), 1\} \geq \min\{\upsilon(x^{(n)}), \upsilon(y^{(n)})\}$. This immediately gives us that υ is an ultrametric valuation on R.

To prove that R is a complete ring, we observe that *ker* $(\theta_n : R \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}) = \{x \in R \mid \upsilon(x) \geq p^n\}$. This is indeed the case since $\upsilon(x) \geq p^n$ if and only if $\upsilon(x^{(0)}) \geq p^n$ i.e., $\upsilon(x^{(n)}) \geq 1$. Since $x^{(n)} = x_n \mod p$ and $x^{(n)} \in \mathcal{O}_{\mathbb{C}_K}, \upsilon(x^{(n)}) \geq 1$ if and only if $x_n = 0$. So, $\theta_n(x) = 0$ i.e., $x \in$ *ker* (θ_n). In the equality that we just proved, considering for all $n \in \mathbb{N}$ the terms on left form a basis for the inverse limit topology on R and the terms on the right forms a basis for the p-adic topology on R. Since both are equal, we conclude that they induce the same topology on R. Since inverse limit topology is complete, we have that R is a complete valuation ring with the valuation given by υ.

In Proposition 1.2, we have already established that R is a perfect ring of characteristic p. Since R is a domain we have Fr $R = \{x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in \mathbb{C}_K$ and $(x^{(n+1)})^p = x^{(n)}\}$ and the valuation map extends to Fr R by $\upsilon(x) = \upsilon(x^{(0)})$. Since R is complete, perfect and of characteristic p, therefore so is Fr R. The ring of integers of Fr R is $R = \{\mathrm{Fr}\ R \mid \upsilon(x) \geq 0\}$ with the maximal ideal given by $\mathfrak{m}_R = \{x \in \mathrm{Fr}\ R \mid \upsilon(x) > 0\}$. To compute the residue field of Fr R, we see that $R \twoheadrightarrow \overline{k}$ by reducing for any $x = (x_n)_{n \in \mathbb{N}} \in R$ to $x_0 \mod \mathfrak{m}_{\overline{K}}$ where $x_n \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Now $y \in$ *ker* $(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \twoheadrightarrow \overline{k})$ if and only if $\upsilon(x^{(0)}) > 0$ where $x^{(0)} \in \mathcal{O}_{\overline{K}}$ is the lift of x_0 as defined in Proposition 1.4. But since $\upsilon_R(x) = \upsilon(x^{(0)})$, we conclude that *ker* $(R \twoheadrightarrow \overline{k}) = \{x \in R \mid \upsilon_R(x) > 0\} = \mathfrak{m}_R$. Hence $R/\mathfrak{m}_R \simeq \overline{k}$. ■

Proposition 1.10. *There exists a unique section $s : \overline{k} \to R$ of the map $R \to \overline{k}$ which is also a homomorphism of rings. The section s is given by*

$$\begin{aligned} s : \overline{k} &\longrightarrow R \\ a &\longmapsto ([a^{p^{-n}}])_{n \in \mathbb{N}} \end{aligned}$$

where $[a^{p^{-n}}] = (a^{p^{-n}}, 0, 0, \ldots) \in \mathcal{O}_{K_0^{\mathrm{un}}}$ is the Teichmüller representative of $a^{p^{-n}}$.

Proof. By Witt vector construction it is obvious that $([a^{p^{-(n+1)}}])^p = ([a^{p^{-n}}])$ for every $n \in \mathbb{N}$. Let $\tilde{a} = ([a^{p^{-n}}])_{n \in \mathbb{N}} \in R$ and $\theta_0(\tilde{a}) = [a]$ with the $\mod p$ reduction being equal to a. s as defined above is a homomorphism of rings since $s(0) = ([0])_{n \in \mathbb{N}}, s(1) = ([1])_{n \in \mathbb{N}}$ and since char $k = p$, $s(a+b) = ([(a+b)^{p^{-n}}])_{n \in \mathbb{N}} = ([a^{p^{-n}} + b^{p^{-n}}])_{n \in \mathbb{N}} = ([a^{p^{-n}}] + [b^{p^{-n}}])_{n \in \mathbb{N}} = s(a) + s(b)$. Similarly, $s(ab) = s(a)s(b)$.

For uniqueness of s, let $s' : \overline{k} \to R$ be another section. Let $x \in \overline{k}^{\times}$ such that $s(x) \neq s'(x)$. Since $s(x) = s'(x) \mod \mathfrak{m}_R$, we have that $\upsilon(s(x) - s'(x)) > 0$. If $s(x) \neq us'(x)$ for some $u \in R^{\times}$, then we get $\upsilon(s(x) - s'(x)) = \min\{\upsilon(s(x)), \upsilon(s'(x))\} > 0$. But $s(x), s'(x) \notin \mathfrak{m}_R$ since $x \neq 0$. So we conclude that $s(x) = us'(x)$. Also, $s(1) = s'(1) = 1$ which gives $u = 1$ and we are done. ■

Fr R is an algebraically closed field, but before proving this we need the following lemma.

Lemma 1.11. *For any $n \in \mathbb{N}$ and $P(X) \in R[X]$, there exists $x \in R$ such that $\upsilon(P(x)) \geq p^n$.*

Proof. For a fixed n, consider $\theta_n : R \twoheadrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. We know that $ker\ \theta_n = \{y \in R \mid \upsilon(y) \geq p^n\}$, so we just need to find an $x \in R$ such that $\theta_n(P(x)) = 0$. Let $Q(X) = X^d + \cdots + \alpha_1 X + \alpha_0 \in \mathcal{O}_{\overline{K}}[X]$ where α_i is a lifting of $\theta_n(a_i)$. Since $\overline{K}$ is algebraically closed, let $u \in \mathcal{O}_{\overline{K}}$ be a root of $Q(X)$ and let $\overline{u}$ be its image in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, then any $x \in R$ such that $\theta_n(x) = \overline{u}$ satisfies $\theta_n(P(x)) = 0$. Since θ_n is surjective, we are done. ■

Proposition 1.12. Fr R *is algebraically closed.*

Proof. Since Fr R is perfect, we only need to show that it is separably closed i.e., if a monic polynomial $P(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in R[X]$ is separable, then $P(X)$ has a root in R. Since $P(X)$ is separable, there exist $U_0, V_0 \in (\text{Fr } R)[X]$ such that $U_0 P + V_0 P' = 1$ with $P' = dP(X)/dX$. Let $\pi \in R$ such that $\upsilon(\pi) = 1$, then we can find $m \geq 0$ such that $U = \pi^m U_0 \in R[X]$, $V = \pi^m V_0 \in R[X]$ and $UP + UP' = \pi^m$. Let $n_0 = 2m + 1$, we want to construct a sequence $(x_n)_{n \geq n_0}$ in R such that $\upsilon(x_{n+1} - x_n) \geq n - m$ and $P(x_n) \in \pi^n R$. The limit $\lim\limits_{n \to +\infty} x_n$ exists since R is complete and it will be a root of $P(X)$.

We construct (x_n) inductively. Note that $n \geq n_0$. From Lemma 1.11 we find x_{n_0}. Suppose x_n has already been constructed. Let

$$P^{[j]} = \sum_{i \geq j} \binom{i}{j} a_i X^{i-j},$$

then $P(X+Y) = P(X) + YP'(X) + \sum_{j \geq 2} Y^j P^{[j]}(X)$. Write $x_{n+1} = x_n + y$, then $P(x_{n+1}) = P(x_n) + yP'(x_n) + \sum_{j \geq 2} y^j P^{[j]}(x_n)$. If $\upsilon(y) \geq n - m$ then, $\upsilon(y^j P^{[j]}(x_n)) \geq 2(n-m) \geq n+1$ for $j \geq 2$. So, we are reduced to finding a y such that $\upsilon(y) \geq n - m$ and $\upsilon(P(x_n) + yP'(x_{n+1})) \geq n + 1$. By construction, $\upsilon(U(x_n)P(x_n)) \geq n > m$, so $\upsilon(V(x_n)P'(x_n)) = \upsilon(\pi^m - U(x_n)P(x_n)) = m$ which implies that $\upsilon(P'(x_n)) \leq m$. Set $y = -P(x_n)/P'(x_n)$, then $\upsilon(y) \geq n - m$ and we get x_{n+1} as desired. ■

1.2 The multiplicative group (Fr $R)^\times$

In this section we introduce the multiplicative group (Fr $R)^\times$ and prove certain isomorphisms results. These results will be helpful while defining a logarithm for the elements of (Fr $R)^\times$ which as we shall see is a crucial step in the definition of one of the Fontaine's period ring B_{st}. We begin with a simple lemma.

Lemma 1.13. *There is a canonical isomorphism of* $\mathbb{Z}$*-modules* (Fr $R)^\times \simeq \text{Hom}(\mathbb{Z}[1/p], \mathbb{C}_K^\times)$.

Proof. Let

$$\begin{aligned} \alpha : \text{Hom}\Big(\mathbb{Z}[1/p], \mathbb{C}_K^\times\Big) &\longrightarrow (\text{Fr}R)^\times \\ f &\longmapsto (f(p^{-n}))_{n \in \mathbb{N}}. \end{aligned}$$

For $f : \mathbb{Z}[1/p] \to \mathbb{C}_K^\times$ a homomorphism, we clearly have $f(p^{-n}) = f(p \cdot p^{-(n+1)}) = f(p^{-(n+1)})^p$. To see that α is injective, let $f \in \text{Hom}(\mathbb{Z}[1/p], \mathbb{C}_K^\times)$ such that $f(p^{-n}) = 1$ for every $n \in \mathbb{N}$. Then $f(1) = f(p \cdot 1/p) = f(1/p)^p = 1$. This gives $f(\mathbb{Z}[1/p]) = \{1\}$ i.e., ψ is injective. For surjection, let $(x^{(n)})_{n \in \mathbb{N}} \in (\text{Fr } R)^\times$. Let us define $f : \mathbb{Z}[1/p] \to \mathbb{C}_K^\times$ by setting $f(0) = 1, f(1) = (x^{(1)})^p$ and $f(p^{-n}) = x^{(n)}$ for every $n > 0$. This gives α as surjective and since it is a homomorphism of $\mathbb{Z}$-modules, we have that it is indeed an isomorphism.

■

Let $U_R \subset (\text{Fr } R)^\times$ be the group of units of R. For $x \in R, x$ is in U_R if and only if $x^{(0)} \in \mathcal{O}_{\mathbb{C}_K}^\times$ and therefore by previous lemma $U_R \simeq \text{Hom}(\mathbb{Z}[1/p], \mathcal{O}_{\mathbb{C}_K}^\times)$. For $W(\overline{k})$, the ring of Witt vectors of $\overline{k}$, we have that $W(\overline{k}) \subset \mathcal{O}_{\mathbb{C}_K}$ and therefore $\overline{k}^\times \hookrightarrow \mathcal{O}_{\mathbb{C}_K}^\times$. Let $U_{\mathbb{C}_K}^+ = 1 + \mathfrak{m}_{\mathbb{C}_K}$, then $\mathcal{O}_{\mathbb{C}_K}^\times = \overline{k}^\times \times U_{\mathbb{C}_K}^+$ and therefore $U_R \simeq \text{Hom}(\mathbb{Z}[1/p], \mathcal{O}_{\mathbb{C}_K}^\times) = \text{Hom}(\mathbb{Z}[1/p], \overline{k}^\times) \times \text{Hom}(\mathbb{Z}[1/p], U_{\mathbb{C}_K^+})$. Since $x \to x^p$ is an automorphism for $\overline{k}^{\times}$ i.e., every element has exactly one p-th root, therefore $\mathrm{Hom}(\mathbb{Z}[1/p], \overline{k}^{\times}) \simeq \overline{k}^{\times}$. Also $U_R^+ = \{x \in R \mid x^{(n)} \in U_{\mathbb{C}_K}^+ = \mathrm{Hom}(\mathbb{Z}[1/p], U_{\mathbb{C}_K}^+)\}$, therefore $U_R \simeq \overline{k}^{\times} \times U_R^+$.

Let $U_R^1 = \{x \in R \mid v(x-1) \geq 1\}$, so we get $(U_R^1)^{p^n} = \{x \in U_R^1 \mid v(x-1) \geq p^n\}$ and $U_R^1 \simeq \varprojlim_n U_R^1/(U_R^1)^{p^n}$ is an isomorphism and homeomorphism of topological groups. So we can consider U_R^1 as a torsion free $\mathbb{Z}_p$-module. Now for $x \in U_R^1$ such that $v(x-1) > 0$, there exists $n \in \mathbb{N}$ large enough such that $p^n(v(x-1)) \geq 1$. Conversely, any $x \in U_R^1$ has a unique p^n-th root in U_R^+. Therefore, we have the following isomorphism

$$\begin{aligned} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_R^1 &\longrightarrow U_R^+ \\ p^{-n} \otimes u &\longmapsto u^{p^{-n}}. \end{aligned}$$

In summary, we can write the following proposition,

Proposition 1.14. *The sequence* $0 \to U_R \to (\mathrm{Fr}\ R)^{\times} \xrightarrow{v} \mathbb{Q} \to 0$ *is exact and we have following natural identifications*

(i) $(\mathrm{Fr}\ R)^{\times} \simeq \mathrm{Hom}(\mathbb{Z}[1/p], \mathbb{C}_K^{\times})$.

(ii) $U_R \simeq \mathrm{Hom}(\mathbb{Z}[1/p], \mathcal{O}_{\mathbb{C}_K}^{\times}) \simeq \overline{k}^{\times} \times U_R^+$.

(iii) $U_R^+ \simeq \mathrm{Hom}(\mathbb{Z}[1/p], U_{\mathbb{C}_K}^+) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_R^1$.

(iv) $U_R^1 = \{x \in R \mid v(x-1) \geq 1\} \simeq \varprojlim_n U_R^1/(U_R^1)^{p^n}$.

Proof. Immediate from the discussion above. ∎

Now we give an explicit element of R which is also a unit. Let $(\varepsilon^{(n)})_{n \geq 0}$ such that $(\varepsilon^{(0)}) = 1$, $\varepsilon^{(1)} \neq 1$, and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ i.e., $\varepsilon^{(n)}$ is a primitive p^n-th root of unity for every $n \geq 1$.

Lemma 1.15. *The element $\varepsilon = (\varepsilon^{(n)})_{n \in \mathbb{N}}$ is a unit of R.*

Proof. Let $\varepsilon_n = \varepsilon^{(n)} \mod p\mathcal{O}_{\mathbb{C}_K}$ for every $n \in \mathbb{N}$. Let $\pi = \varepsilon - 1$, then $\pi^{(0)} = \lim_{m \to +\infty} (\varepsilon^{(m)} + (-1)^{(m)})^{p^m}$. Since $\varepsilon^{(n)}$ is a primitive p^m-th root of unity in $\overline{K}$ and $(-1)^{(n)} = -1$ if $p \neq 2$ whereas $(-1)^{(n)} = 1$ if $p = 2$. We shall treat the cases of $p = 2$ and odd p separately.

If $p = 2$ then

$$v(\varepsilon - 1) = \lim_{m \to +\infty} 2^m v(\varepsilon^{(m)} + 1) = \lim_{m \to +\infty} 2^m v((\varepsilon^{(m)} - 1) + 2).$$

Since $v(\varepsilon^{(m)} - 1) = \frac{1}{2^{m-1}} < v(2)$ for $m > 1$, we have $v((\varepsilon^{(m)} - 1) + 2) = v(\varepsilon^{(m)} - 1)$ for $n > 1$. Therefore $v(\varepsilon - 1) = 2$.

If p is odd then

$$v(\varepsilon - 1) = \lim_{m \to +\infty} p^m v(\varepsilon^{(m)} - 1) = \lim_{m \to +\infty} \frac{1}{p^{m-1}(p-1)} = \frac{p}{p-1}.$$

So for any p we conclude that, $v(\pi^{(0)}) = p/p - 1 > 1$. Hence, $\varepsilon = 1 + \pi = (\varepsilon^{(n)})_{n \in \mathbb{N}}$ is a unit in R. ∎

1.3 The homomorphism θ

Let $W(R)$ be the ring of Witt vectors with coefficients in R, which is a complete discrete valuation ring with the maximal ideal generated by p and residue field $W(R)/(p) = R$. We know that $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ is not a perfect ring. Indeed this is true because the Frobenius endomorphism is not an automorphism (in particular, it is not injective). So there is no evident way of lifting the G_K-equivariant map $R \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ to a ring map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$. Therefore, we seek to construct in a different manner, such a canonical G_K-equivariant map θ.

Let $a = (a_0, a_1, \ldots, a_m, \ldots) \in W(R)$, where $a_n \in R$ for every $m \in \mathbb{N}$. a_m can be written in two different ways

(a) $(a_m^{(r)})_{r\in\mathbb{N}}$ such that $a_m^{(r)} \in \mathcal{O}_{\mathbb{C}_K}$ for every $r \in \mathbb{N}$ and $(a_m^{(r+1)})^p = a_m^{(r)}$, or

(b) $(a_{m,r})_{r\in\mathbb{N}}$ such that $a_{m,r} \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ for every $r \in \mathbb{N}$ and $(a_{m,r+1})^p = a_{m,r}$.

So we have a natural map

$$\begin{aligned} W(R) &\longrightarrow W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \\ a &\longmapsto (a_{0,n}, a_{1,n}, \ldots, a_{n-1,n}) \end{aligned}$$

which gives us a commutative diagram

$$\begin{tikzcd} & W_{n+1}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \arrow[d, "f_n"] \\ W(R) \arrow[ur] \arrow[r] & W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \end{tikzcd}$$

with $f_n((x_0, x_1, \ldots, x_n)) = (x_0^p, x_1^p, \ldots, x_{n-1}^p)$. From this it is immediate that $W(R) \simeq \varprojlim_n W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$. It is a homeomorphism of topological groups if the object on the right hand side of the isomorphism is equipped with the inverse limit topology of discrete topology on each component. For the surjective map

$$\begin{aligned} \psi_{n+1} : W_{n+1}(\mathcal{O}_{\mathbb{C}_K}) &\longrightarrow W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \\ (a_0, \ldots, a_n) &\longmapsto (\overline{a}_0, \ldots, \overline{a}_{n-1}), \end{aligned}$$

let I be its kernel, then $I = \{(pb_0, \ldots, pb_{n-1}, a_n) \mid b_i, a_n \in \mathcal{O}_{\mathbb{C}_K}\}$. Let $\omega_{n+1} : W_{n+1}(\mathcal{O}_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}$ such that $\omega_{n+1}((a_0, \ldots, a_n)) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$. We compose this map with the quotient map $\pi_n : \mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$. This gives a natural map $W_{n+1}(\mathcal{O}_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$. Now, since $\omega_{n+1}(pb_0, \ldots, pb_{n-1}, a_n) = (pb_0)^{p^n} + \cdots + p^{n-1}(pb_{n-1})^p + p^n a_n \in p^n\mathcal{O}_{\mathbb{C}_K}$, therefore the map $W_{n+1}(\mathcal{O}_{\mathbb{C}_K}) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$ factors through $W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ i.e., there exists a unique homomorphism $\theta_n : W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \to \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}$ such that the diagram

$$\begin{tikzcd} W_{n+1}(\mathcal{O}_{\mathbb{C}_K}) \arrow[r, "\omega_{n+1}"] \arrow[d, "\psi_{n+1}"] & \mathcal{O}_{\mathbb{C}_K} \arrow[d, "\pi_n"] \\ W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \arrow[r, "\theta_n"] & \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K} \end{tikzcd}$$

commutes. By construction of θ_n, it is evident that we have another commutative diagram

$$\begin{tikzcd} W_{n+1}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \arrow[r, "\theta_{n+1}"] \arrow[d, "f_n"] & \mathcal{O}_{\mathbb{C}_K}/p^{n+1}\mathcal{O}_{\mathbb{C}_K} \arrow[d] \\ W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \arrow[r, "\theta_n"] & \mathcal{O}_{\mathbb{C}_K}/p^n\mathcal{O}_{\mathbb{C}_K}. \end{tikzcd}$$

Passing to the limit induces a homomorphism of rings, $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$.

Lemma 1.16. *If $x = (x_0, x_1, \ldots, x_n, \ldots) \in W(R)$ for $x_n \in R$ and $x_n = (x_n^{(m)})_{m\in\mathbb{N}}$ with $x_n^{(m)} \in \mathcal{O}_{\mathbb{C}_K}$, then $\theta(x) = \sum_n p^n x_n^{(n)}$.*

Proof. From the map $W(R) \to W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ defined above where $(x_0, x_1, x_2, \ldots) \mapsto (x_{0,n}, x_{1,n}, \ldots, x_{n-1,n})$ we take $x_i^{(n)}$ to be a lifting in $\mathcal{O}_{\mathbb{C}_K}$ of $x_{i,n}$, then

$$\theta_n(x_{0,n}, \ldots, x_{n-1,n}) = \sum_{i=0}^{n-1} p^i \overline{(x_i^{(n)})^{p^{n-i}}} = \sum_{i=0}^{n-1} p^i \overline{x_i^{(n-(n-i))}} = \sum_{i=0}^{n-1} p^i \overline{x_i^{(i)}}.$$

By passing to the limit we get, $\theta(x) = \sum_n p^n x_n^{(n)}$. ∎

For $x \in W(R)$ let $x_n \in R$ such that $x = \sum_{n\in\mathbb{N}} p^n[x_n]$ where $[x_n] \in W(R)$ is the Teichmüller representative of x. Since θ is a homomorphism, we get $\theta(x) = \theta(\sum_n p^n[x_n]) = \sum_n p^n\theta([x_n]) = \sum_n p^n x_n^{(0)}$.

Proposition 1.17. *The homomorphism θ defined above is surjective.*

Proof. For any $a \in \mathcal{O}_{\mathbb{C}_K}$, there exists an $x \in R$ such that $x^{(0)} = a$, since $\mathbb{C}_K$ is algebraically closed. Let $[x] = (x, 0, 0, \ldots)$ be the Teichmüller lift in $W(R)$ of x. Then, $\theta([x]) = x^{(0)} = a$. ∎

Proposition 1.18. *The continuous surjective G_K-equivariant map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$ constructed above is open. Also, using the canonical k-algebra map $s : \overline{k} \to R$ to make $W(R)$ into a $W(\overline{k})$-algebra via $W(s)$, θ is $W(\overline{k})$-algebra map via the natural $W(\overline{k})$-algebra structure on $\mathcal{O}_{\mathbb{C}_K}$.*

Proof. On $W(R)$ we have the product topology of the valuation topology on R and on $\mathcal{O}_{\mathbb{C}_K}$ we have the p-adic topology. So to prove openness we just have to show that if J is an open ideal in R then the image under θ of the additive subgroup of vectors (r_i) with $r_0, r_1, \cdots, r_n \in J$ (for a fixed n) is open in $\mathcal{O}_{\mathbb{C}_K}$. Let $J^{(m)}$, $m \geq 0$ be the image of J under the map of sets $R \to \mathcal{O}_{\mathbb{C}_K}$ defined by $r \mapsto r^{(m)}$, the the image of (r_i) is $J^{(0)} + pJ^{(1)} + \cdots + p^{n-1}J^{(n-1)}$. Since $\mathcal{O}_{\mathbb{C}_K}$ has the p-adic topology, it suffices to show that $J^{(m)}$ is open in $\mathcal{O}_{\mathbb{C}_K}$ for each $m \geq 0$. But $J^{(m)} = (J^{p^m})^{(0)}$, so we only need to prove that $J^{(0)}$ is open in $\mathcal{O}_{\mathbb{C}_K}$ is J is open in R. Now, it is enough to work with J's from a base of open ideals, so we take $J = \{r \in R \mid v_R(x) \geq c\}$ with $c \in \mathbb{Q}$. Since $v_R(r) = v(r^{(0)})$ and the map $r \mapsto r^{(0)}$ is a surjection from R onto $\mathcal{O}_{\mathbb{C}_K}$, for such J we have that $J^{(0)} = \{x \in \mathcal{O}_{\mathbb{C}_K} \mid v(x) \geq c\}$, which is certainly open in $\mathcal{O}_{\mathbb{C}_K}$. Hence θ is an open map.

Next, we prove that θ is a map of $W(\overline{k})$-algebras. $\mathcal{O}_{\mathbb{C}_K}$ is a $W(\overline{k})$-algebra because we have a continuous $W(k)$-algebra map $h : W(\overline{k}) \to \mathcal{O}_{\mathbb{C}_K}$ lifting the identity map on residue field $\overline{k}$. Using p-adic continuity, it is enough to look at Teichmüller representatives. So now we only need to show that for each $\in \overline{k}$ the image $h([c])$ is equal to $\theta([s(c)])$. c is viewed in $\mathcal{O}_{\mathbb{C}_K}$ as $h([c])$, so $s(c) = (h([c]), h([c^{-p}]), \ldots)$. Now $\theta([s(c)]) = s(c)^{(0)} = h([c])$, hence we conclude. ∎

Let $\varpi \in R$ such that $\varpi^{(0)} = -p$. Let $\xi = [\varpi] + p \in W(R)$. Then $\xi = (\varpi, 1, 0, 0, \ldots)$. By Lemma 1.16, $\theta(\xi) = \sum_n \xi_n^{(n)} = \varpi^{(0)} + p = 0$ i.e., $\xi \in ker\ \theta$. We make few observations about $ker\ \theta$.

Proposition 1.19. (i) *$ker\ \theta$ is the principal ideal generated by ξ.*

(ii) *$\cap_n(ker\ \theta)^n = 0$.*

(iii) *An element $r = (r_0, r_1, \ldots) \in ker\ \theta$ is a generator of $ker\ \theta$ if and only if $r_1 \in R^\times$.*

Proof. (i) Let $x \in ker\ \theta$. If we write $x = \xi y_0 + px_1$ with $y_0, x_1 \in W(R)$ then we notice that $\theta(x) = p\theta(x_1) = 0$. Since $\mathcal{O}_{\mathbb{C}_K}$ has no p-torsion elements and $W(R)$ is p-adically separated and complete, we conclude that $x_1 \in ker\ \theta$. From this, if we are able to show that $ker\ \theta \subset (\xi, p)$ then we can conclude $ker\ \theta = (\xi)$ because then we can write a sequence $x_{n-1} = \xi y_{n-1} + x_n$ which would give $x = \xi(\sum_n p^n y_n)$. Assume $x = (x_0, x_1, \ldots, x_n, \ldots) \in ker\ \theta$, then $0 = \theta(x) = x_0^{(0)} + p\sum_{n\in\mathbb{N}_{>0}} p^{n-1}x_n^{(n)}$. Since $v(0) = +\infty$, then we conclude that $v(x_0^{(0)}) \geq 1 = v_p(p)$. This also implies that $v(x_0) \geq 1 = v(-p) = v(\varpi)$. Therefore there exists $b_0 \in R$ such that $x = b_0\varpi$. Let $b = [b_0]$ be the Teichmüller lift of b_0 to $W(R)$. Then $x - b\xi = (x_0, x_1, \ldots) - (b_0, 0, 0, \ldots)\cdot(\varpi, 1, 0, 0, \ldots) = (x_0, x_1, \ldots) - (b_o\varpi, b_0^p, 0, 0, \ldots) = (x_0 - b_0\varpi, y_1, y_2, \ldots) = (0, y_1, y_2, \ldots) = p(y_1', y_2', \ldots) \in pW(R)$ with $(y_i')^p = y_i$. Hence $ker\ \theta = (\xi)$.

(ii) Let $x \in (ker\ \theta)^n$ for every $n \in \mathbb{N}$. Then $v_R(\overline{x}) \geq v_R(\overline{\xi}^n) \geq n$ for every $n \in \mathbb{N}$ where $\overline{x} = x \mod p$. From this we get $\overline{x} = 0$ i.e., there exists $y \in W(R)$ such that $x = py$. Now, $p(\theta(y)) = \theta(x) = 0$, so $\theta(y) = 0$ i.e., $y \in ker\ \theta$. Replacing x by x/ξ^n we have $\theta(x/\xi^n) = p\theta(y/\xi^n)$ and therefore $y/\xi^n \in ker\ \theta$ for every $n \in \mathbb{N}$. Hence $y \in \cap_n(ker\ \theta)^n$. So we can write $x = py = p(pz) = \cdots$. Since $W(R)$ is p-adically separated, we get that $x = 0$.

(iii) A general element $r = (r_0, r_1, \ldots) \in \mathit{ker}\ \theta$ has the form $w = \xi \cdot (s_0, s_1, \ldots) = (\varpi, 1, \ldots)(s_0, s_1, \ldots) = (\varpi s_0, \varpi^p s_1 + s_0^p, \ldots)$, so $r_1 = \varpi^p s_1 + s_0^p$. Hence $r_1 \in R^\times$ if and only if $s_0 \in R^\times$, and this final unit condition is equivalent to the multiplier $(s_0, s_1, \ldots)$ being a unit in $W(R)$, which amounts to r being a principal generator of $\mathit{ker}\ \theta$.

■

Example 1.20. Using the criteria in Proposition 1.19 (iii) and from the proof of Lemma 1.15 we see that the element $\varepsilon - 1 \in \mathit{ker}\ \theta$ is a generator when $p = 2$ whereas for $p > 2$ this is not true since $v(\varepsilon) = p/p - 1 > 0$ for all p.

We recall that $K_0 = \operatorname{Frac} W(k) = W(k)[1/p]$. Let $W(R)[1/p] = K_0 \otimes_{W(k)} W(R)$ and therefore $W(R) \hookrightarrow W(R)[1/p]$ by sending $x \mapsto 1 \otimes x$. Since

$$W(R)[1/p] = \bigcup_{n=0}^{+\infty} W(R)p^{-n} = \varinjlim_n W(R)p^{-n},$$

the homomorphism $\theta : W(R) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ extends to the homomorphism $\theta_{\mathbb{Q}} : W(R)[1/p] \to \mathbb{C}_K$ which is again surjective, continuous and G_K-equivariant. Continuity of this extended map is obvious since for any $x \in \mathbb{C}_K$ it is enough to check the continuity at any $p^n x$ for some $n \in \mathbb{N}$ but there does exist some $n \in \mathbb{N}$ such that $p^n x \in \mathcal{O}_{\mathbb{C}_K}$. The rest follows from the continuity of the map $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$. The kernel of $\theta_{\mathbb{Q}} : W(R)[1/p] \to \mathbb{C}_K$ is a principal ideal generated by ξ.

Corollary 1.21. *For all $n \geq 1$, $W(R) \cap (\mathit{ker}\ \theta_{\mathbb{Q}})^n = (\mathit{ker}\ \theta)^n$. Also, $\cap_n (\mathit{ker}\ \theta)^n = \cap_n (\mathit{ker}\ \theta_{\mathbb{Q}})^n = 0$.*

Proof. It is enough to prove the equality for $n = 1$, the rest follows by induction. But this holds since $W(R)/(\mathit{ker}\ \theta) = \mathcal{O}_{\mathbb{C}_K}$ has no nonzero p-torsion. Since any element of $W(R)[1/p]$ admits a p-power multiple in $W(R)$, we conclude that $\cap_n (\mathit{ker}\ \theta_{\mathbb{Q}})^n = (\cap_n (\mathit{ker}\ \theta)^n)[1/p]$. The rest follows from Proposition 1.19. ■

1.4 The rings B_{dR}^+ and B_{dR}

In this section we introduce Fontaine's de Rham period ring.

Definition 1.22. The de Rham ring B_{dR}^+ is defined as

$$B_{\mathrm{dR}}^+ = \varprojlim_n W(R)[1/p]/(\mathit{ker}\ \theta)^n = \varprojlim_n W(R)[1/p]/(\xi)^n$$

where the transition maps are G_K-equivariant. B_{dR}^+ is ξ-adic completion of $W(R)[1/p]$.

Since B_{dR}^+ is ξ-adically separated i.e., by Corollary 1.21 $\cap_n \xi^n W(R)[1/p] = 0$, there is an injection $W(R) \hookrightarrow W(R)[1/p] \hookrightarrow B_{\mathrm{dR}}^+$ as subrings. B_{dR}^+ admits a G_K-action that is compatible with the action on its subring $W(R)[1/p]$. The inverse limit B_{dR}^+ maps G_K-equivariantly onto each quotient $W(R)[1/p]/(\mathit{ker}\ \theta)^n$ via the evident natural map, and in particular for $n = 1$ the map $\theta_{\mathbb{Q}}$ induces a natural G_K-equivarint surjective map $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbb{C}_K$. Also we have $\mathit{ker}\ \theta_{\mathrm{dR}}^+ \cap W(R) = \mathit{ker}\ \theta$, and $\mathit{ker}\ \theta_{\mathrm{dR}}^+ \cap W(R)[1/p] = \mathit{ker}\ \theta_{\mathbb{Q}}$ since θ_{dR}^+ restricts to $\theta_{\mathbb{Q}}$ on the subring $W(R)[1/p]$.

Proposition 1.23. *The ring B_{dR}^+ is a complete discrete valuation ring with the residue field $\mathbb{C}_K$, and any generator of $\mathit{ker}\ \theta_{\mathbb{Q}}$ in $W(R)[1/p]$ is a uniformizer of B_{dR}^+. Moreover, the natural map $B_{\mathrm{dR}}^+ \to W(R)[1/p]/(\mathit{ker}\ \theta_{\mathbb{Q}})^n$ is identified with the projection to the quotient modulo the n-th power of the maximal ideal for all $n \geq 1$.*

Proof. $\mathit{ker}\ \theta_{\mathbb{Q}}$ is a nonzero principal maximal ideal (with residue field $\mathbb{C}_K$) in the domain $W(R)[1/p]$. For $j \geq 1$, the only ideals of $W(R)[1/p]/(\mathit{ker}\ \theta_{\mathbb{Q}})^j$ are $(\mathit{ker}\ \theta_{\mathbb{Q}})^i/(\mathit{ker}\ \theta_{\mathbb{Q}})^j$ where $0 \leq i \leq j$, therefore it is an artin local ring. In particular, an element of B_{dR}^+ is a unit if and only if it has nonzero image under B_{dR}^+ is a unit if and only if it has nonzero image under θ_{dR}^+. In other words, the maximal ideal $\mathit{ker}\ \theta_{\mathrm{dR}}^+$

consists of precisely the non-units, so B_{dR}^+ is a local ring. Consider a non-unit $b \in B_{\mathrm{dR}}^+$, so its image in each $W(R)[1/p]/(\mathit{ker}\ \theta_{\mathbb{Q}})^j$ has the form $b_j\xi$ with b_j uniquely determined modulo $(\mathit{ker}\ \theta_{\mathbb{Q}})^{j-1}$. In particular, the residue classes $b_j \mod (\mathit{ker}\ \theta_{\mathbb{Q}})^{j-1}$ are a compatible sequence and so define an element $b' \in B_{\mathrm{dR}}^+$ with $b = \xi b'$. By construction, b' is unique. Hence, the maximal ideal of B_{dR}^+ has the principal generator ξ, and ξ is not a zero divisor in B_{dR}^+. It now follows that for each $j \geq 1$ the multiples ξ^j in B_{dR}^+ are the elements sent to zero in the projection $W(R)[1/p]/(\mathit{ker}\ \theta_{\mathbb{Q}})^j$. In particular, B_{dR}^+ is ξ-adically separated, so it is a discrete valuation ring with uniformizer ξ. By construction of B_{dR}^+, we know that it is the inverse limit of its artinian quotients, hence it is a complete discrete valuation ring. ■

Remark 1.24. There are at least two different topologies on B_{dR}^+,

(i) The topology of the discrete valuation ring, i.e., ξ-adic topology. This induces a discrete topology on the residue field $\mathbb{C}_K$.

(ii) There is a topological ring structure on $W(R)[1/p]$ that induces the natural v_R-adic topology on the subring $W(R)$. We can give the inverse limit topology to B_{dR}^+ coming from the topology induced on each quotient by the topology on $W(R)[1/p]$. This induces the natural topology on its residue field $\mathbb{C}_K$. In further discussions this topology would be named *natural*.

Below we mention some of the consequences of the topological ring structure on $W(R)[1/p]$ from Remark 1.24(ii). Details for this can be found in [BC09, Exer. 4.5.3].

(1) $W(R)$ endowed with its product topology using the v_R-adic topology on R is a closed topological subring of $W(R)[1/p]$. Moreover, $K_0 = W(k)[1/p] \subset W(R)[1/p]$ is a closed subfield with its usual p-adic topology.

(2) $\theta_{\mathbb{Q}} : W(R)[1/p] \to \mathbb{C}_K$ is a continuous open map.

(3) The multiplication map $\xi : W(R)[1/p] \to W(R)[1/p]$ is a closed embedding so all ideals $(\mathit{ker}\ \theta_{\mathbb{Q}})^j = \xi^j W(R)[1/p]$ are closed.

(4) Give the quotient topology on each $W(R)[1/p]/(\mathit{ker}\ \theta_{\mathbb{Q}})^j$, the inverse limit topology on B_{dR}^+ makes it a Hausdorff topological ring relative to which

   (a) The powers of the maximal ideal are closed;
   (b) $W(R)$ is a closed subring (with its natural topology as subspace topology);
   (c) The G_K-action on B_{dR}^+ is continous;
   (d) The multiplication map by ξ on B_{dR}^+ is a closed embedding;
   (e) The residue field $\mathbb{C}_K$ inherits its valuation topology as quotient topology.

(5) This topology on B_{dR}^+ is complete.

Definition 1.25. The *field of p-adic periods* (or the *de Rham period ring*) is $B_{\mathrm{dR}} := \mathrm{Frac}\ B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}[1/\xi]$ equipped with its natural G_K-action and G_K-stable filtration via the $\mathbb{Z}$-powers of the maximal ideal of B_{dR}^+.

The Frobenius automorphism φ of $W(R)[1/p]$ does not naturally extend to B_{dR}^+ since it does not preserve $\mathit{ker}\ \theta_{\mathbb{Q}}$; for example, $\varphi(\xi) = [\varpi^p] + p \notin \mathit{ker}\ \theta_{\mathbb{Q}}$. There is no natural Frobenius structure on B_{dR}^+. Nonetheless, we do have a filtration via powers of the maximal ideal, and this is a G_K-stable filtration. For any $i \in \mathbb{Z}$, let $\mathrm{Fil}^i B_{\mathrm{dR}}^+ = \mathfrak{m}^i_{B_{\mathrm{dR}}^+}$ for $i \geq 0$ and for $i < 0$, $\mathrm{Fil}^i B_{\mathrm{dR}}^+$ is the free B_{dR}^+ module generated by ξ^i, i.e.,

$$\mathrm{Fil}^i B_{\mathrm{dR}} = \begin{cases} B_{\mathrm{dR}}^+ & \text{if } i = 0 \\ B_{\mathrm{dR}}^+ \xi^i & \text{if } i \neq 0. \end{cases}$$

Next, we record an important property of B_{dR}^+.

Proposition 1.26. *The K_0-algebra B_{dR}^+ contains a unique copy of $\overline{K}$ as a subfield over K_0, and this lifting from the residue field is compatible with the action of G_{K_0}. Moreover, any extension K'/K_0 inside $\overline{K}$ with finite ramification index gets its valuation topology as the subspace topology from B_{dR}^+. In particular, K' is closed in B_{dR}^+ if it is complete.*

Proof. B_{dR}^+ is a complete discrete valuation ring over K_0, and $\overline{K}$ is a subfield of the residue field $\mathbb{C}_K$ that is separable and algebraic over K_0, it follows from Hensel's Lemma [Mil08, Thm. 7.33] that $\overline{K}$ uniquely lifts to a subfield over K_0 in B_{dR}^+. The uniqueness ofthe lifting ensures that this is a G_K-equivariant lifting.

Next, we take an algebraic extension K'/K_0 with finite ramification index. We need to check that K' gets its valuation topology as the subspace topology. We recall that B_{dR}^+ only depends on $\mathbb{C}_K$, so we can construct it from the view of completion $\widehat{\overline{K_0}} = W(\overline{k})[1/p]$. In particular, B_{dR}^+ contains $\widehat{K_0}K'$ over K' and from the natural topology on B_{dR}^+, the induced topology on $\widehat{K_0}$ is the usual one. Therefore, to check that the topology on K' is as expected it suffices to replace K' with0 $\widehat{K_0}K'$ which we may take as K (upon replacing k with $\overline{k}$). In other words, we only need to show that K gets its valuation topology as subspace topology.

B_{dR}^+ is a topological K_0-algebra and the valuation topology on K is its product topology for a K_0-basis. So, if we give K its valuation topology then the natural map $K \to B_{\mathrm{dR}}^+$ is continuous. To see that this is an embedding it suffices to compare convergent sequences. The map $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \to \mathbb{C}_K$ is continuous for $\mathbb{C}_K$ with its valuation topology. Since $K \hookrightarrow \mathbb{C}_K$ continuously, we get that $K \to B_{\mathrm{dR}}^+$ is an embedding. ∎

We state the following proposition without proof and some remarks about its consequences.

Proposition 1.27. *For the homomorphism $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \to \mathbb{C}_K$ from a complete discrete valuation ring to the residue field of characteristic 0, there exists a section $s : \mathbb{C}_K \to B_{\mathrm{dR}}^+$ which is a homomorphism of rings such that $\theta(s(c)) = c$ for every $c \in \mathbb{C}_K$.*

Remark 1.28.

(i) The section s is not unique. The proof for this is non-trivial and uses axiom of choice. There is no such s which is either continuous for the natural topology or G_K-equivariant.

(ii) For $\overline{K} \subset \mathbb{C}_K$ an algebraic closure of K inside $\mathbb{C}_K$, there exists a unique continuous homomorphism $s : \overline{K} \to B_{\mathrm{dR}}^+$ commuting with the action of G_K such that $\theta(s(a)) = a$ for every $a \in \overline{K}$. This alongwith Proposition 1.26 helps us in viewing $\theta : B_{\mathrm{dR}}^+ \to \mathbb{C}_K$ as a homomorphism of $\overline{K}$-algebras.

(iii) A theorem by Colmez [Fon94, §A2] says that $\overline{K}$ is dense in B_{dR}^+ with the subspace topology induced by the natural topology on B_{dR}^+. Notice that this subspace topology on $\overline{K}$ is not its valuation topology.

1.5 The element t

In this section we construct an explicit element of B_{dR} which we call t. From Lemma 1.15 we know that $\varepsilon \in R$ is a unit given by $\varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. $[\varepsilon] - 1 \in W(R)$ where $[\varepsilon]$ is the Teichmüller representative of ε. Since $\theta([\varepsilon] - 1) = \varepsilon^{(0)} - 1 = 0$, therefore $[\varepsilon] - 1 \in \mathit{ker}\ \theta = \mathrm{Fil}^1 B_{\mathrm{dR}}$. This gives us that $(-1)^{n+1}([\varepsilon] - 1)^n/n \in W(R)[1/p]\xi^n$ and therefore

$$t := \log[\varepsilon] = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \in B_{\mathrm{dR}}^+.$$

Proposition 1.29. *The element $t \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ but $t \notin \mathrm{Fil}^2 B_{\mathrm{dR}}$. Equivalently, t generates the maximal ideal of B_{dR}^+.*

Proof. Since $([\varepsilon]-1)^n/n \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ for every $n \geq 1$, we have that $t \in \mathrm{Fil}^1 B_{\mathrm{dR}}$. Also, $([\varepsilon]-1)^n/n \in \mathrm{Fil}^2 B_{\mathrm{dR}}$ for every $n \geq 2$. So it is enough to show that $[\varepsilon]-1 \notin \mathrm{Fil}^2 B_{\mathrm{dR}}$. From the discussion above, $[\varepsilon]-1 \in \mathit{ker}\ \theta$ i.e., there exists $\lambda \in W(R)$ such that $[\varepsilon]-1=\lambda\xi$. Now $[\varepsilon]-1 \notin \mathrm{Fil}^2 B_{\mathrm{dR}}$ if and only if $\theta(\lambda) \neq 0$ i.e., $\lambda \notin W(R)\xi$. So, if $[\varepsilon]-1 \notin W(R)\xi^2$, we will be done. To show this, we proceed by contradiction. Assume that $[\varepsilon]-1=\lambda\xi^2$ with $\lambda \in W(R)$. We can write $\lambda=(\lambda_0,\lambda_1,\dots)$ and we have $\xi=(\varpi,1,0,0,\dots)$ so $\xi^2=(\varpi^2,\dots)$. Therefore, $\lambda\xi^2=(\lambda_0\varpi^2,\dots)$. On the other hand, $[\varepsilon]-1=(\varepsilon-1,\dots)$. So we must have, $\varepsilon-1=\lambda_0\varpi^2$. Now, $\upsilon(\lambda_0\varpi^2)=\upsilon(\lambda_0)+2\upsilon(\varpi)\geq 2$ i.e., $\upsilon(\varepsilon-1)\geq 2$. But from Lemma 1.15 we have $\upsilon(\varepsilon-1)=p/p-1<2$ for $p\neq 2$. This is a contradiction. Hence $t \notin \mathrm{Fil}^2 B_{\mathrm{dR}}$ for $p \neq 2$.

For $p=2$ we notice that $\xi^2=(\varpi^2,0,\dots)$ in $W(R)$ and for any $\lambda=(\lambda_0,\lambda_1,\dots)\in W(R)$ we have $\lambda\xi^2=(\lambda_0\varpi^2,\lambda_1\varpi^4,\dots)$. However, for $p=2$ we also have $-1=(1,1,\dots)\in\mathbb{Z}_2=W(\mathbb{F}_2)$ since $-1=1+2\cdot 1 \mod 4$, so $[\varepsilon]-1=(\varepsilon-1,\varepsilon-1,\dots)$ in $W(R)$. Thus if $[\varepsilon]-1$ were a $W(R)$-multiple of ξ^2 then $\varepsilon-1=\lambda_1\varpi^4$ for some $\lambda_1\in R$. So we get $\upsilon(\varepsilon-1)\geq\upsilon(\varpi^4)=4$ but from Lemma 1.15, for $p=2$ we have $\upsilon(\varepsilon-1)=2$. Therefore, we reach a contradiction in case $p=2$. ■

Note that $t=\log[\varepsilon]$ depends on our choice of ε i.e., on the choice of primitive p^n-th roots of unity. So if we make another choice ε' then $\varepsilon'=\varepsilon^a$ for a unique $a\in\mathbb{Z}_p^\times$ using the natural $\mathbb{Z}_p$-module structure on units in R. Hence by the continuity of the Teichmüller map $R\to W(R)$ relative to the υ_R-adic topology on R we have $[\varepsilon']=[\varepsilon^a]$ in $W(R)$. Thus $t'=\log[\varepsilon']=\log[\varepsilon^a]$. For the natural topology on B^+_{dR}, it can be shown that $\log[\varepsilon^a]=a\cdot\log[\varepsilon]$. So we get, $t'=at$. In other words, the line $\mathbb{Z}_p t$ in the maximal ideal of B^+_{dR} is intrinsic i.e., independent of the choice of ε and making a choice of $\mathbb{Z}_p$-basis of this line is the same as making a choice of ε. Also, choosing ε is exactly a choice of $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1)=\varprojlim_n \mu_{p^n}(\overline{K})$. From $\mathbb{Z}_p(1)=\mathbb{Z}_p t$, the action of $g\in G_K$ is then given as

$$g(t)=\log(g[\varepsilon])=\log([g(\varepsilon)])=\log([\varepsilon^{\chi(g)}])=\log([\varepsilon]^{\chi(g)})=\chi(g)t,$$

where χ is the p-adic cyclotomic character. We conclude that $\mathbb{Z}_p t$ is a canonical copy of $\mathbb{Z}_p(1)$ as a G_K-stable line in B^+_{dR}. We can also write $\mathrm{Fil}^i B_{\mathrm{dR}}=B_{\mathrm{dR}}t^i=B_{\mathrm{dR}}(i)$ and $B_{\mathrm{dR}}=B^+_{\mathrm{dR}}[1/t]$. Then,

$$\begin{aligned}\mathrm{gr}B_{\mathrm{dR}}&=\bigoplus_{i\in\mathbb{Z}}\mathrm{gr}^i B_{\mathrm{dR}}=\bigoplus_{i\in\mathbb{Z}}\mathrm{Fil}^i B_{\mathrm{dR}}/\mathrm{Fil}^{i+1}B_{\mathrm{dR}}\\&=\bigoplus_{i\in\mathbb{Z}}B^+_{\mathrm{dR}}(i)/tB^+_{\mathrm{dR}}(i)=\bigoplus_{i\in\mathbb{Z}}\mathbb{C}_K(i).\end{aligned}$$

Proposition 1.30. $\mathrm{gr}B_{\mathrm{dR}}=B_{\mathrm{HT}}=\mathbb{C}_K(t,1/t)\subset\widehat{B_{\mathrm{HT}}}=\mathbb{C}_K((t))$.

Remark 1.31. We could choose a section $s:\mathbb{C}_K\to B^+_{\mathrm{dR}}$ to identify $\mathbb{C}_K$ with a subring of B^+_{dR} and then $B_{\mathrm{dR}}\simeq\mathbb{C}_K((t))$. However, as pointed out earlier s would not be G_K-equivariant and continuous.

Now we discuss an important result for the G_K-invariant elements of B_{dR}.

Proposition 1.32. $B_{\mathrm{dR}}^{G_K}=K$.

Proof. We clearly have $K\subset B_{\mathrm{dR}}^{G_K}$ since $K\subset\overline{K}\subset B^+_{\mathrm{dR}}$. Now to prove the equality, let $0\neq b\in B_{\mathrm{dR}}^{G_K}$. Then there is an $i\in\mathbb{Z}$ such that $b\in\mathrm{Fil}^i B_{\mathrm{dR}}$ but $b\notin\mathrm{Fil}^{i+1}B_{\mathrm{dR}}$. Let $\bar{b}$ be the image of b in $\mathrm{Fil}^i B_{\mathrm{dR}}/\mathrm{Fil}^{i+1}B_{\mathrm{dR}}=\mathbb{C}_K(i)$. From Tate-Sen Theorem A.13

$$\mathbb{C}_K(i)^{G_K}=\begin{cases}K & \text{if } i=0\\ 0 & \text{if } i\neq 0.\end{cases}$$

Since $\bar{b}\in\mathbb{C}_K(i)^{G_K}$ and $b\neq 0$, we have that $i=0$. Then $\bar{b}\in K\subset B^+_{\mathrm{dR}}$. So we get that $b-\bar{b}=\alpha\xi^k$ for some $k>0,\alpha\in B^+_{\mathrm{dR}}$ i.e., $b-\bar{b}\in\mathrm{Fil}^i B_{\mathrm{dR}}$ for some $i\geq 1$. But $b\in B^+_{\mathrm{dR}}$ and $\bar{b}\in K$. So it is not possible that $b-\bar{b}\in\mathrm{Fil}^i B_{\mathrm{dR}}$ unless $b-\bar{b}=0$. Hence we have $B_{\mathrm{dR}}^{G_K}=K$. ■

An inspection of the construction shows that B_{dR}^+ depends solely on $\mathcal{O}_{\mathbb{C}_K}$ and not on the particular p-adic field $K \subseteq \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K$ whose algebraic closure is dense in $\mathbb{C}_K$. More specifically, B_{dR}^+ depends functorially on $\mathcal{O}_{\mathbb{C}_K}$, and the action of $\mathrm{Aut}(\mathcal{O}_{\mathbb{C}_K})$ on B_{dR}^+ via functoriality induces the action of G_K. Hence, if $K \to K'$ is a map of p-adic fields and we select a compatible embedding $\overline{K} \to \overline{K'}$ of algebraic closures then the induced map $\mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_{K'}}$ induces a map $B_{\mathrm{dR},K}^+ \to B_{\mathrm{dR},K'}^+$ that is equivariant relative to the corresponding map of Galois groups $G_{K'} \to G_K$. In particular, if the induced map $\mathbb{C}_K \to \mathbb{C}_{K'}$ is an isomorphism then we have $B_{\mathrm{dR},K}^+ = B_{\mathrm{dR},K'}^+$ (compatibly with the inclusion $G_{K'} \hookrightarrow G_K$) and the same works for fraction fields i.e., $B_{\mathrm{dR},K} = B_{\mathrm{dR},K'}$. We apply this in two different scenarios: K'/K a finite extension and $K' = \widehat{K^{\mathrm{un}}}$. So B_{dR}^+ and B_{dR} are naturally insensitive to replacing K with a finite extension or with a completed maximal unramified extension. These changes are important in practice while replacing G_K with an open subgroup or I_K in the context of studying de Rham representations.

1.6 The ring B_{cris}

One defect of B_{dR}^+ is that the Frobenius automorphism of $W(R)[1/p]$ does not preserve $ker\ \theta_{\mathbb{Q}}$, so there is no natural Frobenius endomorphism of $B_{\mathrm{dR}} = \mathrm{Frac}\ B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+[1/t]$. To remedy this defect, in this section we will introduce another ring of periods, namely B_{cris}. From previous sections, we know that

$$\begin{tikzcd} W(R) \arrow[r, "\theta"] \arrow[d, hook] & \mathcal{O}_{\mathbb{C}_K} \arrow[d, hook] \\ W(R)[1/p] \arrow[r, "\theta"] & \mathbb{C}_K, \end{tikzcd}$$

with $ker\ \theta = (\xi)$ where $\xi = [\varpi] + p = (\varpi, 1, 0, 0, \ldots), \varpi \in R$ such that $\varpi^{(0)} = -p$.

Definition 1.33. (i) The module A_{cris}^0 is defined to be the divided power envelope of $W(R)$ with respect to $ker\ \theta$ i.e., it is a G_K-stable $W(R)$-subalgebra in $W(R)[1/p]$, $A_{\mathrm{cris}}^0 = W(R)[a^m/m!]_{n\in\mathbb{N}}$ for every $a \in ker\ \theta$. We can also write $A_{\mathrm{cris}}^0 = W(R)[\xi^n/n!]_{n\in\mathbb{N}}$.

(ii) The ring $A_{\mathrm{cris}} := \varprojlim_n A_{\mathrm{cris}}^0/p^n A_{\mathrm{cris}}^0$ is an abstract p-adic completion of A_{cris}^0.

The description of A_{cris} is complicated and verifying even its basic properties requires a lot of effort. So, we describe some of the properties of A_{cris}^0 and A_{cris} without much explanation (some useful techniques for studying A_{cris} are contained in [Fon82b] and [Fon94]).

Remark 1.34. (i) A_{cris}^0 is naturally a ring since for $\gamma_m(\xi) = \xi^m/m!, m \in \mathbb{N}$ we have $\gamma_m(\xi)\cdot\gamma_n(\xi) = \binom{m+n}{n}\xi^{m+n}/(m+n)!$.

(ii) A_{cris}^0 is a $\mathbb{Z}$-flat domain.

(iii) The natural map $A_{\mathrm{cris}}^0/p^n A_{\mathrm{cris}}^0 \to A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}$ is an isomorphism for all $n \geq 1$.

It can be shown that there exists a unique, continuous, injective and G_K-equivariant map $j : A_{\mathrm{cris}} \to B_{\mathrm{dR}}^+$ such that the diagram

$$\begin{tikzcd} A_{\mathrm{cris}} \arrow[r, "j"] & B_{\mathrm{dR}}^+ \\ A_{\mathrm{cris}}^0 \arrow[u] \arrow[r] & W(R)[1/p] \arrow[u] \end{tikzcd}$$

commutes. From the diagram above and injectivity of j, it is clear that $A_{\mathrm{cris}}^0 \hookrightarrow A_{\mathrm{cris}}$ as a subring. Concretely, the image of A_{cris} in B_{dR}^+ is the subring of elements

$$\Big\{ \sum_{n\geq 0} \alpha_n \frac{\xi^n}{n!} \mid \alpha_n \in W(R), \alpha_n \to 0 \text{ for the } p\text{-adic topology} \Big\} \tag{1.1}$$

in which the infinite sums are taken with respect to the discretely valued topology on B^+_{dR}. A_{cris} is a $\mathbb{Z}_p$-flat domain. The ring homomorphism $\theta : W(R) \to \mathcal{O}_{\mathbb{C}_K}$ can be extended to A^0_{cris}, and therefore to A_{cris}.

$$\begin{tikzcd} W(R) \arrow[d, hook] \arrow[dr, "\theta"] & \\ A^0_{\mathrm{cris}} \arrow[r, "\theta_{\mathrm{cris}}"] \arrow[d, hook] & \mathcal{O}_{\mathbb{C}_K}. \\ A_{\mathrm{cris}} \arrow[ur, "\theta_{\mathrm{cris}}"] & \end{tikzcd}$$

Proposition 1.35. *The G_K-action on A_{cris} is continuous for the p-adic topology. Equivalently, for any $n \geq 1$, the G_K-action on $A_{\mathrm{cris}}/(p^r)$ has open stabilizers.*

Proof. [BC09, Prop. 9.1.2]. ■

Proposition 1.36. *The kernel of $\theta_{\mathrm{cris}} : A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ is a divided power ideal, i.e., if $a \in A_{\mathrm{cris}}$ such that $\theta_{\mathrm{cris}}(a) = 0$, then for all $m \in \mathbb{N}_{>0}, a^m/m! \in A_{\mathrm{cris}}$ and $\theta_{\mathrm{cris}}(a^m/m!) = 0$.*

Proof. For $a = \sum a_n \gamma_n(\xi) \in A^0_{\mathrm{cris}}$, we have

$$\begin{aligned} \frac{a^m}{m!} &= \sum_{\text{sum over } j_n = n} \prod_n a_n \frac{\xi^{n j_n}}{(n!)^{j_n}(j_n)!} \\ &= \sum_{\text{sum over } j_n = n} \prod_n a_n \frac{(n j_n)!}{(n!)^{j_n}(j_n)!} \gamma_{n, j_n}(\xi) \in A^0_{\mathrm{cris}}. \end{aligned}$$

Also, it is immediate that $\theta_{\mathrm{cris}}(a^m/m!) = 0$. The case of A_{cris} is obvious by continuity. ■

By reducing $\bmod\, p$ the image of θ_{cris}, we have a ring homomorphism $\overline{\theta}_{\mathrm{cris}} : A_{\mathrm{cris}} \xrightarrow{\theta_{\mathrm{cris}}} \mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$.

Proposition 1.37. *The kernel $ker\ \overline{\theta}_{\mathrm{cris}} = (ker\ \theta_{\mathrm{cris}}, p)$ is a divided power ideal, i.e., for any $a \in ker\ \overline{\theta}_{\mathrm{cris}}$, we have that for all $m \in \mathbb{N}_{>0}, a^m/m! \in A_{\mathrm{cris}}$ and $\overline{\theta}_{\mathrm{cris}}(a^m/m!) = 0$.*

Proof. For $a \in ker\ \overline{\theta}_{\mathrm{cris}}$, we write $a = x + \beta p$ for some $x \in ker\ \theta_{\mathrm{cris}}$ and $\beta \in A_{\mathrm{cris}}$. Then we have

$$\frac{a^m}{m!} = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} x^k (\beta p)^{m-k} = \sum_{k=0}^{m} \frac{x^k}{k!} \frac{(\beta p)^{m-k}}{(m-k)!}.$$

Clearly, $p^k/k!$ is divisible by p for any $k \geq 1$. Also, $x^k/k! \in A_{\mathrm{cris}}$ for every $k \geq 1$ and $\overline{\theta}_{\mathrm{cris}}(x^k/k!) = 0$. So, $a^m/m! \in A_{\mathrm{cris}}$ and $\overline{\theta}_{\mathrm{cris}}(a^m/m!) = \overline{\theta}_{\mathrm{cris}}(p^m/m!) = 0$. ■

If $\alpha \in A_{\mathrm{cris}}, \alpha$ can be written, though not in a unique way as, $\alpha = \sum_n \alpha_n \xi^n/n!, \alpha_n \in W(R)$ and $\alpha_n \to 0$ p-adically. From Section 1.5, we know that $t = \sum_n (-1)^{n+1}([\varepsilon] - 1)^n/n \in B^+_{\mathrm{dR}}$. Since $[\varepsilon] - 1 \in ker\ \theta$, we have $[\varepsilon] - 1 = b\xi$ for some $b \in W(R)$. Then $([\varepsilon] - 1)^n/n = (n-1)! b^n \gamma_n(\xi)$. Also $(n-1)! \to 0$ p-adically, therefore $t \in A_{\mathrm{cris}}$.

Proposition 1.38. $t^{p-1} \in pA_{\mathrm{cris}}$.

Proof. We only need to show that $([\varepsilon] - 1)^{p-1} \in pA_{\mathrm{cris}}$. $[\varepsilon] - 1 = (\varepsilon - 1, \ldots)$ and $(\varepsilon - 1)^{(n)} = \lim\limits_{m \to +\infty} (\zeta_{p^{n+m}} - 1)^{p^m}$ where $\zeta_{p^n} = \varepsilon^{(n)}$ is a primitive p^n-th root of unity. $\pi = \varepsilon - 1$, so

$$v((\varepsilon - 1)^{(n)}) = p^{-n} v(\pi^{(0)}) = p^{-n} \frac{p}{p-1} = \frac{1}{p^{n-1}(p-1)}.$$

Also, $(\varepsilon - 1)^{p-1} = \pi^{p-1}$. Since $v(\pi^{p-1}) = p = p \cdot v(\varpi)$, so we must have $(\varepsilon - 1)^{p-1} = \varpi^p$, where u is a unit in R. Now, $([\varepsilon] - 1)^{p-1} \equiv [\varpi^p] \cdot a \equiv (\varpi)^p \cdot a' \equiv (\xi - p)^p \cdot a' \equiv \xi^p \cdot a' \mod pA_{\mathrm{cris}}$. Since $\xi^p = p(p-1)! \gamma_p(\xi) \in pA_{\mathrm{cris}}$, we have that $([\varepsilon] - 1)^{p-1} \in pA_{\mathrm{cris}}$. So we conclude by looking at the expression for t. ■

Definition 1.39. (i) Define the G_K-stable $W(R)[1/p]$-subalgebra $B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p] \subset B^+_{\mathrm{dR}}$.

(ii) The *crystalline period ring* for K is the G_K-stable $W(R)[1/p]$-subalgebra, $B_{\mathrm{cris}} := B^+_{\mathrm{cris}}[1/t] = A_{\mathrm{cris}}[1/t]$ inside $B^+_{\mathrm{dR}}[1/t] = B_{\mathrm{dR}}$.

Theorem 1.40. *The natural G_K-equivariant map $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective, and if we give $K \otimes_{K_0} B_{\mathrm{cris}}$ the subspace filtration then the induced map between the associated graded algebras is an isomorphism.*

Proof. Proof of injectivity can be given by direct calculations based on [Fon82b, Prop. 4.7]. We omit the details here.

For the isomorphism property on associated graded objects, since $t \in B_{\mathrm{cris}}$ and A_{cris} map onto $\mathcal{O}_{\mathbb{C}_K}$, we get the isomorphism result since $\mathrm{gr} B_{\mathrm{dR}} = B_{\mathrm{HT}}$ has its graded components of dimension 1 over $\mathrm{gr}^0 B_{\mathrm{dR}} = \mathbb{C}_K$. ∎

Frobenius endomorphism φ of B_{cris}

First we examine how φ on $W(R)$ acts on the subring A^0_{cris}.

Lemma 1.41. *The $W(R)$-subalgebra $A^0_{\mathrm{cris}} \subset W(R)[1/p]$ is φ-stable.*

Proof. On $W(R)$, the Frobenius map is given as $\varphi((a_1, a_1, \dots, a_n, \dots)) = (a_0^p, a_1^p, \dots, a_n^p, \dots)$. For each $b \in W(R)$, $\varphi(b) \equiv b^p \mod p$, therefore $\varphi(\xi) = \xi^p + p\eta = p((p-1)!\gamma_p(\xi) + \eta)$ with $\eta \in W(R)$ and $\varphi(\xi^m) = p^m((p-1)!\gamma_p(\xi) + \eta)^m$. Since $p^m/m! \in \mathbb{Z}_p$ for all $m \geq 1$, we have that $\varphi(\gamma_m(\xi)) = (\eta + (p-1)!\gamma_p(\xi))^m \cdot p^m/m! \in A^0_{\mathrm{cris}}$. This shows that A^0_{cris} is stable under the action of φ. ∎

The endomorphism of A^0_{cris} induced by φ on $W(R)[1/p]$ extends uniquely to a continuous endomorphism of the p-adic completion A_{cris}, and hence an endomorphism φ of $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$ that extends the Frobenius automorphism φ of the subring $W(R)[1/p]$.

Lemma 1.42. $\varphi(t) = pt$ *for* $t = \log[\varepsilon]$ *as in Section 1.5.*

Proof. We first recall that to prove $t \in A_{\mathrm{cris}}$ we showed that the summation $\sum_{n\geq 1}(-1)^{n+1}([\varepsilon]-1)^n/n$ initially defining t in B^+_{dR} actually made sense as a convergent sum in the p-adic topology of A_{cris}. This sum defines the element of A_{cris} that "is" t via the embedding $A_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$. So we may use p-adic continuity to compute

$$\varphi(t) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)}{n},$$

since φ on A_{cris} extends the usual Frobenius map on $W(R)$. Thus $\varphi(t) = \log[\varepsilon^p] = \log[\varepsilon]^p = p\log[\varepsilon] = pt$. ∎

So we can extend φ uniquely to an endomorphism of B_{cris} by setting $\varphi(1/t) = 1/pt$. We mention a theorem without proof. The original proof was omitted from [Fon94].

Theorem 1.43. *The Frobenius endomorphism $\varphi : A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is injective. In particular, the induced Frobenius endomorphism of $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ is injective.*

Remark 1.44. The action of φ commutes with G_K i.e., for every $g \in G_K$ and $b \in B_{\mathrm{cris}}$ we have $\varphi(g(b)) = g(\varphi(b))$.

Remark 1.45. We give B_{cris} the subspace filtration via $B_{\mathrm{cris}} \hookrightarrow K \otimes_{K_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ i.e., for $i \in \mathbb{Z}$, define $\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap \mathrm{Fil}^i B_{\mathrm{dR}}$. One should be aware that the Frobenius operator on B_{cris} does not preserve the subspace filtration. The basic notion for this incompatibility is that *ker* θ is not stable by the Frobenius. More specifically, $\xi = [\varpi] + p$ is killed by θ whereas $\varphi(\xi) = [\varpi^p] + p$ is not $(\theta(\varphi(\xi)) = (-p)^p + p \neq 0)$, so $\xi \in \mathrm{Fil}^1 B_{\mathrm{cris}}$ and $\varphi(\xi) \notin \mathrm{Fil}^1 B_{\mathrm{cris}}$.

1.7 The logarithm map

In this section we define a logarithm for the elements in $(\mathrm{Fr}\ R)^\times$ taking values in B_{dR}. The classical G_K-equivariant homomorphism, the p-adic logarithm on $\log : \mathbb{C}_K^\times \to \mathbb{C}_K$ serves as our motivation. Using the important fact $\log(xy) = \log x + \log y$, we review the construction in classical case below. Let $x \in \mathbb{C}_K^\times$.

(a) For every x satisfying $v(x-1) \geq 1$, set $\log x := \sum_n (-1)^{n+1}(x-1)^n/n$.

(b) For every $x \in 1 + \mathfrak{m}_{\mathbb{C}_K} = \{x \in \mathbb{C}_K \mid v(x-1) > 0\}$, there exists $m \in \mathbb{N}$ such that $v(x^{p^m} - 1) \geq 1$, then we set $\log x := \log(x^{p^m})/p^m$.

(c) For $x \in \mathcal{O}_{\mathbb{C}_K}$, then $\overline{x} \in \overline{k}$ and $\overline{x} \neq 0$. We have a decomposition $x = [\overline{x}]a$, where $\overline{x} \in \overline{k}^\times, [\overline{x}] \in W(\overline{k})$ and $a \in 1 + \mathfrak{m}_{\mathbb{C}_K}$. Then we set $\log x := \log a$.

(d) For any $x \in \mathbb{C}_K$ with $v(x) = \frac{r}{s}, r, s \in \mathbb{Z}, s \geq 1$, we have $v(x^s) = r = v(p^r)$ and $x^s/p^r = y \in \mathcal{O}_{\mathbb{C}_K}^\times$. We must also have, $\log(x^s/p^r) = s \log x - r \log p = \log y$. So, to define $\log x$ for any $x \in \mathbb{C}_K$, it is enough to define $\log_p p$. In particular, if take the convention of $\log_p p = 0$, then $\log_p x := \frac{1}{s} \log y$.

Employing similar ideas for $(\mathrm{Fr}\ R)^\times$ we would like to define a logarithm map taking values in B_{dR}^+. The important rule to note again is that $\log[xy] = \log[x] + \log[y]$. From Proposition 1.14 we know that $U_R^+ = 1 + \mathfrak{m}_R = \{x \in R \mid v(x-1) > 0\}$ and $U_R^+ \supset U_R^1 = \{x \in R \mid v(x-1) \geq 1\}$. We give the following construction,

(a) First of all we define the logarithm for elements in U_R^1. Let $x \in U_R^1$, then the Teichmüller representative $[x] = (x, 0, 0, \dots) \in W(R)$. We define $\log[x] := \sum_n (-1)^{n+1}([x]-1)^n/n, x \in U_R^1$. The series above converges in A_{cris}. This is indeed the case because, $\theta([x]-1) = x^{(0)} - 1$ and $x \in U_R^1$ or equivalently $\overline{\theta}([x]-1]) = 0$. From Proposition 1.37, we have $\gamma_n([x]-1) = ([x]-1)^n/n! \in A_{\mathrm{cris}}$. So $\log[x] = \sum_n (-1)^{n+1}(n-1)!\gamma_n([x]-1)$ converges since $(n-1)! \to 0$ p-adically as $n \to \infty$.

(b) After defining logarithm on U_R^1, we want to extend this uniquely to a map $\log_{\mathrm{cris}} : U_R^+ \to B_{\mathrm{cris}}^+$. Notice that for any $x \in U_R^+$, there exists $m \in \mathbb{N}, m \geq 1$ such that $x^{p^m} \in U_R^1$ i.e., $v(x-1) \geq 1$. From this we can easily define $\log[x] := (1/p^m)\log[x^{p^m}]$ for some m large enough. This definition is clearly independent of the choice of such an m.

(c) Next we move on to $R^\times = U_R$. From Proposition 1.14 we have, $U_R = \overline{k}^\times \times U_R^\times$. For $x \in U_R$, we write $x = x_0 a$ for $x_0 \in \overline{k}^\times$ and $x \in U_R^+$, and define $\log[x] := \log_{\mathrm{cris}}[a]$.

(d) At last we look at $(\mathrm{Fr}\ R)^\times$. From discussions in Proposition 1.19 we know that $\varpi \in R$ given by $\varpi^{(0)} = -p$ and $v(\varpi) = 1$. For $x \in (\mathrm{Fr}\ R)^\times$ with $v(x) = r/s$, we must have $x^s/\varpi^r = y \in U_R$. Also, the following relation must hold for log, $\log(x^s/\varpi^r) = s \log x - r \log \varpi$. So, from this we can define $\log[x] := (\log[y] + r\log[\varpi])/s$ and if we define $\log[\varpi]$, we should be done. We have $[\varpi] \in W(R) \subset W(R)[1/p]$. Note that $\theta([\varpi]/(-p)) = -p/(-p) - 1 = 0$ i.e., $[\varpi](-p) \in ker\ \theta$. Therefore,
$$\log\left(\frac{[\varpi]}{-p}\right) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\left(\frac{[\varpi]}{-p} - 1\right)^n}{n} = -\sum_{n=1}^{+\infty} \frac{\xi^n}{p^n n} \in B_{\mathrm{dR}}^+$$
is well defined. So we set $\log[\varpi] := \log([\varpi]/(-p)) \in B_{\mathrm{dR}}^+$. Hence we have the desired G_K-equivariant logarithm map $\log : (\mathrm{Fr}\ R)^\times \to B_{\mathrm{dR}}^+$.

Remark 1.46. For every $g \in G_K$, we have $g\varpi = \varpi\varepsilon^{\chi(g)}$, so $g(\log[\varpi]) = \log([g\varpi]) = \log[\varpi] + \chi(g)t$ where $t = \log[\varepsilon]$.

1.8 The ring B_{st}

In this section we introduce a yet another period ring called B_{st}. This ring would play an important role while studying p-adic representations coming from elliptic curves over $\mathbb{Q}_p$. Also, we have more structure on this ring compared to B_{cris} as we shall see.

Definition 1.47. The ring B_{st} is defined to be the sub B_{cris}-algebra of B_{dR} generated by $\log[\varpi]$, i.e. $B_{\mathrm{st}} = B_{\mathrm{cris}}[\log[\varpi]]$.

B_{st} is stable under the action of G_K and G_{K_0}. Let C_{cris} and C_{st} denote the field of fractions of B_{cris} and B_{st} respectively. Both these fields are then stable under the action of G_K and G_{K_0}. Also, the Frobenius map φ on B_{cris} extends to C_{cris}. Next, we show that infact C_{st} is not an algebraic extension of C_{cris}.

Lemma 1.48. *The element* $\log[\varpi]$ *is not contained in* C_{cris}.

Proof. Let $\beta = \xi/p$, then ξ and β are in $\mathrm{Fil}^1 B_{\mathrm{dR}}$ but not in $\mathrm{Fil}^2 B_{\mathrm{dR}}$. Let $S = W(R)[[\beta]] \subset B^+_{\mathrm{dR}}$ be the subring of power series $\sum_n a_n \beta^n$ with coefficients $a_n \in W(R)$. For each $r \in \mathbb{N}$, let $\mathrm{Fil}^r S = S \cap \mathrm{Fil}^r B_{\mathrm{dR}}$, then $\mathrm{Fil}^r S$ is a principal ideal of S generated by β^i. We denote $\theta_i : \mathrm{Fil}^i B_{\mathrm{dR}} \to \mathbb{C}_K$ the map sending $\beta^i \alpha$ to $\theta(\alpha)$. It is obvious that $\theta_i(\mathrm{Fil}^i S) = \mathcal{O}_{\mathbb{C}_K}$. By description of A_{cris} in (1.1), we see that $A_{\mathrm{cris}} \subset S$ and hence $C_{\mathrm{cris}} = \mathrm{Fr}\ A_{\mathrm{cris}} \subset \mathrm{Fr}\ S$. We will show that if $\alpha \in S$ is not zero, then $\alpha \log[\varpi] \notin S$, which is sufficient to claim the lemma.

Since S is separated by the p-adic topology, it suffices to show that if $r \in \mathbb{N}$ and $\alpha \in S - pS$, then $p^r \alpha \log[\varpi] \notin S$. If $a \in W(R)$ satisfying $\theta(a) \in p\mathcal{O}_{\mathbb{C}_K}$, then $a \in (p, \xi)W(R)$ and hence $a \in pS$. Therefore, one can find $b_i \in W(R)$ such that $\theta(b_i) \notin p\mathcal{O}_{\mathbb{C}_K}$ and $\alpha = A + B$ with $A = p(\sum_{0 \le n < i} b_n \beta^n)$ and $B = \sum_{n \ge i} b_n \beta^n$. Note that $\log[\varpi] = -\sum_n \beta^n/n$. Suppose $j > r$ is an integer such that $p^j > i$. If $p^r \alpha \log[\varpi] \in S$, one has $\alpha \cdot \sum_{n \ge 0} p^{j-1}\beta^n/n \in S$. Since $\alpha \cdot \sum_{0<n<p^j} p^{j-1}\beta^n/n \in S$, therefore

$$A \cdot \sum_{n \ge p^i} p^{j-1}\frac{\beta^n}{n} \in \mathrm{Fil}^{p^j} B_{\mathrm{dR}}, \;\; B \cdot \sum_{n > p^j} p^{j-1}\frac{\beta^n}{n} \in \mathrm{Fil}^{i+p^j+1} B_{\mathrm{dR}} \;\text{ and }\; \frac{\beta^{p^j}}{p}\sum_{n>i} b_n \beta^n \in \mathrm{Fil}^{i+p^j+1} B_{\mathrm{dR}}.$$

Thus,

$$b_i \frac{\beta^{i+p^j}}{p} \in \mathrm{Fil}^{i+p^j} B_{\mathrm{dR}} \cap (S + \mathrm{Fil}^{i+p^j+1} B_{\mathrm{dR}}) = \mathrm{Fil}^{i+p^j} S + \mathrm{Fil}^{i+p^j+1} B_{\mathrm{dR}}.$$

Now, on one hand we have $\theta_{i+p^j}(b_i \beta^{i+p^j}/p) = \theta(b_i)/p \notin \mathcal{O}_{\mathbb{C}_K}$; while on the other hand $\theta_{i+p^j}(\mathrm{Fil}^{i+p^j} S + \mathrm{Fil}^{i+p^j+1} B_{\mathrm{dR}}) = \mathcal{O}_{\mathbb{C}_K}$. This gives us a contradiction, therefore $p^r \alpha \log[\varpi] \notin S$ which implies $\log[\varpi] \notin C_{\mathrm{cris}}$. ■

Proposition 1.49. $\log[\varpi]$ *is transcendental over* C_{cris}.

Proof. Let $\log[\varpi]$ be algebraic over C_{cris} and $c_0 + c_1 X + \cdots + c_{d-1}X^{d-1} + X^d$ be its minimal polynomial. For $g \in G_{K_0}$, we have $g([\varpi]/p) = ([\varpi]/p)[\varepsilon]^{\chi(g)}$ where χ is the cyclotomic character. Therefore, $g \log[\varpi] = \log[\varpi] + \chi(g)t$. C_{cris} is stable under the action of G_{K_0}. So for $g \in G_{K_0}$, $g(c_0 + c_1(\log[\varpi]) + \cdots + c_{d-1}(\log[\varpi])^{d-1} + (\log[\varpi])^d) = 0$. By uniqueness of the minimal polynomial for $\log[\varpi]$ and comparing the coefficients of $(\log[\varpi])^{d-1}$ in the expression above and the minimal polynomial assumed in the beginning, we get $c_{d-1} = g(c_{d-1}) + d \cdot \chi(g)t$. Let $c = g(c_{d-1}) + d \cdot \log[\varpi]$. Then $g(c) = g(c_{d-1}) + d \cdot g(\log[\varpi]) = g(c_{d-1}) + d(\log[\varpi] + \chi(g)t) = g(c_{d-1}) + d \cdot \chi(g)t + d \cdot \log[\varpi] = c_{d-1} + d \cdot \log[\varpi] = c$. Since $B_{\mathrm{dR}}^{G_{K_0}} = K_0$, we have that $c \in K_0 \subset B_{\mathrm{cris}}$. Therefore, $\log[\varpi] = (c - c_{d-1})/d \in C_{\mathrm{cris}}$ contradicting Lemma 1.48. Hence $\log[\varpi]$ is transcendental over C_{cris}. ■

As a consequence of this proposition we get,

Theorem 1.50. *The homomorphism of B_{cris}-algebras*

$$\begin{aligned} B_{\mathrm{cris}}[x] &\longrightarrow B_{\mathrm{st}} \\ x &\longmapsto \log[\varpi] \end{aligned}$$

is an isomorphism.

Theorem 1.51. (i) *The G_K-equivariant map $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ sending $\lambda \otimes b \mapsto \lambda b$ is injective.*

(ii) *$(C_{\mathrm{st}})^{G_K} = K_0$ and therefore $(B^+_{\mathrm{cris}})^{G_K} = (B_{\mathrm{cris}})^{G_K} = (B_{\mathrm{st}})^{G_K} = K_0$.*

Proof. (i) Frac $K \otimes_{K_0} B_{\mathrm{cris}}$ is a finite extension over C_{cris}. So $\log[\varpi]$ is transcendental over Frac $K \otimes_{K_0} B_{\mathrm{cris}}$. Therefore $K \otimes_{K_0} B_{\mathrm{st}} = K \otimes_{K_0} B_{\mathrm{cris}}[\log[\varpi]] = (K \otimes_{K_0} B_{\mathrm{cris}})[\log[\varpi]]$. Injection of the map immediately follows from this since $K \otimes_{K_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ from Theorem 1.40 and also $\log[\varpi] \in B_{\mathrm{dR}}$ but $\log[\varpi] \notin B_{\mathrm{cris}}$.

(ii) We know that $W(R)^{G_K} = W(R^{G_K}) = W(k)$, $(W(R)[1/p])^{G_K} = K_0 = W(k)[1/p]$ and $W(R)[1/p] \subset B^+_{\mathrm{cris}}$. So, $K_0 \subset (B^+_{\mathrm{cris}})^{G_K} \subset (B_{\mathrm{cris}})^{G_K} \subset (B_{\mathrm{st}})^{G_K} \subset (C_{\mathrm{st}})^{G_K} \subset (B_{\mathrm{dR}})^{G_K} = K$. Since $K_0 \subset B_{\mathrm{st}}^{G_K}$, from (i) the injection $K \otimes_{K_0} B_{\mathrm{st}}^{G_K} \to B_{\mathrm{dR}}^{G_K} = K$ is possible if and only if $B_{\mathrm{st}}^{G_K} = K_0$. Now the rest of the equalities are obvious.

∎

The operators φ and N on B_{st}

Since φ is injective on B_{cris} by Theorem 1.43, we can canonically extend it to an endomorphism of B_{st} by setting $\varphi(\log[\varpi]) = p \log[\varpi]$. From this it is immediate that φ commutes with the action of G_K.

Definition 1.52. The monodromy operator

$$\begin{aligned} N : B_{\mathrm{st}} &\longrightarrow B_{\mathrm{st}} \\ \sum_{k=0}^{n} b_k (\log[\varpi])^k &\longmapsto \sum_{k=1}^{n} k b_k (\log[\varpi])^{k-1} \end{aligned}$$

is the unique derivation such that $N(\log[\varpi]) = 1$.

Remark 1.53. Setting $N(\log[\varpi]) = 1$ is a matter of convention and adopted by Fontaine in [FO08]; some authors choose the convention to be $N(\log[\varpi]) = -1$.

From Theorem 1.50 we can now write,

Proposition 1.54. *The sequence*

$$0 \longrightarrow B_{\mathrm{cris}} \longrightarrow B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \longrightarrow 0$$

is exact.

Proposition 1.55. *The monodromy operator N satisfies*

(i) *$gN = Ng$ for all $g \in G_{K_0}$.*

(ii) *$N\varphi = p\varphi N$.*

Proof. Let $b \in B_{\mathrm{st}}, b \neq 0$ then we can write $b = \sum_{0 \le k \le n} b_k (\log[\varpi])^k$.

(i) Let $g \in G_{K_0}$. On one hand we have

$$g(N(b)) = g(\sum_{k=1}^{n} kb_k(\log[\varpi])^{k-1}) = \sum_{k=1}^{n} kb_k(\log[\varpi] + \chi(g)t)^{k-1},$$

while on the other hand

$$N(g(b)) = N(\sum_{k=0}^{n} b_k(\log[\varpi] + \chi(g)t)^k) = \sum_{k=1}^{n} kb_k(\log[\varpi] + \chi(g)t)^{k-1}.$$

Therefore $gN = Ng$.

(ii) In this case

$$N(\varphi(b)) = N(\sum_{k=0}^{n} \varphi(b_k)(p\log[\varpi])^k) = \sum_{k=1}^{n} kp^k\varphi(b_k)(\log[\varpi])^{k-1},$$

whereas

$$\varphi(N(b)) = \varphi(\sum_{k=1}^{n} kb_k(\log[\varpi])^{k-1}) = \sum_{k=1}^{n} k\varphi(b_k)(p\log[\varpi])^{k-1}.$$

Just by comparing the two expressions we get $N\varphi = p\varphi N$.

■

Remark 1.56. In [BC09, §9.2] we find an abstract construction of B_{st} from B_{cris}. To get the injection of B_{cris}-algebras $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ they make some choices. It has been made clear that while the injective map may depend on some choices, the image of B_{st} inside B_{dR} and thereby the filtration structure on B_{st} is not dependent on choices once we make the convention $\log_p(p) = 0$. So the construction yields the same result as ours.

Chapter 2

Filtered (φ, N)-modules

In the next chapter we will study construction of functors D_{dR}, $\mathrm{D}_{\mathrm{cris}}$ and D_{st}, respectively from the category of de Rham, crystalline and semistable p-adic Galois representations. These functors would give us certain linear algebra objects which we discuss below.

2.1 Category of filtered vector spaces

Let K be any field.

Definition 2.1. Fil_K is defined as the category of finite dimensional K-vector spaces D equipped with a decreasing filtration indexed by $\mathbb{Z}$ which is exhaustive and separated. This means

(i) $\mathrm{Fil}^i D$ are sub K-vector spaces of D;

(ii) $\mathrm{Fil}^{i+1} \subset \mathrm{Fil}^i D$;

(iii) $\mathrm{Fil}^i D = 0$ for $i \gg 0$ and $\mathrm{Fil}^i D = D$ for $i \ll 0$.

Morphism: For $D_1, D_2 \in \mathrm{Fil}_K$, a morphism between them $f : D_1 \to D_2$ is a K-linear map such that $f(\mathrm{Fil}^i D_1) \subset \mathrm{Fil}^i D_2$ for every $i \in \mathbb{Z}$.

In the category Fil_K there are good functorial notions of kernel and cokernel of a map $f : D_1 \to D_2$ between objects, namely the usual K-linear kernel and cokernel endowed respectively with the *subspace filtration* $\mathrm{Fil}^i(\mathit{ker}\ f) := \mathit{ker}\ f \cap \mathrm{Fil}^i D_1 \subset \mathit{ker}\ f$ and the *quotient filtration* $\mathrm{Fil}^i(\mathit{coker}\ f) := (\mathrm{Fil}^i D_1 \cap f(D_1))/f(D_1) \subset \mathit{coker}\ f$. These have the expected universal properties but one should be careful that Fil_K is an *additive* category but not an *abelian* category.
In Fil_K we have following three objects,

(a) If D_1 and D_2 are two objects in Fil_K, we define $D_1 \otimes D_2$ with

   (1) $D_1 \otimes D_2 = D_1 \otimes_K D_2$ as K-vector spaces;

   (2) $\mathrm{Fil}^i(D_1 \otimes D_2) = \sum\limits_{i_1+i_2=i} \mathrm{Fil}^{i_1} D_1 \otimes_K \mathrm{Fil}^{i_2} D_2$.

(b) The unit object $K[0]$ is K as a vector space with $\mathrm{Fil}^i K[0] = K$ for $i \leq 0$ and $\mathrm{Fil}^i K[0] = 0$ for $i > 0$. Canonically, $D \otimes K[0] \simeq K[0] \otimes D \simeq D$ in Fil_K for all D.

(c) The dual object $D^\vee$ of $D \in \mathrm{Fil}_K$ is defined as $D^\vee = \mathrm{Hom}_K(D, K)$ as a K-vector space with $\mathrm{Fil}^i D^\vee = (\mathrm{Fil}^{-i+1} D)^\perp = \{f : D \to K \mid f(x) = 0 \text{ for every } x \in \mathrm{Fil}^{-i+1} D\}$. The reason we use $\mathrm{Fil}^{1-i} D$ rather than $\mathrm{Fil}^{-i} D$ is to ensure that $K[0]^\vee = K[0]$.

Example 2.2. The unit object $K[0]$ is naturally self-dual in Fil_K, and there is a natural isomorphism $D_1^\vee \otimes D_2^\vee \simeq (D_1 \otimes D_2)^\vee$ in Fil_K induced by the usual K-linear isomorphism. Likewise we have the usual double-duality isomorphism $D \simeq D^{\vee\vee}$ in Fil_K and the evaluation isomorphism $D \otimes D^\vee \to K[0]$ is a map in Fil_K.

For a map $f : D_1 \to D_2$ in Fil_K there are two notions of "image" that are generally distinct in Fil_K but have the same underlying space. We define the *image* of f to be $f(D_1) \subset D_2$ with the subspace filtration from D_2; define the *coimage* of f to be $f(D_1)$ with the quotient filtration from D_2. Equivalently, $coim\ f = D_2/ker\ f$ with the quotient filtration and $im\ f = ker\ (D_2 \to coker\ f)$ with the subspace filtration. There is a canonical map $coim\ f \to im\ f$ in Fil_K that is a linear bijection, and it is generally not an isomorphism in Fil_K.

Definition 2.3. A morphism $f : D_1 \to D_2$ in Fil_K is *strict* if the canonical map $coim\ f \to im\ f$ is an isomorphism, which is to say that the quotient filtration and the subspace filtration on $f(D_1)$ coincide.

Definition 2.4. A short exact sequence in Fil_K is a sequence $0 \to D' \xrightarrow{\alpha} D \xrightarrow{\beta} D'' \to 0$ such that

(i) α and β are strict morphisms;

(ii) α is injective, β is surjective and $\alpha(D') = \{x \in D \mid \beta(x) = 0\}$.

Remark 2.5. There is a natural functor $\mathrm{gr} = \mathrm{gr}^\bullet : \mathrm{Fil}_K \to \mathrm{Gr}_{K,f}$ to the category of finite-dimensional graded K-vector spaces via $\mathrm{gr}(D) = \oplus_{i\in\mathbb{Z}} \mathrm{Fil}^i D/\mathrm{Fil}^{i+1} D$. This functor is dimension preserving, and it is exact in the sense that it carries short exact sequences in Fil_K to short exact sequences in $\mathrm{Gr}_{K,f}$. By choosing bases compatible with filtrations we can see that the functor gr is compatible with tensor products in the sense that there is a natural isomorphism $\mathrm{gr}(D_1) \otimes \mathrm{gr}(D_2) \simeq \mathrm{gr}(D_1 \otimes D_2)$ in $\mathrm{Gr}_{K,f}$ for any $D_1, D_2 \in \mathrm{Fil}_K$, using the tensor product grading on the left side and the tensor product filtration on $D_1 \otimes D_2$ on the right side.

2.2 Category of (φ, N)-modules

Definition 2.6. A (φ, N)-*module* over K_0 (or equivalently over k) is a K_0-vector space D equipped with two maps
$$\varphi, N : D \to D$$
with following properties:

(i) φ is injective and semilinear with respect to the absolute Frobenius σ on K_0;

(ii) N is a K_0-linear map, called the *monodromy* map;

(iii) $N\varphi = p\varphi N$.

Remark 2.7. The map $\varphi : D \to D$ is additive and $\varphi(\lambda d) = \sigma(\lambda)\varphi(d)$ for every $\lambda \in K_0$ and $d \in D$.

In particular, a (φ, N)-module over K_0 is a K_0-vector space. We consider the module $D_\varphi = K_0\ {}_\varphi\!\otimes_{K_0} D$ where K_0 is viewed as a K_0-module by the Frobenius $\sigma : K_0 \to K_0$. More explicitly, it means that for any $\lambda, \mu \in K_0$ and $x \in D$
$$\begin{aligned} \lambda(\mu \otimes x) &= \lambda\mu \otimes x \text{ and,} \\ \lambda \otimes \mu x &= \sigma(\mu)\lambda \otimes x. \end{aligned}$$

D_φ is a K_0-vector space, and if $\{e_1, \dots, e_d\}$ is a basis of D over K_0, then $\{1 \otimes e_1, \dots, 1 \otimes e_d\}$ is a basis of D_φ over K_0. Hence we have, $\dim_{K_0} D_\varphi = \dim_{K_0} D$. Also, giving a semilinear map $\varphi : D \to D$ is equivalent to giving a linear map
$$\begin{aligned} \Phi : D_\varphi &\longrightarrow D \\ \lambda \otimes x &\mapsto \lambda\varphi(x). \end{aligned}$$

This is indeed the case; looking at the composition of maps below makes it clear that equality holds in each column of the last two rows,

$$\begin{array}{lclcl}
K_0 \otimes_{K_0} D & \xrightarrow{\sigma\otimes id} & K_0 \ {}_{\varphi}\!\otimes_{K_0} D & \xrightarrow{\Phi} & D \\
\lambda \otimes x & \longmapsto & \sigma(\lambda)\otimes x & \longmapsto & \sigma(\lambda)\varphi(x) \\
\mu\lambda \otimes x & \longmapsto & \sigma(\mu)\sigma(\lambda)\otimes x & \longmapsto & \sigma(\mu)\sigma(\lambda)\varphi(x) \\
\lambda \otimes \mu x & \longmapsto & \sigma(\lambda)\otimes \mu x & \longmapsto & \sigma(\lambda)\sigma(\mu)\varphi(x).
\end{array}$$

Morphism: A morphism $f : D_1 \to D_2$ between two (φ, N)-modules is a K_0-linear map commuting with φ and N, i.e., if D_1, D_2 are two such modules with respective Frobenius and monodromy maps given by φ_1, N_1 and φ_2, N_2, then we must have $f\varphi_1 = \varphi_2 f$ and $fN_1 = N_2 f$.

Remark 2.8. The category of (φ, N)-modules is the category of left modules over the non-commutative ring generated by K_0, φ and N with relations given by

(i) For each $\lambda \in K_0$, $\varphi\lambda = \sigma(\lambda)\varphi$ and $N\lambda = \lambda N$;

(ii) $N\varphi = p\varphi N$.

Analogous to the category of left modules over a commutative ring with unity, we can define the following objects in the category of (φ, N)-modules.

1. *Tensor Product*: There is a tensor product in this category given by $D_1 \otimes D_2 = D_1 \otimes_{K_0} D_2$ as K_0-vector spaces. The Frobenius semilinear structure is given by $\varphi(d_1 \otimes d_2) = \varphi_1 d_1 \otimes \varphi_2 d_2$ and, the monodromy structure is defined as $N(d_1 \otimes d_2) = N_1 d_1 \otimes d_2 + d_1 \otimes N_2 d_2$ for all $d_1 \in D_1, d_2 \in D_2$.

2. *Unit Object*: K_0 has a structure of (φ, N)-module with $\varphi = \sigma$ and $N = 0$. Moreover for any such (φ, N)-module D, $K_0 \otimes D = D \otimes K_0 = D$.

3. *Dual Object*: Assume that φ is bijective on D and $\dim_{K_0} D < +\infty$. We may define the dual object in this category as $D^\vee = \mathrm{Hom}(D, K_0)$, the set of K_0-linear maps $\eta : D \to K_0$ such that $\varphi^\vee = \sigma \circ \eta \circ \varphi^{-1}$ and $N^\vee = -\eta \circ N$.

The definition of $\varphi^\vee$ and $N^\vee$ above seems rather strange and we would like to give some motivation for this. In the category of (φ, N)-modules, it is natural to expect the usual Tensor-Hom adjunction. A special case of that would be to expect the following bijection of sets where the Hom-sets are homomorphisms are taken to be that of (φ, N)-modules and not just K_0-vector spaces,

$$\mathrm{Hom}(D^\vee \otimes D, K_0) \xrightarrow{\simeq} \mathrm{Hom}(D^\vee, \mathrm{Hom}(D, K_0)) \simeq \mathrm{Hom}(D^\vee, D^\vee).$$

Since $D^\vee \otimes D$ is a (φ, N)-module and therefore for any $f \in \mathrm{Hom}(D^\vee \otimes D, K_0)$ such that $f : D^\vee \otimes D \to K_0$ with $f(\eta \otimes d) = \eta(d)$ we must have, $f \circ \tilde{\varphi}(\eta \otimes d) = \sigma \circ f(\eta \otimes d)$ where $\tilde{\varphi}$ and σ are respective semilinear maps of $D^\vee \otimes D$ and K_0. Therefore $\varphi^\vee \circ \eta \circ \varphi(d) = \sigma \circ \eta(d)$ and since φ is bijective on D, $\varphi^\vee \circ \eta(d) = \sigma \circ \eta \circ \varphi^{-1}(d)$ i.e., $\varphi^\vee(\eta) = \sigma \circ \eta \circ \varphi^{-1}$ which matches with our definition above.

Similarly, for $N^\vee$ notice that $N_{K_0} = 0$ for K_0, i.e., $\tilde{N} \circ f(\eta \otimes d) = N_{K_0} \circ \eta(d) = 0$ for all $\eta \in D^\vee$ and $d \in D$. Here $\tilde{N}$ is the monodromy map of $D^\vee \otimes D$. Also, $\tilde{N} \circ f(\eta \otimes d) = f \circ \tilde{N}(\eta \otimes d) = f(N^\vee(\eta) \otimes d + \eta \otimes N(d)) = N^\vee(\eta)(d) + \eta \circ N(d)$ for all $\eta \in D^\vee$ and $d \in D$. From both these equalities we recover, $N^\vee(\eta) = -\eta \circ N$. Please note that this is not a formal argument in any way. The definition of $\varphi^\vee$ and $N^\vee$ have been rigged so that we have bijection of sets as mentioned above.

Remark 2.9. In the argument above, we were working under the assumption that φ is bijective on D and $\dim_{K_0} D < +\infty$. It is obvious that these objects form a full-subcategory of the category of (φ, N)-module over K_0. Also, this subcategory is an *abelian category* which is stable under tensor product.

2.3 Category of filtered (φ, N)-modules

In previous sections we have studied two different categories with objects having respective structures filtration and (φ, N), i.e., Frobenius and monodromy; here we combine both these structures on the objects and define a new category of filtered (φ, N)-modules.

Definition 2.10. A *filtered* (φ, N)*-module* over K consists of a (φ, N)-module D over K_0 and a filtration on the K-vector space $D_K = K \otimes_{K_0} D$ such that for any $i \in \mathbb{Z}$, $\mathrm{Fil}^i D_K$ the sub K-vector spaces of D_K satisfy

(i) $\mathrm{Fil}^{i+1} D_K \subset \mathrm{Fil}^i D_K$ (decreasing);

(ii) $\cap_{i\in\mathbb{Z}} \mathrm{Fil}^i D_K = 0$ (separated);

(iii) $\cup_{i\in\mathbb{Z}} \mathrm{Fil}^i D_K = D_K$ (exhaustive).

Morphism: A morphism $f : D_1 \to D_2$ of filtered (φ, N)-modules is a morphism of (φ, N)-modules such that the induced K-linear map $f_K : K \otimes D_1 \to K \otimes_{K_0} D_2$ satisfies $f_K(\mathrm{Fil}^i D_{1K}) \subset \mathrm{Fil}^i D_{2K}$ for every $i \in \mathbb{Z}$.

The filtered (φ, N)-modules over K form a category which we denote as $\mathrm{MF}_K(\varphi, N)$. The full subcategory of objects for which $N = 0$ is denoted by $\mathrm{MF}_K(\varphi)$. $\mathrm{MF}_K(\varphi, N)$ and $\mathrm{MF}_K(\varphi)$ are additive categories but *not* abelian. This is illustrated with help of the following example.

Example 2.11. [Sta18, Example 0108] Let $D_1, D_2 \in \mathrm{MF}_K(\varphi, N)$ with $D_1 = D_2 = K_0 = K$ as K-vector spaces and

$$\mathrm{Fil}^i D_1 = \begin{cases} D_1, & \text{if} \quad i < 0 \\ 0, & \text{if} \quad i \geq 0 \end{cases} \quad \text{and} \quad \mathrm{Fil}^i D_2 = \begin{cases} D_2, & \text{if} \quad i \leq 0 \\ 0, & \text{if} \quad i > 0. \end{cases}$$

Consider the map $id_K : D_1 \to D_2$ on the underlying vector spaces. Set $f := id_K$ and observe that f has trivial kernel, cokernel and $f(\mathrm{Fil}^i D_1) \subset \mathrm{Fil}^i D_2$ for all $i \in \mathbb{Z}$ but f is not an ismorphism. Also $coim\ f = D_1$ while $im\ f = D_2$ i.e., $coim\ f \not\simeq im\ f$. As a consequence, $\mathrm{MF}_K(\varphi, N)$ is not an abelian category.

Analogous to the category of (φ, N)-module over K_0, we can define following objects in $\mathrm{MF}_K(\varphi, N)$.

1. *Tensor Product*: For (φ, N)-module D_1, D_2 we have $D_1 \otimes D_2 = D_1 \otimes_{K_0} D_2$ with φ and N as before and the filtration on $(D_1 \otimes D_2)_K = K \otimes_{K_0} (D_1 \otimes D_2) = (K \otimes_{K_0} D_1) \otimes_K (K \otimes_{K_0} D_2) = D_{1K} \otimes_K D_{2K}$ given by
$$\mathrm{Fil}^i (D_{1K} \otimes_K D_{2K}) = \sum_{i_1 + i_2 = i} \mathrm{Fil}^{i_1} D_{1K} \otimes_K \mathrm{Fil}^{i_2} D_{2K}.$$

2. *Unit Object*: $K_0[0]$ can be viewed as a filtered (φ, N)-module with the underlying space K_0, $\varphi = \sigma$, $N = 0$ and $(K_0[0])_K = K \otimes_{K_0} K_0 = K$ with filtration
$$\mathrm{Fil}^i K = \begin{cases} K, & \text{if} \quad i \leq 0 \\ 0, & \text{if} \quad i > 0. \end{cases}$$
Then for any filtered (φ, N)-module D, we have $K_0[0] \otimes D \simeq D \otimes K_0[0] \simeq D$.

3. *Dual Object*: Assume that φ is bijective on D and $\dim_{K_0} D < +\infty$. We define the dual object $D^\vee$ of D by $(D^\vee)_K = K \otimes D^\vee = (D_K)^\vee \simeq \mathrm{Hom}(D_K, K)$ and $\mathrm{Fil}^i (D^\vee)_K = (\mathrm{Fil}^{-i+1} D_K)^\perp$.

2.4 Newton and Hodge Numbers

Let D be a (φ, N)-module over K_0 such that $\dim_{K_0} D < +\infty$ and φ is bijective on D. We associate an integer $t_N(D)$, the *Newton number* to D.

Definition 2.12. If D is a (φ, N)-module over K_0 of dimension 1 such that φ is bijective, then set

$$t_N(D) := v_p(\lambda)$$

where $\lambda \in \mathrm{GL}_1(K_0) = K_0^\times$ is the matrix of φ under some basis.

We should check that this number is well-defined and does not depend on the choice of λ. Indeed, if $\dim_{K_0} = 1$, then $D = K_0 d$ for some $0 \neq d \in D$ with $\varphi(d) = \lambda d$ for some $\lambda \in K_0$. φ is bijective and therefore, $\lambda \neq 0$. Let $d' = ad$ with a nonzero and $a \in K_0, d' \in D$ such that $\varphi(d') = \lambda' d'$ with $\lambda' \in K_0$. Observe that $\varphi(d') = \varphi(ad) = \sigma(a)\varphi(d) = \sigma(a)\lambda d = (\sigma(a)/a)\lambda d'$ i.e., $\lambda' = (\sigma(a)/a)\lambda$. As $\sigma : K_0 \to K_0$ is an automorphism, $v(\lambda) = v(\lambda') \in \mathbb{Z}$ is independent of the choice of basis for D. Hence, $t_N(D)$ is well-defined in case $\dim_{K_0} D = 1$.

Definition 2.13. If D is a (φ, N)-module over K_0 of dimension h such that φ is bijective, then let $\{e_1, \dots, e_h\}$ be a basis of D over K_0, such that $\varphi(e_i) = \sum_{1 \le j \le h} a_{ij} e_j$. Set $A = (a_{ij})_{1 \le i,j \le h}$. Then

$$t_N(D) := v_p(\det A).$$

Again, we need to check that $t_N(D)$ is well-defined. Let $\{e'_1, \dots, e'_h\}$ be another basis for D. Write $\varphi(e'_i) = \sum_{1 \le j \le h} a'_{ij} e'_j$ with $e_k = \sum_{1 \le i \le h} p_{ki} e'_i$ and $e'_j = \sum_{1 \le l \le h} q_{jl} e_l$. Set $P = (p_{ki})_{1 \le i,k \le h}, A' = (a'_{ij})_{1 \le i,j \le h}$ and $Q = (q_{jl})_{1 \le j,l \le h}$. Obviously, $Q = P^{-1}$. Now,

$$\begin{aligned}
\varphi(e_k) = \varphi\Big(\sum_{i=1}^{h} p_{ki} e'_i\Big) &= \sum_{i=1}^{h} \sigma(p_{ki})\varphi(e'_i) = \sum_{i=1}^{h}\sum_{j=1}^{h} \sigma(p_{ki}) a'_{ij} e'_j \\
&= \sum_{i=1}^{h}\sum_{j=1}^{h}\sum_{k=1}^{h} \sigma(p_{ki}) a'_{ij} q_{jl} e_l = \sum_{i=1}^{h}\sum_{j=1}^{h}\sum_{k=1}^{h} (\sigma(P)A'Q)_{kl} e_l.
\end{aligned}$$

So $A = \sigma(P)A'P^{-1}$ and therefore $v_p(\det A) = v_p(\det \sigma(P)A'P^{-1}) = v_p(\det \sigma(P)/\det P) + v_p(\det A') = v_p(\sigma(\det P)/\det P) + v_p(\det A') = v_p(\det A')$ since $\sigma(x)/x$ is a unit in $W(k) = \mathcal{O}_{K_0}$ for all $x \neq 0$.

Now we give an alternative characterization of the Newton numbers for a (φ, N)-module via its top exterior power. Let D is a (φ, N)-module with $\dim_{K_0} = h$, then $\wedge^h D$ is a one-dimensional K_0-vector space. Moreover, if φ is bijective over D then it is bijective over $\wedge^h D$. If $\{e_1, \dots, e_n\}$ a basis for D, then $e_1 \wedge \cdots \wedge e_n$ is a basis for $\wedge^h D$. Writing $\varphi(e_i) = \sum_{1 \le j \le h} a_{ij}$ for each $i \in \{1, \dots, h\}$ as before and setting $A = (a_{ij})_{1 \le i,j \le h}$ gives $\varphi(e_1 \wedge \cdots \wedge e_n) = (\det A) e_1 \wedge \cdots \wedge e_n$. Therefore we have our next definition.

Definition 2.14. If D is a (φ, N)-module over K_0 of dimension h such that φ is bijective over D, then set

$$t_N(D) := t_N(\wedge^h D).$$

From the discussion above it is clear that both definitions for the Newton number coincide. Now we study certain properties that Newton numbers satisfy.

Proposition 2.15. (i) *Given a short exact sequence of (φ, N)-modules*

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

we have $t_N(D) = t_N(D') + t_N(D'')$.

(ii) *Let D_1, D_2 be (φ, N)-modules , then*

$$t_N(D_1 \otimes D_2) = \dim_{K_0}(D_2) \cdot t_N(D_1) + \dim_{K_0}(D_1) \cdot t_N(D_2).$$

In particular, $t_N(D_1^{\otimes r}) = r \cdot \dim_{K_0}(D)^{r-1} t_N(D)$ for each $r \geq 1$.

(iii) *If D is a (φ, N)-module, then $t_N(D^\vee) = -t_N(D)$.*

Proof. (i) Let $\{e_1, \ldots, e_{h'}\}$ be a basis for D', and let A be the matrix for $\varphi_{D'}$ under this basis. Extend this to a basis $\{e_1, \ldots, e_{h'}, e_{h'+1}, \ldots, e_h\}$ for D. Therefore, we will have $\{\overline{e_{h'+1}}, \ldots, \overline{e_h}\}$ as a basis for D''. Let B be the matrix for $\varphi_{D''}$ under this basis. Then the matrix for φ_D is given as

$$C = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}.$$

Clearly, $\det C = \det A \cdot \det B$ which gives, $t_N(D) = \upsilon_p(\det C) = \upsilon_p(\det A) + \upsilon_p(\det B) = t_N(D') + t_N(D'')$.

(ii) Let $\{e_i\}_{1 \leq i \leq h_1}$(resp. $\{f_j\}_{1 \leq j \leq h_2}$) be a basis for D_1 (resp. D_2) and let A (resp. B) be the matrix for φ_{D_1} (resp. φ_{D_2}) under this basis. Clearly, $\{e_i \otimes f_j\}_{1 \leq i \leq h_1, 1 \leq j \leq h_2}$ is a basis for $D_1 \otimes D_2$ and the matrix C for $\varphi_{D_1 \otimes D_2}$ is given by $C = A \otimes B$. Since $\det C = (\det A)^{\dim_{K_0} D_2} \cdot (\det B)^{\dim_{K_0} D_1}$, therefore $t_N(D_1 \otimes D_2) = \upsilon_p(\det C) = \upsilon_p((\det A)^{\dim_{K_0} D_2}) + \upsilon_p((\det B)^{\dim_{K_0} D_1}) = \dim_{K_0} D_2 \cdot t_N(D_1) + \dim_{K_0} D_1 \cdot t_N(D_2)$. The conclusion for $t_N(D^{\otimes r})$ follows immediately.

(iii) If the matrix for φ_D under the basis $\{e_i\}_{1 \leq i \leq h_1}$ for D is given by A, then the matrix for $\varphi^\vee$ under the basis $\{e_i^\vee\}_{1 \leq i \leq h_1}$ for $D^\vee$ is given by $\sigma(A^{-1})$. Clearly, $t_N(D^\vee) = \upsilon_p(\det \sigma(A^{-1})) = \upsilon_p(\det A^{-1}) = -\upsilon_p(\det A) = -t_N(D)$.

■

Next we discuss a classification theorem of Dieudonné-Manin [Man63] which would provide us with a more concrete way of looking at Newton numbers.

Definition 2.16. Let k be a perfect field of characteristic $p > 0$, and let $\sigma : W(k) \simeq W(k)$ be the Frobenius automorphism lifting the p-power map on k. The *Dieudonné ring* of k is the associative ring $\mathscr{D}_k = W(k)[\mathscr{F}, \mathscr{V}]$ subject to the relations $\mathscr{F}\mathscr{V} = \mathscr{V}\mathscr{F} = p$, $\mathscr{F}c = \sigma(c)\mathscr{F}$, and $c\mathscr{V} = \mathscr{V}\sigma(c)$ for $c \in W(k)$.

This ring is a non-commutative ring when $k \neq \mathbb{F}_p$ and is $\mathbb{Z}_p[xy]/(xy - p)$ when $k = \mathbb{F}_p$. Also, $\mathscr{D}_k[1/p]$ has a much simpler structure than $\mathscr{D}_k$: if we let $K_0 = W(k)[1/p]$ then $\mathscr{D}_k[1/p]$ is the twisted polynomial ring $K_0[\mathscr{F}]$ in a variable $\mathscr{F}$ satisfying the commutation relation $\mathscr{F}c = \sigma(c)\mathscr{F}$ for all $c \in K_0$. Moreover, we observe that a left $\mathscr{D}_k$-module is the same thing as a $W(k)$-module D equipped with a σ-semilinear endomorphism $\mathscr{F} : D \to D$ and a σ^{-1}-semilinear endomorphism $\mathscr{V} : D \to D$ such that $\mathscr{F}\mathscr{V} = \mathscr{V}\mathscr{F} = [p]_D$.

Definition 2.17. An *isocrystal* over K_0 is a finite-dimensional K_0-vector space D equipped with a bijective Frobenius-semilinear endomorphism $\varphi_D : D \to D$.

The abelian category of isocrystals over K_0 is denoted $\mathrm{Mod}_{K_0}(\varphi)$, with evident notions of tensor product and dual.

Example 2.18. Let $K_0[\varphi] = \mathscr{D}_k[1/p]$ (with $\varphi = \mathscr{F}$ from Definition 2.16) be the twisted polynomial ring satisfying $\varphi c = \sigma(c)\varphi$ for $c \in K_0$. A class of isocrystals over K_0 is given by the quotients

$$D_{r,s} = K_0[\varphi]/(K_0[\varphi](\varphi^r - p^s))$$

for any integers $r > 0$ and s (possibly < 0). The Frobenius structure on $D_{r,s}$ is defined by left multiplication by φ. By a "divison algorithm" argument we see that $D_{r,s}$ has finite dimension r over K_0, and it is an isocrystal over K_0. Although it does not make sense to speak of eigenvalues for the φ-operator on $D_{r,s}$ when $k \neq \mathbb{F}_p$ (since this operator is just semilinear rather than linear), it is good to imagine that φ should have "eigenvalues" on $D_{r,s}$ that are integral unit multiples of $p^{s/r}$.

Let $\overline{k}$ be an algebraic closure of k. For any isocrystal D over K_0 we get an isocrystal over $\widehat{K_0^{\mathrm{un}}} = W(\overline{k})[1/p]$ by scalar extension: $\widehat{D} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ endowed with the bijective semilinear tensor-product Frobenius structure $\varphi_{\widehat{D}}(c \otimes d) = \sigma(c) \otimes \varphi_D(d)$. The Dieudonné-Manin classification [Man63, II, §4.1] describes the possibilities for $\widehat{D}$. A simpler proof could also be found in [DO12].

Theorem 2.19. *For an algebraically closed field k of characteristic $p > 0$, the category $\mathrm{Mod}_{K_0}(\varphi)$ of isocrystals over $K_0 = W(k)[1/p]$ is semisimple (i.e., all objects are finite direct sums of simple objects and all short exact sequences are split). Moreover, the simple objects are given upto isomorphism (without repetition) by the isocrystals $D_{r,s}$ in Example 2.18 with $gcd(r, s) = 1$.*

This theorem says that if $k = \overline{k}$ then the isomorphism classes of simple isocrystals over K_0 are in natural bijection with $\mathbb{Q}$, where a rational number α expressed uniquely in reduced form s/r with $r > 0$ corresponds to $D_{r,s}$. We use Δ_α to denote $D_{r,s}$; this is called the simple object with *pure slope* α in $\mathrm{Mod}_{K_0}(\varphi)$ (when $k = \overline{k}$).

For any perfect field k with characteristic $p > 0$ and any isocrystal D over $K_0 = W(k)[1/p]$, the Dieudonné-Manin classification provide a unique decomposition of $\widehat{D} := \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ in the form

$$\widehat{D} = \bigoplus_{\alpha \in \mathbb{Q}} \widehat{D}(\alpha) \tag{2.1}$$

for subobjects $\widehat{D}(\alpha) \simeq \Delta_\alpha^{e_\alpha}$ having "pure slope α" (and $\widehat{D}(\alpha) = 0$ for all but finitely many α). For each $\alpha = s/r \in \mathbb{Q}$ in reduced form (with $r > 0$), the integer $\dim_{\widehat{K_0^{\mathrm{un}}}} \widehat{D}(\alpha) = re_\alpha$ is the number (with multiplicity) of "eigenvalues" of φ_D with slope α.

Definition 2.20. The $\alpha \in \mathbb{Q}$ for which $\widehat{D}(\alpha) \neq 0$ are the *slopes* of D, and $\dim_{\widehat{K_0^{\mathrm{un}}}} \widehat{D}(\alpha)$ is called the multiplicity of this slope. We say that D is *isoclinic* (with slope α_0) if $D \neq 0$ and $\widehat{D} = \widehat{D}(\alpha_0)$ for some $\alpha_0 \in \mathbb{Q}$ (i.e., $\widehat{D} \simeq \Delta_{\alpha_0}^e$ for some $e \geq 1$).

We discuss an example where we explicitly determine the slopes for some isocrystal over K_0.

Example 2.21. Let $K_0 = W(\mathbb{F}_{p^2})[1/p]$ with $p \equiv 3 \mod 4$, and let $i = \sqrt{-1} \in K_0$. Let $D = K_0 e_1 \oplus K_0 e_2$ and define $\varphi_D : D \to D$ by the matrix

$$\begin{pmatrix} p-1 & (p+1)i \\ (p+1)i & -(p-1)i \end{pmatrix}.$$

That is we set $\varphi_D(e_1) = (p-1)e_1 + (p+1)ie_2$ and $\varphi_D(e_2) = (p+1)ie_1 - (p-1)e_2$ and extend φ_D uniquely by Frobenius-semilinearity. The characteristic polynomial of the matrix for φ_D above, is therefore $X^2 - 4p$, so its roots are $\pm 2\sqrt{p}$. The p-adic valuation of these roots is $1/2$. However if we make a change of basis to $e_1' = e_1 + ie_2$ and $e_2' = ie_1 + e_2$ then since the Frobenius of K_0 takes i to $-i$ (as $p \equiv 3 \mod 4$), we compute that $\varphi_D(e_1') = 2pe_1'$ and $\varphi_D(e_2') = 2e_2'$. So in this new basis the matrix for φ_D has eigenvalues 2 and $2p$ with respective p-adic valuations 0 and 1.

It is natural to guess that D has slopes $\{0, 1\}$ or the single slope $1/2$ with multiplicity 2. We will check that the first of these two guesses is correct. By using the basis $\{e_1', e_2'\}$ gives us an isomorphism of D with a direct sum of two 1-dimensional object on which Frobenius acts (relative to a suitable basis vector over K_0) via multiplicayion by $2p$ and and 2 respectively. Letting σ denote the absolute Frobenius automorphism of $W(\overline{\mathbb{F}_p})$, the self-map of $W(\overline{\mathbb{F}_p})^\times$ defined by $u \mapsto \sigma(u)/u$ is surjective by Lemma 4.7. In particular, we can find $c \in W(\overline{\mathbb{F}_p})^\times$ such that $\sigma(c)/c = 1/2$, and over $W(\overline{\mathbb{F}_p})[1/p] = \widehat{\mathbb{Q}_p^{\mathrm{un}}}$ we compute that φ fixes ce_2' and multiplies ce_1' by p. Thus, we get an isomorphism $\widehat{\mathbb{Q}_p^{\mathrm{un}}} \otimes_{\mathbb{Q}_p} D \simeq \Delta_1 \oplus \Delta_0$, so the slopes are as claimed.

Although the Dieudonné-Manin classification does not extend to the case when k is not assumed to be algebraically closed, the "slope decomposition" (2.1) into isoclinic parts does uniquely descend [BC09, 8.1.11, pg. 107]

Proposition 2.22. *Let D be an isocrystal over K_0.*

(i) *There exist $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ with $\alpha_i \in \mathbb{Q}$ for $i \in \{1, 2, \ldots, s\}$ called the slopes of φ, and φ-stable K_0-vector subspaces $D(\alpha_i)$ of D such that*

$$D = \bigoplus_{i=1}^{s} D(\alpha_i).$$

Moreover, each $\widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D(\alpha_i)$ has a basis $\{e_1, \ldots, e_m\}$ such that for each $j \in \{1, 2, \ldots, m\}$ there exists $\lambda_j \in \overline{K}$ with $\upsilon_p(\lambda_j) = \alpha$ and $\varphi(e_j) = \lambda_j e_j$.

(ii) $\sum_{i=1}^{s} \alpha_i \cdot \dim_{K_0} D(\alpha_i) = t_N(D)$.

(iii) $\alpha_i \cdot \dim_{K_0} D(\alpha_i) \in \mathbb{Z}$ *for each* $i \in \{1, 2, \ldots, s\}$.

Using the previous result, we can prove some properties for the monodromy operator N.

Proposition 2.23. *If D is a (φ, N)-module such that $\dim_{K_0} < +\infty$ and φ is bijective then*

(i) *N decreases slopes by 1, i.e., $N(D(\alpha)) \subset D(\alpha - 1)$.*

(ii) *N is nilpotent.*

Proof. (i) For $D \in \mathrm{MF}_K(\varphi, N)$, consider the isoclinic decomposition $D = \oplus_{\alpha \in \mathbb{Q}} D(\alpha)$ of the underlying isocrystal. By the definition of $D(\alpha)$, its scalar extension $\widehat{D}(\alpha)$ over $\widehat{K_0^{\mathrm{un}}}$ is spanned by vectors v such that $\varphi^r_{\widehat{D}}(v) = p^s v$ for s/r the reduced form of α, so

$$\varphi^r_{\widehat{D}}(Nv) = p^{-r} N \varphi^r_{\widehat{D}}(v) = p^{s-r} Nv.$$

But $(s-r)/r = \alpha - 1$, so $Nv \in \widehat{D}(\alpha - 1)$. Hence, by descent from $\widehat{K_0^{\mathrm{un}}}$, we get $N(D(\alpha)) \subset D(\alpha - 1)$.

(ii) Let us suppose N is not nilpotent. Since N decreases slope by 1, from (i), let h be an integer such that $N^h(D) = N^m(D)$ for all $m \geq h$. Let $D' = N^h(D)$, so D' is invariant by N. φ is bijective on D and therefore $\varphi(D) = D$. For $m \geq h$, we get $D' = N^m(D) = N^m(\varphi(D)) = p^m \varphi(N^m D) = p^m \varphi(D')$ i.e., D' is invariant under φ. Therefore, φ and N are both surjective on the (φ, N)-module D'. Let us choose a basis for D' and let A and B be the respective matrices for φ and N under this basis. We have the relation $N\varphi = p\varphi N$ which gives $BA = pA\sigma(B)$. Observe that $\upsilon_p(\det BA) = 1 + \upsilon_p(\det A\sigma(B))$ which gives $\upsilon_p(\det B) = 1 + \upsilon_p(\det B)$, a contradiction. Hence N must be nilpotent.

∎

Our next goal is to define Hodge number for filtered vector spaces. We denote by Fil_K, the category of finite dimensional filtered K-vector spaces.

Definition 2.24. Suppose $D \in \mathrm{Fil}_K$ is a finite dimensional K-vector space. If $\dim_K D = 1$, define

$$t_H(D) := \max\{i \in \mathbb{Z} : \mathrm{Fil}^i D = D\}.$$

Thus it is the integer i such that $\mathrm{Fil}^i D = D$ and $\mathrm{Fil}^{i+1} = 0$.

Example 2.25. From Example 2.11, we can easily see that $t_H(D_{1K}) = -1$ while $t_H(D_{2K}) = 0$.

Similar to the Definition 2.14 for Newton number, we define Hodge number for higher dimensions.

Definition 2.26. Let $D \in \mathrm{Fil}_K$ and $\dim_K D = h$, define

$$t_H(D) := t_H(\wedge^h D)$$

where $\wedge^h D$ is the top-exterior power of D equipped with the quotient filtration from $D \otimes D \otimes \cdots \otimes D$ (h times).

Now we want to give an alternative description of $t_H(D)$. For filtration $\mathrm{Fil}^i D$ ($i \in \mathbb{Z}$), we say that there is a jump in the filtration at j-th position if $\mathrm{Fil}^j D \neq \mathrm{Fil}^{j+1} D$. For a given filtration on D let $j_1 < j_2 < \cdots < j_s$ be the jumps. Also, in the filtration of the top exterior power there must only be one jump. Since char $K = 0$ we have that $\wedge^h D \subset \otimes^h D$, and the subspace filtration here coincides with the quotient filtration on the top-exterior power. The filtration on the tensor product is given as

$$\mathrm{Fil}^i(\otimes^h D) = \sum_{i_1+i_2+\cdots+i_h=i} \mathrm{Fil}^{i_1} D \otimes_K \mathrm{Fil}^{i_2} D \otimes_K \cdots \otimes_K \mathrm{Fil}^{i_h} D.$$

Since there is only one jump in the filtration of the top exterior power, we are looking for largest possible choices for $i_1, i_2, \ldots, i_h$ such that $\mathrm{Fil}^{i_1} D \wedge \mathrm{Fil}^{i_2} D \wedge \cdots \wedge \mathrm{Fil}^{i_h} D \neq 0$. Also, since this product is symmetric in all terms, we can arrange i_r's such that $i_1 \leq i_2 \leq \cdots \leq i_h$. Since j_s is the largest index for a jump to occur in the filtration of D, this index could be "assigned" to as many i_r's as possible. We can always choose a basis for $\mathrm{Fil}^{j_s} D$ and extend it all the way down to $\mathrm{Fil}^{j_1} D$. This makes sure that we have exactly h basis vectors in the end. Now, we set $i_h = i_{h-1} = \cdots = i_{h-d} = j_s$ where $d = \dim_K D$. By doing this we are making sure that we have the maximum possible sum for $\sum_{r=1}^{h} i_r$.

Next, we extend the previous basis of $\mathrm{Fil}^{j_s} D$ to a basis for $\mathrm{Fil}^{j_s-1} D$. So we can set $i_{h-d-1} = i_{h-d-2} = \cdots = i_{h-d-m} = j_{s-1}$ where $m + \dim_K \mathrm{Fil}^{j_s} D = \dim_K \mathrm{Fil}^{j_{s-1}} D$ i.e., $m = \dim_K \mathrm{Fil}^{j_{s-1}} D - \dim_K \mathrm{Fil}^{j_{s-1}} D = \dim_K(\mathrm{Fil}^{j_{s-1}} D / \mathrm{Fil}^{j_s} D)$. Also note that $\mathrm{Fil}^{j_{s-1}+1} D = \mathrm{Fil}^{j_s} D$. So we have $\mathrm{gr}^i D = \mathrm{Fil}^i D / \mathrm{Fil}^{i+1} D$, $i \in \mathbb{Z}$ and $\mathrm{gr}^i D \neq 0$ precisely when i is a jump position. We can continue this process of extending the basis all the way down to j_1 and "assigning" values to i_r's. This would be an exhaustive process since there are only finitely many jumps and hence we get the equality of following two finite sums.

$$\sum_{r=1}^{h} i_r = \sum_{d=1}^{d} j_k \cdot \dim_K \mathrm{gr}^{j_k} D = \sum_{i \in \mathbb{Z}} i \cdot \dim_K \mathrm{gr}^i D$$

where the last sum is finite since $\mathrm{gr}^i D = 0$ if $i \notin \{j_1, j_2, \ldots, j_s\}$.

We know that, $t_H(D) = \max\{i \in \mathbb{Z} : \mathrm{Fil}^i \wedge^h D = \wedge^h D\}$. From the discussion above, such an i would be an index such that $\mathrm{Fil}^i(\wedge^h D) \neq 0$ i.e., for largest possible i and $i_1 \leq i_2 \leq \cdots \leq i_h$ such that $\sum_{r=1}^{h} = i_r$. From the above equality of sums, we conclude that

$$t_H(D) = t_H(\wedge^h{}_K D) = \sum_{i \in \mathbb{Z}} i \cdot \dim_K \mathrm{gr}^i D. \tag{2.2}$$

With this alternate description, we prove certain results for Hodge Numbers as in Proposition 2.15

Proposition 2.27. (i) *Given a short exact sequence of filtered K-vector spaces*

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

we have $t_H(D) = t_H(D') + t_H(D'')$.

(ii) *Let $D_1, D_2 \in \mathrm{Fil}_K$, then*

$$t_H(D_1 \otimes D_2) = \dim_K(D_2) \cdot t_H(D_1) + \dim_K(D_1) \cdot t_H(D_2).$$

In particular, $D^{\otimes r} = r \cdot (\dim_K D)^{r-1} t_H(D)$ for each $r \geq 1$.

(iii) *If D is a filtered (φ, N)-module such that $\dim_{K_0} D < +\infty$ and φ is bijective on D, then $t_H(D^\vee) = -t_H(D)$.*

Proof. (i) From the exact sequence we have an isomorphism of K-vector spaces $D'' \simeq D/D'$. The filtration on D' is the subspace filtration from D and the filtration on D'' is the quotient filtration from D. From this we get that, $\mathrm{Fil}^i D'' = \mathrm{Fil}^i D / \mathrm{Fil}^i D'$ for all $i \in \mathbb{Z}$ and therefore $\dim_K \mathrm{gr}^i D'' = \dim_K \mathrm{gr}^i D - \dim_K \mathrm{gr}^i D'$. Thus from (2.2), we get $t_H(D') + t_H(D'') = t_H(D)$.

(ii) We know that $\mathrm{Fil}^i(D_1 \otimes D_2) = \sum_{i_1+i_2=i} \mathrm{Fil}^{i_1} D_1 \otimes_K \mathrm{Fil}^{i_2} D_2$. Let the jump positions for D_1(resp. D_2) be $j_1 < j_2 < \cdots < j_s$ (resp. $m_1 < m_2 < \cdots < m_t$). So the jump positions for $D_1 \otimes_K D_2$ would be $j_1 + m_1 < \cdots < j_\alpha + m_\beta < \cdots < j_s + m_t$. Therefore, we have

$$\begin{aligned} t_H(D_1 \otimes_K D_2) &= \sum_{\alpha=1}^{s} \sum_{\beta=1}^{t} (j_\alpha + m_\beta) \cdot \dim_K \mathrm{gr}^{j_\alpha + m_\beta}(D_1 \otimes_K D_2) \\ &= \sum_{j,m \in \mathbb{Z}} (j+m) \cdot \dim_K \mathrm{gr}^{j+m}(D_1 \otimes_K D_2). \end{aligned}$$

Now we observe a simple fact, $\dim_K \mathrm{gr}^{j+m}(D \otimes_K D_2) = (\dim_K \mathrm{gr}^j D_1) \cdot (\dim_K \mathrm{gr}^m D_2)$. Therefore,

$$\begin{aligned} t_H(D_1 \otimes_K D_2) &= \sum_{j,m,\in \mathbb{Z}} (j+m) \cdot \dim_K \mathrm{gr}^j D_1 \cdot \dim_K \mathrm{gr}^m D_2 \\ &= \sum_{j \in \mathbb{Z}} j \cdot \dim_K \mathrm{gr}^j D_1 \cdot \sum_{m \in \mathbb{Z}} \dim_K \mathrm{gr}^m D_2 + \sum_{m \in \mathbb{Z}} m \cdot \dim_K \mathrm{gr}^m D_2 \cdot \sum_{j \in \mathbb{Z}} \dim_K \mathrm{gr}^j D_1 \\ &= \dim_K(D_2) \cdot t_H(D_1) + \dim_K(D_1) \cdot t_H(D_2). \end{aligned}$$

(iii) From general theory about dual of vector spaces, we know that, $\wedge^h D_K^\vee = (\wedge^h D_K)^\vee$. Therefore by considering 1-dimensional case we have, $t_H(D_K^\vee) = -t_H(D_K)$.

■

2.5 Admissible filtered (φ, N)-modules

Let D be a filtered (φ, N)-module over K_0, we set $t_H(D) = t_H(D_K)$. A subobject D' of D is a K_0-vector subspace stable under φ and N and with filtration given by $\mathrm{Fil}^i D'_K = D'_K \cap \mathrm{Fil}^i D_K$.

Definition 2.28. A filtered (φ, N)-module D over K is called *admissible* if $\dim_{K_0} D < +\infty$, φ is bijective on D and

(i) $t_H(D) = t_N(D)$;

(ii) For any subobject $D' \subset D$, $t_H(D') \leq t_N(D')$.

Remark 2.29. The additivity of t_N and t_H implies that condition (ii) in Definition 2.28 is equivalent to $t_H(D'') \geq t_N(D'')$, for any quotient object D'' of D. Indeed, since there is a 1-1 bijection between subobjects and quotient objects of $D \in \mathrm{MF}_K(\varphi, N)$ where the correspondence is easy to see

$$\begin{aligned} \{\text{subobjects of } D\} &\longleftrightarrow \{\text{quotient objects of } D\} \\ D' &\longmapsto D/D' \\ ker\ \pi &\longleftarrow\!\!\shortmid D'' \end{aligned}$$

where $\pi : D \twoheadrightarrow D''$. Thus we have by additivity of t_N and t_H that, $t_N(D) = t_N(D') + t_N(D/D')$ and $t_H(D) = t_H(D') + t_H(D/D')$ with t_N and t_H being constant for D irrespective of the subobject $D' \subset D$ and the equivalence follows.

We denote by $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ the full subcategory of $\mathrm{MF}_K(\varphi, N)$ consisting of admissible filtered (φ, N)-modules. It turns out that $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ is an *abelian* category. Our next goal is to prove this claim but first we note some results.

Proposition 2.30. *Let $D \in \mathrm{MF}_K(\varphi, N)$. Then D is admissible if and only if $D^\vee$ is admissible.*

Proof. The claim easily follows from the remark made above and the fact that $t_H(D^\vee) = -t_H(D)$ and $t_N(D^\vee) = -t_N(D)$ and the additivity of Newton and Hodge numbers. ■

Recall that Fil_K is the category of finite-dimensional filtered K-vector spaces.

Lemma 2.31. *If* $f : D_1 \to D_2$ *is a bijective morphism in* Fil_K*, then* $t_H(D_1) \le t_H(D_2)$ *with equality if and only if* f *is an isomorphism in* Fil_K *(i.e., it is a* strict *morphism).*

Proof. We know that $t_H(D_1) = t_H(\wedge^{h_1} D_1)$ and $t_H(D_2) = t_H(\wedge^{h_2} D_2)$. It is clear that if f is an isomorphism in Fil_K then the induced map $\hat{f} : \wedge^{h_1} D_1 \to \wedge^{h_2} D_2$ is an isomorphism. Also, since f is bijective, $h_1 = h_2 = h$. Now the claim is that if the induced morphism $\hat{f} : \wedge^h D_1 \to \wedge^h D_2$ is an isomorphism in Fil_K then f should be an isomorphism to begin with. Let us choose a basis $\{d_i\}$ for D_1 such that $e_i := f(d_i)$ for all $1 \le i \le h$ is a basis for D_2. We can also assume that $\mathrm{Fil}^0 D_1 = D_1$ and $\mathrm{Fil}^0 D_2 = D_2$. Let i be the smallest positive integer such that $\mathrm{Fil}^i D_1 \not\simeq \mathrm{Fil}^i D_2$. Since $f(\mathrm{Fil}^i D_1) \subset \mathrm{Fil}^i D_2$ and f is injective, we get $\mathrm{Fil}^i D_1 \hookrightarrow \mathrm{Fil}^i D_2$ is injective but not surjective. Let $\dim_K \mathrm{Fil}^i D_1 = d_1$ and $\dim_K \mathrm{Fil}^i D_2 = d_2$ with $d_1 < d_2$. Now consider $\mathrm{Fil}^{i \cdot d} \wedge^h D_1 = 0$ but $\mathrm{Fil}^{i \cdot d} \wedge^h D_2 \neq 0$, since $\mathrm{Fil}^i D_1$ has less than d vectors in its basis. Therefore, $\wedge^h D_1 \not\simeq \wedge^h D_2$ contratry to our assumption. Hence $f : D_1 \to D_2$ is an isomorphism. Now looking at $f : \wedge^h D_1 \to \wedge^h D_2$, we know that $t_H(\wedge^h D_1) \le t_H(\wedge^h D_2)$ and equality holds if and only if $\hat{f}$ is an isomorphism in Fil_K if and only if f is an isomorphism in Fil_K. ∎

Using the lemma above, we prove the following proposition.

Proposition 2.32. *If* $0 \to D' \to D \to D'' \to 0$ *is a short exact sequence in* $\mathrm{MF}_K(\varphi, N)$ *and any two of the three terms are admissible then so is the third.*

Proof. If D is admissible then for any subobject D'_1 of D', we may view D'_1 as a subobject of D and therefore $t_H(D'_1) \le t_N(D'_1)$. If in addition D'' is admissible, then $t_H(D'') = t_N(D'')$ and therefore $t_H(D') = t_H(D) - t_H(D'') = t_N(D) - t_N(D'') = t_N(D')$. Thus D' is admissible if D and D'' are.
Now let us assume that D' and D are admissible. From the given exact sequence we get that $0 \to D'^\vee \to D^\vee \to D''^\vee \to 0$ is exact. Now $D^\vee$ and $D'^\vee$ are admissible by Proposition 2.30. Now from the conclusion above $D''^\vee$ is admissible and therefore D'' is admissible.

Next suppose that D' and D'' are admissible. From the additivity of t_N and t_H it is clear that $t_N(D) = t_H(D)$. Now we have to show that $t_H(D_1) \le t_N(D_1)$ for any subobject $D_1 \subset D$. Let $D'_1 := D' \cap D_1$ and give $(D'_1)_K$ the subspace filtration from $(D_1)_K$ which coincides with the one from D'_K. Let $D''_1 := D_1/D'_1$ with the quotient filtration from $(D''_1)_K$. Naturally, there is an injection $j : D''_1 \hookrightarrow D'' = D/D'$ in $\mathrm{MF}_K(\varphi, N)$ but a priori it may not be strict (i.e., the quotient filtration from $(D''_1)_K$ may be finer than subspace filtration from D''_K). We know that $t_H(D'_1) \le t_N(D'_1)$ since D'_1 is a subobject of D'. Therefore, $t_H(D_1) = t_H(D'_1) + t_H(D''_1) \le t_N(D'_1) + t_H(D''_1)$. So we will be done if $t_H(D''_1) \le t_N(D''_1)$.

Let $j(D''_1)$ be the image of D''_1 inside D''. $D''_1 \to j(D''_1)$ is an isomorphism in the category $\mathrm{Mod}_{K_0}(\varphi)$ of isocrystals over K_0 (but may not be as filtered spaces). Therefore $t_N(D''_1) = t_N(j(D''_1))$. So we are reduced to proving that $t_H(D''_1) \le t_N(j(D''_1))$. $j(D''_1)$ is also a subobject of D'' and therefore $t_H(j(D''_1)) \le t_N(j(D''_1))$. From Lemma 2.31 for the bijective morphism $j : D''_1 \to j(D''_1) \subset D''$ we have that $t_H(D''_1) \le t_H(D''_1)$ with equality if and only if j is an isomorphism in $\mathrm{MF}_K(\varphi, N)$. Thus we have $t_H(D''_1) \le t_N(j(D''_1))$ and therefore $t_H(D_1) \le t_N(D_1)$. Hence D is admissible and we are done. ∎

Now we are ready to prove that $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ is an abelian category.

Theorem 2.33. *Let* $f : D \to D'$ *be a map in* $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$*. The map* f *is strict and ker* f *(resp. coker* f*) is admissible with the subspace (resp. quotient) filtration. In particular, the object im* $f \simeq$ *coim* f *is admissible and* $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ *is abelian.*

Proof. Consider the maps in $\mathrm{MF}_K(\varphi, N)$

$$\mathit{ker}\ f \hookrightarrow D \twoheadrightarrow \mathit{coim}\ f \longrightarrow \mathit{im}\ f \hookrightarrow D' \twoheadrightarrow \mathit{coker}\ f$$

with *coim* $f := D/\mathit{ker}\ f$ given quotient filtration. *ker* f has subspace filtration from D, *im* f has subspace filtration from D' and *coker* f has quotient filtration from D'. As modules *coim* $f \to$ *im* f is a bijective morphism and by Lemma 2.31 we get $t_H(\mathit{coim}\ f) \le t_H(\mathit{im}\ f)$ with equality if and only if f is a strict morphism (i.e., *coim* $f \simeq$ *im* f).

Admissibility of D implies, $t_N(coim\ f) \leq t_H(coim\ f)$ and admissibility of D' implies, $t_H(im\ f) \leq t_N(im\ f)$. Putting it all together gives us $t_N(coim\ f) \leq t_H(coim\ f) \leq t_H(im\ f) \leq t_N(im\ f)$. But $coim\ f \xrightarrow{\sim} im\ f$ is an isomorphism as (φ, N)-modules i.e., $t_N(coim\ f) = t_N(im\ f)$ and hence $t_H(coim\ f) = t_H(im\ f)$. So we get that f is a strict morphism or equivalently $coim\ f \xrightarrow{\sim} im\ f$ in $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$. From the above argument it is also clear that $t_H(coim\ f) = t_N(coim\ f)$ and since it is a quotient object (of D) or a subobject($\simeq im\ f$ of D'), we conclude that $coim\ f \simeq im\ f$ is admissible. Now, from Proposition 2.32 we have the admissibility of $ker\ f$ and $coker\ f$ because of the exact sequences,

$$0 \longrightarrow ker\ f \longrightarrow D \longrightarrow coim\ f \longrightarrow 0,$$
$$0 \longrightarrow im\ f \longrightarrow D' \longrightarrow coker\ f \longrightarrow 0.$$

■

2.5.1 Tate's Twist

We introduce a twisting operation on filtered (φ, N)-modules that, under the contravariant functors $\mathrm{D}_B^* = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\cdot, B)$ (to be discussed in Chapter 3), corresponds to the operation $V \rightsquigarrow V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(i)$ on the Galois side for $i \in \mathbb{Z}$. Suppose $B \subset B_{\mathrm{dR}}$ is a $\mathbb{Q}_p[G_K]$-subalgebra containing the canonical $\mathbb{Z}_p(1)$ (of which the two most important examples are B_{cris} and B_{st}). For any basis t of $\mathbb{Z}_p(1)$, elements of $D' = \mathrm{Hom}_{\mathbb{Q}_p}(V(i), B)$ can be written as $d' = t^{-i}d$ for $d \in D := \mathrm{Hom}_{\mathbb{Q}_p}(V, B)$, so $d' \in D'$ is G_K-invariant if and only if $d \in D$ is G_K-invariant. Clearly $d' \in \mathrm{Hom}_{\mathbb{Q}_p}(V, t^r B_{\mathrm{dR}}^+)$ if and only if $d \in \mathrm{Hom}_{\mathbb{Q}_p}(V, t^{r+i} B_{\mathrm{dR}}^+)$. Since $t \in B_{\mathrm{cris}}$, where the Frobenius acts on t^{-i} as multiplication by p^{-i}, and $N(t) = 0$, so we give the following definition.

Definition 2.34. Let $D \in \mathrm{MF}_K(\varphi, N)$. For $i \in \mathbb{Z}$, let us define $D\langle i \rangle$ such that $D\langle i \rangle = D$ as a K_0-vector space. Let $\mathrm{Fil}^r(D\langle i \rangle)_K = \mathrm{Fil}^{r+i} D_K$ for every $r \in \mathbb{Z}$ and where N' and φ' on $D\langle i \rangle$ are given by $N' = N$ and $\varphi' = p^{-i}\varphi$. $D\langle i \rangle$ is called the *i-fold Tate twist* of D. Clearly, $D\langle i \rangle \in \mathrm{MF}_K(\varphi, N)$.

Proposition 2.35. *D is admissible if and only if $D\langle i \rangle$ is admissible.*

Proof. By the decomposition in Proposition 2.22, let $D = \oplus_{\alpha \in \mathbb{Q}} D(\alpha)$ and $D\langle i \rangle = \oplus_{\beta \in \mathbb{Q}} D(\beta)$ such that $D(\alpha)$ (resp. $D(\beta)$) is nonzero for finitely many α's (resp. β's). We know that if there is $d \in \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D(\alpha)$ and $\lambda \in \overline{K}$ such that $\varphi(d) = \lambda d$ then $v_p(\lambda) = \alpha$. Now $\varphi(d) = \lambda d$ is equivalent to $p^{-i}\varphi(d) = p^{-i}\lambda d$ i.e., $\varphi'(d) = \lambda' d$ with $\lambda' = p^{-i}\lambda$. Therefore, $v_p(\beta) = -i + v_p(\lambda)$. Hence $t_N(D\langle i \langle) = \sum_{j=1}^{s} \beta_j \cdot \dim_{K_0} D(\beta_j) = -ih + \sum_{j=1}^{s} \alpha \cdot \dim_{K_0} D(\alpha_j) = t_N(D) - ih$.

For Hodge number we notice, $t_H(D\langle i \rangle) = \sum_{r \in \mathbb{Z}} r \cdot \dim_K \mathrm{gr}^r D\langle i \rangle$. Also,

$$\mathrm{gr}^r D\langle i \rangle = \mathrm{Fil}^r D\langle i \rangle / \mathrm{Fil}^{r+1} D\langle i \rangle = \mathrm{Fil}^{r+i} D / \mathrm{Fil}^{r+i+1} D = \mathrm{gr}^{r+i} D$$

and $\dim_K \mathrm{gr}^r D\langle i \rangle = \dim_K \mathrm{gr}^{r+i} D$. So,

$$\begin{aligned} t_H(D\langle i \rangle) &= \sum_{r \in \mathbb{Z}} (r+i) \cdot \dim_K \mathrm{gr}^{r+i} D - \sum_{r \in \mathbb{Z}} i \cdot \dim_K \mathrm{gr}^{r+i} D \\ &= \sum_{s \in \mathbb{Z}} s \cdot \dim_K \mathrm{gr}^s D - i \cdot \sum_{s \in \mathbb{Z}} \dim_K \mathrm{gr}^s D = t_H(D) - ih. \end{aligned}$$

These two computations hold for any $D \in \mathrm{MF}_K(\varphi, N)$ and $i \in \mathbb{Z}$. Hence we conclude that D is admissible if and only if $D\langle i \rangle$ is admissible. ■

2.6 Newton and Hodge Polygons

Let D be a filtered (φ, N)-module. We have defined the Newton number $t_N(D)$ and Hodge number $t_H(D)$ in previous sections. There is a useful visualization tool for these invariants namely the Newton

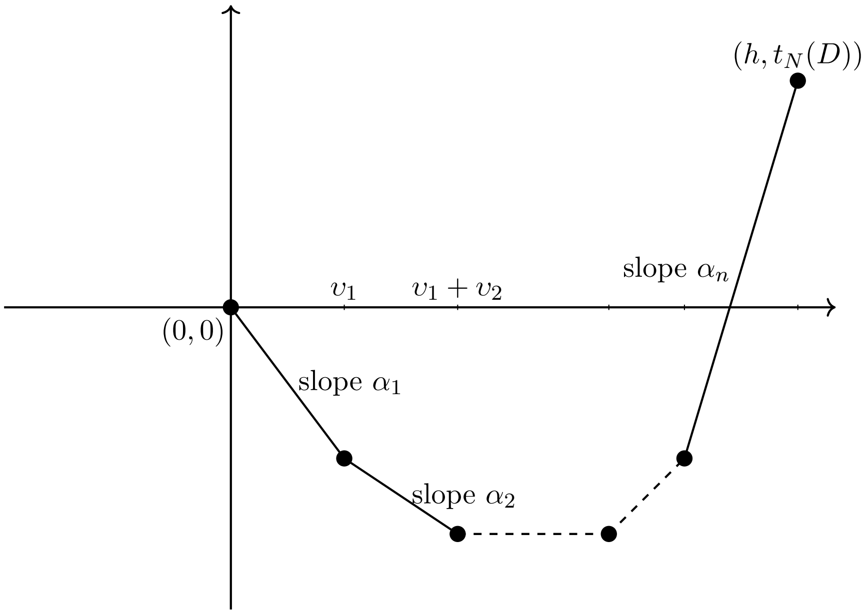

Figure 2.1: A typical Newton polygon $P_N(D)$

polygon $P_N(D)$ and Hodge polygon $P_H(D)$. By Dieudonné-Manin classification in Lemma 2.22, for a nonzero isocrystal D over K_0 has a unique decomposition $D = \oplus_{\alpha\in\mathbb{Q}} D(\alpha)$, where $D(\alpha)$ is the part of D of slope $\alpha \in \mathbb{Q}$. Suppose $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ are all α's such that $D(\alpha) \neq 0$. We write $v_j = \dim_{K_0} D(\alpha_j)$.

Definition 2.36. The *Newton polygon* $P_N(D)$ is the polygon with break points $(0,0)$ and $(v_1 + \cdots + v_j, \alpha_1 v_1 + \cdots + \alpha_j v_j)$ for $j \in \{1, 2, \ldots, s\}$. Thus the end point of $P_N(D)$ is just $(h, t_N(D))$. See Figure 2.1.

Remark 2.37. A nonzero isocrystal D over K_0 is isoclinic of slope α if and only if $P_N(D)$ is a segment with slope α, which is to say that $\widehat{D} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ is isoclinic of slope α.

Let $i_1 < i_2 < \cdots < i_s$ such that $\mathrm{Fil}^{i_j} D_K / \mathrm{Fil}^{i_j+1} D_K \neq 0$ and let $h_j = \dim_K(\mathrm{Fil}^{i_j} D_K / \mathrm{Fil}^{i_j+1} D_K)$ for each $j \in \{1, 2, \ldots, s\}$.

Definition 2.38. The *Hodge Polygon* $P_N(D)$ is the polygon with break points $(0,0)$ and $(h_1 + \cdots + h_j, i_1 h_1 + \cdots + i_j h_j)$ for $j \in \{1, 2, \ldots, s\}$. Thus the end point of $P_H(D)$ is just $(h, t_H(D))$. See Figure 2.2.

Remark 2.39. The formation of P_N and P_H is unchanged by the scalar extension $K_0 \to \widehat{K_0^{\mathrm{un}}}$.

Observe that the admissibility condition of Definition 2.28 says that D is admissible if and only if for all subobjects $D' \subset D$, we have $P_H(D') \leq P_N(D')$ i.e., $P_N(D')$ lies above $P_H(D')$. We now prove an equivalent statement originally given by Fontaine.

Proposition 2.40. *Let $D \in \mathrm{MF}_K(\varphi)$ be a filtered (φ, N)-module such that $\dim_{K_0} D < +\infty$ and φ is bijective on D. The following two conditions are equivalent,*

(i) *For all subobjects $D' \subset D$, $P_H(D') \leq P_N(D')$ i.e., $P_N(D')$ lies above $P_H(D')$.*

(ii) *For all subobjects $D' \subset D$, the right most endpoint of $P_N(D')$ lies on or above the one for $P_H(D')$ i.e., $t_N(D') \geq t_H(D')$.*

Moreover, these properties hold for $D \in \mathrm{MF}_K(\varphi)$ if and only if they hold for $\widehat{D} := \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} D$ in $\mathrm{MF}_{\widehat{K_0^{\mathrm{un}}}}(\varphi)$.

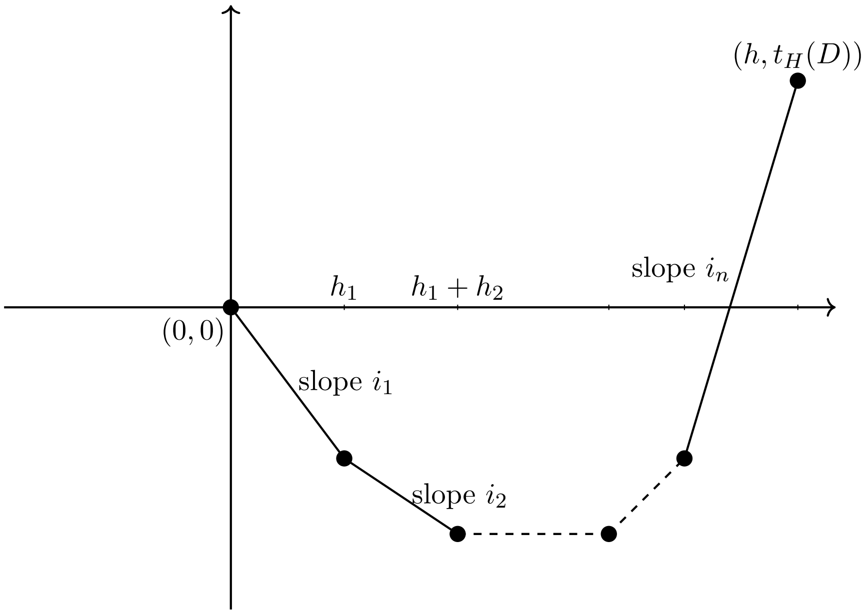

Figure 2.2: A typical Hodge polygon $P_H(D)$

Proof. It is immediate that the first condition implies the second. For the converse, let us assume that there is some subobject $D' \subset D$ such that $P_N(D')$ contains a point lying strictly below $P_H(D')$ on the same vertical line. We seek to construct a subobject $D'' \subset D$ violating the second condition i.e., $t_N(D'') < t_H(D'')$. It must be that $D' \neq 0$. Both polygons $P_N(D')$ and $P_H(D')$ are convex with common left endpoint $(0,0)$, and by hypothesis the right endpoint of $P_N(D')$ lies on or above that of $P_H(D')$. So, there is some $0 < x_0 < \dim D'$ such that the line $x = x_0$ meets $P_N(D')$ and $P_H(D')$ at the respective points (x_0, y_N) and (x_0, y_H) where $y_N < y_H$.

By small deformation of x_0 and continuity considerations, we can arrange that none of these two points on $x = x_0$ are corner of their respective polygons and keep the condition of $y_N < y_H$. Therefore, there is a well defined slope of the polygons at such points. Depending on which of the two slopes is larger, by convexity we can move either forwards or backwards to get to the case when (x_0, y_N) is the final point of the part of $P_N(D')$ with some slope α_0. We still have $0 < x_0 < \dim_{K_0} D$ because the left endpoint $P_N(D)$ and $P_H(D)$ is $(0,0)$ and their respective right endpoints are $(\dim_{K_0} D, t_N(D))$ and $(\dim_{K_0} D, t_H(D))$ where $t_N(D) \geq t_H(D)$ by hypothesis on D.

Consider the isoclinic decomposition $\widehat{D} = \oplus_{\alpha \in \mathbb{Q}} \widehat{D}(\alpha)$ of $\widehat{D} \in \mathrm{MF}_K(\varphi)$ from Proposition 2.22. Let $\widehat{D}' = \oplus_{\alpha \leq \alpha_0} \widehat{D}(\alpha)$ and endow $\widehat{D}'_K$ with the subspace filtration from $\widehat{D}_K$. So $\widehat{D}'$ is a subobject of $\widehat{D}$ in $\mathrm{MF}_K(\varphi)$. By construction, $P_N(\widehat{D}')$ is the subset of $P_N(\widehat{D})$ since it consists of all slopes upto α_0. We see that its right endpoint is therefore (x_0, y_N) which gives $t_N(\widehat{D}') = y_N$. Since $\widehat{D}'_K$ has the subspace filtration from $\widehat{D}_K$, the filtration jumps for $\widehat{D}'_K$ stay on or ahead of those of $\widehat{D}_K$ for the first $\dim_{\widehat{K_0^{\mathrm{un}}}} \widehat{D}'$ segments of the Hodge polygons. This means, $P_H(\widehat{D}')$ lies on or above $P_H(\widehat{D})$ for $0 \leq x \leq \dim_{\widehat{K_0^{\mathrm{un}}}} \widehat{D}'$. Thus, $t_H(\widehat{D}') \geq y_H > y_N = t_N(\widehat{D}')$, contradicting our hypothesis about right endpoints of Newton and Hodge polygons of all subobjects of D.

Finally, it remains to check that the scalar extension by $K_0 \to \widehat{K_0^{\mathrm{un}}}$ does not affect whether or not the equivalent properties (i) and (ii) hold. This is not immediately clear because $\widehat{D}'$ may have subobjects that do not arise from subobject of D. When $\widehat{D}$ satisfies these conditions in $\mathrm{MF}_{\widehat{K^{\mathrm{un}}}}(\varphi)$ then so does D in $\mathrm{MF}_K(\varphi)$ by Remark 2.39. Conversely, suppose $\widehat{D}$ violates these conditions in $\mathrm{MF}_{\widehat{K}^{\mathrm{un}}}(\varphi)$, we would show the same for D in $\mathrm{MF}_K(\varphi)$. The argument above gives us a slope α_0 such that the subobject $\widehat{\Delta} = \oplus_{\alpha \leq \alpha_0} \widehat{D}(\alpha)$ of $\widehat{D}$ has Newton polygon $P_N(\widehat{\Delta})$ that does not lie on or above the Hodge polygon $P_H(\widehat{\Delta})$. But then $\Delta = \oplus_{\alpha \leq \alpha_0} D(\alpha)$ is a subobject of D (with Δ_K given the subspace filtration from D_K) such that $\widehat{\Delta} = \widehat{K_0^{\mathrm{un}}} \otimes_{K_0} \Delta$ as subobjects of $\widehat{D}$, so $P_N(\Delta) = P_N(\widehat{\Delta})$ does not lie on or above

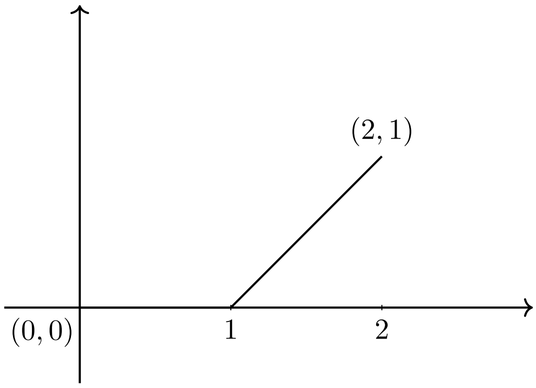

Figure 2.3: Hodge polygon of an elliptic curve

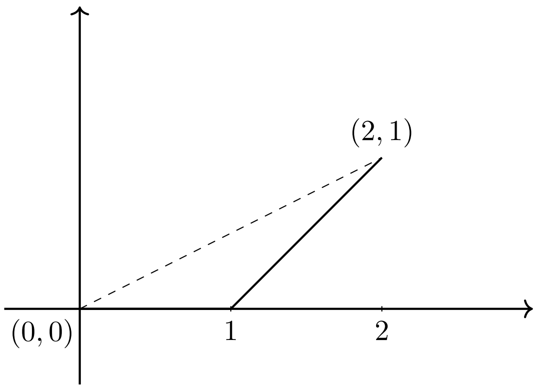

Figure 2.4: Newton polygon of an elliptic curve

$P_H(\widehat{\Delta}) = P_H(\Delta)$. ■

We now discuss an interesting case of elliptic curves with good reduction and draw the respective Newton and Hodge polygons.

Example 2.41. Let E be an elliptic curve over K with good reduction, say with $\mathscr{E}$ the unique elliptic scheme over $\mathcal{O}_K$ having generic fiber E and with $\mathscr{E}_0$ denoting its special fiber. Let D be the filtered φ-module over K associated to E, with its natural Frobenius structure and with D_K filtered. The object D_K in Fil_K is the same for all E, a 2-dimensional K-vector space with gr^0 and gr^1 each 1-dimensional, so the Hodge polygon $P_H(D)$ is the same for all E. See Figure 2.3

In contrast, the structure of D as an isocrystal depends on whether the reduction $\mathscr{E}_0$ over k is ordinary or supersingular. From [BBM82, 2.5.6, 2.5.7, 3.3.7, 4.2.14] we see that $P_N(D)$ looks as in Figure 2.4 where the solid diagram is for the ordinary case while the dashed diagram is for the supersingular case. In particular, for all E with good reduction we see that $P_N(D)$ lies on or above $P_H(D)$ and their right endpoints coincide.

2.6.1 Trivial Filtration

Let $D \in \mathrm{MF}_K(\varphi, N)$ such that $n = \dim_{K_0} D \geq 1$ and φ is bijective on D. We consider the special case when the filtration structure is trivial, i.e., $\mathrm{Fil}^1 D = 0$ (this can be achieved via Tate twist). Removing the effect of the Tate twist at the start (i.e., assume $\mathrm{Fil}^r D = D$ and $\mathrm{Fil}^{r+1} D = 0$ for some r), these are the cases in which the Hodge polygon is a straight line. In this case by convexity and agreement of both endpoints it follows that $P_N(D) = P_H(D)$, so in terms of isoclinic decomposition there is only one slope. Hence, without any hypotheses on $\dim_K D$ we must have $N = 0$ and $\varphi : D \simeq D$ with pure slope 0.

The subobjects are the φ-stable subspaces, each of which has Hodge and Newton polygons that coincide (as segments along x-axis). Hence, admissibility criterion is always satisfied. Also, there is always a lattice $\Lambda \subset D$ that is φ-stable and on which φ acts as an automorphism.

To summarize, when $K = \mathbb{Q}_p$ and the filtration structure is trivial, we are simply studying $\mathbb{Q}_p$-isogeny classes of pairs (Λ, T) consisting of a lattice Λ over $\mathbb{Z}_p$ and a linear automorphism T of Λ. In other words, this is the study of $\mathrm{GL}_n(\mathbb{Q}_p)$-conjugacy classes of elements of $\mathrm{GL}_n(\mathbb{Z}_p)$.

2.7 Admissible filtered (φ, N)-modules of dimension 1

Let $D \in \mathrm{MF}_K(\varphi, N)$ with $\dim_{K_0} D = 1$ such that φ is bijective on D. We can write $D = K_0 d$ for some $d \in D$ and $\varphi(d) = \lambda d$ for some $\lambda \in K_0^\times$. N must be zero since N is nilpotent.
$D_K = K \otimes_{K_0} D = Kd$ is 1-dimesional over K and there exists $r \in \mathbb{Z}$ such that

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } \ i \leq r \\ 0, & \text{if } \ i > r. \end{cases}$$

Clearly, $t_N(D) = \upsilon_p(\lambda)$ and $t_H(D) = r$. Therefore, D is admissible if and only if $\upsilon_p(\lambda) = r$.

Conversely, given $\lambda \in K_0^\times$, we can associate to it $D_\lambda \in \mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ of dimension 1 given by $D_\lambda = K_0, \varphi = \lambda\sigma, N = 0$ and

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } \ i \leq \upsilon_p(\lambda) \\ 0, & \text{if } \ i > \upsilon_p(\lambda). \end{cases}$$

Proposition 2.42. *If $\lambda, \lambda' \in K_0^\times$, then $D_\lambda \simeq D_{\lambda'}$ if and only if there exists $u \in W(k)^\times$ such that $\lambda' = \lambda \cdot \sigma(u)/u$.*

Proof. Let us assume $f : D_\lambda \xrightarrow{\sim} D_{\lambda'}$ and for some $d \in D_\lambda$ and $d' \in D_{\lambda'}$, we have $\varphi(d) = \lambda d$ and $\varphi'(d') = \lambda' d'$ for $\lambda, \lambda' \in K_0^\times$. Now $D_\lambda \xrightarrow{\sim} D_{\lambda'}$ if and only if $d' = ud$ for some $u \in W(k)^\times$. So, $\lambda' d' = \varphi'(d') = \varphi(ud) = \sigma(u)\varphi(d) = \sigma(u)\lambda d = (\sigma(u)/u)\lambda d'$. Therefore $D_\lambda \xrightarrow{\sim} D_{\lambda'}$ if and only $\lambda' = (\sigma(u)/u)\lambda$. The important point to note here is that the underlying vector space for D_λ and $D_{\lambda'}$ are the same which we can fix as K_0. ∎

In the special case of $K = \mathbb{Q}_p$, then $K_0 = \mathbb{Q}_p$ and $\sigma = id$. Therefore from Proposition 2.42 above, $D_\lambda \simeq D_{\lambda'}$ if and only if $\lambda = \lambda'$.

2.8 Admissible filtered (φ, N)-modules of dimension 2

Let $D \in \mathrm{MF}_K(\varphi, N)$ such that $\dim_{K_0} D = 2$ and φ is bijective. Then there exists a unique $i \in \mathbb{Z}$ such that $\mathrm{Fil}^i D_K = D_K$ and $\mathrm{Fil}^{i+1} D_K \neq D_K$. Replacing D with $D\langle i \rangle$ (Tate twist), we may assume that $i = 0$. Now we consider 2 separate cases.

Case 1: $\mathrm{Fil}^1 D_K = 0$.
This means the filtration is trivial. This case has already been discussed in Subsection 2.6.1.

Case 2: $\mathrm{Fil}^1 D_K \neq 0$.
Then $\mathrm{Fil}^1 D_K = L$ is a 1-dimensional K-vector subspace of D_K. So, there exists a unique $r \geq 1$ such that

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } \ i \leq 0 \\ L, & \text{if } \ 1 \leq i \leq r \\ 0, & \text{if } \ i > r. \end{cases}$$

The Hodge polygon $P_H(D)$ is shown in Figure 2.5.

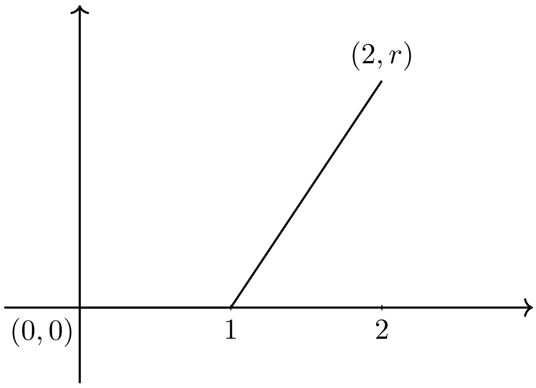

Figure 2.5: The Hodge polygon $P_H(D)$

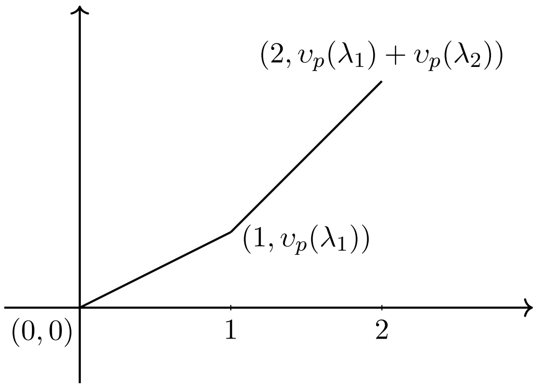

Figure 2.6: The Newton polygon $P_N(D)$

For the rest of this section we assume that $K = \mathbb{Q}_p$. Then $K_0 = \mathbb{Q}_p, D_K = D, \sigma = id$ and φ is $\mathbb{Q}_p$-linear and bijective on D. Let $P_\varphi(X)$ be the characteristic polynomial of φ on D. Then

$$P_\varphi(X) = X^2 + aX + b = (X - \lambda_1)(X - \lambda_2)$$

for some $a, b \in \mathbb{Q}_p$ and $\lambda_1, \lambda_2 \in \overline{\mathbb{Q}_p}$. We may assume that $\upsilon_p(\lambda_1) \leq \upsilon_p(\lambda_2)$. Then the Newton polygon $P_N(D)$ is shown in Figure 2.6 The admissibility condition implies that, $\upsilon_p(b) = \upsilon_p(\lambda_1) + \upsilon_p(\lambda_2) = r \geq 0$ and $\upsilon_p(\lambda_1) \geq 0$ which also gives that $\upsilon_p(a) \geq 0$.

Now we consider different cases for N.

$\mathbf{N \neq 0}$ (The non-crystalline case) : We know that N is a nilpotent operator, so over the completed maximal unramified extension $W(\overline{\mathbb{F}_p})[1/p]$ the relation $N\varphi = p\varphi N$ and the Dieudonné-Manin classification of Proposition 2.19 imply that there are two distinct slopes and they differ by 1. More precisely this means, $\upsilon_p(\lambda_2) \neq \upsilon_p(\lambda_1)$ and $\upsilon_p(\lambda_2) = \upsilon_p(\lambda_1) + 1$. Hence, P_φ cannot be irreducible over $\mathbb{Q}_p$ which means $\lambda_1, \lambda_2 \in \mathbb{Q}_p^\times$.

Let us assume that $\upsilon_p(\lambda_1) = m$. Then from above we have, $m \geq 0$ and $r = 2m + 1$. Now let e_2 be an eigenvector for λ_2, i.e., $\varphi(e_2) = \lambda_2 e_2$. Let $e_1 = N(e_2)$, which is nonzero since $N \neq 0$. Using $N\varphi = p\varphi N$ we see that λ_2/p is the eigenvalue of eigenvector e_1 of φ. Therefore, $\lambda_2 = p\lambda_1$. We set $\lambda_1 = \lambda$ and therefore $\lambda_2 = p\lambda$. So we have $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\lambda \in \mathbb{Z}_p$ (since $\upsilon_p(\lambda) \geq 0$) and $\varphi(e_1) = \lambda e_1, \varphi(e_2) = p\lambda e_2$ with $N(e_1) = 0, N(e_2) = e_1$, i.e.,

$$[\varphi] = \begin{pmatrix} \lambda & 0 \\ 0 & p\lambda \end{pmatrix} \quad \text{and} \quad [N] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now we want to investigate what could L be. For this, we look at the admissibility condition from definition 2.28. We have $t_H(D) = t_N(D)$ and for any subobject D' of $D, t_H(D') \leq t_N(D')$. Let

$D' = \mathbb{Q}_p d$ for some $d \in D$ as we only need to check nontrivial subobjects of dimension 1. So let $d = \alpha e_1 + \beta e_2$ for some $\alpha, \beta \in \mathbb{Q}_p$. If D' is a subobject of D then it should be stable under φ. For this consider three different cases,

1. Assume $\alpha \neq 0, \beta \neq 0$. Then we should have $\varphi(D') \subset D'$ i.e., $\varphi(\alpha e_1 + \beta e_2) = \gamma(\alpha e_1 + \beta e_2)$ for some $\gamma \in \mathbb{Q}_p$. Upon simplification and using the fact that $\{e_1, e_2\}$ form a basis of D, we get $\gamma = \lambda = p\lambda$ which is not possible.

2. Assume $\alpha = 0, \beta \neq 0$. Then $D' = \mathbb{Q}_p e_2$. But then D' would not be stable under the action of N since $N(e_2) = e_1 \notin D'$.

3. Assume $\beta = 0, \alpha \neq 0$. Then $D' = \mathbb{Q}_p e_1$. Clearly, $\varphi(\alpha e_1) = \alpha\lambda e_1 \in D'$ and $N(e_1) = 0 \in D'$. So, D' is a subobject of D.

So looking at $D' = \mathbb{Q}_p e_1$ we have, $t_N(D') = v_p(\lambda) = m$ and,

$$t_H(D') = \begin{cases} r, & \text{if } L = D' \\ 0, & \text{otherwise.} \end{cases}$$

Since $r > m$, the admissibility condition implies that $t_H(D') = 0$ i.e., L can be any 1-dimenional subspace of D except $D' = \mathbb{Q}_p e_1$. So, there exists a unique $\alpha \in \mathbb{Q}_p$ such that $L = \mathbb{Q}_p(e_2 + \alpha e_1)$.

Now we look at the converse. Let $\lambda \in \mathbb{Z}_p$ and $\alpha \in \mathbb{Q}_p$. We can associate a 2-dimensional filtered (φ, N)-module $D_{\lambda,\alpha}$ of $\mathbb{Q}_p$ to the pair (λ, α), where $D_{\lambda,\alpha} \simeq \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\varphi(e_1) = \lambda e_1, \varphi(e_2) = p\lambda e_2$ and $N(e_1) = 0, N(e_2) = e_1$ and filtration given by,

$$\mathrm{Fil}^i D_{\lambda,\alpha} = \begin{cases} D_{\lambda,\alpha}, & \text{if } i \leq 0 \\ (e_2 + \alpha e_1)\mathbb{Q}_p, & \text{if } 1 \leq i \leq 2v_p(\lambda_1) + 1 \\ 0, & \text{otherwise} \end{cases}$$

After choosing e_1 and e_2 we define φ and N such that we can easily write down $\mathrm{Fil}^i D_{\lambda,\alpha}$ using the discussion above. If we replace the initial choice of e_2 with a $\mathbb{Q}_p^\times$-multiple then $e_1 = N(e_2)$ is scaled in the same way and so α does not change. Thus, α is intrinsic to $D_{\lambda,\alpha}$. Now we look an important lemma towards classification of all modules of the type discussed above.

Lemma 2.43. *$D_{\lambda,\alpha} \simeq D_{\lambda',\alpha'}$ if and only if $\lambda = \lambda'$ and $\alpha = \alpha'$.*

Proof. By construction it is obvious that if given λ and α there is a unique such $D_{\lambda,\alpha}$. We need to show the converse. Suppose $D_{\lambda,\alpha} = \mathbb{Q}_p e_1 \oplus \mathbb{Q} e_2$ and $D_{\lambda',\alpha'} = \mathbb{Q}_p e_1' \oplus \mathbb{Q}_p e_2'$ for some choice of e_1, e_2, e_1', e_2' and φ, N, φ', N' and the respective filtrations defined as before. Let,

$$\begin{aligned} f : D_{\lambda,\alpha} = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 &\xrightarrow{\sim} \mathbb{Q}_p e_1' \oplus \mathbb{Q}_p e_2' = D_{\lambda',\alpha'} \\ e_1 &\longmapsto x e_1' + y e_2' \\ e_2 &\longmapsto z e_1' + w e_2'. \end{aligned}$$

We know that $\varphi' \circ f = f \circ \varphi$ and $N' \circ f = f \circ N$. This gives us, $f(\varphi(e_1)) = f(\lambda e_1) = x\lambda e_1' + y\lambda e_2'$ and $\varphi'(f(e_1)) = \varphi'(xe_1' + ye_2') = x\lambda' e_1' + yp\lambda' e_2'$. Since both are equal, we have $x\lambda = x\lambda'$ and $\psi\lambda = yp\lambda'$. Also, $f(\varphi(e_2)) = f(\lambda e_2) = pz\lambda e_1' + pw\lambda e_2'$ and $\varphi'(f(e_2)) = \varphi'(ze_1' + we_2') = z\lambda' e_1' + pw\lambda' e_2'$. And again since both are equal, we have $pz\lambda = z\lambda'$ and $pw\lambda = pw\lambda'$ i.e., either $z = 0$ and $\lambda = \lambda'$ or $w = 0$ and $p\lambda = \lambda'$. From the computations above, the only possibility that we get is $y = 0, z = 0$ and $\lambda = \lambda'$. Now we also have, $N'(f(e_2)) = N'(we_2') = we_1'$ and $f(N(e_2)) = f(e_1) = xe_1'$. And again we have equal quatitites, which gives $x = w$. Since, $D_{\{\lambda,\alpha\}} \simeq D_{\{\lambda,\alpha\}}$ as filtered (φ, N)-modules,restriction of f gives,

$$\begin{aligned} \tilde{f} : (e_1 + \alpha e_2)\mathbb{Q}_p &\xrightarrow{\simeq} (e_1' + \alpha' e_2')\mathbb{Q}_p \\ e_1 + \alpha e_2 &\longmapsto xe_1' + \alpha x e_2'. \end{aligned}$$

So, we must have $xe_1' = \beta(e_1' + \alpha e_2')$ for some $\beta \in \mathbb{Q}_p$. This gives $\beta = x$ and $\alpha = \alpha'$ since $x \neq 0$. Hence, we have $D_{\lambda,\alpha} \simeq D_{\lambda',\alpha'}$ if and only if $\lambda = \lambda'$ and $\alpha = \alpha'$. ∎

Remark 2.44. $D_{\lambda,\alpha}$ is irreducible if and only if $\upsilon_p(\lambda) > 0$. Indeed, $D_{\lambda,\alpha}$ is not irreducible if and only if there exists a nontrivial subobject of it in the category of admissible filtered (φ, N)-modules. From above we see that the only candidate is $D' = \mathbb{Q}_p e_1$. Now, D' is admissible if and only if $t_H(D') = t_N(D') = \upsilon_p(\lambda)$. But $t_H(D') = 0$. Hence D' is admissible i.e., $D_{\lambda,\alpha}$ is reducible if and only if $\upsilon_p(\lambda) = 0$.

Next, we consider the 2-dimensional admissible filtered (φ, N)-modules which are reducible. From Remark 2.44, necessarily $\lambda \in \mathbb{Z}_p^\times$. In this case we have $\mathrm{Fil}^0 D_K = D_K$, $\mathrm{Fil}^1 D_K = L \neq D_K$ and $\mathrm{Fil}^i D_K = 0$ for $i \geq 2$ for $\upsilon_p(\lambda) = 0$ i.e., objects with Hodge-Tate weight (0,1) from Definition 3.19. We recall that we made this assumption by applying Tate's twist. It is clear that for each $i \in \mathbb{Z}$, we could apply the Tate twist and get $D_{\lambda,\alpha}\langle i \rangle$. Conversely, for any 2-dimensional reducible $D \in \mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ there is a unique $i \in \mathbb{Z}$ such that $\mathrm{Fil}^i D_K = D_K$ and $\mathrm{Fil}^{i+1} D_K \neq D_K$. By applying Tate twist, we may reduce to the case of Hodge-Tate weight $(0, 1)$ and in such a situation $D\langle -i \rangle \simeq D_{\lambda,\alpha}$ for some unique $\lambda \in \mathbb{Z}_p^\times$ and $\alpha \in \mathbb{Q}_p$ from the Lemma 2.43. Hence we have a 1-1 correspondence as stated below.

Proposition 2.45. *The map*

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z}_p^\times \times \mathbb{Q}_p &\longrightarrow \left\{ \begin{array}{l} \textit{set of isomorphism classes of 2-} \\ \textit{dimensional reducible admissible filtered} \\ (\varphi, N)\textit{-modules over } \mathbb{Q}_p \textit{ with } N \neq 0 \end{array} \right\} \\ (i, \lambda, \alpha) &\longmapsto D_{\lambda,\alpha}\langle i \rangle \end{aligned}$$

is bijective.

N = 0 (The crystalline case): For any subobject D' of D, due to admissibility condition, we need to check that $t_H(D') \leq t_N(D')$.

Lemma 2.46. *Let $a, b \in \mathbb{Z}_p$ with $r = \upsilon_p(b) > 0$ such that $P_\varphi(X) = X^2 + aX + b$ is irreducible over $\mathbb{Q}_p$. Set $D_{a,b} = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\varphi(e_1) = e_2, \varphi(e_2) = -be_1 - ae_2$ and $N = 0$, i.e.,*

$$[\varphi] = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix} \quad \textit{and} \quad [N] = 0.$$

Let the filtration be given by,

$$\mathrm{Fil}^i D_{a,b} = \begin{cases} D_{a,b}, & \textit{if } i \leq 0 \\ \mathbb{Q}_p e_1, & \textit{if } 1 \leq i \leq r \\ 0, & \textit{otherwise.} \end{cases}$$

Then $D_{a,b}$ is admissible and irreducible.

Proof. We do a step-by-step verfication of each condition for admissibility. By description of $D_{a,b}$, it is clear that $\dim_{K_0} D_{a,b} = 2 < +\infty$.

φ is bijective on $D_{a,b}$: For injection, let $\alpha, \beta \in \mathbb{Q}_p$ such that $\varphi(\alpha e_1 + \beta e_2) = 0$ with $\alpha e_1 + \beta e_2 \in D_{a,b}$. By the action of φ as defined above and using the fact that $\{e_1, e_2\}$ is a basis for $D_{a,b}$, we get $b\beta = 0$ and $\alpha = a\beta$. Since $b \neq 0$ ($X^2 + aX + b$ is irreducible), therefore $\beta = 0$ and $\alpha = 0$. For surjection, again let $\alpha e_1 + \beta e_2 \in D_{a,b}$ as before. Now let $x = \beta - a\alpha/b$ and $y = -\alpha/b$. From this it is immediate that $\varphi(xe_1 + ye_2) = \alpha e_1 + \beta e_2$. Hence, φ is bijective on $D_{a,b}$.

For Newton and Hodge numbers we know that $t_N(D_{a,b}) = \upsilon_p(b)$ and $t_H(D_{a,b}) = r$. Therefore, $t_N(D_{a,b}) = t_H(D_{a,b})$.

Next, we consider nontrivial subobjects of $D_{a,b}$. Let $D' = \mathbb{Q}_p(\alpha e_1 + \beta e_2)$ with $\alpha, \beta \in \mathbb{Q}_p$. We consider different cases in which D' could be a subobject of $D_{a,b}$. Clearly, D' is stable under N. For D' to be a subobject of $D_{a,b}$, it should be stable under φ, i.e., $\varphi(D') \subset D'$. We see that, $\varphi(\alpha e_1 + \beta e_2) = \alpha e_2 + \beta(-be_1 - ae_2) = -b\beta e_1 + (\alpha - a\beta)e_2$. Now,

(i) If $\alpha = 0$ and $\beta \neq 0$. Then, $\varphi(\beta e_2) = -b\beta e_1 - a\beta e_2 \notin D' = \mathbb{Q}_p e_2$.

(ii) If $\alpha \neq 0$ and $\beta = 0$. Then, $\varphi(\alpha e_1) = \alpha e_2 \notin D' = \mathbb{Q}_p e_1$.

(iii) If $\alpha \neq 0$ and $\beta \neq 0$. Then, there must exist $x \in \mathbb{Q}_p$ such that $-b\beta e_1 + (\alpha - a\beta)e_2 = x(\alpha e_1 + \beta e_2)$ which is possible if and only if $x \neq 0$ and $x^2 + ax + b = 0$. But then, $x = \lambda_1$ or λ_2 and $x \notin \mathbb{Q}_p$, since $X^2 + aX + b$ is irreducible over $\mathbb{Q}_p$.

Thus, none of these cases are possible and therefore there are no nontrivial subobjects of $D_{a,b}$. Hence, $D_{a,b}$ is admissible and irreducible. ■

Lemma 2.47. *Let $\lambda_1, \lambda_2 \in \mathbb{Z}_p$, nonzero $\lambda_1 \neq \lambda_2$ and $v_p(\lambda_1) \leq v_p(\lambda_2)$. Let $r = v_p(\lambda_1) + v_p(\lambda_2)$. Set $D'_{\lambda_1,\lambda_2} = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with $\varphi(e_1) = \lambda_1 e_1, \varphi(e_2) = \lambda_2 e_2$ and $N = 0$, i.e.,*

$$[\varphi] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \textit{and} \quad [N] = 0.$$

Let the filtration be given by,

$$\mathrm{Fil}^i D'_{\lambda_1,\lambda_2} = \begin{cases} D'_{\lambda_1,\lambda_2}, & \textit{if } i \leq 0 \\ (e_1 + e_2)\mathbb{Q}_p, & \textit{if } 1 \leq i \leq r \\ 0, & \textit{otherwise.} \end{cases}$$

Then D_{λ_1,λ_2} is admissible. Moreover, it is irreducible if and only if $v_p(\lambda_1) > 0$.

Proof. In this proof, we would again do a step-by-step verification as in previous lemma. Clearly, $\dim_{K_0} D'_{\lambda_1,\lambda_2} = 2 < +\infty$.

Next, φ should be bijective on D'_{λ_1,λ_2}. For injection, let $\alpha, \beta \in \mathbb{Q}_p$ such that $\alpha e_1 + \beta e_2 \in D'_{\{\lambda_1,\lambda_2\}}$ and $\varphi(\alpha e_1 + \beta e_2) = 0$. From the action of φ it is straightforward that $\alpha = \beta = 0$. For surjection, again let $\alpha e_1 + \beta e_2 \in D'_{\lambda_1,\lambda_2}$ as before. Let $x = \alpha/\lambda_1$ and $y = \beta/\lambda_2$. From this it is obvious that, $\varphi(xe_1 + ye_2) = \alpha e_1 + \beta e_2$. Hence, φ is bijective on D'_{λ_1,λ_2}.

Moving on to admissibility conditions, we have $t_N(D'_{\lambda_1,\lambda_2}) = v_p(\lambda_1) + v_p(\lambda_2) = r = t_H(D'_{\lambda_1,\lambda_2})$.

For any subobject D'' of D'_{λ_1,λ_2} we need to check that $t_H(D'') \leq t_N(D'')$. Let $D'' = \mathbb{Q}_p(\alpha e_1 + \beta e_2)$ with $\alpha, \beta \in \mathbb{Q}_p$ be a nontrivial subobject of D'_{λ_1,λ_2}. D'' is stable under N. D'' should be stable under φ as well. Observe that, $\varphi(\alpha e_1 + \beta e_2) \in D''$ if and only if there is an $x \in \mathbb{Q}_p$ such that $\varphi(\alpha e_1 + \beta e_2) = x(\alpha e_1 + \beta e_2)$ which gives $\alpha(\lambda_1 - x) = 0$ and $\beta(\lambda_2 - x) = 0$. Now we consider different situations,

(i) If $\alpha = 0$ and $\beta \neq 0$. Then $x = \lambda_2$ and D'' is stable under φ.

(ii) If $\alpha \neq 0$ and $\beta = 0$. Then $x = \lambda_1$ and D'' is stable under φ.

(iii) If $\alpha \neq 0$ and $\beta \neq 0$. Then $x = \lambda_1 = \lambda_2$. But this violates our assumption.

So we get that the only possibilities for D'' are $D'' = \mathbb{Q}_p e_1$ or $D'' = \mathbb{Q}_p e_2$. By filtration on D'_{λ_1,λ_2} the following holds,

$$t_H(D'') = \begin{cases} 0, & \text{if } D'' \neq (e_1 + e_2)\mathbb{Q}_p \\ r, & \text{if } D'' = (e_1 + e_2)\mathbb{Q}_p. \end{cases}$$

Clearly in (i) above $t_H(D'') = 0$ and $t_N(D'') = v_p(\lambda_2)$ and we know that $t_H(D'') = 0 \leq v_p(\lambda_2) = t_N(D'')$. Moreover, D'' is admissible if and only if $v_p(\lambda_2) = 0$. Also, $0 \leq v_p(\lambda_1) \leq v_p(\lambda_2)$. Therefore, $v_p(\lambda_1) = 0$ which gives $r = 0$. But $r \geq 1$. So, it turns out that $D'' = \mathbb{Q}_p e_2$ is not admissible. In (ii) we have $t_H(D'') = 0$ and $t_N(D'') = v_p(\lambda_1)$ and $v_p(\lambda_1) \geq 0$. Therefore, D'_{λ_1,λ_2} is admisible since both nontrivial subobjects meet the admissibility conditions.

Moreover, $D'' = \mathbb{Q}_p e_1$ is admissible if and only if $v_p(\lambda_1) = 0$. Hence D'_{λ_1,λ_2} is irreducible if and only if $v_p(\lambda_1) > 0$. ■

Conversely, we have the following proposition.

Proposition 2.48. *Assume D is an admissible filtered (φ, N)-module over $\mathbb{Q}_p$ of dimension 2 with $N = 0$ such that $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^1 D \neq D, 0$. Assume D is not a direct sum of two admissible (φ, N)-modules of dimension 1. Then either $D \simeq D_{a,b}$ for uniquely determined (a, b) or $D \simeq D'_{\lambda_1,\lambda_2}$ for uniquely determined (λ_1, λ_2).*

Proof. We are given $D \in \mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ with

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ L, & \text{if } 1 \leq i \leq r \\ 0, & \text{if } i > r. \end{cases}$$

We write down the characteristic polynomial for φ which is $P_\varphi(X) = X^2 + aX + b$ for some $a, b \in \mathbb{Q}_p$. The condition $v_p(b) = r = t_H(D) = t_N(D)$ implies that $b \in p^r \mathbb{Z}_p^\times$. Now we have two different cases,

(i) $P_\varphi(X)$ is irreducible. Then the roots of $P_\varphi(X)$ are $\lambda_1, \lambda_2 \in \overline{\mathbb{Q}_p}$ with $v_p(\lambda_1) = v_p(\lambda_2) = r/2 > 0$. Since L is 1-dimensional, so we choose a basis vector e_1 for L. Also $v_p(a) = v_p(\lambda_1 + \lambda_2) \geq r/2$ i.e., $a \in p^{\lfloor r/2 \rfloor}\mathbb{Z}_p$, so we conclude that there are no nontrivial subobjects of D since any such subobject would have its Newton number smaller than $r/2$ while its Hodge number would be r. Since L is stable under N trivially, and $t_H(L) \leq t_N(L)$, we conclude that L is not stable under the action of φ because otherwise it would be a subobject of D. We set $e_2 := \varphi(e_1) \notin L$ and so we can fix a basis $\{e_1, e_2\}$ for D. Now it easily follows that we are in the setting of lemma 2.46 and therefore, $D \simeq D_{a,b}$ where a and b are uniquely determined by the characteristic polynomial of φ.

(ii) $P_\varphi(X)$ is reducible. Let $P_\varphi(X) = (X - \lambda_1)(X - \lambda_2)$ for $\lambda_1, \lambda_2 \in \mathbb{Q}_p^\times$ and $v_p(\lambda_2) \geq v_p(\lambda_1)$ with $v_p(\lambda_1) + v_p(\lambda_2) = r \geq 1$ so $v_p(\lambda_2) \geq 1$.

First assume that $\lambda_1 \neq \lambda_2$. Let e_1 and e_2 be the corresponding eigenvectors respectively. For any subobject D' of D,

$$t_H(D') = \begin{cases} 0, & \text{if } D' \neq L \\ r, & \text{if } D' = L. \end{cases}$$

In particular, $t_H(\mathbb{Q}_p e_1) \geq 0$ which means $v_p(\lambda_1) \geq 0$. By conditions on valuation on λ_1 and λ_2, $v_p(\lambda_1) < r$. Since $t_N(\mathbb{Q}_p e_1) = v_p(\lambda_1)$, we conclude $L \neq \mathbb{Q}_p e_1$. If $L = \mathbb{Q}_p e_2$, then $t_N(\mathbb{Q}_p e_2) = v_p(e_2) \geq t_H(\mathbb{Q}_p e_2) = r$, since L is a subobject of D. This is only possible if $v_p(\lambda_1) = 0$ and $v_p(\lambda_2) = r$. But in this case $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ is a direct sum of two admissible filtered (φ, N)-modules contrary to our assumption. Therefore, $L \neq \mathbb{Q}_p e_2$. After scaling e_i's we may assume that $L = (e_1 + e_2)\mathbb{Q}_p$. Now we observe that we are in the setting of Lemma 2.47. Therefore $D \simeq D'_{\lambda_1,\lambda_2}$. Since λ_1, λ_2 are roots of characteristic polynomial of φ, they are uniquely determined.

Next we assume that $\lambda_1 = \lambda_2 = \lambda$. This means $r = v_p(\lambda)$. Since $r > 0$, r is even and $\lambda \in p\mathbb{Z}_p$. φ cannot be a scalar. Suppose on the contrary that it is, then since L is a subobject of D, it is stable under φ. But $t_H(L) = r > \frac{r}{2} = v_p(\lambda) = t_N(L)$ violating the weak admissibility of D. This gives us that λ-eigenspace is 1-dimensional. We choose an eigenvector $\{e_1\}$ corresponding to λ and from the reasoning as in (i), a basis $\{e_1, e_2\}$ for D with scaling such that $L = (e_1 + e_2)\mathbb{Q}_p$. So the matrix for φ is

$$[\varphi] = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda. \end{pmatrix}$$

Therefore, as before we get that $D \simeq D'_{\lambda_1,\lambda_2}$ with uniquely determined $\lambda_1 = \lambda_2$. The only difference is that D in this case would always be irreducible, since there are no admissible subobjects of D.

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Remark 2.49. (i) In the first case of the proof above, we have classified the cases with irreducible P_φ up to isomorphism in terms of the parameters $(a, b) \in p^{\lfloor r/2 \rfloor}\mathbb{Z}_p \times p^r\mathbb{Z}_p^\times$ (subject to the condition that $b^2 - 4a$ is a nonsquare in $\mathbb{Q}_p^\times$). The filtration jumps for D are in degrees 0 and r. Removing the effect of the initial Tate twist on these examples amounts to allowing the smaller of the two distinct Hodge-Tate weights to be an arbitrary integer.

(ii) In the second case of the proof above, we may assume that $L = (\alpha e_1 + e_2)\mathbb{Q}_p$ with $\alpha \in \mathbb{Q}_p^\times$. However, a simple computation shows that for any two $\alpha, \alpha' \in \mathbb{Q}^\times$, the admissible filtered (φ, N)-module D with $L = (\alpha e_1 + e_2)\mathbb{Q}_p$ and D' with $L = (\alpha' e_1 + e_2)\mathbb{Q}_p$ are isomorphic in $\mathrm{MF}_K(\varphi, N)$.

(iii) In case $\lambda_1 \neq \lambda_2$, these modules are parametrized by unordered pairs of distinct nonzero $\lambda_1, \lambda_2 \in \mathbb{Z}_p$ such that $v_p(\lambda_1) + v_p(\lambda_2) = r \geq 1$.

(iv) In case $\lambda_1 = \lambda_2 = \lambda$, the explicit description shows that up to isomorphism such examples are completely determined by $\lambda \in p^{r/2}\mathbb{Z}_p^\times$. Note that we can remove the effect of the initial Tate twist by allowing any $\lambda \in \mathbb{Q}_p^\times$ (in which case $v_p(\lambda) \in \mathbb{Z}$ is the average of the two distinct Hodge-Tate weights).

(v) Also, in the second case of the proof above, $L = \mathbb{Q}_p e_2$ can only occur if $v_p(\lambda_1) = 0$ and $v_p(\lambda_2) = r$, in which case it corresponds to D that is a direct sum of the 1-dimensional objects $\mathbb{Q}_p e_1$ and $L = \mathbb{Q}_p e_2$, with these subobjects having respective filtration jumps in degrees 0 and r.

Chapter 3

p-adic Galois representations

3.1 B-representations and regular G-rings

In this section we introduce the formalism needed to define the functors which establishes the equivalence of categories between certain classes of p-adic representations and semilinear algebra objects. We will elaborate on these notions in following sections.

3.1.1 B-representations

Let G be a topological group and B be a topological commutative ring equipped with a continuous action of G compatible with the structure of the ring, i.e., for every $g \in G, b_1, b_2 \in B$ we should have $g(b_1 + b_2) = g(b_1) + g(b_2)$ and $g(b_1 b_2) = g(b_1)g(b_2)$.

Definition 3.1. A B-representation X of G is a finitely generated B-module equipped with a semi-linear and continuous action of G where semi-linear means that for every $g \in G, \lambda \in B$ and $x, x_1, x_2 \in X$ we have $g(x_1 + x_2) = g(x_1) + g(x_2)$ and $g(\lambda x) = g(\lambda)g(x)$.

If G acts trivially on B, we just have a linear representation. If $B = \mathbb{Q}_p$ with the p-adic topology, we say that it is a p-adic representation.

Definition 3.2. A free B-representation of G is a B-representation such that the underlying B-module is free.

Example 3.3. Let $F \subset B^G$ be a closed subfield and V be an F-representation of G. Let $X = B \otimes_F V$ be equipped with G-action given by $g(\lambda \otimes x) = g(\lambda) \otimes g(x)$ where $g \in G, \lambda \in B, x \in X$. Then X is a free B-representation.

Definition 3.4. A free B-representation X of G is trivial if,

(i) There exists a basis of X consisting of elements of X^G, or

(ii) $X \simeq B^d$ with the natural action of G.

Next, we discuss a classification of free B-representation of G with $\{e_1, e_2, \ldots, e_d\}$ as a basis. For every $g \in G$, let $g(e_j) = \sum_{1 \leq j \leq d} a_{ij}(g)e_i$, then we have a map

$$\begin{aligned} \alpha : G &\longrightarrow \mathrm{GL}_d(B) \\ g &\longmapsto (a_{ij}(g))_{1 \leq i,j \leq d}. \end{aligned}$$

For any $g_1, g_2 \in G$, we see that $\alpha(g_1 g_2) = \alpha(g_1)g_1(\alpha(g_2))$ i.e., α is a 1-cochain in $Z^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$. Moreover, if $\{e'_1, e'_2, \ldots, e'_d\}$ is another basis and if P is the base change matrix, we write $g(e'_j) = \sum_{1 \leq i \leq d} a'_{ij}(g)e'_i$ and $\alpha'(g) = (a'_{ij}(g))_{1 \leq i,j, \leq d}$. So we get $\alpha'(g) = P^{-1}\alpha(g)g(P)$. Therefore, α and α' are cohomologous to each other. Hence, the class of α in $\mathrm{H}^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$ is independent of the choice of basis of X and we denote this chomology class by $[X]$. Conversely, given a 1-cocycle $\alpha \in \mathrm{Z}^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$, there is a unique semi-linear action of G on $X = B^d$ such that for every $g \in G$, $g(e_j) = \sum_{i=1}^{d} \alpha_{ij}(g)e_i$ and $[X]$ is the class of α. Hence we have the following proposition,

Proposition 3.5. *Let $d \in \mathbb{N}$. The correspondence $X \longmapsto [X]$ defines a bijection between the set of equivalence classes of free B-representations of G of rank d and $\mathrm{H}^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$. Moreover, X is trivial if and only if $[X]$ is the distinguished point in $\mathrm{H}^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$.*

3.1.2 (F, G)-regular rings

Let B be a (topological) ring and G a (topological) group acting (continuously) on B. Set $E = B^G$ and assume that it is a field. Let F be a closed subfield of E. If B is a domain then the action of G extends to $C = \mathrm{Frac}\, B$ by $g(b_1/b_2) = g(b_1)/g(b_2)$ for every $g \in G$ and $b_1, b_2 \in B$.

Definition 3.6. B is said to be (F, G)-regular if the following conditions hold,

(i) B is a domain.

(ii) $B^G = C^G$.

(iii) If $b \in B - \{0\}$ such that its F-linear span Fb is G-stable i.e., for every $g \in G$, there exists $\lambda \in F$ depending on g with $g(b) = \lambda b$, then b is a unit in B.

Remark 3.7. If B is a field then it is always (F, G)-regular.

Example 3.8. Let K be a p-adic field with a fixed algebraic closure $\overline{K}$ and let $\mathbb{C}_K = \widehat{\overline{K}}$. Let $G = G_K = \mathrm{Gal}(K/\overline{K})$. Let $B = B_{\mathrm{HT}} = \oplus_{n\in\mathbb{Z}}\mathbb{C}_K(n)$ endowed with its natural G_K-action. Non-canonically, $B = \mathbb{C}_K[T, 1/T]$ with G acting through the p-adic cyclotomic character $\chi : G_K \to \mathbb{Z}_p^\times$ via $g(\sum a_n T^n) = \sum g(a_n)\chi(g)^n T^n$. In this case, obviously we have $C = \mathbb{C}_K(T)$. We will show that B_{HT} is $(\mathbb{Q}_p, G_K)$-regular (with $E = B^G = K$).

By the Tate-Sen Theorem A.13, $B_{\mathrm{HT}}^G = \oplus_{n\in\mathbb{Z}}\mathbb{C}_k(n)^G = K$. To show that C^G is also equal to K, consider the G_K-equivariant inclusion of $C = \mathbb{C}_K(T)$ into the formal Laurent series field $\mathbb{C}_K((T))$ equipped with its evident G-action. Its enough to show that $\mathbb{C}_K((T))^G = K$. The action of $g \in G$ on a formal Laurent series $\sum c_n T^n$ is given by $g(\sum c_n T^n) = \sum g(c_n)\chi(g)^n T^n$, so G-invariance amounts to the condition $c_n \in \mathbb{C}_K(n)^G$ for all $n \in \mathbb{Z}$. Hence, by the Tate-Sen Theorem A.13 we get $c_n = 0$ for $n \neq 0$ and $c_0 \notin K$.

For condition (iii), if $b \in B - \{0\}$ spans a G_K-stable $\mathbb{Q}_p$-line then G_K acts on the line $\mathbb{Q}_p b$ by some character $\psi : G_K \to \mathbb{Q}_p^\times$. It is a crucial fact that ψ must be continuous (so it takes value in $\mathbb{Z}_p$). Writing the Laurent polynomial b as $b = \sum c_i T^i$, we have $\psi(g)b = g(b) = \sum g(c_i)\chi(g)^i T^i$, so for each i we have $(\psi^{-1}\chi^i)(g) \cdot g(c_i) = c_i$ for all $g \in G_K$. That is, each c_i is G_K-invariant in $\mathbb{C}_K(\psi^{-1}\chi^i)$. But by the Tate-Sen Theorem A.13, for a $\mathbb{Z}_p^\times$-valued continuous character η of G_K, if $\mathbb{C}_K(\eta)$ has a nonzero G_K-invariant element then $\eta|_{I_K}$ has finite order. Hence, $(\psi^{-1}\chi^i)|_{I_K}$ has finite order whenever $c_i \neq 0$. It follows that we cannot have $c_i, c_{i'} \neq 0$ for some $i \neq i'$, for taking the ratio of the associated finite-order characters would give that $\chi^{i-i'}|_{I_K}$ has finite order, so $\chi|_{I_K}$ has finite order, but this is a contradiction since χ cuts out an infinitely ramified extension of K. It follows that there is atmost one i such that $c_i \neq 0$, and there is a nonzero c_i since $b \neq 0$. Hence, $b = cT^i$ for some i and some $c \in \mathbb{C}_K^\times$, so $b \in B^\times$.

Example 3.9. From Remark 3.7, $B = B_{\mathrm{dR}}$ is trivially $(\mathbb{Q}_p, G)$-regular with $G = G_K$. Consider $B = B_{\mathrm{dR}}^+$ equipped with its natural action by $G = G_K$. In this case, the $(\mathbb{Q}_p, G)$-regularity fails since $t \in B$ spans a G-stable $\mathbb{Q}_p$-line but $t \notin B^\times$.

Let $\mathrm{Rep}_F(G)$ denote the category of continuous F-representations of G. This is an abelian category with additional structures,

(a) *Tensor product*: If V_1, V_2 are F-representations of G, we set $V_1 \otimes V_2 = V_1 \otimes_F V_2$ as F-vector spaces and the action of G is given by $g(v_1 \otimes v_2) = g(v_1) \otimes g(v_2)$.

(b) *Unit representation*: F with the trivial G-action.

(c) *Dual representation*: If V is an F-representation of G, we set $V^\vee = \mathrm{Hom}_F(V,F) = \{\text{linear maps } V \to F\}$, with the G-action given by $(gf)(v) = f(g^{-1})(v)$.

With these additional structures, $\mathrm{Rep}_F(G)$ is a *Tannakian category* over F.

Definition 3.10. A category $\mathcal{C}'$ is a strictly full sub-category of a category $\mathcal{C}$ if it is a full sub-category and if $X \in \mathcal{C}$ is isomorphic to an object of $\mathcal{C}'$, then $X \in \mathcal{C}'$.

Definition 3.11. A sub-Tannakian category of $\mathrm{Rep}_F(G)$ is a strictly full subcategory $\mathcal{C}$, such that

(i) The unit representation F is an object of $\mathcal{C}$.

(ii) If $V \in \mathcal{C}$ and V' is a sub-representation of V, then V' and V/V' are all in $\mathcal{C}$.

(iii) If V is an object in $\mathcal{C}$, so is $V^\vee$.

(iv) If $V_1, V_2 \in \mathcal{C}$, then so are $V_1 \oplus V_2$ and $V_1 \otimes V_2$.

Definition 3.12. Let V be an F-representation of G. V is said to be B-admissible if $B \otimes_F V$ is a trivial B-representation of G.

We now discuss the general construction of the functor that we mentioned in the beginning and prove an important theorem which would be key in all that follows.

Let V be any F-representation of G, then $B \otimes_F V$, equipped with the G-action by $g(\lambda \otimes x) = g(\lambda) \otimes g(x)$, is free B-representation of G. Let $\mathrm{D}_B(V) = (B \otimes_F V)^G$. $\mathrm{D}_B(V)$ could be seen as a functor from $\mathrm{Rep}_F(G)$ to the category of E-vector spaces. We also get a map,

$$\begin{aligned} \alpha_V : B \otimes_E \mathrm{D}_B(V) &\longrightarrow B \otimes_F V \\ \lambda \otimes x &\longmapsto \lambda x \end{aligned}$$

α_V is B-linear and commutes with the action of G, where G acts on $B \otimes_E \mathrm{D}_B(V)$ via $g(\lambda \otimes x) = g(\lambda) \otimes x$.

Theorem 3.13. *Assume B is (F,G)-regular. Then,*

(i) *For any F-representation V of G, the map α_V is injective and* $\dim_E \mathrm{D}_B(V) \leq \dim_F V$. *Moreover, the following are equivalent,*

  (a) $\dim_E \mathrm{D}_B(V) = \dim_F V$.
  (b) *α_V is an isomorphism.*
  (c) *V is B-admissible.*

(ii) *Let $\mathrm{Rep}_F^B(G)$ be the full subcategory of $\mathrm{Rep}_F(G)$ consisting of the representations V which are B-admissible. The restriction of D_B to $\mathrm{Rep}_F^B(G)$ is an exact and faithful functor. In $\mathrm{Rep}_F^B(G)$, we have the following,*

  (a) *Any subrepresentation or quotient of a B-admissible representation is B-admissible.*
  (b) *For V_1 and $V_2 \in \mathrm{Rep}_F^B(G)$, there is a natural isomorphism $\mathrm{D}_B(V_1) \otimes \mathrm{D}_B(V_2) \simeq \mathrm{D}_B(V_1 \otimes V_2)$, so $V_1 \otimes_F V_2 \in \mathrm{Rep}_F^B(G)$.*
  (c) *B-admissibility is preserved under the formation of exterior amd symmetric powers, and D_B naturally commutes with both such constructions.*
  (d) *For $V \in \mathrm{Rep}_F^B(G)$ the natural map, $\mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V^\vee) \simeq \mathrm{D}_B(V \otimes_F V^\vee) \to \mathrm{D}_B(F) = E$ is a perfect duality between $\mathrm{D}_B(V)$ and $\mathrm{D}_B(V^\vee)$.*

(iii) *$\mathrm{Rep}_F^B(G)$ is a sub-Tannakian category of $\mathrm{Rep}_F(G)$.*

Proof. First we prove the equivalence of (a) and (b) in (i). Let $C := \mathrm{Fr}\ B$. Since B is (F, G)-regular, we have that $C^G = B^G = E$. For $\mathrm{D}_C(V) := (C \otimes_F V)^G$ we have the following commutative diagram,

$$\begin{tikzcd} B \otimes_E \mathrm{D}_B(V) \arrow[r] \arrow[d] & B \otimes_F V \arrow[dd] \\ B \otimes_E \mathrm{D}_C(V) \arrow[d] & \\ C \otimes_E \mathrm{D}_C(V) \arrow[r] & C \otimes_F V \end{tikzcd}$$

where the vertical arrows are obviously injective. To prove injectitivity of the top arrow it suffices to prove it for the bottom arrow. Hence, we can replace B with C i.e., we can reduce to the case when B is a field. In this case, the injectivity amounts to the claim that α_V carries an E-basis of $\mathrm{D}_B(V)$ to a B-linearly independent set $B \otimes_F V$, so it suffices to show that if $x_1, x_2, \dots, x_r \in B \otimes_F V$ are E-linearly independent and G-invariant then they are B-linearly independent. Assume on the contrary that there is a nontrivial B-linear dependence relation among the x_i's and consider such a relation of minimal length. We may assume it to have the form $x_r = \sum_{i<r} b_i x_i$ for some $r \geq 2$ since B is a field and all x_i are nonzero. Application of any $g \in G$ gives

$$x_r = g(x_r) = \sum_{i<r} g(b_i) g(x_i) = \sum_{i<r} g(b_i) x_i.$$

Thus, minimal length for the relation forces the equality of coefficients i.e., $b_i = g(b_i)$ for all $i < r$. So $b_i \in B^G = E$ for all $i < r$. Hence, we have a nontrivial E-linear dependence relation among $x_1, \dots, x_r$. This is a contradiction. Therefore we conclude that α_V is indeed injective.

Extending scalars from B to C preserves injectivity, so $C \otimes_E D_B(V)$ is a C-subspace of $C \otimes_F V$. Comparing C-dimensions then gives $\dim_E \mathrm{D}_B(V) \leq \dim_F V$. Now we will show that in case of equality of dimensions, say with common dimension d, the map α_V is an isomorphism. Let $\{e_j\}$ be an E-basis of $\mathrm{D}_B(V)$ and let $\{v_i\}$ be an F-basis of V, so relative to these bases we can express α_V using a $d \times d$ matrix (b_{ij}) over B. In other words, $e_j = \sum b_{ij} \otimes v_i$. The determinant $\det \alpha_V := \det(b_{ij}) \in B$ is nonzero due to the isomorphism property over $C = \mathrm{Fr}\ B$ (as scalar extension of α_V to a C-linear injection between C-vector spaces with the same finite dimension d must be an isomorphism). We want that $\det(\alpha_V) \in B^\times$, so then α_V is an isomorphism over B. Since B is an (F, G)-regular ring, to show that nonzero $\det(\alpha_V) \in B$ is a unit it suffices to show that it spans a G-stable F-line in B. The vectors $e_j = \sum b_{ij} \otimes v_i \in \mathrm{D}_B(V) \subset B \otimes_F V$ are G-invariant, so passing to d-th exterior powers on α_V gives that

$$\wedge^d(\alpha_V)(e_1 \wedge \cdots \wedge e_d) = \det(b_{ij}) v_1 \wedge \cdot \wedge v_d$$

is a G-invariant vector in $B \otimes_F \wedge^d V$. But G acts on $v_1 \wedge \cdots \wedge v_d$ by some character $\eta : G \to F^\times$, so G must act on $\det(b_{ij}) \in B - \{0\}$ through the $F^\times$-valued η^{-1}. Hence, $\det(b_{ij})$ is invertible in B and therefore α_V is an isomorphism. For the converse, if α_V is an isomorphism, then $\dim_E \mathrm{D}_B(V) = \dim_F V = \mathrm{rank}_B(B \otimes_F V)$.

To prove the equivalence of (b) and (c) in (i), we observe that the condition V is B-admissible is nothing but that there exists a B-basis $\{x_1, x_2, \dots, x_r\}$ of $B \otimes_F V$ such that each x_i is G-invariant. Since $\alpha_V(1 \otimes x_i) = x_i$, and α_V is always injective, the condition is equivalent to α_V being an isomorphism.

Next we move on to (ii). For any B-admissible V we have a natural isomorphism $B \otimes_E \mathrm{D}_B(V) \simeq B \otimes_F V$, so D_B is exact and faithful on the category $\mathrm{Rep}_F^B(G)$. To show (a) i.e., the subrepresentations and quotients of a B-admissible V are B-admissible, consider a short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

of $F[G]$-modules with B-admissible V. We need to show that V' and V'' are B-admissible. From the definition, D_B is left-exact, so we have a left-exact sequence of E-vector spaces,

$$0 \longrightarrow \mathrm{D}_B(V') \longrightarrow \mathrm{D}_B(V) \longrightarrow \mathrm{D}_B(V'')$$

with $\dim_E \mathrm{D}_B(V) = d$ by B-admissibility of V, so $d \leq \dim_E \mathrm{D}_B(V') + \dim_E \mathrm{D}_B(V'')$. From (i), we also know that the outer terms have respective E-dimension at most $d' = \dim_F V'$ and $d'' = \dim_F V''$. But $d = d' + d''$ from the given short exact sequence of $F[G]$-modules, so all these inequalities are in fact equalities, and in particular V' and V'' are B-admissible.

For (b) let V_1 and V_2 in $\mathrm{Rep}_F(G)$ be B-admissible with $d_i = \dim_F V_i$, then there is an evident natural map

$$\mathrm{D}_B(V_1) \otimes_E \mathrm{D}_B(V_2) \longrightarrow (B \otimes_F V_1) \otimes_E (B \otimes_F V_2) \longrightarrow B \otimes_F (V_1 \otimes V_2)$$

that is seen to be invariant under the G-action on the target, so we obtain a natural E-linear map

$$t_{V_1,V_2} : \mathrm{D}_B(V_1) \otimes_E \mathrm{D}_B(V_2) \longrightarrow \mathrm{D}_B(V_1 \otimes V_2),$$

with source having E-dimension $d_1 d_2$ (because of B-admissibility of V_i's) and target having E-dimension at most $\dim_F(V_1 \otimes_F V_2) = d_1 d_2$ by using (i) for $V_1 \otimes_F V_2$. Hence, if we can show that t_{V_1,V_2} is injective then it would be forced to be an isomorphism and $V_1 \otimes_F V_2$ would become B-admissible. To show that t_{V_1,V_2} is injective it suffices to check injectivity after composing with the inclusion of $\mathrm{D}_B(V_1 \otimes_F V_2)$ into $B \otimes_F (V_1 \otimes_F V_2)$ and by construction this composite coincides with the composition of the injective map

$$\mathrm{D}_B(V_1) \otimes_E \mathrm{D}_B(V_2) \longrightarrow B \otimes_E (\mathrm{D}_B(V_1) \otimes_E \mathrm{D}_B(V_2)) = (B \otimes_E \mathrm{D}_B(V_1)) \otimes_B (B \otimes_E \mathrm{D}_B(V_2))$$

and the isomorphism $\alpha_{V_1} \otimes_B \alpha_{V_2}$ (using again that V_i are B-admissible). Thus, we have that D_B naturally commutes with the formation of tensor products.

Now we take a look at (c). As a special case of (b), we see that if V is B-admissible then so is $V^{\otimes r}$ for any $r \geq 1$ with $\mathrm{D}_B(V)^{\otimes r} \simeq \mathrm{D}_B(V^{\otimes r})$. The quotient $\wedge^r V$ of $V^{\otimes r}$ is therefore also B-admissible, and there is an analogous map $\wedge^r \mathrm{D}_B(V) \longrightarrow \mathrm{D}_B(\wedge^r V)$ that fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{D}_B(V)^{\otimes r} & \xrightarrow{\simeq} & \mathrm{D}_B(V^{\otimes r}) \\ \downarrow & & \downarrow \\ \wedge^r \mathrm{D}_B(V) & \longrightarrow & \mathrm{D}_B(\wedge^r V) \end{array}$$

in which the left arrow is canonically surjective and the right arrow is surjective because it is D_B applied to a surjection between B-admissible representations. And therefore, we get that the bottom arrow is also surjective. But the left and right terms on the bottom have the same dimension (since V and $\wedge^r V$ are B-admissible with $\dim_F V = \dim_E \mathrm{D}_B(V)$), so the bottom arrow is an isomorphism. The same method works with symmetric powers in place of exterior powers.

The last point to prove is (d). For this, let $V \in \mathrm{Rep}_B^F(G)$. We need to show that $V^\vee$ is B-admissible and the resulting natural pairing between $\mathrm{D}_B(V)$ and $\mathrm{D}_B(V^\vee)$ is perfect. For any fnite-dimensional vector space W over a field with $\dim W = d \geq 1$ there is a natural isomorphism

$$\det(W^\vee) \otimes \wedge^{d-1} W \simeq W^\vee$$

defined by

$$(l_1 \wedge \cdots \wedge l_d) \otimes (\omega_2 \wedge \cdots \wedge \omega_d) \mapsto (\omega_1 \mapsto \det(l_i(w_j))),$$

and this is equivariant for the naturally induced group actions in case W is a linear representation space for a group. Hence, to show that $V^\vee$ is a B-admissible F-linear representation space for G we are reduced to proving B-admissibility for $\det(V^\vee) = (\det V)^\vee$. Since $\det V$ is B-admissible, we are reduced to showing the 1-dimensional case.

Now assume that V is B-admissible with $\dim_F V = 1$, and let v_0 be an F-basis of V, so B-admissibility gives that $\mathrm{D}_B(V)$ is 1-dimensional. Hence, $\mathrm{D}_B(V) = E(b \otimes v_0)$ for some nonzero $b \in B$. The isomorphism $\alpha_V : B \otimes_E \mathrm{D}_B(V) \simeq B \otimes_F V = B(1 \otimes v_0)$ between free B-modules of rank 1 carries the B-basis $b \otimes v_0$ of the left side to $b \otimes v_0 = b \cdot (1 \otimes v_0)$ on the right side, so $b \in B^\times$. The G-invariance of

$b \otimes v_0$ gives $g(b) \otimes g(v_0) = b \otimes v_0$ and we have $g(v_0) = \eta(g)v_0$ for some $\eta(g) \in F^\times$ (as V is a 1-dimensional representation space of G over F, say with character η), so $\eta(g)g(b) = b$. Thus, $b/g(b) = \eta(g) \in F^\times$. Letting $v_0^\vee$ be the dual basis of $V^\vee$, we can then see that $\mathrm{D}_B(V^\vee)$ contains the nonzero vector $b^{-1} \otimes v_0^\vee$, so it is a nonzero space. The 1-dimensional $V^\vee$ is therefore B-admissible, as required.

Now that we know that duality preserves B-admissibility in general, we fix a B-admissible V and aim to prove the perfectness of the pairing defined by

$$\langle \cdot, \cdot \rangle : \mathrm{D}_B(V) \otimes_E \mathrm{D}_B(V^\vee) \simeq \mathrm{D}_B(V \otimes_F V^\vee) \longrightarrow \mathrm{D}_B(F) = E.$$

From $\dim_F V = 1$, this is immediate from the explicit description of $\mathrm{D}_B(V)$ and $\mathrm{D}_B(V^\vee)$. In the general case, since V and $V^\vee$ are both B-admissible, for any $r \geq 1$ we have natural isomorphisms $\wedge^r \mathrm{D}_B(V) \simeq \mathrm{D}_B(\wedge^r V)$ and $\wedge^r \mathrm{D}_B(V^\vee) \simeq \mathrm{D}_B(\wedge^r V^\vee) \simeq \mathrm{D}_B((\wedge^r V)^\vee)$ with respect to which the pairing

$$\wedge_E^r \mathrm{D}_B(V) \otimes_E \wedge_E^r \mathrm{D}_B(V^\vee) \longrightarrow E$$

induced by $\langle \cdot, \cdot \rangle_V$ on r-th exterior powers is identified with $\langle \cdot, \cdot \rangle_{\wedge^r V}$. Since perfectness of a bilinear pairing between finite-dimesnional vector spaces of the same dimension is equivalent to perfectness of the induced bilinear pairing between their top exterior powers, by taking $r = \dim_F V$ we see that the perfectness of the pairing $\langle \cdot, \cdot \rangle_V$ for the B-admissible V is equivalent to prefectness of the pairing associated to the B-admissible 1-dimensional $\det V$. The 1-dimensional case was proved above, so this settles our claim.

For (iii), we see that it is obvious from Definition 3.10 and (ii) above. ∎

3.2 de Rham representations

Now that we are comfortable with the formalism of (F, G)-regular rings and having understood the category Fil_K, we move on to study the p-adic Galois representations. Here we set $F = \mathbb{Q}_p$ and $G = G_K$ and we look at de Rham representations. From Example 3.9 we have that B_{dR} is $(\mathbb{Q}_p, G_K)$-regular. Let $\mathrm{D}_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. From discussions leading to Theorem 3.13, we have an injective map $\alpha_{\mathrm{dR}} : B_{\mathrm{dR}} \otimes \mathrm{D}_{\mathrm{dR}}(V) \longrightarrow B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$.

Definition 3.14. A p-adic representation V of G_K is called *de Rham* if it is B_{dR}-admissible, equivalently if α_{dR} is an isomorphism or if $\dim_K \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$.

Example 3.15. For $n \in \mathbb{Z}$, $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(n))$ is 1-dimensional with its unique filtration jump in degree $-n$ (i.e., gr^{-n} is nonzero).

For a p-adic representation V of G_K, $\mathrm{D}_{\mathrm{dR}}(V)$ is finite-dimensional filtered K-vector space, with $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ for every $i \in \mathbb{Z}$. Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$ be the category of p-adic de Rham representation of G_K. So we have a covariant functor,

$$\begin{aligned} \mathrm{D}_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) &\longrightarrow \mathrm{Fil}_K \\ V &\longmapsto (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}. \end{aligned}$$

Next we prove the following imporant theorem,

Proposition 3.16. $\mathrm{D}_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \longrightarrow \mathrm{Fil}_K$ *is an exact, faithful and tensor functor.*

Proof. First of all, we check for exactness. For an exact sequence $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ of de Rham representations, we always have exactness on left, $0 \longrightarrow \mathrm{D}_{\mathrm{dR}}(V') \longrightarrow \mathrm{D}_{\mathrm{dR}}(V) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V'')$ while the exactness on the right is not clear a priori. But we know that $\dim_K \mathrm{D}_{\mathrm{dR}}(V') + \dim_K \mathrm{D}_{\mathrm{dR}}(V'') = \dim_{\mathbb{Q}_p} V' + \dim_{\mathbb{Q}_p} V'' = \dim_{\mathbb{Q}_p} V = \dim_K \mathrm{D}_{\mathrm{dR}}(V)$. Thus the sequence is in fact right exact. The functor D_{dR} is obviously faithful since $\mathrm{D}_{\mathrm{dR}}(V) \neq 0$ whenever $V \neq 0$.

Next, we need to check that if V_1, V_2 are de Rham representations then $\mathrm{D}_{\mathrm{dR}}(V_1) \otimes_K \mathrm{D}_{\mathrm{dR}}(V_2) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$ as filtered K-vector spaces. We have obvious injections $V_1 \hookrightarrow V_1 \otimes V_2$ and $V_2 \hookrightarrow V_1 \otimes V_2$

which naturally induce $\mathrm{D}_{\mathrm{dR}}(V_1) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$ and $\mathrm{D}_{\mathrm{dR}}(V_2) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$ respectively in Fil_K. Therefore, we also have the injection $\mathrm{D}_{\mathrm{dR}}(V_1) \otimes_K \mathrm{D}_{\mathrm{dR}}(V_2) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$ in Fil_K. Now from Theorem 3.13 (i), it must be that $\dim_K(\mathrm{D}_{\mathrm{dR}}(V_1) \otimes_K \mathrm{D}_{\mathrm{dR}}(V_2)) = \dim_{\mathbb{Q}_p}(V_1 \otimes V_2) \geq \dim_K \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$. Hence $\mathrm{D}_{\mathrm{dR}}(V_1) \otimes \mathrm{D}_{\mathrm{dR}}(V_2) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{dR}}(V_1 \otimes V_2)$ in Fil_K.

At last we need to check that the dual object is carried over by the functor i.e., for a de Rham representation V, its dual $V^\vee = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ should be such that $\mathrm{D}_{\mathrm{dR}}(V^\vee) \simeq (\mathrm{D}_{\mathrm{dR}}(V))^\vee$ as filtered K-vector spaces. Now $\mathrm{D}_{\mathrm{dR}}(V^\vee) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p))^{G_K} \simeq (\mathrm{Hom}_{B_{\mathrm{dR}}}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V, B_{\mathrm{dR}}))^{G_K} \simeq \mathrm{Hom}_K((B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}, K) = (D_{\mathrm{dR}}(V))^\vee$. ∎

Corollary 3.17. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $n \in \mathbb{Z}$, V is de Rham if and only if $V(n)$ is de Rham.*

Proof. By Example 3.15, this follows from the tensor compatibility in Theorem 3.16 and the isomorphism $V \simeq (V(n))(-n)$. ∎

B_{dR} has more structure than B_{HT} and it turns out that de Rham representations are Hodge-Tate as well.

Proposition 3.18. *Let V be a p-adic de Rham representation, then V is also a Hodge-Tate representation. Moreover,* $\mathrm{gr}\, \mathrm{D}_{\mathrm{dR}}(V) = \mathrm{D}_{\mathrm{HT}}(V)$ *(or* $\dim_K D_{\mathrm{dR}}(V) = \sum_{i\in\mathbb{Z}} \dim_K \mathrm{gr}^i D_{\mathrm{dR}}(V)$ *where* $\mathrm{gr}^i D_{\mathrm{dR}}(V) = \mathrm{Fil}^i D_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1} D_{\mathrm{dR}}(V)$*). In general there is an injection* $\mathrm{gr}\, \mathrm{D}_{\mathrm{dR}}(V) \hookrightarrow \mathrm{D}_{\mathrm{HT}}(V)$ *(or* $\dim_K D_{\mathrm{dR}}(V) \leq \sum_{i\in\mathbb{Z}} \dim_K \mathrm{gr}^i D_{\mathrm{dR}}(V)$ *) is an equality of $\mathbb{C}_K$-vector spaces when V is de Rham.*

Proof. For V a de Rham representation, the filtration on $\mathrm{D}_{\mathrm{dR}}(V)$ is given as $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) \otimes_{\mathbb{Q}_p} V)^{G_K}$ for every $i \in \mathbb{Z}$. We have a short exact sequence

$$0 \longrightarrow \mathrm{Fil}^{i+1} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^i B_{\mathrm{dR}} \longrightarrow \mathbb{C}_K(i) \longrightarrow 0.$$

On tensoring it with V, we get

$$0 \longrightarrow \mathrm{Fil}^{i+1} B_{\mathrm{dR}} \longrightarrow \mathrm{Fil}^i B_{\mathrm{dR}} \longrightarrow \mathbb{C}_K(i) \otimes_{\mathbb{Q}_p} V \longrightarrow 0.$$

Taking the G_K-invariant gives,

$$0 \longrightarrow \mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V) \longrightarrow \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) \longrightarrow (\mathbb{C}_K(i) \otimes_{\mathbb{Q}_p})^{G_K}.$$

Therefore,

$$\mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) = \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V) \hookrightarrow (\mathbb{C}_K(i) \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Hence,

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) \hookrightarrow \bigoplus_{i\in\mathbb{Z}} (\mathbb{C}_K(i) \otimes_{\mathbb{Q}_p} V)^{G_K} = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathrm{D}_{\mathrm{HT}}(V).$$

Now,

$$\sum_{i\in\mathbb{Z}} \dim_K \mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) = \sum_{i\in\mathbb{Z}} \dim_K \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) - \dim_K \mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V).$$

Since, $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = \mathrm{D}_{\mathrm{dR}}(V)$ for $i \ll 0$ and $\mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = 0$ for $i \gg 0$, we have that $\sum_{i\in\mathbb{Z}} \dim_K \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) - \dim_K \mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V) = \dim_K \mathrm{D}_{\mathrm{dR}}(V)$. Therefore we conclude $\sum_{i\in\mathbb{Z}} \dim_K \mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) = \dim_K \mathrm{D}_{\mathrm{dR}}(V)$. From the inclusion above, we also have that $\dim_{\mathbb{Q}_p} V \geq \dim_K \mathrm{D}_{\mathrm{HT}}(V) \geq \sum_{i\in\mathbb{Z}} \dim_K \mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) = \dim_K \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$. So we get equality everywhere from which it follows that V is indeed a Hodge-Tate representation.

∎

Definition 3.19. The *Hodge-Tate weights* of a de Rham representation V are those i for which the filtration on $\mathrm{D}_{\mathrm{dR}}(V)$ "jumps" from degree i to degree $i+1$, which is to say $\mathrm{gr}^i \mathrm{D}_{\mathrm{dR}}(V) \neq 0$. The multiplicity of such an i as a Hodge-Tate weight is the K-dimension of the filtration jump, i.e, $\dim_K \mathrm{gr}^i(\mathrm{D}_{\mathrm{dR}}(V)$.

Since $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(n))$ is a line with nontrivial gr^{-n}, we have that $\mathbb{Q}_p(n)$ has Hodge-tate weight $-n$ (with multiplicity 1). Thus, sometimes it is more convenient to define Hodge-Tate weight using the same filtration condition ($\mathrm{gr}^i \neq 0$) applied to the contravariant functor $\mathrm{D}^*_{\mathrm{dR}}(V) = \mathrm{D}_{\mathrm{dR}}(V^\vee) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{dR}})$ so as to negate things (so that $\mathbb{Q}_p(n)$ acquires Hodge-Tate weight n instead).

An important refinement of Proposition 3.16 is that the de Rham comparison isomorphism is also filtration-compatible.

Proposition 3.20. *For $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$, the G_K-equivariant B_{dR}-linear comparison isomorphism*

$$\alpha_{\mathrm{dR}} : B_{\mathrm{dR}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V) \simeq B_{\mathrm{dR}} \otimes V$$

respects the filtrations and its inverse does too.

Proof. By construction α_{dR} is filtration-compatible, so we need to show that its inverse is filtration-compatible as well. This statement is equivalent to showing that the induced B_{HT}-linear map $\mathrm{gr}(\alpha_{\mathrm{dR}})$ on associated graded objects is an isomorphism. We know that $\mathrm{gr}(B_{\mathrm{dR}} \otimes V) = B_{\mathrm{HT}} \otimes V$. From Remark 2.5, we see that $\mathrm{gr}(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) = B_{\mathrm{HT}} \otimes_K \mathrm{gr}\, \mathrm{D}_{\mathrm{dR}}(V)$. From Proposition 3.18, for the de Rham representation V of G_K, there is a natural isommorphism $\mathrm{gr}\, \mathrm{D}_{\mathrm{dR}}(V) \simeq \mathrm{D}_{\mathrm{HT}}(V)$. In this manner, $\mathrm{gr}(\alpha_{\mathrm{dR}})$ is naturally identified with the graded comparison morphism

$$\alpha_{\mathrm{HT}} : B_{\mathrm{HT}} \otimes_K \mathrm{D}_{\mathrm{HT}}(V) \longrightarrow B_{\mathrm{HT}} \otimes V$$

that is a graded isomorphism since V is Hodge-Tate. ∎

From the discussions at the end of the Section 1.5, we recall that the construction of B^+_{dR} as a topological ring with G_K-action only depends on $\mathcal{O}_{\mathbb{C}_K}$ endowed with its G_K-action. Thus, replacing K with a dicretely-valued complete subfield $K' \subset \mathbb{C}_K$ has no effect on the construction (aside from replacing G_K with the closed subgroup $G_{K'}$ within the isometric automorphism group of $\mathbb{C}_K$). It therefore makes sense to ask if the property of $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ being de Rham is insensitive to replacing K with such a K'.

To avoid any confusion, we write $\mathrm{D}_{\mathrm{dR},K}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, so for a discretely-valued complete extension K'/K inside of $\mathbb{C}_K$ we have $\mathrm{D}_{\mathrm{dR},K'}(V) = (B_{\mathrm{dR}} \otimes V)^{G_{K'}}$. There is a natural map $K' \otimes \mathrm{D}_{\mathrm{dR},K}(V) \longrightarrow \mathrm{D}_{\mathrm{dR},K'}(V)$ in $\mathrm{Fil}_{K'}$ for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ via the canonical embeddings of K and K' into the same B^+_{dR}.

Proposition 3.21. *For any complete discretely-valued extension K'/K inside of $\mathbb{C}_K$ and any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, the natural map $K' \otimes \mathrm{D}_{\mathrm{dR},K}(V) \longrightarrow \mathrm{D}_{\mathrm{dR},K'}(V)$ is an isomorphism in $\mathrm{Fil}_{K'}$. In particular, V is de Rham as a G_K-representation if and only if V is de Rham as a $G_{K'}$-representation.*

Proof. [BC09, Prop. 6.3.8]. ∎

Remark 3.22. In the 1-dimensional case, the Hodge-Tate and de Rham representations are equivalent. Indeed, we know that de Rham representations are always Hodge-Tate (in any dimension), and for the converse suppose that V is a 1-dimensional Hodge-Tate representation. Thus, it has some Hodge-Tate weight i, so if we replace V with $V(-i)$ (as we may without loss of generality since every $\mathbb{Q}_p(n)$ is de Rham) we may reduce to the case when the continuous character $\psi : G_K \longrightarrow \mathbb{Z}_p^\times$ of V is Hodge-Tate with Hodge-Tate weight 0. Hence, $\mathbb{C}_K(\psi)^{G_K} \neq 0$, so by Tate-Sen Theorem A.13 $\psi(I_K)$ is finite. By choosing a sufficiently ramified finite extension K'/K we can thereby arrange that $\psi(K') = 1$. Since the de Rham property is insensitive to replacing K with $\widehat{K'^{\mathrm{un}}}$, we thereby reduce to the case of the trivial character, which is de Rham.

The argument above can be used to show that the exact faithful tensor functor $\mathrm{D}_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p} \longrightarrow \mathrm{Fil}_K$ is not fully faithful. This is a serious deficiency, akin to losing information between good reduction and potentially good reduction. To improve on this situation we have already introduced rings B_{cris} and B_{st} with “finer structure”. Moreover, we will see that the functor $\mathrm{D}_{\mathrm{cris}} := \mathrm{D}_{B_{\mathrm{cris}}}$ takes values in a richer linear algebra category than filtered vector spaces.

3.3 Crystalline and Semistable p-adic representations

In this section we are going to introduce crystlline and semistable p-adic Galois representations studying which which is our central objective. As one might guess, the construction would include the ring B_{cris} and B_{st}, and follow the general steps that we discussed for de Rham representations.

Proposition 3.23. *The rings B_{cris} and B_{st} are $(\mathbb{Q}_p, G_K)$-regular which means that,*

(i) *B_{cris} and B_{st} are domains.*

(ii) $B_{\text{cris}}^{G_K} = B_{\text{st}}^{G_K} = C_{\text{st}}^{G_K} = K_0$.

(iii) *If $b \in B_{\text{cris}}$ (resp. B_{st}), $b \neq 0$ such that $\mathbb{Q}_p \cdot b$ is stable under G_K then b is invertible in B_{cris} (resp. B_{st}).*

Proof. (i) This is obvious since $B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}$.

(ii) This follows from Theorem 1.51.

(iii) We know that $\bar{k}$ is the residue field of R so $W(R)$ contains $W(\bar{k})$. Let $K' := \widehat{K_0^{\text{un}}} = W(\bar{k})[1/p] \subset W(R)[1/p]$. Then B_{cris} contains K'. Let $\overline{K'}$ be the algebraic closure of K' in $\mathbb{C}_K$, then B_{dR} is a $\overline{K'}$-algebra. There exists some $i \in \mathbb{Z}$ such that up to multiplication by t^{-i}, $b \in B_{\text{dR}}^+ = \text{Fil}^0 B_{\text{dR}}$ and $b \notin \text{Fil}^1 B_{\text{dR}}$. Suppose $g(b) = \psi(g)b$ where $\psi(g) \in \mathbb{Q}_p$ for $g \in G$. Let $\bar{b} = \theta(b) \in \mathbb{C}_K$ where $\theta : B_{\text{dR}} \longrightarrow \mathbb{C}_K$, then $\mathbb{Q}_p\bar{b} \simeq \mathbb{Q}_p(\psi)$ is a one dimensional $\mathbb{Q}_p$-subspace of $\mathbb{C}_K$ stable under G_K. By the Tate-Sen Theorem A.13 $\psi(I_K)$ is finite and $\bar{b} \in \overline{P} \subset B_{\text{dR}}^+$. So we have that $b' = b - \bar{b} \in \text{Fil}^i B_{\text{dR}}, b' \notin \text{Fil}^{i+1} B_{\text{dR}}$ for some $i \geq 1$. $\mathbb{Q}_p b'$ is also stable by G_K whose action is again defined by ψ. Now, the G_K-action on $\mathbb{Q}_p\theta(t^{-i}b')$ is defined by $\chi^{-i}\psi$ where χ is the cyclotomic character. But now $\chi^{-i}\psi(I_K)$ is not finite, so the only possibility we have is $b' = 0$ i.e., $b = \bar{b} \in \overline{K'}$.

We are interested in the case when $b \in B_{\text{st}}$. Then we have $b \in \overline{K'} \cap B_{\text{st}}$. But $\overline{K'} \cap B_{\text{st}} = K' \subset B_{\text{cris}}$. This is indeed the case because if not then assume $K' \subset M = \overline{K'} \cap B_{\text{st}}$. Then $K' \subset L \subset \text{Frac}(M)$ such that L is a non-trivial finite extension of K'. Let L_0 be the maximal unramified extension of K_0 inside L. Then $L_0 = K' = \widehat{K_0^{\text{un}}}$ and by (b) $B_{\text{st}}^{G_L} = K'$. But $\text{Frac}(M)^{G_L} = L$ which contradicts (b). So $\overline{K'} \cap B_{\text{st}} = K'$ and therefore $b \in K' \in B_{\text{cris}}$, i.e., b is invertible in B_{cris}. ∎

For a p-adic representation V, let $\mathrm{D}_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $\mathrm{D}_{\text{st}}(V) := (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. By the general formalism of Section 3.1, we have functors D_{cris} and D_{st} from the category $\text{Rep}_{\mathbb{Q}_p}(G_K)$ defined respectively by $V \rightsquigarrow (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $V \rightsquigarrow (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. On $\mathrm{D}_{\text{cris}}(V)$ and $\mathrm{D}_{\text{st}}(V)$ we have a lot of structure because of the structure on rings B_{cris} and B_{st}. The map φ on $B_{\text{cris}} \otimes V$ is given as $\varphi(b \otimes v) = \varphi(b) \otimes v$ for $b \in B_{\text{cris}}$, $v \in V$ and the two maps φ and N on $B_{\text{st}} \otimes_{\mathbb{Q}_p} V$ are defined as $\varphi(b \otimes v) = \varphi(b) \otimes v$ and $N(b \otimes v) = N(b) \otimes v$ for $b \in B_{\text{st}}$, $v \in V$. These maps obviously commute with the G_K-action respectively on $B_{\text{cris}} \otimes_{\mathbb{Q}_p} V$ and $B_{\text{st}} \otimes_{\mathbb{Q}_p} V$ where in the latter case we have $N\varphi = p\varphi N$. There are natural descending, exhaustive and separated filtration on $K \otimes_{K_0} \mathrm{D}_{\text{cris}}(V)$ and $K \otimes_{K_0} \mathrm{D}_{\text{st}}(V)$ via their injection into $\mathrm{D}_{\text{dR}}(V)$. From [BC09, Exercise 7.4.10] and Theorem 1.43, we conclude that D_{cris} is naturally valued in the category $\text{MF}_K(\varphi)$. Also, from Definition 1.52 we conclude that D_{st} is naturally valued in the category $\text{MF}_K(\varphi, N)$.

$\mathrm{D}_{\text{cris}}(V)$ and $\mathrm{D}_{\text{st}}(V)$ are K_0-vector spaces and the maps

$$\alpha_{\text{cris}} : B_{\text{cris}} \otimes_{K_0} \mathrm{D}_{\text{cris}}(V) \longrightarrow B_{\text{cris}} \otimes_{\mathbb{Q}_p} V$$
$$\alpha_{\text{st}} : B_{\text{st}} \otimes_{K_0} \mathrm{D}_{\text{st}}(V) \longrightarrow B_{\text{st}} \otimes_{\mathbb{Q}_p} V$$

are always injective by Theorem 3.13.

crystalline representationssemistable representations

(i) A p-adic representation V of G_K is called *crystalline* if it is B_{cris}-admissible i.e., α_{cris} above is an isomorphism. The full subcategory of crystalline representations is denoted as $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K)$.

(ii) A p-adic representation V of G_K is called *semistable* if it is B_{st}-admissible i.e., α_{st} above is an isomorphism. The full subcategory of semistable representations is denoted as $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$.

By Section 3.1 and Proposition 3.23, these full subcategories are stable under duality and tensor products. Moreover, similar arguments as used earlier for D_{dR} show that the following covariant functors are faithful, exact and naturally commute with the formation of tensor products and duals

$$\mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K(\varphi) \tag{3.1}$$
$$\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K(\varphi, N). \tag{3.2}$$

We can also write the contravariant functors,

$$\mathrm{D}^*_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K(\varphi) \tag{3.3}$$
$$\mathrm{D}^*_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K(\varphi, N). \tag{3.4}$$

where $\mathrm{D}^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{cris}})$ for any $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)$ and similarly $\mathrm{D}^*_{\mathrm{st}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{st}})$ for semistable representation V of G_K.

For V a p-adic representation $\mathrm{D}_{\mathrm{cris}}(V)$ is a K_0-vector subspace of $\mathrm{D}_{\mathrm{st}}(V)$ and $\dim_{K_0}\mathrm{D}_{\mathrm{cris}}(V) \leq \dim_{K_0}\mathrm{D}_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p}(V)$. From the (B, G)-regular ring formalism 3.13, we have

Proposition 3.24. (i) *A p-adic representation V is crystalline if and only if $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$.*

(ii) *A p-adic representation V is semi-stable if and only if $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p}(V)$.*

Now we prove an important result relating different subcategories of representations we have studied so far.

Proposition 3.25. (i) *If $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)$ then the natural map $K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V)$ is an isomorphism in Fil_K. In particular, crystalline representations are de Rham. Moreover, the B_{cris}-linear, Frobenius-compatible and G_K-equivariant crystalline comparison isomorphism*

$$\alpha_{\mathrm{cris}} : B_{\mathrm{cris}} \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) \simeq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$$

satisfies the property that $\alpha_{\mathrm{cris},K}$ is a filtered isomorphism.

(ii) *If $V \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ then the natural map $K \otimes_{K_0} \mathrm{D}_{\mathrm{st}}(V) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V)$ is an isomorphism in Fil_K. In particular, semistable representations are de Rham. Moreover, the B_{st}-linear, Frobenius-compatible, N-compatible and G_K-equivariant semistable comparison isomorphism*

$$\alpha_{\mathrm{st}} : B_{\mathrm{st}} \otimes_{K_0} \mathrm{D}_{\mathrm{st}}(V) \simeq B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

satisfies the property that $\alpha_{\mathrm{st},K}$ is a filtered isomorphism.

(iii) *Crystalline representations are semistable, and $\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{st}}(V)$ in $\mathrm{MF}_K(\varphi, N)$ for all V. If V is semistable and $\mathrm{D}_{\mathrm{st}}(V)$ has vanishing monodromy operator then V is crystalline.*

Proof. (i) Let V be a crystalline p-adic representation of G_K. Since $K \otimes_{K_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ and $[K : K_0] < +\infty$, by Theorem 1.40, we have a natural map $K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) = K \otimes_{K_0} (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} = (K \otimes_{K_0} (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V))^{G_K} = ((K \otimes_{K_0} B_{\mathrm{cris}}) \otimes_{\mathbb{Q}_p} V)^{G_K} \hookrightarrow (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \mathrm{D}_{\mathrm{dR}}(V)$. So $K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V)$ is a subobject of $\mathrm{D}_{\mathrm{dR}}(V)$ in Fil_K. Since V is crystalline, $\dim_{K_0}\mathrm{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$, therefore $\dim_K K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V \geq \dim_K \mathrm{D}_{\mathrm{dR}}(V)$. Hence $\dim_K \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$, i.e., V is de Rham. To show that the K-linear inverse $\alpha^{-1}_{\mathrm{cris},K}$ is filtration-compatible too, or in other words that the filtration-compatible $\alpha_{\mathrm{cris},K}$ is a filtered isomorphism, it is equivalent to show that $\mathrm{gr}(\alpha_{\mathrm{cris},K})$ is an isomorphism. Since $K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) \simeq \mathrm{D}_{\mathrm{dR}}(V)$ and $\mathrm{gr}(K \otimes_{K_0} B_{\mathrm{cris}}) = \mathrm{gr}\ B_{\mathrm{dR}} = B_{\mathrm{HT}}$, the method of the proof of Proposition 3.20 adapts to show that $\mathrm{gr}(\alpha_K)$ is identified with the Hodge-Tate isomorphism for V.

(ii) By replacing the crystalline representation in (i) by a semistable representation and B_{cris} by B_{st}, the same argument works. Therefore, we conclude that a semistable representation is de Rham and $\alpha_{\mathrm{st},K}$ is a filtered isomorphism.

(iii) Since $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$, we see that $D_{\mathrm{st}}(V)^{N=0} = D_{\mathrm{cris}}(V)$ in $\mathrm{MF}_K(\varphi)$ for every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. In particular, if V is crystalline then for dimension reasons the K_0-linear inclusion $\mathrm{D}_{\mathrm{cris}}(V) \subset \mathrm{D}_{\mathrm{st}}(V)$ is an isomorphism in $\mathrm{MF}_K(\varphi, N)$. Thus crystalline representations are semistable. If V is semistable but $\mathrm{D}_{\mathrm{st}}(V)$ has vanishing monodromy operator then $\mathrm{D}_{\mathrm{cris}}(V) = \mathrm{D}_{\mathrm{st}}(V)$ and this has K_0-dimension $\dim_{\mathbb{Q}_p} V$, so V is crystalline.

■

To summarize:

$$\text{crystalline} \implies \text{semistable} \implies \text{de Rham} \implies \text{Hodge-Tate}.$$

3.4 An example: Tate's elliptic curve

In this section we explicitly construct a p-adic Galois representation using Tate's curve. As we would see later, this curve would play an imporatnt role in the classification of Galois representations of certain type coming from elliptic curves over $\mathbb{Q}_p$. For the remiander of this section we let $K = K_0$ i.e., K is an unramified extension of $\mathbb{Q}_p$.

The analytic theory of elliptic curves over complex numbers says that any complex elliptic curve has a parametrization $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. One may suppose to do the same with $\overline{K}$. However, such an attempt fails because $\overline{K}$ has no non-trivial discrete subgroups. Tate used a different approach of parametrization of elliptic curves in the complex case. For the lattice Λ, we may choose $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{C}^\times$. Setting $q := e^{2\pi\iota\tau}$, the exponential map $e : z \to e^{2\pi\iota z}$ induces an isomorphism $\mathbb{C}/\Lambda \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ where $q^{\mathbb{Z}}$ is the free multiplicative group generated by $e(\tau)$. Now the analogous situation over $\overline{K}$ is much more promising, since $\overline{K}^\times$ has lots of discrete subgroups. Since, the formulas defining the coefficients in the complex-analytic case are given by power series in q, the same could be done for K. Indeed, it turns out that we get a p-adic analytic isomorphism of $\overline{K}^\times/q^{\mathbb{Z}}$ with a certain curve E_q. The results could be made more precise in the following theorem which we state without proof. A detailed account could be found in the book of Silverman [Sil94, V.3].

Theorem 3.26. *Let K be a p-adic field with absolute value $|\cdot|$ and let $q \in K^\times$ satisfying $|q| < 1$. Also let*

$$s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n}, \quad a_4(q) = -s_3(q) \ \text{ and } \ a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}.$$

(i) *The series $a_4(q)$ and $a_6(q)$ converge in K. Define the Tate curve E_q by the equation $E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q)$.*

(ii) *The Tate curve is an elliptic curve defined over K with discriminant $\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$ and j-invariant $j(E_q) = \frac{1}{q} + 744 + 196884q + \cdots$.*

(iii) *The series*

$$X(u, q) = \sum_{n \in \mathbb{Z}} \frac{q^n u}{(1 - q^n)^2} - 2s_1(q) \ \text{ and } \ Y(u, q) = \sum_{n \in \mathbb{Z}} \frac{(q^n u)^2}{(1 - q^n u)^3} + s_1(q)$$

converge for all $u \in \overline{K}, u \notin q^{\mathbb{Z}}$. They define a surjective homomorphism

$$\begin{aligned} \alpha : \overline{K}^\times &\longrightarrow E_q(\overline{K}) \\ u &\longmapsto \begin{cases} (X(u, q), Y(u, q)) & \text{if } u \notin q^{\mathbb{Z}} \\ 0 & \text{if } u \in q^{\mathbb{Z}} \end{cases} \end{aligned}$$

The kernel of α is $q^{\mathbb{Z}}$.

(iv) *The map* α *in* (iii) *is compatible with the action of the Galois group* $G_{\overline{K}/K}$ *in the sense that* $\alpha(g(g)u) = g(\alpha(u))$ *for every* $u \in \overline{K}^{\times}$ *and* $g \in G_{\overline{K}/K}$. *In particular for any algebraic extension* L/K, α *induces an isomorphism*

$$\alpha : L^{\times}/q^{\mathbb{Z}} \longmapsto E_q(L).$$

With the help of this theorem, we want to construct a p-adic representation. Let T_pE_q be the usual Tate module attached to the Tate curve E_q. Let $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ with $\varepsilon^{(n)} \in \overline{K}$ be a p-power compatible sequence of primitive p^n-th roots of unity (as chosen in Section 1.5). Let $q^{(0)} = q$, $(q^{(n+1)})^p = q^{(n)}$ be another sequence with q as chosen in the definition of E_q. The isomorphism α induces the following isomorphisms,

$$\overline{K}^{\times}/q^{\mathbb{Z}} \longrightarrow E_q(\overline{K})$$
$$\{x \in \overline{K}^{\times}/q^{\mathbb{Z}}, x^{p^n} \in q^{\mathbb{Z}}\} \longrightarrow E_q(\overline{K})[p^n].$$

An exact sequence: If $x \in E_q[p^n]$ and $\widehat{x}$ is any lift of x to $\overline{K}^{\times}$, then there exists an integer $N_q(\widehat{x})$ such that $\widehat{x}^{p^n} = q^{N_q(\widehat{x})}$ and

$$E_q[p^n] \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$
$$x \mapsto N_q(\widehat{x}) \mod p^n\mathbb{Z}$$

defines a surjective morphism. Indeed this map is well defined, for if $\widehat{x_1}$ and $\widehat{x_2}$ are two different lifts of $x \in E_q[p^n]$, then $\widehat{x_1} = \widehat{x_2}q^m$ for some $m \in \mathbb{Z}$. Now, since $\widehat{x_1}^{p^n} = \widehat{x_2}^{p^n} q^{tp^n}$, therefore $N_q(\widehat{x_1}) = N_q(\widehat{x_2}) + tp^n$, so they are equal modulo $p^n\mathbb{Z}$. We see that surjectivity is immediate from the definition. The kernel of this map is $\mu_{p^n}(\overline{K})$ i.e., p^n-th roots of unity. So we have an exact sequence

$$0 \longrightarrow \mu_{p^n}(\overline{K}) \longrightarrow E_q[p^n] \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0$$

Passing to the limit and identifying $\varprojlim_n \mu_{p^n} = \mathbb{Z}_p(1)$, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T_pE_q \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{3.5}$$

Action of $\mathbf{G_K}$: Coming back to the morphism α, it is clear that $\{x \in \overline{K}^{\times}/q^{\mathbb{Z}}, x^{p^n} \in q^{\mathbb{Z}}\} = \{(\varepsilon^{(n)})^i(q^{(n)})^j, 0 \leq i,j < p^n - 1\}$, so $\alpha(\varepsilon^{(n)}), \alpha(q^{(n)})$ form a basis of $E_q(\overline{K})[p^n]$. Therefore a basis of T_pE_q is given by $e := \alpha(\varprojlim_n \varepsilon^{(n)})$ and $f = \alpha(\varprojlim_n q^{(n)})$. Since the G_K-action is compatible with isomorphisms above, we get

$$g(e) = g(\alpha(\varprojlim_n \varepsilon^{(n)})) = \alpha(g(\varprojlim_n \varepsilon^{(n)})) = \alpha(\varprojlim_n g(\varepsilon^{(n)})) = \alpha(\varprojlim_n (\varepsilon)^{\chi(g)}) = \alpha(\varprojlim_n \varepsilon^{(n)})^{\chi(g)} = \chi(g)e$$

and

$$g(f) = g(\alpha(\varprojlim_n q^{(n)})) = \alpha(g(\varprojlim_n q^{(n)})) = \alpha(\varprojlim_n g(q^{(n)}\varepsilon^{(n)}) = \alpha(\varprojlim_n q^{(n)} + c(g)\varprojlim_n \varepsilon^{(n)}) = f + c(g)e$$

for some p-adic integer $c(g)$. Hence the matrix for g acting on (e, f) is given by

$$\begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}.$$

p-adic periods of E_q: To determine p-adic periods of $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pE_q$ we look at the elements of $\mathrm{D}_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes V)^{G_K}$. We know that $t = \log([\varepsilon]) \in B_{\mathrm{dR}}$, and $g(t) = \chi(g)t$. So an obvious choice for basis element would be $x := t^{-1} \otimes e \in \mathrm{D}_{\mathrm{dR}}(V)$ since $g(t^{-1} \otimes e) = g(t^{-1}) \otimes g(e) = \chi(g)^{-1}\chi(g)(t \otimes e) = t \otimes e$ for every $g \in G_K$. We know that e and f form a basis for T_pE_q, so we take $y := a \otimes e + 1 \otimes f$ to be another term which would linearly independent from x and hence form a basis of $\mathrm{D}_{\mathrm{dR}}(V)$ together with x. Also, y should be stable under the action of G_K i.e., for any $g \in G_K$ we must have $g(y) = y$. Since

$g(y) = g(a\otimes e+1\otimes f) = g(a)\otimes g(e)+1\otimes g(f) = g(a)\otimes\chi(g)e+1\otimes(f+c(g)e) = (g(a)\chi(g)+c(g))\otimes e+1\otimes f$, therefore we must have that $g(a)\chi(g) + c(g) = a$.

Let $\tilde{q} \in R(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$ such that $\tilde{q} = (q^{(0)}, q^{(1)}, \ldots)$. So $g(\tilde{q}) = (g(q^{(0)}), g(q^{(1)}), \ldots) = q\varepsilon^{c(g)}$. From section 1.7 we know how to define $u := \log[\tilde{q}]$. For the G_K-action we observe that for any $g \in G_K$, $g(u) = g(\log[\tilde{q}]) = \log[g(\tilde{q})] = \log[\tilde{q}\varepsilon^{c(g)}] = \log[\tilde{q}] + c(g)\log[\varepsilon] = u + c(g)t$. And for $a = -ut^{-1}$, we see that $g(a)\chi(g) + c(g) = g(-ut^{-1})\chi(g) + c(g) = -ut^{-1} = a$. Hence, a basis of $D_{\mathrm{dR}}(V)$ could be given by $\{x = t^{-1} \otimes e, y = -ut^{-1} \otimes e + 1 \otimes f\}$. Thus, we conclude that V is a de Rham representation.

Filtration on $\mathrm{D}_{\mathrm{dR}}(V)$: For the filtration on $\mathrm{D}_{\mathrm{dR}}(V)$, we recall that $\mathrm{Fil}^i\mathrm{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^i B_{\mathrm{dR}}\otimes V)^{G_K}$ with $\mathrm{Fil}^0 B_{\mathrm{dR}} = B_{\mathrm{dR}}^+$ and $\mathrm{Fil}^i B_{\mathrm{dR}} = B_{\mathrm{dR}}^+ t^i$ for $i \neq 0$. From Proposition 1.29, we also know that t generates the maximal ideal of B_{dR}^+. Since $\theta_{\mathrm{dR}}(u - \log_p(q)) = 0$ we therefore conclude that $u - \log_p(q) = bt$ for some $b \in B_{\mathrm{dR}}^+$. Now $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V)$ is given by $(B_{\mathrm{dR}}^+ \otimes V)^{G_K}$ which we would like to determine explicitly. Since $t^{-1} \notin B_{\mathrm{dR}}^+$, we conclude that $(B_{\mathrm{dR}}^+ \otimes V)^{G_K}$ could not be Kx or Ky. It is also clear that $(B_{\mathrm{dR}}^+ \otimes V)^{G_K} \neq 0$. From the remark in previous paragraph we see that $y = -(bt + \log_p(q))t^{-1} \otimes e + 1 \otimes f = -b \otimes e + 1 \otimes f - \log_p(q)t^{-1} \otimes e$. Therefore $y + \log_p(q)x$ has no term involving t and therefore $(B_{\mathrm{dR}}^+ \otimes V)^{G_K} = (y + \log_p(q)x)K$. Hence,

$$\mathrm{Fil}^i\mathrm{D}_{\mathrm{dR}}(V) = \begin{cases} \mathrm{D}_{\mathrm{dR}}(V) & \text{if } i < 0 \\ (y + \log_p(q)x)K & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Next, we show that V is semi-stable. For this it is enough to show that u and t are in B_{st}^+. Clearly, $t \in B_{\mathrm{cris}}^+ \subset B_{\mathrm{st}}^+$. Also, since $q^{(0)}/p^{\upsilon_p(q)}$ is a unit in $\mathcal{O}_K$, we get that $\log[\tilde{q}/\tilde{p}^{\upsilon_p(q)}]$ converges in B_{cris}^+ by the construction of logarithm map in Section 1.7. Now, since $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[\log[\varpi]]$ writing $u = \upsilon_p(q)\log[\varpi] + \log[\tilde{q}/\tilde{p}^{\upsilon_p(q)}]$ shows that $u \in B_{\mathrm{st}}^+$. Hence V is a semistable representation.

Action of φ **and** N **on** $(B_{\mathrm{st}} \otimes V)^{G_K}$: For the Frobenius action on the basis is given as, $\varphi(x) = \varphi(t^{-1} \otimes e) = p^{-1}t^{-1} \otimes e = p^{-1}x$ and $\varphi(y) = \varphi(-ut^{-1}) \otimes e + \varphi(1) \otimes f = -pu(pt)^{-1} \otimes e + 1 \otimes f = y$. Therefore the matrix of φ is given by

$$[\varphi] = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

For N, since $t \in B_{\mathrm{cris}}$ we get $N(x) = 0$. Also $N(u) = \upsilon_p(q)$ from above, therefore $N(y) = \upsilon_p(q)x$. The matrix for N is then given as

$$[N] = \begin{pmatrix} 0 & \upsilon_p(q) \\ 0 & 0 \end{pmatrix}.$$

If $q = -p$ then $u = \log[\varpi]$ and $N(u) = 1$.

Kummer Theory: Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ be an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$, i.e., there is an exact sequence

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0.$$

All these extensions are classified by the cohomology group $\mathrm{H}^1(K, \mathbb{Q}_p(1))$, which is described by Kummer Theory. Indeed, for every $n \geq 1$, there is an isomorphism $\delta_n : K^\times/(K^\times)^{p^n} \longrightarrow \mathrm{H}^1(K, \mu_{p^n})$. By passing to the limit we get a map $\delta : \widehat{K^\times} \longrightarrow \mathrm{H}^1(K, \mathbb{Z}_p(1))$ because $\varprojlim_n \mu_{p^n} \simeq \mathbb{Z}_p(1)$ after a choice of p-power compatible sequence $\varepsilon^{(n)}$. By tensoring with $\mathbb{Q}_p$, we get an isomorphism $\delta : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \widehat{K^\times} \longrightarrow \mathrm{H}^1(K, \mathbb{Q}_p(1))$ which is defined as follows: if $q = q^{(0)} \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \widehat{K^\times}$, choose a sequence $q^{(n)}$ such that $(q^{(n+1)})^p = q^{(n)}$ and define $c = \delta(q)$ by $(\varepsilon^{(n)})^{c(g)} = g(q^{(n)})/(q^{(n)})$. Of course, this depends on the choice of $q^{(n)}$, but two different choices give cohomologous cocycles.

Proposition 3.27. *Every extension of* $\mathbb{Q}_p$ *by* $\mathbb{Q}_p(1)$ *is semistable.*

Proof. From the notations as described above, we get $tc(g) = g(\log[\tilde{q}]) - \log[\tilde{q}]$ where $\tilde{q} = (q^{(n)})$. If $q \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_K^\times}$, then the series defining $u = \log[\tilde{q}]$ converges in B_{cris}^+ and therefore the extension V is crystalline. In general, if $q \in \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \widehat{K^\times}$, then $\log[\tilde{q}]$ will be in $B_{\text{cris}}^+ + v_p(q)\log[\varpi]B_{\text{cris}}^+ \subset B_{\text{st}}^+$. Hence, the extension V is always semistable. ∎

In Proposition 3.27, for such an extension V the K-vector space $\mathrm{D}_{\text{st}}(V)$ will have a basis $x = t^{-1} \otimes e$ and $y = -\log[\tilde{q}]t^{-1} \otimes e + 1 \otimes f$ so that $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. Moreover, the filtration on $\mathrm{D}_{\text{dR}}(V)$ given by $\mathrm{Fil}^0\mathrm{D}_{\text{dR}}(V) = (y + \log_p(q)x)K$ by taking the convention $\log_p(p) = 0$.

3.5 p-adic de Rham comparison theorem: Two examples

The goal of this section is to understand in special cases the following complicated looking theorem (stated in its full generality).

Theorem 3.28. (Beilinson, [Bei12]) *Let X/K be a scheme of finite type and separated. There is a natural isomorphism*

$$\rho_{\text{dR}} : \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \xrightarrow{\simeq} \mathrm{H}^n_{\text{dR}}(X/K) \otimes_K B_{\text{dR}}$$

compatible with respective G_K-action on $\mathrm{H}^n_{\text{ét}}$ and B_{dR} as well as respective filtrations on H^n_{dR} and B_{dR}. Moreover, if X has a semistable (resp. crystalline) model over $\mathcal{O}_K$, we can replace B_{dR} with the smaller ring B_{st} (resp. B_{cris}).

Proving this theorem would require a lot of work and we would not attempt to do so here. However, we would illustrate this theorem via two examples for the case $n = 1$. We would not attempt to invoke the general understanding of étale cohomology, but rather give a direct description in the cases we consider. Here we only consider commutative group schemes. Note that for G a commutative group scheme $\mathrm{Hom}(T_pG, \mathbb{Z}_p(1)) \simeq \mathrm{H}^1_{\text{ét}}(G_{\overline{K}}, \mathbb{Z}_p)$ where T_pG is the p-adic Tate module of G. Our objective is to construct a duality pairing, $\mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)^\vee \times \mathrm{H}^1_{\text{dR}}(X_K) \longrightarrow B_{\text{dR}}$ in two specific cases. Some basic facts on Kähler differentials and algebraic de Rham cohomology is part of Appendix B.

3.5.1 1-dimensional torus $\mathbb{G}_{m,K}$

For the first example, we look at the case of 1-dimesional torus $\mathbb{G}_{m,K}$ or by abuse of notation, we write $\mathbb{G}_m$. As a group, $\mathbb{G}_m(\overline{K}) = \overline{K}^\times$ and as a scheme $\mathbb{G}_{m,K} = \mathrm{Spec}\ K[T, T^{-1}]$. Multiplicative structure on $\mathbb{G}_{m,K}$ is induced by the following morphism of rings

$$\begin{aligned} m : K[T, T^{-1}] &\longrightarrow K[X, X^{-1}, Y, Y^{-1}] \\ T &\longmapsto XY. \end{aligned}$$

Lemma 3.29 (The Tate module)**.** *$\mathbb{G}_m(\overline{K})$ is an abelian group. The Tate module of $\mathbb{G}_m$ is given as $T_p\mathbb{G}_m = \varprojlim_n \mathbb{G}_m(\mathcal{O}_{\overline{K}})[p^n]$.*

Proof. For ε_n (resp. ε_{n+1}) primitive p^n-th (resp. p^{n+1}-th) root of unity, we have that $\varepsilon_{n+1}^p = \varepsilon_n$ i.e., $\mathbb{G}_m(\overline{K})[p^{n+1}] \xrightarrow{(\cdot)^p} \mathbb{G}_m(\overline{K})[p^n]$ is a morphism of multiplicative abelian groups for all $n \in \mathbb{N}$. Hence, we could write the Tate module of $\mathbb{G}_m$ as $T_p\mathbb{G}_m = \varprojlim_n \mathbb{G}_m(\overline{K})[p^n] \cong \varprojlim_n \mathbb{G}_m(\mathcal{O}_{\overline{K}})[p^n]$. Here, we can take $\mathcal{O}_{\overline{K}}$ points instead of $\overline{K}$ points because all points in $\mathbb{G}_m(\overline{K})[p^n]$ are roots of unity and therefore they are obviously inside $\mathcal{O}_{\overline{K}}$. ∎

Lemma 3.30 (de Rham cohomology)**.** *For $\mathbb{G}_m = \mathrm{Spec}\ K[T, T^{-1}]$,*

$$\mathrm{H}^0_{\text{dR}}(\mathbb{G}_m) = K \quad \textit{and} \quad \mathrm{H}^1_{\text{dR}}(\mathbb{G}_m) = K \cdot \frac{dT}{T}.$$

Proof. Before, we compute the de Rham cohomology group for $\mathbb{G}_{m,K}$ we need to know the module of Kähler differentials for $K[T, T^{-1}]$. For this, using the Definition B.2 it is obvious that

$$\Omega^1_{K[T,T^{-1}]/K} = K[T, T^{-1}]dT = \mathcal{O}_K[T, T^{-1}]dT \otimes_{\mathcal{O}_K} K = \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K.$$

Now, using the Definition B.9, we want to compute $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,K})$. Since $\mathbb{G}_m = \mathrm{Spec}\ K[T, T^{-1}]$, we look at the de Rham complex $K[T, T^{-1}] \xrightarrow{d} K[T, T^{-1}]dT$. Also,

$$\mathrm{H}^0_{\mathrm{dR}}(K[T, T^{-1}]/K) = K \quad \text{and} \quad \mathrm{H}^1_{\mathrm{dR}}(K[T, T^{-1}]/K) = \frac{ker(\Omega^1_{K[T,T^{-1}]/K} \to \Omega^2_{K[T,T^{-1}]/K})}{im(\Omega^0_{K[T,T^{-1}]/K} \to \Omega^1_{K[T,T^{-1}]/K})}.$$

We have that $\Omega^2_{K[T,T^{-1}]/K} = 0$. So the numerator in the expression of $\mathrm{H}^1_{\mathrm{dR}}(K[T, T^{-1}]/K)$ is $K[T, T^{-1}]dT$. For the denominator, we observe that given any $P'(T) = a_{-m}T^{-m} + \cdots + a_{-2}T^{-2} + a_0 + a_1T + \cdots + a_nT^n \in K[T, T^{-1}]$ we have that

$$P(T) = \frac{a_{-m}}{-m+1}T^{-m+1} + \cdots - a_{-2}T^{-1} + a_0T + \frac{a_1}{2}T^2 + \cdots + \frac{a_n}{n+1}T^{n+1} \in K[T, T^{-1}]$$

such that $d(P(T)) = P'(T)$. Moreover, for $P'(T) \in K[T, T^{-1}]$ such that $a_{-1} \neq 0$ there is no $P(T) \in K[T, T^{-1}]$ such that $d(P(T)) = P'(T)dT$. Therefore in the quotient $K[T, T^{-1}]/d(K[T, T^{-1}]) \cdot dT$ we are left with $K \cdot dT/T$, i.e., $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m) = K \cdot dT/T$. ∎

For $\mathbb{G}_{m,K}$, we have $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mathbb{Q}_p) \simeq \mathrm{Hom}(T_p\mathbb{G}_m, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. So, we want to look at the pairing $T_p\mathbb{G}_m \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m) \to B_{\mathrm{cris}}$. Here we replace B_{dR} with B_{cris}, reasons for which will become clear later on. For remainder of this section we fix a choice of p-power compatible primitive p^n-th roots of unity i.e., a choice of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ where ε_n is primitive p^n-th root of unity and $\varepsilon_{n+1}^p = \varepsilon_n$.

Before we define the pairing map, we mention without proof the following lemma from [Fon82a, Thm. 1]

Lemma 3.31. *There exists an isomorphism*

$$f : \mathbb{C}_K(1) \xrightarrow{\sim} \mathbb{Q}_p \otimes T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$$

where $1 \otimes \{\varepsilon_n\}_{n\in\mathbb{N}} \in \mathbb{C}_K(1)$ *is carried by* f *to* $1 \otimes \left\{\frac{d\varepsilon_n}{\varepsilon_n}\right\}_{n\in\mathbb{N}} \in \mathbb{Q}_p \otimes T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}$.

Proposition 3.32 (A duality pairing)**.** *There exists a duality pairing bilinear map,*

$$\psi : T_p\mathbb{G}_m \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m) \longrightarrow B_{\mathrm{cris}}$$
$$\left(\{\varepsilon_n\}_{n\in\mathbb{N}}, \frac{dT}{T}\right) \mapsto t.$$

where $t \in B_{\mathrm{cris}}$ *as defined in Section 1.5.*

Proof. To properly define this pairing, we make the following observation. Let u_n be a collection of K-algebra maps defined as

$$u_n : \mathcal{O}_K[T, T^{-1}] \longrightarrow \mathcal{O}_{K_n}$$
$$T \longmapsto \varepsilon_n,$$

where $\mathcal{O}_{K_n}$ is the ring of integers of the field $K(\varepsilon_n)$ and ε_n is a primitive p^n-th root of unity (chosen above). Also, $\varepsilon_{n+1}^p = \varepsilon_n$. From the map above and the discussion preceding Proposition B.6, we get an $\mathcal{O}_{K_n}$-linear map between the module of Kähler differentials

$$u_n^* : \mathcal{O}_{K_n} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K} \longrightarrow \Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K} \subset \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n]$$
$$a \otimes dT \longmapsto a d\varepsilon_n.$$

where $\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n]$ denotes its p^n-torsion elements. Since $dT/T \in \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K}$ is the element of our interest as it generates $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)$, we observe that the image of $1 \otimes dT/T$ under u_n^* is a p^n-torsion element

$$p\frac{d\varepsilon_{n+1}}{\varepsilon_{n+1}} = p\frac{\varepsilon_{n+1}^{p-1}d\varepsilon_{n+1}}{\varepsilon_{n+1}^p} = \frac{d\varepsilon_{n+1}^p}{\varepsilon_{n+1}^p} = \frac{d\varepsilon_n}{\varepsilon_n}.$$

Equivalently, this means

$$pu_{n+1}^*\Big(1 \otimes \frac{dT}{T}\Big) = u_n^*\Big(1 \otimes \frac{dT}{T}\Big).$$

Now, $d\varepsilon_n/\varepsilon_n$ is a generator of $\Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K}$ and $u_n^*(1 \otimes dT/T) = d\varepsilon_n/\varepsilon_n$ so we look at the submodule $\Gamma_n = (1 \otimes dT/T)\mathcal{O}_{K_n}$ of $\mathcal{O}_{K_n} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K}$ generated by $1 \otimes dT/T$. Therefore, we can draw a commutative diagram

$$\begin{tikzcd} \Gamma_{n+1} \arrow[r, "u_{n+1}^*"] \arrow[d] & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^{n+1}] \arrow[d, "\cdot p"] \\ \Gamma_n \arrow[r, "u_n^*"] & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n] \end{tikzcd}$$

where the left vertical arrow is the restriction of $\mathcal{O}_{K_{n+1}} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K} \to \mathcal{O}_{K_n} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K}$ induced by the natural map $\varepsilon_{n+1} \mapsto \varepsilon_{n+1}^p$. Now, by passing to the limit and setting $\Gamma = (1 \otimes dT/T)\mathcal{O}_{\overline{K}}$ for the submodule of $\mathcal{O}_{\overline{K}} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K}$, we get a map of $\mathcal{O}_{\overline{K}}$-modules

$$\begin{aligned} u : \Gamma &\longrightarrow T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \\ 1 \otimes \frac{dT}{T} &\longmapsto \Big\{\frac{d\varepsilon_n}{\varepsilon_n}\Big\}_{n\in\mathbb{N}}. \end{aligned}$$

Using Lemma 3.31 we have an $\mathcal{O}_{\overline{K}}$-linear composition of maps

$$\begin{array}{ccccc} \Gamma & \xrightarrow{u} & \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} & \xrightarrow{f^{-1}} & \mathbb{C}_K(1) = \mathbb{C}_K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \\ 1 \otimes \dfrac{dT}{T} & \longmapsto & 1 \otimes \Big\{\dfrac{d\varepsilon_n}{\varepsilon_n}\Big\}_{n\in\mathbb{N}} & \longmapsto & 1 \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}. \end{array}$$

From Proposition 1.36 we get a surjective map $\theta^+_{\mathrm{cris}} : B_{\mathrm{cris}} \twoheadrightarrow \mathbb{C}_K$ of $\mathcal{O}_K$-algebras. By the filtration on B_{dR} we also have an exact sequence

$$0 \longrightarrow t^2B^+_{\mathrm{dR}} \longrightarrow tB^+_{\mathrm{dR}} \longrightarrow \mathbb{C}_K(1) \longrightarrow 0.$$

Restricting to B^+_{cris} we get a surjective map of $\mathcal{O}_K$-modules $\vartheta^+_{\mathrm{cris}} : tB^+_{\mathrm{cris}} \to \mathbb{C}_K(1)$ with $\vartheta^+_{\mathrm{cris}}(x) = \theta^+_{\mathrm{cris}}(x/t) \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}$. In particular, $\vartheta^+_{\mathrm{cris}}(t) = 1 \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}$. So we can draw a commutative diagram

$$\begin{tikzcd} & tB^+_{\mathrm{cris}} \arrow[d, "\vartheta^+_{\mathrm{cris}}"] \\ \Gamma \arrow[ur, dashed, "h"] \arrow[r, "f^{-1}\circ u"'] & \mathbb{C}_K(1), \end{tikzcd}$$

where the lift h is an $\mathcal{O}_K$-linear map and it exists (though may not uniquely) because M is a free (therefore projective) $\mathcal{O}_{\overline{K}}$-module and hence a free $\mathcal{O}_K$-module[1]. One such h can be given as $h(b \otimes dT/T) = \log[\tilde{b}]t$ for $b \in \mathcal{O}_{\overline{K}}$ but not in $\mathcal{O}_K$ and $\tilde{b} = (b, b^{1/p}, b^{1/p^2}, \ldots)$. By the commutativity of the

[1] This is true because $\mathcal{O}_{\overline{K}}$ is projective over $\mathcal{O}_K$ since $\mathcal{O}_K$ is local and perfect and $\mathcal{O}_{\overline{K}}$ is flat over $\mathcal{O}_K$ [Liu06, Chap. 1, Cor. 2.14].

diagram, we see that $h(1 \otimes dT/T) = t$. Also, there is an obvious inclusion $tB^+_{\text{cris}} \hookrightarrow B^+_{\text{cris}}$. Therefore, we can write a duality pairing map,

$$\psi : T_p\mathbb{G}_m \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m) \longrightarrow tB^+_{\text{cris}} \hookrightarrow B_{\text{cris}} \subset B_{\mathrm{dR}}$$
$$\Big(\{\varepsilon_n\}_{n\in\mathbb{N}}, \frac{dT}{T}\Big) \longmapsto t.$$

■

Proposition 3.33 (The action of φ and G_K)**.** *The pairing in Proposition 3.32 is perfect and ψ is a G_K-equivariant and Frobenius-compatible map. Moreover, we have the following G_K-equivariant and Frobenius-compatible isomorphism*

$$\rho_{\text{cris}} : \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,K}) \otimes_K B_{\text{cris}}.$$

Proof. Since t is nonzero and $T_p\mathbb{G}_m$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)$ are finite-dimensional, we conclude that the pairing is perfect. To check G_K-equivariance of ψ, let $g \in G_K$. Then, on one hand we have $\psi(g(\varepsilon, dT/T)) = \psi(g(\varepsilon), dT/T) = \psi(\chi(g)\cdot\varepsilon, dT/T) = \chi(g)t$. On the other hand we have $g(\psi(\varepsilon, dT/T)) = g(t) = \chi(g)t$. Hence, ψ is G_K-equivariant.

For Frobenius-compatibility, on one side we have $\psi(\varphi(\varepsilon, dT/T)) = \psi(\varepsilon, \varphi(dT/T)) = \psi(\varepsilon, p\cdot dT/T) = pt$, since the Frobenius φ on $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,K})$ is given by multiplication by p. Also, on the other side we have $\varphi(\psi(\varepsilon, dT/T)) = \varphi(t) = pt$. Therefore ψ is a Frobenius-compatible map.

Since we know that $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mathbb{Q}_p) \simeq \mathrm{Hom}(T_p\mathbb{G}_m, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the pairing defined is perfect, we immediately get an isomorphism

$$\rho_{\text{cris}} : \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{m,K}) \otimes_K B_{\text{cris}}$$

which is given as multiplication by $1/t$. ■

3.5.2 1-dimensional non-split torus $\mathbb{U}_{\alpha,K}$

Let $\alpha \in \mathbb{Z}$ such that $\alpha \neq 0$, it is squarefree and $(\alpha, p) = 1$. Also, assume that $p \geq 3$. In this example, we look at the case of commutative group scheme $\mathbb{U}_{\alpha,K}$ or by abuse of notation, written as $\mathbb{U}_\alpha$ which is a 1-dimesional non-split torus and it splits over the quadratic extension $K(\sqrt{\alpha})$ of K. As a group, $\mathbb{U}_\alpha(\overline{K}) = \{(x, y) \in (\overline{K}^\times)^2 \mid y^2 - \alpha x^2 = 1\}$. And, as a scheme $\mathbb{U}_{\alpha,K} = \mathrm{Spec}\, K[X, Y]/(Y^2 - \alpha X^2 - 1)$. Multiplicative structure on $\mathbb{U}_{\alpha,K}$ is induced by the morphism of rings

$$m : \frac{K[X, Y]}{(Y^2 - \alpha X^2 - 1)} \longrightarrow \frac{K[X_1, Y_1, X_2, Y_2]}{(Y_1^2 - \alpha X_1^2 - 1, Y_2^2 - \alpha X_2^2 - 1)}$$
$$X \longmapsto Y_1X_2 + Y_2X_1$$
$$Y \longmapsto Y_1Y_2 + \alpha X_1X_2.$$

We give another description of $\mathbb{U}_\alpha(\overline{K})$ via the isomorphism

$$\mathbb{U}_\alpha(\overline{K}) \xrightarrow{\ \simeq\ } \Big\{ \begin{pmatrix} y & \alpha x \\ x & y \end{pmatrix} \in \mathrm{GL}_2(\overline{K}) \mid y^2 - \alpha x^2 = 1 \Big\}$$
$$(x, y) \longmapsto \begin{pmatrix} y & \alpha x \\ x & y \end{pmatrix} = yI + xA,$$

where I is the 2×2 identity matrix and $A = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$. Notice that $A^2 = I$. This is a G_K-equivariant isomorphism and it is also immediate that $\mathbb{U}_\alpha(\overline{K})$ is an abelian group. From now on we identify $\mathbb{U}_\alpha(\overline{K})$ via this isomorphism.

Lemma 3.34 (The Tate module)**.** *$\mathbb{U}_\alpha(\overline{K})$ is an abelian group. The Tate module of $\mathbb{U}_\alpha$ is given as $T_p\mathbb{U}_\alpha \simeq \varprojlim_n \mathbb{U}_\alpha(\mathcal{O}_{\overline{K}})[p^n]$.*

Proof. For each $n \in \mathbb{N}$, let ε_n be a primitive p^n-th root of unity. Set $y_n = (\varepsilon_n + \varepsilon_n^{-1})/2$ and $x_n = (\varepsilon_n - \varepsilon_n^{-1})/2\sqrt{\alpha}$ where clearly $y_n I + x_n A \in \mathbb{U}_\alpha(\overline{K})$. Since $\varepsilon_{n+1}^p = \varepsilon_n$ we have $(y_{n+1}I + x_{n+1}A)^p = y_n I + x_n A$ and therefore $\mathbb{U}_\alpha(\overline{K})[p^{n+1}] \xrightarrow{(\cdot)^p} \mathbb{U}_\alpha(\overline{K})[p^n]$ is a morphism of multiplicative abelian groups for each $n \in \mathbb{N}$. Hence, we could write the Tate module of $\mathbb{U}_\alpha$ as $T_p\mathbb{U}_\alpha = \varprojlim_n \mathbb{U}_\alpha(\overline{K})[p^n]$. Now, our claim is that whenever $(\alpha, p) = 1$ we have $T_p\mathbb{U}_\alpha \simeq \varprojlim_n \mathbb{U}_\alpha(\mathcal{O}_{\overline{K}})[p^n]$ i.e., we can take $\mathcal{O}_{\overline{K}}$ points instead of $\overline{K}$ points. This is because for any $yI + xA \in \mathbb{U}_\alpha(\overline{K})[p^n]$, from the equalities $(yI + xA)^{p^n} = I$ and $y^2 - \alpha x^2 = 1$ and the assumption that $p \neq 2$ and $(\alpha, p) = 1$, we get that y is a root of an integral polynomial with coefficients in $\overline{K}$ and the same is true for x. ■

Lemma 3.35 (de Rham cohomology)**.** *For $\mathbb{U}_\alpha = \operatorname{Spec} K[X,Y]/(Y^2 - \alpha X^2 - 1)$,*

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbb{U}_\alpha) = K \quad \textit{and} \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_\alpha) = K \cdot \omega$$

where $\omega = YdX - XdY \in \Omega^1_{R/K}$ with $R = K[X,Y]/(Y^2 - \alpha X^2 - 1)$.

Proof. First, we take a look at the module of Kähler differentials for R. Let $f(X) := \alpha X^2 + 1 \in K[X]$ and $f'(X) := df(X)/dX = 2\alpha X + 1$, the derivative of $f(X)$. Then the de Rham complex is

$$R = \frac{K[X]}{(Y^2 - f(X))} = \Omega^0_{R/K} \xrightarrow{d} \Omega^1_{R/K} = \frac{RdX \oplus RdY}{R(2YdY - f'(X)dX)}.$$

We note that any element of R could be written as $g(X) + Yh(X)$ with $g(X), h(X) \in K[X]$. Moreover, $f(X)$ is coprime to $f'(X)$ and therefore by Bezout's identity there exist $\lambda_1(X), \lambda_2(X) \in K[X]$ such that $\lambda_1(X)f(X) + \lambda_2 f'(X) = 1$. In our case, it is immediately clear that $\lambda_1(X) = 1$ and $\lambda_2(X) = -X/2$.

Let $\omega := \lambda_1(X)YdX + 2\lambda_2(X)dY = YdX - XdY$. In $\Omega^1_{R/K}$ we also have that $YdY = \alpha XdX$, therefore $Y\omega = Y^2dX - XYdY = (\alpha X^2 + 1)dX - XYdY = X(\alpha XdX - YdY) + dX = dX$ and $f'(X)\omega/2 = \alpha X(YdX - XdY) = \alpha XYdX - \alpha X^2dY = Y(\alpha XdX - YdY) + dY = dY$. Hence, from the expression for $\Omega^1_{R/K}$ it is clear that any differential form could be written as $(g(X) + Yh(X))\omega$ i.e., $\Omega^1_{R/K} = R \cdot \omega = O_R \cdot \omega \otimes_{\mathcal{O}_K} K = \Omega^1_{O_R/\mathcal{O}_K} \otimes_{\mathcal{O}_K} K$ where $O_R := \mathcal{O}_K[X,Y]/(Y^2 - \alpha X^2 - 1)$.

Now we want to compute the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_{\alpha,K})$. Since $\mathbb{U}_{\alpha,K} = \operatorname{Spec} K[X,Y]/(Y^2 - \alpha X^2 - 1)$ we have that

$$\mathrm{H}^0_{\mathrm{dR}}(R/K) = K \quad \text{and} \quad \mathrm{H}^1_{\mathrm{dR}}(R/K) = \frac{ker(\Omega^1_{R/K} \to \Omega^2_{R/K})}{im(\Omega^0_{R/K} \to \Omega^1_{R/K})}.$$

Note that $\Omega^2_{R/K} = 0$ since $\omega \wedge \omega = 0$. So the numerator in the expression of $\mathrm{H}^1_{\mathrm{dR}}(R/K)$ is $R\omega$. For the denominator, we observe that $(g(X) + h(X)Y)\omega \in \Omega^1_{R/K}$, is in the image of $\Omega^0_{R/K}$ if and only if there exist $a(X), b(X) \in K[X]$ such that $d(a(X) + Yb(X)) = (g(X) + Yh(X))\omega$. Since $d(a + Yb) = a'dX + Yb'dX + bdY = Ya'\omega + Y^2b'\omega + bf'\omega/2 = (b'(\alpha X^2 + 1) + bf'/2 + Ya')\omega$ from the equality before, we get a set of equalities

$$h(X) = a'(X) \quad \text{and,}$$
$$g(X) = b'(X)f(X) + \frac{1}{2}b(X)f'(X).$$

It is always possible to find an $a(X)$ such that the first expression is satisfied. For the second expression, let βx^r be the leading term of $b(X)$ then the leading term of $b'f + bf'/2$ is $(r+1)\beta\alpha X^{r+1}$. By equating this with the leading term of $g(X)$ and proceeding recursively, we see that $b(X)$ can be determined, upto its constant term, such that the second equation above is satisfied. Hence $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_{\alpha,K}) = \mathrm{H}^1_{\mathrm{dR}}(R/K) = K \cdot \omega$. ■

For $\mathbb{U}_{\alpha,K}$, we have $\mathrm{H}^1_{\text{ét}}(\mathbb{U}_{\alpha,\overline{K}}, \mathbb{Q}_p) \simeq \mathrm{Hom}(T_p\mathbb{U}_\alpha, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. So we look at the pairing $T_p\mathbb{U}_\alpha \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_\alpha) \longrightarrow B_{\mathrm{cris}}$. To properly define this pairing, we make the following observation. First of all, we fix some p-power compatible primitive p^n-th roots of unity i.e., $\{\varepsilon_n\}_{n\in\mathbb{N}}$ where ε_n is a primitive p^n-th root of unity and $\varepsilon_{n+1}^p = \varepsilon_n$. Now, from before we have $O_R = \mathcal{O}_K[X,Y]/(Y^2 - \alpha X^2 - 1)$ and for $n \in \mathbb{N}_{>0}$ let v_n be a collection of maps defined as,

$$\begin{aligned} v_n : O_R &\longrightarrow \mathcal{O}_{K_n} \\ Y &\longmapsto y_n = \frac{\varepsilon_n + \varepsilon_n^{-1}}{2} \\ X &\longmapsto x_n = \frac{\varepsilon_n - \varepsilon_n^{-1}}{2\sqrt{\alpha}} \end{aligned}$$

where $\mathcal{O}_{K_n}$ is the ring of integers of the field $K(\varepsilon_n)$ and ε_n is a primitive p^n-th root of unity as chosen before. Also, $\varepsilon_{n+1}^p = \varepsilon_n$. So, from the map above we get an $\mathcal{O}_{K_n}$-linear map between the modules of Kähler differentials

$$\begin{aligned} v_n^* : \mathcal{O}_{K_n} \otimes \Omega^1_{O_R/\mathcal{O}_K} &\longrightarrow \Omega^1_{\mathcal{O}_{K_n}/\mathcal{O}_K} \\ a \otimes \omega &\longmapsto a(x_n dy_n - y_n dx_n). \end{aligned} \tag{3.6}$$

To avoid cumbersome notation, for each $n \in \mathbb{N}$ we write $v_n^*(1 \otimes \omega) = v_n^*\omega = x_n dy_n - y_n dx_n$.

Lemma 3.36. *Using the notation as above, $pv_{n+1}^*\omega = v_n^*\omega$.*

Proof. We know that $(y_{n+1}I + x_{n+1}A)^p = y_nI + x_nA$ and therefore,

$$d((y_{n+1}I + x_{n+1}A)^p) = p(y_{n+1}I + x_{n+1}A)^{p-1}(dy_{n+1}I + x_{n+1}A)$$

Equating this to $d(y_nI + x_nA)$ we get a set of equivalent expressions

$$\begin{aligned} d(y_nI + x_nA) &= p(y_{n+1}I + x_{n+1}A)^{p-1}(dy_{n+1}I + x_{n+1}A), \\ (y_{n+1}I + x_{n+1}A)(dy_nI + dx_nA) &= p(y_{n+1}I + x_{n+1}A)^p(dy_{n+1}I + dx_{n+1}A) \quad \text{and,} \\ (y_{n+1}I + x_{n+1}A)(dy_nI + dx_nA) &= p(y_nI + x_nA)(dy_{n+1}I + dx_{n+1}A). \end{aligned}$$

From this we obtaine the equalities

$$\begin{aligned} p(y_n dy_{n+1} + \alpha x_n dx_{n+1}) &= y_{n+1}dy_n + \alpha x_{n+1}dx_n \quad \text{and,} \\ p(x_n dy_{n+1} + y_n dx_{n+1}) &= y_{n+1}dx_n + x_{n+1}dy_n. \end{aligned}$$

Upon multiplying the first expression by x_n and the second by y_n and subtracting, we get

$$\begin{aligned} pdx_{n+1} &= (x_n y_{n+1} - y_n x_{n+1})dy_n + (\alpha x_n x_{n+1} - y_n y_{n+1})dx_n \quad \text{and therefore,} \\ pdy_{n+1} &= (y_n y_{n+1} - \alpha x_n x_{n+1})dy_n - (\alpha x_n x_{n+1} + y_n x_{n+1})dx_n. \end{aligned}$$

Now upon multiplication and subtraction, we are reduced to

$$p(x_{n+1}dy_{n+1} - y_{n+1}dx_{n+1}) = x_n dy_n - y_n d_n$$

or equivalently

$$pv_{n+1}^*\omega = v_n^*\omega$$

■

Proposition 3.37 (A duality pairing)**.** *There is a duality pairing bilinear map,*

$$\begin{aligned} \tau : T_p\mathbb{U}_\alpha \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_\alpha) &\longrightarrow B_{cris} \\ \left(\{y_nI + x_nA\}_{n\in\mathbb{N}}, \omega\right) &\mapsto \frac{t}{\sqrt{\alpha}}. \end{aligned}$$

where $t \in B_{\mathrm{cris}}$ as defined in Section 1.5.

Proof. Following the construction in Proposition 3.32 we look at the submodule $\Sigma_n = (1 \otimes \omega)\mathcal{O}_{K_n}$ of $\mathcal{O}_{K_n} \otimes \Omega^1_{\mathcal{O}_R/\mathcal{O}_K}$ generated by $1 \otimes \omega$. Using (3.6) and Lemma 3.36, we can draw a commutative diagram

$$\begin{tikzcd} \Sigma_{n+1} \arrow[r, "v^*_{n+1}"] \arrow[d] & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^{n+1}] \arrow[d, "\cdot p"] \\ \Sigma_n \arrow[r, "v^*_n"] & \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}[p^n] \end{tikzcd}$$

where the left vertical arrow is the restriction of $\mathcal{O}_{K_{n+1}} \otimes \Omega^1_{\mathcal{O}_R/\mathcal{O}_K} \to \mathcal{O}_{K_n} \otimes \Omega^1_{\mathcal{O}_R/\mathcal{O}_K}$ induced by the natural map $\varepsilon_{n+1} \mapsto \varepsilon^p_{n+1}$. Now, by passing to the limit and setting $\Gamma = (1 \otimes \omega)\mathcal{O}_{\overline{K}}$ for the submodule of $\mathcal{O}_{\overline{K}} \otimes \Omega^1_{\mathcal{O}_R/\mathcal{O}_K}$, we get a map of $\mathcal{O}_{\overline{K}}$-modules

$$\begin{aligned} v : \Sigma &\longrightarrow T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \\ 1 \otimes \omega &\longmapsto \{v^*_n\omega\}_{n\in\mathbb{N}}. \end{aligned}$$

Now we make an important observation that

$$\begin{aligned} f : \mathcal{O}_K[T, T^{-1}] &\longrightarrow \frac{\mathcal{O}_K[X, Y]}{(Y^2 - \alpha X^2 - 1)} \\ T &\longmapsto Y + \sqrt{\alpha}X \\ T^{-1} &\longmapsto Y - \sqrt{\alpha}X \end{aligned}$$

is an morphism of $\mathcal{O}_K$-algebras which is in fact an isomorphism over $\mathcal{O}_{\overline{K}}$. Therefore, on the differential forms, we have an induced map of $\mathcal{O}_K$-modules

$$\begin{aligned} f^*_K : \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K} &\longrightarrow \Omega^1_{\mathcal{O}_R/\mathcal{O}_K} \\ \frac{dT}{T} &\longmapsto (Y - \sqrt{\alpha}X)(dY + \sqrt{\alpha}dX). \end{aligned}$$

Tensoring with $\mathcal{O}_{\overline{K}}$ gives a morphism of $\mathcal{O}_{\overline{K}}$-modules and restricting to respective submodules of $\mathcal{O}_{\overline{K}} \otimes \Omega^1_{\mathcal{O}_K[T,T^{-1}]/\mathcal{O}_K}$ and $\mathcal{O}_{\overline{K}} \otimes \Omega^1_{\mathcal{O}_R/\mathcal{O}_K}$ generated by $1 \otimes dt/T$ and $1 \otimes \omega$, we get a morphism between these $\mathcal{O}_{\overline{K}}$-submodules. Thus, we can immediately draw a commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, "u"] \arrow[d, "\mathrm{Id}\otimes f^*_K"] & T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \arrow[d, equal] \\ \Sigma \arrow[r, "v"] & T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}. \end{tikzcd}$$

Also,

$$\begin{aligned} v^*_n(Y - \sqrt{\alpha}X)(dY + \sqrt{\alpha}dX) &= (y_n - \sqrt{\alpha}x_n)(dy_n + \sqrt{\alpha}dx_n) \\ &= y_n dy_n - \alpha x_n dx_n + \sqrt{\alpha}(y_n dx_n - x_n dy_n) \\ &= d(y^2_n - \alpha x^2_n) + \sqrt{\alpha}v^*_n\omega = \sqrt{\alpha}v^*_n\omega. \end{aligned}$$

Therefore, from the commutative diagram above $\{u^*_n(1 \otimes dT/T)\}_{n\in\mathbb{N}} = \{v^*_n(\mathrm{Id} \otimes f^*_K(1 \otimes dT/T))\}_{n\in\mathbb{N}} = \sqrt{\alpha}\{v^*_n\omega\}_{n\in\mathbb{N}}$. But $\{u^*_n dT/T\}_{n\in\mathbb{N}} = \{d\varepsilon_n/\varepsilon_n\}_{n\in\mathbb{N}}$, so

$$\{v^*_n\omega\}_{n\in\mathbb{N}} = \frac{1}{\sqrt{\alpha}}\Big\{\frac{d\varepsilon_n}{\varepsilon_n}\Big\}_{n\in\mathbb{N}}.$$

Now, using Lemma 3.31 we have an $\mathcal{O}_{\overline{K}}$-linear composition of maps

$$\begin{aligned} \Sigma &\xrightarrow{v} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p\Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \xrightarrow{f^{-1}} \mathbb{C}_K(1) = \mathbb{C}_K \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \\ 1 \otimes \omega &\longmapsto 1 \otimes \frac{1}{\sqrt{\alpha}}\Big\{\frac{d\varepsilon_n}{\varepsilon_n}\Big\}_{n\in\mathbb{N}} \longmapsto \frac{1}{\sqrt{\alpha}} \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}. \end{aligned}$$

From Proposition 3.32 we have a surjective map of $\mathcal{O}_K$-modules $\vartheta^+_{\text{cris}} : tB^+_{\text{cris}} \to \mathbb{C}_K(1)$ with $\vartheta^+_{\text{cris}}(x) = \theta^+_{\text{cris}}(x/t) \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}$. In particular, $\vartheta^+_{\text{cris}}(t) = 1 \otimes \{\varepsilon_n\}_{n\in\mathbb{N}}$. So we can draw a commutative diagram

$$\begin{tikzcd} & tB^+_{\text{cris}} \arrow[d, "\vartheta^+_{\text{cris}}"] \\ \Sigma \arrow[ur, dashed, "h"] \arrow[r, "f^{-1}\circ v"'] & \mathbb{C}_K(1), \end{tikzcd}$$

where the lift h is an $\mathcal{O}_K$-linear map and it exists (though may not uniquely) because Σ is a free (therefore projective) $\mathcal{O}_{\overline{K}}$-module and hence a free $\mathcal{O}_K$-module. One such h can be given as $h(b\otimes\omega) = \log[\tilde{b}]t/\sqrt{\alpha}$ for $b \in \mathcal{O}_{\overline{K}}$ but not in $\mathcal{O}_K$ and $\tilde{b} = (b, b^{1/p}, b^{1/p^2}, \ldots)$. By the commutativity of the diagram, we see that $h(1\otimes\omega) = t/\sqrt{\alpha}$. Also, there is an obvious inclusion $tB^+_{\text{cris}} \hookrightarrow B^+_{\text{cris}}$. Therefore, we can write a duality pairing map,

$$\begin{aligned} \tau : T_p\mathbb{U}_\alpha \times \mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_\alpha) &\longrightarrow tB^+_{\text{cris}} \hookrightarrow B_{\text{cris}} \subset B_{\mathrm{dR}} \\ \Big(\{y_nI + x_nA\}_{n\in\mathbb{N}}, \omega\Big) &\longmapsto \frac{t}{\sqrt{\alpha}}. \end{aligned}$$

■

Proposition 3.38 (The action of φ and G_K)**.** *The pairing in Proposition 3.37 is perfect and τ is a G_K-equivariant and Frobenius-compatible map. Moreover, we have a G_K-equivariant and Frobenius-compatible isomorphism*

$$\rho_{\text{cris}} : \mathrm{H}^1_{\text{ét}}(\mathbb{U}_{\alpha,\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_{\alpha,K}) \otimes_K B_{\text{cris}}.$$

Proof. It is immediately clear that the pairing is perfect. To check G_K-equivariance of τ, let $g \in G_K$ and $\gamma = (y_nI + x_nA)_{n\in\mathbb{N}}$. To avoid cumbersome notation, below we write $(y_nI + x_nA)$ instead of $(y_nI + x_nA)_{n\in\mathbb{N}}$. First we look at

$$\begin{aligned} g(\gamma) &= g(y_n)I + g(x_n)A \\ &= \frac{g(\varepsilon_n) + g(\varepsilon_n^{-1})}{2}I + \frac{g(\varepsilon_n) - g(\varepsilon_n^{-1})}{2g(\sqrt{\alpha})}A \\ &= \frac{\chi(g)\cdot(\varepsilon_n + \varepsilon_n^{-1})}{2}I + \frac{\chi(g)\cdot(\varepsilon_n - \varepsilon_n^{-1})}{2\eta(g)\sqrt{\alpha}}A \\ &= \chi(g)\cdot(y_nI + \eta(g)x_nA) \end{aligned}$$

where $\chi(g)$ is the usual p-adic cyclotomic character and $\eta(g) = g(\sqrt{\alpha})/\sqrt{\alpha} \in \{\pm 1\}$. Note that $\eta(g)\cdot(y_nI+x_nA) = (y_nI+x_nA)^{\eta(g)} = (y_n+\eta(g)x_nA)$. Hence $g(\gamma) = (\eta(g)\chi(g))\cdot(y_nI+x_nA) = (\eta(g)\chi(g))\cdot\gamma$.

Now, on one hand we have $\tau(g(\gamma,\omega)) = \tau(g(\gamma),\omega) = \tau(\eta(g)\chi(g)\cdot\gamma, dT/T) = \eta(g)\chi(g)t/\sqrt{\alpha}$. On the other hand, we have $g(\tau(\gamma,\omega)) = g(t/\sqrt{\alpha}) = \chi(g)t/\eta(g)\sqrt{\alpha} = \eta(g)\chi(g)t/\sqrt{\alpha}$. Hence, τ is G_K-equivariant.

For Frobenius-compatibility, since the Frobenius φ on $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{U}_{\alpha,K})$ is given by multiplication by $(\frac{\alpha}{p})p$ where $(\frac{\alpha}{p})$ is the usual Legendre symbol, therefore on one side we have $\tau(\varphi(\gamma,\omega)) = \tau(\gamma,\varphi(\omega)) = \tau(\gamma, p(\frac{\alpha}{p})\omega) = p(\frac{p}{\alpha})t/\sqrt{\alpha}$. Also, on the other side we have $\varphi(\tau(\gamma,\omega)) = \varphi(t/\sqrt{\alpha}) = pt/\varphi(\sqrt{\alpha})$. Therefore τ is φ-compatible if and only if $\varphi(\sqrt{\alpha}) = (\frac{\alpha}{p})\sqrt{\alpha}$. We consider two different cases,

(1) Suppose $\sqrt{\alpha} \in \mathbb{Z}_p$ then $\sqrt{\alpha} \mod p\mathbb{Z}_p \in \mathbb{F}_p$ which means that $(\frac{\alpha}{p}) = 1$. Also, φ acts as identity on $\mathbb{Z}_p$ since the absolute Frobenius map on $\mathbb{F}_p$ is identity. So, in this case we indeed have that $\varphi(\sqrt{\alpha}) = (\frac{\alpha}{p})\sqrt{\alpha}$.

(2) Next we assume $\sqrt{\alpha} \notin \mathbb{Z}_p$ then $\sqrt{\alpha} \in \mathbb{Q}_p(\sqrt{\alpha})$, a degree 2 extension of $\mathbb{Q}_p$.

(a) If $\mathbb{Q}_p(\sqrt{\alpha})$ is an unramified extension of $\mathbb{Q}_p$ then $\left(\frac{\alpha}{p}\right) = -1$. Also, the residue field of $\mathbb{Q}_p(\sqrt{\alpha})$ is a degree 2 extension of $\mathbb{F}_p$ and therefore the absolute Frobenius on $\sqrt{\alpha}$ i.e., $(\sqrt{\alpha})^p = \alpha^{\frac{p}{2}} = \alpha^{\frac{p-1}{2}}\sqrt{\alpha} = \left(\frac{\alpha}{p}\right)\sqrt{\alpha} = -\sqrt{\alpha} \mod p$. Since φ is the lift of absolute Frobenius to $\mathbb{Q}_p(\sqrt{\alpha})$, we get $\varphi(\sqrt{\alpha}) = -\sqrt{\alpha} = \left(\frac{\alpha}{p}\right)\sqrt{\alpha}$.

(b) If $\mathbb{Q}_p(\sqrt{\alpha})$ is a ramified extension of $\mathbb{Q}_p$ then $\left(\frac{\alpha}{p}\right) = 1$. But then the residue field of $\mathbb{Q}_p(\sqrt{\alpha})$ is $\mathbb{F}_p$ and therefore the absolute Frobenius on $\sqrt{\alpha}$ is identity. Since φ is the lift of absolute Frobenius to $\mathbb{Q}_p(\sqrt{\alpha})$, we get $\varphi(\sqrt{\alpha}) = \sqrt{\alpha} = \left(\frac{\alpha}{p}\right)\sqrt{\alpha}$.

Since we know that $\mathrm{H}^1_{\text{ét}}(\mathbb{U}_{\alpha,\overline{K}}, \mathbb{Q}_p) \simeq \mathrm{Hom}(T_p\mathbb{U}_\alpha, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the pairing defined is perfect, we immediately get an isomorphism

$$\rho_{\text{cris}} : \mathrm{H}^1_{\text{ét}}(\mathbb{U}_{\alpha,\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \xrightarrow{\sim} \mathrm{H}^1_{\text{dR}}(\mathbb{U}_{\alpha,K}) \otimes_K B_{\text{cris}}$$

which is given as multiplication by $\sqrt{\alpha}/t$. ■

Chapter 4

An equivalence of categories

In the last chapter we saw the following faithful, exact and tensor functors,

$$\begin{aligned} \mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) &\longrightarrow \mathrm{MF}_K(\varphi) \\ \mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) &\longrightarrow \mathrm{MF}_K(\varphi, N). \end{aligned}$$

In the first section our aim is to establish the full faithfulness of these functors by describing respective quasi inverse functors. Later on in the chapter, we will also give a description of their respective essential images. As it turns out, the essential image of $\mathrm{D}_{\mathrm{cris}}$ is the abelian category $\mathrm{MF}^{\mathrm{ad}}_K(\varphi)$ whereas the essential image of D_{st} is the abelian category $\mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$. Though establishing these results requires a lot more work, we will prove some partial results and give references for the rest.

4.1 Quasi inverse functors

To get started, we mention the following result where the second exact sequence is sometimes referred to as the *fundamental exact sequence*,

Proposition 4.1. *For $r \geq 0$, the following sequence is exact*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow (\mathrm{Fil}^{-r} B_{\mathrm{cris}})^{\varphi=1} \longrightarrow \mathrm{Fil}^{-r} B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \longrightarrow 0.$$

Passing to the direct limit as $r \to +\infty$, we get that the following sequence is exact

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow (B_{\mathrm{cris}})^{\varphi=1} \longrightarrow B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \longrightarrow 0.$$

Proof. [CF00, Prop. 1.3]. ■

Remark 4.2. From the first exact sequence in Proposition 4.1, by setting $r = 0$, we get $(\mathrm{Fil}^0 B_{\mathrm{cris}})^{\varphi=1} = \{b \in \mathrm{Fil}^0 B_{\mathrm{cris}} \mid \varphi(b) = b\} = \mathbb{Q}_p$.

Our first goal is to establish the full faithfulness of D_{st} and $\mathrm{D}_{\mathrm{cris}}$. We start with semistable representations of G_K.

Let V be any semistable p-adic representation of G_K of dimension h. Let $D = \mathrm{D}_{\mathrm{st}}(V)$. We want to construct a covariant functor

$$\mathrm{V}_{\mathrm{st}} : \mathrm{MF}_K(\varphi, N) \longrightarrow \mathbb{Q}_p[G_K]\text{-modules}$$

such that $\mathrm{V}_{\mathrm{st}}(\mathrm{D}_{\mathrm{st}}(V)) \simeq V$. Recall that we have the natural semistable isomorphism from Proposition 3.25 (ii)

$$\alpha_{\mathrm{st}} : B_{\mathrm{st}} \otimes_{K_0} D \longrightarrow B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V.$$

α_{st} is B_{cris}-linear, G_K-equivariant, Frobenius- and monodromy-compatible. Also, $\alpha_{\mathrm{st},K}$ is a filtered isomorphism. Let us identify these isomorphic objects as X. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$

be a basis of V over $\mathbb{Q}_p$ and D over K_0 respectively. Identify v_i with $1 \otimes v_i$ and w_i with $1 \otimes w_i$, so then $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ are both bases of X over B_{st}. Any element of X can be written as a sum of $b \otimes w$ where $b \in B_{\mathrm{st}}$, $w \in D$ and also as a sum $c \otimes v$ where $c \in B_{\mathrm{st}}$ and $v \in V$. The actions of G_K, φ and N on X are as follows,

$$\begin{aligned} &G_K\text{-action}: \ g(b \otimes w) = g(b) \otimes w, && g(c \otimes v) = g(c) \otimes g(v); \\ &\varphi\text{-action}: \ \varphi(b \otimes w) = \varphi(b) \otimes \varphi(w), && \varphi(c \otimes v) = \varphi(c) \otimes v; \\ &N\text{-action}: \ N(b \otimes w) = N(b) \otimes N(w), && N(c \otimes v) = N(c) \otimes v. \end{aligned}$$

X also has a filtration. By the map $x \mapsto 1 \otimes x$, we also have the inlusion

$$X \subset X_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}} X \simeq B_{\mathrm{dR}} \otimes_K D_K \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V.$$

Then the filtration on X is induced by

$$\mathrm{Fil}^i X_{\mathrm{dR}} = \mathrm{Fil}^i (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = \sum_{r+s=i} \mathrm{Fil}^r B_{\mathrm{dR}} \otimes \mathrm{Fil}^s D_K.$$

We set

$$\begin{aligned} \mathrm{V}_{\mathrm{st}}(D) &= \{x \in X \mid \varphi(x) = x, \ N(x) = 0, \ x \in \mathrm{Fil}^0 X\} \\ &= \{x \in X \mid \varphi(x) = x, \ N(x) = 0, \ x \in \mathrm{Fil}^0 X_{\mathrm{dR}}\}. \end{aligned}$$

Notice that $V \hookrightarrow X$ where $v \mapsto 1 \otimes v$ satisfies these conditions.

Next we need to check that $\mathrm{V}_{\mathrm{st}}(\mathrm{D}_{\mathrm{st}}(V)) \simeq V$. Write $x = \sum_{1 \le n \le h} b_n \otimes v_n \in \mathrm{V}_{\mathrm{st}}(D)$, where $b \in B_{\mathrm{st}}$ and $D = \mathrm{D}_{\mathrm{st}}(V)$. First of all $N(x) = 0$, i.e., $\sum_{1 \le n \le h} N(b_n) \otimes v_n = 0$. Therefore, $N(b_n) = 0$ which means $b_n \in B_{\mathrm{cris}}$ for each $1 \le n \le h$. Secondly, we have $\varphi(x) = x$, i.e., $\sum_{1 \le n \le h} \varphi(b_n) \otimes v_n = \sum_{1 \le n \le h} b_n \otimes v_n$. Therefore, $\varphi(b_n) = b_n$ for each $1 \le n \le h$. Lastly, the condition $x \in \mathrm{Fil}^0 X_{\mathrm{dR}}$ implies that $b_n \in \mathrm{Fil}^0 B_{\mathrm{dR}} = B_{\mathrm{dR}}^+$ for each $1 \le n \le h$. From the fundamental exact sequence in Proposition 4.1 we have that $b_n \in \mathbb{Q}_p$. Therefore $x \in V$. By using the semistable isomorphism we conclude that $V \simeq \mathrm{V}_{\mathrm{st}}(\mathrm{D}_{\mathrm{st}}(V))$. We can also write the functor V_{st} in the following equivalent manner,

$$\begin{aligned} \mathrm{V}_{\mathrm{st}}(D) &= \mathrm{Fil}^0 (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0, \varphi=1} \\ &= \mathit{ker}\ (\delta(D) : (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0, \varphi=1} \longrightarrow (B_{\mathrm{dR}} \otimes_K D_K)/\mathrm{Fil}^0 (B_{\mathrm{dR}} \otimes_K D_K)) \end{aligned}$$

For any $D \in \mathrm{MF}_K(\varphi, N)$, after applying the functor V_{st}, we do not know if $\mathrm{V}_{\mathrm{st}}(D)$ is finite-dimensional over $\mathbb{Q}_p$ with continuous G_K-action. This will be shown in the next section. But assuming this, we show the full faithfullness of D_{st}.

Proposition 4.3. *The exact tensor functor* $\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K(\varphi, N)$ *is fully faithful with inverse on its essential image given by* V_{st}. *The same holds for the contravariant functor* $\mathrm{D}^*_{\mathrm{st}}$ *using the contravariant functor* $\mathrm{V}^*_{\mathrm{st}}(D) = \mathrm{Hom}_{\mathrm{Fil}, \varphi, N}(D, B_{\mathrm{st}})$.

Proof. From discussions in Section 3.3 we know that D_{st} is a faithful, exact, tensor functor. Now we show that it is fully faithul. Suppose V_1 and V_2 are semistable p-adic representations of G_K and let $D_1 = \mathrm{D}_{\mathrm{st}}(V_1)$ and $D_2 = \mathrm{D}_{\mathrm{st}}(V_2)$ in $\mathrm{MF}_K(\varphi, N)$. For $f : D_1 \to D_2$ a map in $\mathrm{MF}_K(\varphi, N)$ we need to show that there exists map $V_1 \to V_2$ such that D_{st} takes this map to f. Now via the semistable comparison isomorphism in Proposition 3.25 (ii) for V_1 and V_2, the B_{st}-linear extension $1 \otimes f : B_{\mathrm{st}} \otimes_{K_0} D_1 \to B_{\mathrm{st}} \otimes_{K_0} D_2$ of f is identified with a B_{st}-linear, G_K-equivariant, Frobenius- and monodromy-compatible morphim $\tilde{f} : B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V_1 \to B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V_2$. Explicitly, $\tilde{f} = \alpha_{\mathrm{st}}(V_2) \circ (1 \otimes f) \circ \alpha_{\mathrm{st}}(V)^{-1}$. The map $\tilde{f}$ respects the formation of the φ-fixed part in the filtration degree 0, i.e., this B_{st}-linear morphism must carry V_1 into V_2 by a G_K-equivariant map. Hence, $\tilde{f}$ is the B_{st}-scalar extension of some map $g : V_1 \to V_2$ in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. So by functoriality of the semistable comparison isomorphism we get that the map $g : V_1 \to V_2$ in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ induces the map $f : \mathrm{D}_{\mathrm{st}}(V_1) = D_1 \to D_2 = \mathrm{D}_{\mathrm{st}}(V_2)$. Thus we have full faithfulness as desired. ∎

Using the fact that crystalline representations are semistable representations with vanishing monodromy we can define the covariant functor

$$\mathrm{V}_{\mathrm{cris}} : \mathrm{MF}_K(\varphi) \longrightarrow \mathbb{Q}_p[G_K]\text{-modules}$$

with $\mathrm{V}_{\mathrm{cris}}(D) = \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi=1}$. For $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)$, we know that $V \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ and N vanishes for $\mathrm{D}_{\mathrm{st}}(V) = \mathrm{D}_{\mathrm{cris}}(V)$. Therefore, the argument for the covariant functor V_{st} can be adapted to conclude that $\mathrm{V}_{\mathrm{cris}}(\mathrm{D}_{\mathrm{cris}}(V)) \simeq V$ for any $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K)$. Moreover, assuming that for any $D \in \mathrm{MF}_K(\varphi)$, $\mathrm{V}_{\mathrm{cris}}(D)$ is finite-dimensionl over $\mathbb{Q}_p$ with continuous G_K-action, we see that $\mathrm{D}_{\mathrm{cris}}$ is a fully faithful exact tensor functor. The same holds for the contravariant functor $\mathrm{D}^*_{\mathrm{cris}}$ using the contravariant functor $\mathrm{V}^*_{\mathrm{cris}}(D) = \mathrm{Hom}_{\mathrm{Fil},\varphi}(D, B_{\mathrm{cris}})$.

We end this section with an explicit calculation via an example.

Example 4.4. We calculate $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(r)) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(r), B_{\mathrm{cris}})$. Given any $\mathbb{Q}_p[G_K]$-linear map $\mathbb{Q}_p(r) \to B_{\mathrm{cris}}$, if we multiply it by t^{-r} then we get a $\mathbb{Q}_p[G_K]$-linear map $\mathbb{Q}_p \to B_{\mathrm{cris}}$. In other words, $D = \mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(r)) = B^{G_K}_{\mathrm{cris}} \cdot t^r = K_0 t^r$. This has Frobenius-action $\varphi(ct^r) = p^r\sigma(c)t^r$ and the unique filtration jump for D_K occurs in degree r (i.e., $\mathrm{gr}^r(D_K) \neq 0$). In other words, $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(r))$ is the Tate twist $K_0[0]\langle r\rangle$ in the sense of Definition 2.34.

Next we want to compute $\mathrm{V}^*_{\mathrm{cris}}(\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(r))) = \mathrm{V}^*_{\mathrm{cris}}(K_0[0]\langle r\rangle)$. This consists of K_0-linear maps $f : K_0 \to \mathrm{Fil}^r B_{\mathrm{cris}}$ that satisfy $\varphi(f(c)) = f(p^r(\sigma(c)))$ for every $c \in K_0$ or in other words $\sigma(c) \cdot \varphi(f(1)) = p^r\sigma(c)f(1)$ for all $c \in K_0$. This says $\varphi(f(1)) = p^r f(1)$ with $f(1) \in \mathrm{Fil}^r B_{\mathrm{cris}}$. So, if we write $f(1) = bt^r$ with $b \in \mathrm{Fil}^0 B_{\mathrm{cris}}$ (as we may since $t \in B^\times_{\mathrm{cris}}$) then the condition on b exactly says that $b \in (\mathrm{Fil}^0 B_{\mathrm{cris}})^{\varphi=1} = \mathbb{Q}_p$ (by Remark 4.2). Hence, $\mathrm{V}^*_{\mathrm{cris}}(\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(r))) = \mathbb{Q}_p t^r$ is the canonical copy of $\mathbb{Q}_p(r)$ inside B_{cris}. So we saw in this special case that $\mathrm{V}^*_{\mathrm{cris}}$ is "inverse" to $\mathrm{D}^*_{\mathrm{cris}}$.

4.2 Towards the equivalence

The goal of this section is to understand the respective essential images of the functors $\mathrm{D}_{\mathrm{cris}}$ and D_{st}. But first, we record some properties of these covariant functors. Following is a result showing the insensitivity to inertial restriction of the crystalline and semistable property.

Proposition 4.5. *Let $K' = \widehat{K^{\mathrm{un}}}$. The natural map $K'_0 \otimes_{K_0} \mathrm{D}_{\mathrm{st},K}(V) \to \mathrm{D}_{\mathrm{st},K'}(V)$ in $\mathrm{MF}_{K'}(\varphi, N)$ is an isomorphism for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and likewise for the functor $\mathrm{D}_{\mathrm{cris},K'}$ that is valued in $\mathrm{MF}_{K'}(\varphi)$. In particular, V is semistable as a G_K-representation if and only if it is semistable as a representation of $G_{K'} = I_K$, and likewise for the crystalline property.*

Proof. [BC09, Prop. 9.3.1]. ∎

Corollary 4.6. *If $\rho : G_K \to \mathrm{GL}(V)$ is a p-adic representation with open kernel then ρ is semistable if and only if it is crystalline if and only if it is unramified. Also, a continuous character $\eta : G_K \to \mathbb{Q}_p^\times$ is semistable if and only if it is crystalline if and only if it is a Tate twist of an unramified character.*

Proof. By Proposition 4.5 we may replace K with $\widehat{K^{\mathrm{un}}}$ so that k is algebraically closed. For the first part we need to show that if ρ is semistable with $ker\ \rho$ open in G_K then it must be that ρ has trivial action on V. Let L/K be the finite Galois extension corresponding to $ker\ \rho$, so V is a representation space for $\mathrm{Gal}(L/K)$ and it is semistable as a G_K-representation space. Our goal is to prove that $V^{\mathrm{Gal}(L/K)} = V$.

Since k is algebraically closed we have $L_0 = K_0$. Therefore, $B^{\mathrm{Gal}(L/K)}_{\mathrm{st}} = L_0 = K_0$. Now,

$$\mathrm{D}_{\mathrm{st},K}(V) = (\mathrm{D}_{\mathrm{st},L}(V))^{\mathrm{Gal}(L/K)} = (B^{G_L}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(L/K)} = (K_0 \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(L/K)} = K_0 \otimes_{\mathbb{Q}_p} V^{\mathrm{Gal}(L/K)}.$$

But $\dim_{K_0}\mathrm{D}_{\mathrm{st},K}(V) = \dim_{\mathbb{Q}_p} V$, so $\dim_{K_0} V^{\mathrm{Gal}(L/K)} = \dim_{\mathbb{Q}_p} V$ by K_0-dimension reasons. Therefore, $V = V^{\mathrm{Gal}(L/K)}$.

For the second claim about semistable characters η, since semistable representation are Hodge-Tate there is a Hodge-Tate weight $n \in \mathbb{Z}$ for η. We can twist by the crystalline (hence semistable) representation $\mathbb{Q}_p(-n)$ and the twisted character is still semistable, so we may assume that η has Hodge-Tate weight 0. Thus, by the Tate-Sen Theorem A.13 we see that $\eta(G_K)$ is finite (as $G_K = I_K$) which precisley means that *ker* η is open. By the first part, it follows that $\eta = 1$. ∎

Lemma 4.7. *Let k be an algebraically closed field with characteristic $p > 0$. The map $W(k)^\times \longrightarrow W(k)^\times$ defined by $w \mapsto \sigma(w)/w$ is surjective, where σ is the Frobenius automorphism of $W(k)$.*

Proof. [BC09, Lem. 9.3.3]. ∎

Now we come to a key theorem which explains our interest in admissible filtered (φ, N)-modules (in the sense of Definition 2.28).

Theorem 4.8. *If $V \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ then $\mathrm{D}_{\mathrm{st}}(V) \in \mathrm{MF}_K(\varphi, N)$ is admissible. In particular, if V is crystalline then $\mathrm{D}_{\mathrm{cris}}(V) \in \mathrm{MF}_K(\varphi)$ is admissible.*

Proof. Since admissibility condition is insensitive to the scalar extension $K_0 \longrightarrow \widehat{K_0^{\mathrm{un}}}$, by Proposition 4.5 we may assume that k is algebraically closed. Let $D = \mathrm{D}_{\mathrm{st}}(V)$ and let $D' \subset D$ be a subobject. We need to show that $t_H(D') \leq t_N(D')$ where equality holds if and only if $D' = D$. We may assume $D' \neq 0$, so $d' = \dim_{K_0} D' > 0$. For this, we describe D' in concrete terms by reducing to the case $d' = 1$ using determinants arguments.

Note that $\wedge^{d'} V$ is semistable, so $\wedge^{d'} D$ is naturally identified with $\mathrm{D}_{\mathrm{st}}(\wedge^{d'} V)$. Also, $\det D' = \wedge^{d'} D'$ is naturally a 1-dimensional subobject of $\wedge^{d'} D$. Since $t_H(D') = t_H(\det D')$ and $t_N(D') = t_N(\det D')$, we may therefore pass to $\wedge^{d'} V$ to reduce to the case of $\dim_{K_0} D' = 1$. In case $D' = D$ we have $\dim_{\mathbb{Q}_p} V = 1$, so $V = \mathbb{Q}_p(n)$ by Corollary 4.6 for some $n \in \mathbb{Z}$ (as k is algebraically closed). In this case we see with the help of $t^{-n} \in B_{\mathrm{cris}}^\times$ that $t_H(D) = t_N(D) = -n$ (since we are using covariant functors $\mathrm{D}_{\mathrm{cris}}$ and D_{st}). Thus, it remains to show that in general $t_H(D') \leq t_N(D')$. Let $e' \in D'$ be a K_0-basis so $\varphi(e') = \lambda e'$ for some $\lambda \in K_0^\times$ and $t_N(D') = v_p(\lambda)$. Also, $N(e') = 0$ since N is a nilpotent operator and D' is 1-dimenisonal. Let $s = t_H(D')$, so $e' \in \mathrm{Fil}^s(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) = \mathrm{Fil}^s B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ but $e' \notin \mathrm{Fil}^{s+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$.

Pick a basis $\{v_1, v_2, \ldots, v_n\}$ of V, so the inclusion $D' \subset D = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ gives a unique expansion $e' = \sum b_i \otimes v_i$ for $b_i \in B_{\mathrm{st}}$. The equality $\lambda e' = \varphi(e') = \sum \varphi(b_i) \otimes v_i$ gives $\varphi(b_i) = \lambda b_i$ for any $1 \leq i \leq n$ and the vanishing $N(e') = \sum N(b_i) \otimes v_i$ gives $N(b_i) = 0$ for $1 \leq i \leq n$. In particular, $b_i \in B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$ for all $1 \leq i \leq n$. Since $e' \in \mathrm{Fil}^s B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ but $e' \notin \mathrm{Fil}^{s+1} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$, we conclude that $b_i \in \mathrm{Fil}^s B_{\mathrm{cris}}$ for all i but $b_{i_0} \notin \mathrm{Fil}^{s+1} B_{\mathrm{cris}}$ for some i_0. Looking at b_{i_0} it suffices to show generally that if $b \in B_{\mathrm{cris}}$ lies in $\mathrm{Fil}^s B_{\mathrm{cris}}$ but not in $\mathrm{Fil}^{s+1} B_{\mathrm{cris}}$ (so $b \neq 0$) and $\varphi(b) = \lambda b$ for $\lambda \in K_0^\times$ then $s \leq v_p(\lambda)$.

Assume on the contrary that $s \geq v_p(\lambda) + 1$. Let $n = v_p(\lambda)$, so $b \in \mathrm{Fil}^s B_{\mathrm{cris}} \subset \mathrm{Fil}^{n+1} B_{\mathrm{cris}}$. To get a contradiction, it suffices to show that the only $b \in \mathrm{Fil}^{n+1} B_{\mathrm{cris}}$ such that $\varphi(b) = \lambda b$ with $n = v_p(\lambda)$ is $b = 0$. We may replace b with b/t^n to reduce to the case $n = 0$. Hence, $b \in \mathrm{Fil}^1 B_{\mathrm{cris}}$ and $\varphi(b) = ub$ with $u \in W(k)^\times$. But k is algebarically closed, so $u = \sigma(u')/u'$ for some $u' \in W(k)^\times$ from Lemma 4.7. Thus $b/u' \in (\mathrm{Fil}^1 B_{\mathrm{cris}})^{\varphi=1}$. But $(\mathrm{Fil}^1 B_{\mathrm{cris}})^{\varphi=1} = \mathbb{Q}_p$ by Remark 4.2 and this meets $\mathrm{Fil}^1 B_{\mathrm{cris}}$ in 0. ∎

A fundamental result of Colmez and Fontaine [CF00, Thm. A] is that the fully faithful, exact tensor functor $\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$ is an equivalence. That is, every admissible filtered (φ, N)-module D over K is isomorphic as such to $\mathrm{D}_{\mathrm{st}}(V)$ for a semistable p-adic representation of G_K. In principle we know that V is should be such that $V \simeq \mathrm{V}_{\mathrm{st}}(D)$. But it is not a priori obvious that $\mathrm{D}_{\mathrm{st}}(\mathrm{V}_{\mathrm{st}}(D)) \simeq D$ for admissible D. So in the remainder of this section, we take up the work to prove that $\mathrm{V}_{\mathrm{st}}(D)$ is always in $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ for any $D \in \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$. We will also prove the Colmez-Fontaine lemma that says $\dim_{\mathbb{Q}_p} \mathrm{V}_{\mathrm{st}}(D) \leq \dim_{K_0} D$ for any admissible D with equality if and only if $D \simeq \mathrm{D}_{\mathrm{st}}(\mathrm{V}_{\mathrm{st}}(D))$ in $\mathrm{MF}_K(\varphi, N)$. To prove that the inequality is always an equality requires much more work. Assuming this key result, we will have the equivalence of categories stated at the beginning of the paragraph.

The first issue that we encountered was whether the $\mathbb{Q}_p[G_K]$-module $\mathrm{V}_{\mathrm{st}}(D)$ for $D \in \mathrm{MF}_K(\varphi, N)$ is finite-dimensional with continuous G_K-action. We claim that any G_K-stable finite-dimensional $\mathbb{Q}_p$-subspace of $\mathrm{V}_{\mathrm{st}}(D)$ has continuous G_K-action. In particular, if $\mathrm{V}_{\mathrm{st}}(D)$ is finite-dimensional over $\mathbb{Q}_p$ then the natural G_K-action on it is continuous. By definition, $\mathrm{V}_{\mathrm{st}}(D) \subset B_{\mathrm{st}} \otimes_{K_0} D$ with the G_K-action doing nothing to D. So we prove,

Proposition 4.9. *For any $n \geq 1$, any $\mathbb{Q}_p[G_K]$-submodule V of B_{st}^n with finite $\mathbb{Q}_p$-dimension has continuous G_K-action relative to its natural p-adic topology.*

Proof. Consider the non-canonical presentation $B_{\mathrm{st}} \simeq B_{\mathrm{cris}}[X]$ (resting on a choice of $\tilde{q} \in \mathfrak{m}_R - \{0\}$ with $\tilde{q}^{(0)} \in \mathcal{O}_K$). The B_{cris} -submodule $B_{\mathrm{cris}}[X]_{<d}$ of polynomials with degree below a given bound $d \geq 1$ is G_K-stable because $g(X) = X + \eta(g)t$ for a suitable continuous $\eta : G_K \longrightarrow \mathbb{Z}_p$ depending on the choice of $t = \log[\varepsilon]$. The finite-dimensional $\mathbb{Q}_p$-subspace $V \subset B_{\mathrm{st}}^n$ is contained in the finite free B_{cris}-submodule $B_{\mathrm{cris}}[X]_{<d}^n$ for some $d \geq 1$. However, note that $B_{\mathrm{cris}}[X]_{<d}^n$ is not G_K-equivariantly identified with B_{cris}^{nd} via the basis of vectors in standard monomials when $d > 1$. Since $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ and $\dim_{\mathbb{Q}_p} V < +\infty$, the t-denominators needed to describe V are bounded: for $M \gg 0$ we have $V \subset \mathbb{Q}_p \cdot t^{-M} A_{\mathrm{cris}}[X]_{<d}^n$. The action by G_K on t is through $\mathbb{Z}_p^\times$-value continuous χ, so we can replace V with $t^M V$ for some $M \gg 0$ to arrange that V is generated over $\mathbb{Q}_p$ by the G_K-stable $\Lambda := V \cap A_{\mathrm{cris}}[X]_{<d}^n$. This $\mathbb{Z}_p$-submodule of V contains no infinitely p-divisible elements becasue A_{cris} is p-adically separated, so it follows that Λ must be finitely generated over $\mathbb{Z}_p$ and hence is a $\mathbb{Z}_p$-lattice in V. Thus, it suffices to prove that the G_K-action on Λ is continuous for the p-adic topology of Λ.

Let $\Lambda_r = \Lambda \cap (p^r A_{\mathrm{cris}}[X]_{<d}^n)$, so $p^r\Lambda \subset \Lambda_r \subset \Lambda$ and Λ_r is G_K-stable. Since A_{cris} is p-adically separated we have $\cap_r \Lambda_r = 0$, so by [MR87, Exer. 8.7] it follows that Λ_r's give the same topology to Λ as its p-adic topology. Therefore, we are now reduced to showing that for each $r \geq 1$ the G_K-action on each finite quotient Λ/Λ_r is discrete, i.e., the points have open stabilizers. Let us fix such an r. For the finite quotient Λ/Λ_r, there is a natural inclusion into $(A_{\mathrm{cris}}/(p^r))[X]_{<d}^n$, so we need to show that if an element of $(A_{\mathrm{cris}}/(p^r))[X]_{<d}^n$ has a finite G_K-orbit then it has an open stailizer. We will show that all orbits are finite with open stabilizers. By projection to factors of this direct sum of truncated polynomial modules, we can assume $n = 1$.

We may replace K with the finite Galois extension corresponding to *ker* $(\eta \mod p^r)$, which is to say that we can assume that the additive character $\eta \mod p^r$ vanishes. Hence, the G_K-action on $X \mod p^r$ has now been eliminated, so we can project to monomial coefficients in each separated degree less than d, which is to say that we are reduced to proving that every G_K-orbit in $A_{\mathrm{cris}}/(p^r)$ has an open stabilizer (and hence is finite) for each $r \geq 1$. This is true from Proposition 1.35 and so we are done. ∎

Now we look at $\mathrm{V}_{\mathrm{st}}(D)$ when D is admissible. First of all we analyze the case of $\dim_{K_0} D = 1$.

Lemma 4.10. *If D is an arbitrary filtered (φ, N)-module over K with $\dim_{K_0} D = 1$ then $\mathrm{V}_{\mathrm{st}}(D)$ is 1-dimensional when D is admissible (i.e., $t_H(D) = t_N(D)$), it vanishes when $t_H(D) < t_N(D)$, and it is infinite-dimensional over $\mathbb{Q}_p$ when $t_H(D) < t_N(D)$.*

Proof. We have $D = K_0 d$ with $\varphi(d) = \lambda d$ for some $\lambda \in K_0^\times$. The monodromy operator vanishes on D since it is nilpotent and $\dim_{K_0} D = 1$. By definition $t_N(D) = \upsilon_p(\lambda) \in \mathbb{Z}$ and $\mathrm{Fil}^{t_H(D)} D_K = D_K$, $\mathrm{Fil}^{t_H(D)+1} D_K = 0$. Since $\dim_{K_0} D = 1$, D is admissible if and only if $t_H(D) = t_N(D)$. We wish to relate the $\mathbb{Q}_p$-dimension of $\mathrm{V}_{\mathrm{st}}(D)$ (possibly infinite) to the nature of the difference $t_H(D) - t_N(D)$.

Let us compute $\mathrm{V}_{\mathrm{st}}(D)$ in general, using the K_0-basis $\{d\}$ of D. For $x \in \mathrm{V}_{\mathrm{st}}(D)$ we have, $x \in B_{\mathrm{st}} \otimes_{K_0} D$ such that $\varphi(x) = x$, $N(x) = 0$ and $x \in \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D) = \mathrm{Fil}^{-t_H(D)} B_{\mathrm{st}} \otimes_{K_0} D$. In particular, $x \in B_{\mathrm{cris}} \otimes_{K_0} D$ so $x = b \otimes d$ for a unique $b \in \mathrm{Fil}^{-t_H(D)} B_{\mathrm{cris}}$ such that $\varphi(b) = b/\lambda$. We can write $\lambda = p^m u$ for $m = t_N(D)$ and $u \in \mathcal{O}_{K_0}^\times = W(k)^\times$. Letting $b' = t^{t_H(D)} b \in B_{\mathrm{cris}}$, the conditions are that $b' \in \mathrm{Fil}^0(B_{\mathrm{cris}})$ with $\varphi(b') = p^{t_H(D) - t_N(D)}(b'/u)$. By Lemma 4.7 we may choose $w \in W(\bar{k})^\times$ such that $\sigma(w)/w = u$. Replace b' with $b'' = wb'$, so $\mathrm{V}_{\mathrm{st}}(D)$ as a $\mathbb{Q}_p$-vector space is identified as the set of elements $b'' \in \mathrm{Fil}^0 B_{\mathrm{cris}}$ such that $\varphi(b'') = p^{t_H(D) - t_N(D)} b''$. Thus for the admissibility (i.e., $t_H(D) = t_N(D)$) the condition on b'' says exactly that $b'' \in (\mathrm{Fil}^0 B_{\mathrm{cris}})^{\varphi=1} = \mathbb{Q}_p$, so $\dim_{\mathbb{Q}_p} \mathrm{V}_{\mathrm{st}}(D) = 1$ in such cases.

In general, if $r := t_H(D) - t_N(D)$ then $\varphi(b''/t^r) = b''/t^r$, so if $r < 0$ then $b''/t^r \in \mathrm{Fil}^{-r} B_{\mathrm{cris}} \subset \mathrm{Fil}^1 B_{\mathrm{cris}}$ is a φ-invariant vector and thus vanishes (as the only φ-invariant elements of $\mathrm{Fil}^0 B_{\mathrm{cris}}$ are elements of $\mathbb{Q}_p$, none of which lie in $\mathrm{Fil}^1 B_{\mathrm{cris}}$ except for the element 0). Hence, b'' vanishes when $r < 0$. The remaining case is when $r > 0$, in which case $b''/t^r \in \mathrm{Fil}^{-r} B_{\mathrm{cris}}$ is φ-invariant vector, and the space of these is infinite-dimensional due to the exact sequence from Proposition 4.1

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow (\mathrm{Fil}^{-r} B_{\mathrm{cris}})^{\varphi=1} \longrightarrow \mathrm{Fil}^{-r}(B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \longrightarrow 0$$

that is valid for $r \geq 0$. ■

The following preliminary result generalizing Lemma 4.10 is a small part of the proof of the general result that admissible filtered (φ, N)-modules are the essential image of D_{st}.

Proposition 4.11. *Let $D \in \mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$. The vector space $\mathrm{V}_{\mathrm{st}}(D)$ is finite-dimensional over $\mathbb{Q}_p$ with dimension at most $\dim_{K_0} D$, and it is semistable as a p-adic representation of G_K. Moreover, $D' := \mathrm{D}_{\mathrm{st}}(\mathrm{V}_{\mathrm{st}}(D))$ is naturally identified with a subobject of D, and D is in the essential image of D_{st} if and only if $\dim_{\mathbb{Q}_p} \mathrm{V}_{\mathrm{st}}(D) = \dim_{K_0} D$ or equivalently $D' = D$.*

Proof. Let C_{st} denote the fraction field of the domain B_{st} and let $V = V_{\mathrm{st}}(D)$ within the C_{st}-vector space $C_{\mathrm{st}} \otimes_{K_0} D$ of dimension $s = \dim_{K_0} D$, the $\mathbb{Q}_p$-subspace V generates a C_{st}-subspace $V' = C_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ of some dimension $r \leq s$. The case of $r = 0$ (i.e., $\mathrm{V}_{\mathrm{st}}(D) = 0$) is trivial, so we may and do assume $r > 0$. The action by G_K on $C_{\mathrm{st}} \otimes_{K_0} D$ preserves the C_{st}-subspace V'. View V' as a C_{st}-valued point of the Grassmannian variety $\mathrm{G}_r(D)$ over K_0 parametrizing r-dimensional subspaces of D. This point is invariant by G_K and so it descends to a $C_{\mathrm{st}}^{G_K}$-valued point. By the $(\mathbb{Q}_p, G_K)$-regularity of B_{st}, $C_{\mathrm{st}}^{G_K} = B_{\mathrm{st}}^{G_K} = K_0$, so V' corresponds to a K_0-valued point of $\mathrm{G}_r(D)$, which means that $V' = C_{\mathrm{st}} \otimes_{K_0} D'$ for a K_0-subspace $D' \subset D$ with dimension r. Thus $V \subset V' \cap (B_{\mathrm{st}} \otimes_{K_0} D) = B_{\mathrm{st}} \otimes_{K_0} D'$.

The K_0-subspace D' in D is stable by φ and N since this holds after scalar extension from K_0 to C_{st}. Using the subspace filtration on $D'_K \subset D_K$, we thereby make D' into a filetered (φ, N)-module over K that is a subobject of D. Since $V = \mathrm{V}_{\mathrm{st}}(D) = \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1, N=0}$ and $V \subset B_{\mathrm{st}} \otimes_{K_0} D'$, we have $V \subset \mathrm{V}_{\mathrm{st}}(D') \subset \mathrm{V}_{\mathrm{st}}(D) = V$, so $V = \mathrm{V}_{\mathrm{st}}(D')$.

By definition, V' is spanned over C_{st} by V, so we can find a C_{st}-basis $\{v_1, v_2, \ldots, v_r\}$ for V' consistng of elements of V; the v_i's are a maximal C_{st}-linearly independent subset of V. Thus, the map $\wedge^r_{\mathbb{Q}_p} V \longrightarrow \wedge^r_{C_{\mathrm{st}}} V'$ carries $v_1 \wedge v_2 \wedge \cdots \wedge v_r$ to a non-zero element and $\wedge^r_{C_{\mathrm{st}}} V'$ is a C_{st}-subspace of $C_{\mathrm{st}} \otimes_{K_0} \wedge^r_{K_0} D'$, so $v_1 \wedge v_2 \wedge \cdots v_r$ has non-zero iamge in $C_{\mathrm{st}} \otimes_{K_0} \wedge^r_{K_0} D'$. In other words, if we choose a K_0-basis $\{d_1, d_2, \ldots, d_r\}$ of D' and write $v_j = \sum_i b_{ij} d_i$ with $b_{ij} \in B_{\mathrm{st}}$ then $b := \det(b_{ij}) \in B_{\mathrm{st}}$ lies in $C_{\mathrm{st}}^{\times}$; that is $b \neq 0$ in B_{st}. Thus, the element

$$v_1 \wedge v_2 \wedge \cdots v_r = b d_1 \wedge d_2 \wedge \cdots \wedge d_r \in B_{\mathrm{st}} \otimes_{K_0} \wedge^r D' \tag{4.1}$$

lies in the 0-th filtered piece and the action of N vanishes and fixed by φ since each v_j lies in $V = \mathrm{V}_{\mathrm{st}}(D)$. Hence, we have a nonzero element of $\mathrm{V}_{\mathrm{st}}(\wedge^r D')$. But $\wedge^r D'$ is a 1-dimensional filtered (φ, N)-module over K. Since we have exhibited a non-zero element of $\mathrm{V}_{\mathrm{st}}(\wedge^r D')$ by Lemma 4.10 we cannot have $t_H(\wedge^r D') < t_N(\wedge^r D')$, or in other words the case $t_H(D') < t_N(D')$ cannot occur. The admissibility hypothesis on D implies $t_H(D') \leq t_N(D')$ for the subobject $D' \subset D$, so $t_H(D') = t_N(D')$. Hence, D' is admissible (as D is) and $\mathrm{V}_{\mathrm{st}}(\wedge^r D')$ must be exactly 1-dimensional over $\mathbb{Q}_p$.

Any r-fold wedge product of elements of $V = \mathrm{V}_{\mathrm{st}}(D) = \mathrm{V}_{\mathrm{st}}(D')$ is naturally an element of $\mathrm{V}_{\mathrm{st}}(\wedge^r D')$, and so if unique $\mathbb{Q}_p$-mulitple of $v_1 \wedge v_2 \wedge \cdots \wedge v_r$. But we can view this wedge product as being formed over B_{st} within $B_{\mathrm{st}} \otimes_{K_0} \wedge^r D'$, so if an element $v \in V \subset V'$ is arbitrary and we write (as we may) $v = \sum c_i v_i$ with unique $c_i \in C_{\mathrm{st}}$ then $v_1 \wedge \cdots \wedge v_{i-1} \wedge v \wedge v_{i+1} \wedge \cdots \wedge v_r = c_i(v_1 \wedge \cdots \wedge v_r)$. Hence, $c_i \in \mathbb{Q}_p$ for any $1 \leq i \leq r$. This shows that the v_i's span V over $\mathbb{Q}_p$, so they are a basis for V (since they are linearly independent over C_{st}). In other words, V has finite $\mathbb{Q}_p$-dimension that is equal to $r = \dim_{K_0} D' \leq \dim_{K_0} D$ and V then must have continuous G_K-action by Proposition 4.9.

The identity (4.1) now implies that G_K acts on b through a $\mathbb{Q}_p^{\times}$-valued character, so $\mathbb{Q}_p \subset B_{\mathrm{st}}$ is a G_K-stable line. Hence, by $(\mathbb{Q}_p, G_K)$-regularity of B_{st} we must have that $b \in B_{\mathrm{st}}^{\times}$. It therefore follows from (4.1) that the $\mathbb{Q}_p$-basis $\{v_1, v_2, \ldots, v_r\}$ for $V = \mathrm{V}_{\mathrm{st}}(D')$ is also a B_{st}-basis of $B_{\mathrm{st}} \otimes_{K_0} D'$, so the B_{st}-linear map $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \longrightarrow B_{\mathrm{st}} \otimes_{K_0} D'$ induceed by the identtification $V = \mathrm{V}_{\mathrm{st}}(D')$ is actually a linear isomorphism. By G_K-compatibility, we deduce that as K_0-vector spaces

$$\mathrm{D}_{\mathrm{st}}(V) \simeq (B_{\mathrm{st}} \otimes_{K_0} D')^{G_K} = B_{\mathrm{st}}^{G_K} \otimes_{K_0} D' = D'. \tag{4.2}$$

This shows that $\mathrm{D}_{\mathrm{st}}(V)$ has K_0-dimension equal to $\dim_{K_0} D' = r = \dim_{\mathbb{Q}_p} V$, so V is a semistable p-adic representation of G_K with dimension $r \leq \dim_{K_0} D$. The identification $\mathrm{D}_{\mathrm{st}}(V) = D'$ in (4.2) respects the Frobenius and monodromy operators, and carries $\mathrm{Fil}^j\mathrm{D}_{\mathrm{st}}(V)$ into $\mathrm{Fil}^j D'$ for all j. But D' is admissible and so is $\mathrm{D}_{\mathrm{st}}(V)$ by Theorem 4.8. Any morphism of admissible filtered (φ, N)-modules that is a liner isomorphism on K_0-vector spaces is automatically an isomorphism in $\mathrm{MF}_K(\varphi, N)$ (i.e., it is compatible with filtrations in both directions) by Theorem 2.33, so $D' \simeq \mathrm{D}_{\mathrm{st}}(V)$ as filtered (φ, N)-modules. We conclude that $\mathrm{D}_{\mathrm{st}}(V)$ is naturally a subobject of D, with K_0-dimension $\dim_{\mathbb{Q}_p} V$. Hence, $\dim_{\mathbb{Q}_p} V = \dim_{K_0} D$ if and only if the subobject $\mathrm{D}_{\mathrm{st}}(V) \subset D$ has full K_0-dimension, in which case D is in the essential image of D_{st}. Conversely, if D is in the essential image of D_{st}, say $D \simeq \mathrm{D}_{\mathrm{st}}(V_1)$ for $V_1 \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ then $V = \mathrm{V}_{\mathrm{st}}(D) \simeq \mathrm{V}_{\mathrm{st}}(\mathrm{D}_{\mathrm{st}}(V_1)) \simeq V_1$ (the final isomorphism due to the semistability of V_1). Hence, in such cases $\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} V_1 = \dim_{K_0} D$. ∎

Remark 4.12. Suppose $D \in \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$ is a simple object. In particular, $D \neq 0$. If $V := \mathrm{V}_{\mathrm{st}}(D) \neq 0$ then the above proof realizes $\mathrm{D}_{\mathrm{st}}(V)$ as a non-zero subobject of D, in which case it must equal D by simplicity. Hence, an admissible D that is simple in $\mathrm{MF}_K(\varphi, N)$ is in the essential image of D_{st} if and only if $\mathrm{V}_{\mathrm{st}}(D) \neq 0$.

Remark 4.13. As noted earlier, to actually prove that for any $D \in \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$, $\dim_{\mathbb{Q}_p} \mathrm{V}_{\mathrm{st}}(D) = \dim_{K_0} D$ requires a lot more work. Using this equality we immediately get that D is in the essential image of D_{st}. Proof of this claim can be found in [BC09, Pg. 183].

Assuming the claim from Remark 4.13, we summarize the following result. Similar statements are true for crystalline representations by taking semistable representations with vanishing monodromy.

Theorem 4.14. (i) *If V is a semistable p-adic representation of G_K, then $\mathrm{D}_{\mathrm{st}}(V)$ is an admissible filtered (φ, N)-module over K.*

(ii) *If D is an admissible filtered (φ, N)-module over K, then $\mathrm{V}_{\mathrm{st}}(D)$ is a semistable p-adic representation of G_K.*

(iii) *The exact tensor functor $\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N)$ is an equivalence of categories and $\mathrm{V}_{\mathrm{st}} : \mathrm{MF}^{\mathrm{ad}}_K(\varphi, N) \longrightarrow \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K)$ is a quasi-inverse.*

4.3 Crystalline and semistable representations in small dimensions

In the last section we saw an equivalence of categories and in Chapter 2 we have already studied the classification of admissible filetered (φ, N)-modules over $\mathbb{Q}_p$ in dimension 1 and 2. In case of dimension 1 we can study the representations in general for any p-adic field K but for dimension 2 case we take $K = \mathbb{Q}_p$ since in such cases φ is linear over $K_0 = K$ and this makes analyzing objects much easier on the linear algebra side. So, in this section our goal is to understand the crystalline and semistable representations that arise from our classification in Chapter 2.

4.3.1 Unramified Characters

For $n \geq 1$ let $F : W_n(k)^\times \longrightarrow W_n(k)^\times$ be the relative Frobenius morphism of the smooth affine $\mathbb{F}_p$-group of units in the length-n Witt vectors. Also, let $\gamma : W_n(k)^\times \longrightarrow W_n(k)^\times$ with $\gamma(x) := F(x)/x$.

Lemma 4.15. *There is a natural isomorphism*

$$W(k)^\times/\gamma W(k)^\times \simeq \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}_p^\times) = \mathrm{Hom}^{\mathrm{un}}_{\mathrm{cont}}(G_K, \mathbb{Q}_p^\times)$$

onto the group of unramified p-adic characters of G_K.

Proof. [BC09, Pg. 118]. ∎

In other words, the lemma parametrizes such characters by integral units $\lambda \in W(k)^\times$ upto the equivalence relation $\lambda \sim (\sigma(c)/c)\lambda$. In Proposition 2.42 we have seen that such equivalence classes also parametrize isomorphism classes of 1-dimensional admissible filtered (φ, N)-modules D over K with $t_H(D) = 0$. So to each continuous charcter $\eta : G_K \longrightarrow \mathbb{Q}_p^\times$ we can associate the isomorphism class of D_η of a 1-dimensional admissible filtered (φ, N)-module over K.

Lemma 4.16. *The bijective correspondence $\eta \longrightarrow D_\eta$ from continuous unramified character of G_K to isomorphism classes of 1-dimensional admisible filtered (φ, N)-modules over K with $t_H = 0$, is the contravariant functor $\mathrm{D}^*_{\mathrm{cris}} = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\cdot, B_{\mathrm{cris}})$. That is $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(\eta))$ is in the isomorphism class D_η.*

Proof. Let $\eta : G_K \longrightarrow \mathbb{Q}_p^\times$ be an unramified character. Them from Lemma 4.15 we get a $\lambda \in W(k)^\times$ such that for $w \in W(\overline{k})^\times$ satisfying $\gamma(w) = \lambda$, we have $g(w) = \eta(g)w$ for all $g \in G_K$. The choice of w is unique up to a $\mathbb{Z}_p^\times$-multiple, so the line $D = K_0 w \subset W(\overline{k})$ only depends on λ. Now $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(\eta)) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(\eta), B_{\mathrm{cris}})$ contains a nonzero element e corresponding to the map $1 \mapsto w$. But $\dim_{K_0} \mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(\eta)) \leq \dim_{\mathbb{Q}_p}(\mathbb{Q}_p(\eta)) = 1$, so $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(\eta))$ is 1-dimensional over K_0 with basis e. Clearly, the nontrivial gr^i is for $i = 0$ (as $w \in \widehat{K_0^{\mathrm{un}}}^\times$) and $\varphi(e) = \lambda e$ because $\sigma(w) = \lambda w$ by the way we choose w. ■

In Example 4.4 we verified that $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(1))$ is identified with the Tate twist $K_0[0]\langle 1 \rangle$ of the unit object. Hence, in view of the tensor compatibility of the functor and the direct calculation of the filtered (φ, N)-module $\mathrm{D}^*_{\mathrm{cris}}(\mathbb{Q}_p(1))$, it follows from Lemma 4.16 via Tate-twisting that every 1-dimensional admissible filtered (φ, N)-module over K is $\mathrm{D}^*_{\mathrm{cris}}$ applied to the Tate twist of an unramified character. From Corollary 4.6 all continuous unramified characters are crystalline.

4.3.2 Trivial Filtration

Let $K = \mathbb{Q}_p$ and $\mathrm{Gal}(\overline{K}/K) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = G_{\mathbb{Q}_p}$. From Subsection 2.6.1 we recall that for $K = \mathbb{Q}_p$, the study of modules in $\mathrm{MF}_K^{\mathrm{ad}}(\varphi, N)$ with finite dimension, bijective Frobenius-action and trivial filtration, is equivalent to studying, up to isomorphism, the $\mathrm{GL}_n(\mathbb{Q}_p)$-conjugacy classes of elements of $\mathrm{GL}_n(\mathbb{Z}_p)$. Now, from the equivalence of categories $\mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{MF}_K^{\mathrm{ad}}(\varphi)$, we get that $V = \mathrm{V}_{\mathrm{cris}}(D)$ is a crystalline representation. In particular, V is a Hodge-Tate representation and so $\mathrm{gr}\, D = \mathrm{gr}^0 D = \mathrm{Fil}^0 D/\mathrm{Fil}^1 D = D$. Therefore, V must be a $\mathbb{C}_p$-admissible representation of $G_{\mathbb{Q}_p}$. From [FO08, Prop. 3.55] V is $\mathbb{C}_p$-admissible if and only if the action of $I_{\mathbb{Q}_p}$, on V is discrete. Now from Corollary 4.6 it is obvious that V is an unramified representation of $G_{\mathbb{Q}_p}$. In summary we have the following proposition.

Proposition 4.17. *The n-dimensional admissible filtered (φ, N)-modules over $\mathbb{Q}_p$ with a single Hodge-Tate weight have vanishing N, and in case of Hodge-Tate weight 0 are parametrized upto isomorphism by $\mathrm{GL}_n(\mathbb{Q}_p)$-conjugacy classes of elements of $\mathrm{GL}_n(\mathbb{Z}_p)$. In general if the Hodge-Tate weight is r then such objects natyrally correspond under $\mathrm{D}^*_{\mathrm{cris}}$ to χ^r-twists of n-dimensional unramified p-adic representations of $G_{\mathbb{Q}_p}$.*

4.3.3 2-dimensional cases

Let $K = \mathbb{Q}_p$ and $\mathrm{Gal}(\overline{K}/K) = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) = G_{\mathbb{Q}_p}$. Note that for $K = \mathbb{Q}_p$, the Frobenius-action on $D \in \mathrm{MF}_K(\varphi, N)$ is linear. The following proposition and the subsequent remark classifies all 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$.

Proposition 4.18. *The set of isomorphism classes of 2-dimensional crystalline representations V of $G_{\mathbb{Q}_p}$ that have distinct Hodge-Tate weights $\{0, r\}$ with $r > 0$ and are not direct sum of two characters is naturally parametrized by the set of quadratic polynomials $P_\varphi(X) = X^2 + aX + b \in \mathbb{Z}_p[X]$ with $\upsilon_p(b) = r$, where P_φ is the characteristic polynomial of φ on $D = \mathrm{D}^*_{\mathrm{cris}}(V)$.*

(i) *If P_φ is irreducible then D has description as in Lemma 2.46. The crystalline Galois representation $\mathrm{V}^*_{\mathrm{cris}}(D)$ contravariantly associated to D is irreducible.*

(ii) *If P_φ is reducible with distinct roots then D has description as in Lemma 2.47.*

If P_φ is reducible with a repeated root λ (so $r = 2\upsilon_p(\lambda) \in 2\mathbb{Z}^+$) then D has description as in Proposition 2.48(ii) for the case $\lambda_1 = \lambda_2 = \lambda$.

In all these cases, φ does not act as a scalar on D. Also, the associated Galois representation $\mathrm{V}^*_{\mathrm{cris}}(D)$ is reducible if and only if P_φ has a unit root $\mu_1 \in \mathbb{Z}_p^\times$ (so this never occurs when P_φ has repeated roots), in which case the other root is $p^r\mu_2$ for some $\mu_2 \in \mathbb{Z}_p^\times$ and $\mathrm{V}^*_{\mathrm{cris}}(D)$ is an extension of the unramified character ψ_{μ_1} associated to μ_1 by the r-fold Tate twist $\chi^r\psi_{\mu_2}$ of the unramified character ψ_{μ_2} associated to μ_2.

Proof. (i) From Remark 2.49(i) we get the parametrization. These are exactly the 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights 0 and r for which φ acts irreducibly on D. As noted, removing the effect of the initial Tate twist on these examples amounts to allowing the smaller of the two distinct Hodge-Tate weights to be an arbitrary integer.

(ii) For the case when P_φ is reducible with distinct roots, from Proposition 2.48(ii) and Lemma 2.47, we obtain all crystalline representations of $G_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights 0 and $r \geq 1$ (via the contravariant functor $\mathrm{V}^*_{\mathrm{cris}}$) such that the representation is not a direct sum of two characters and the φ-action has distinct eigenvalues. As noted in Remark 2.49(iii), these are parametrized by unordered pairs of distinct non-zero $\lambda_1, \lambda_2 \in \mathbb{Z}_p$ such that $v_p(\lambda_1) + v_p(\lambda_2) = r \geq 1$.

In terms of this parameterization, the reducible Galois representations are exactly those for which exactly one of λ_1 or λ_2 is in $\mathbb{Z}_p^\times$. Moreover, in these reducible (non-decomposble) cases the unique non-trivial admissible subobject of D is the φ-eigenline for the unit eigenvalue, so in terms of the contravariant functor, the Galois representation has the following non-semisimple form,

$$\begin{pmatrix} \psi_2(r) & * \\ 0 & \psi_1 \end{pmatrix}$$

with ψ_1 and ψ_2 unramified characters of $G_{\mathbb{Q}_p}$ (valued in $\mathbb{Z}_p^\times$). These unramified characters correspond respectively to the units λ_1 and λ_2/p^r and our analysis (here and in Lemma 2.46) shows that the knowledge of these eigenvalues determines the Galois representation up to isomorphism.

From this we conclude two important facts. First of all, for any pair of unramified characters $\psi_1, \psi_2 : G_{\mathbb{Q}_p} \rightrightarrows \mathbb{Z}_p^\times$ and any $r \geq 1$ there is exactly one non-semisimple crystalline representation ρ_{ψ_1,ψ_2} containing $\psi_2(r)$ and admitting ψ_1 as a quotient. And the other important conclusion is that there is no non-split crystalline extension of $\psi_2(r)$ by ψ_1 with $r \geq 1$. That is, if $\eta_1, \eta_2 : G_{\mathbb{Q}_p} \rightrightarrows \mathbb{Q}_p^\times$ are crystalline characters (i.e., Tate twist of unramified characters) with respective Hodge-Tate weights h_1 and h_2, then there is no non-split crystalline extension of η_2 by η_1 if $h_2 > h_1$.

Next, we look into the case when P_φ is reducible with repeated roots. The statement about D is clear from Proposition 2.48. The corresponding Galois representations via the contravariant functor $\mathrm{V}^*_{\mathrm{cris}}$ are the irreducible crystalline representations with Hodge-Tate weights 0 and $r \in 2\mathbb{Z}^+$ such that the φ-action has a double root (with slope $r/2$) for its characteristic polynomial. The parametrization is clear from Remark 2.49(iii). ■

Remark 4.19. From Remark 2.49(iii), we notice that if D is a direct sum of two 1-dimensional objects then in contravariant Galois-theoretic terms, by Lemma 4.16 the corresponding representations are direct sums $\psi_1 \oplus \psi_2(r)$ with each ψ_i unramified (and the integral units λ_1 and λ_2/p^r encode the Frobenius-action for ψ_i). Removing the Tate twist makes this into the reducible decomposable crystalline case with distinct Hodge-Tate weights.

Combining Proposition 4.17, Proposition 4.18 and Remark 4.19 we obtain all 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$.

Next, we consider the case of non-crystalline semistable 2-dimensional representations V of $G_{\mathbb{Q}_p}$.

Proposition 4.20. *The non-crystalline semistable 2-dimensional representations V of $G_{\mathbb{Q}_p}$ with smallest Hodge-Tate weight equal to 0 are parametrized as follows: there is a Hodge-Tate weight $r > 0$ of the form $r = 2m+1$ with $m \geq 0$ and V is parametrized up to isomorphism by a pair (λ, c) with $\lambda \in p^m\mathbb{Z}_p^\times$ and $c \in \mathbb{Q}_p$. For a given (λ, c) the contravariantly associated filtered (φ, N)-module $D = \mathrm{D}^*_{\mathrm{st}}(V)$ is* *given explicitly by the non-crystalline case ($N \neq 0$) in Section 2.8. In these cases there is an admissible nontrivial subobject of D if and only if $m = 0$, i.e., $\lambda \in \mathbb{Z}_p^\times$.*

Proof. Given a semistable 2-dimensional representation V of $G_{\mathbb{Q}_p}$, up to Tate-twist we may assume that its smallest Hodge-Tate weight equals 0. By the equivalence of categories in Theorem 4.14 we get that $\mathrm{D}_{\mathrm{st}}^*(V)$ is admissible and has $\mathbb{Q}_p$-dimension 2. By our analysis of non-crystalline case ($N \neq 0$) in Section 2.8, we have the statement of the proposition. The last statement follows from Remark 2.44. ∎

Remark 4.21. According to the parameterization in Proposition 4.20, if $m > 0$ (i.e., the necessarily distinct Hodge-Tate weights 0 and $2m+1$ are not consecutive integers) then the semistable representation is irreducible, whereas if $m = 0$ then it is necessarily reducible and non-semisimple (as $N \neq 0$).

The case of reducible non-crystalline semistable representations from Proposition 4.20 and Remark 4.21 can be analyzed further. Using Lemma 4.16 and the contravariant functor $\mathrm{D}_{\mathrm{st}}^*$, these cases are precisely of non-split extensions of ψ by $\psi(1)$ for the unramified character $\psi : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p^\times$ classified by $\lambda \in \mathbb{Z}_p^\times$. In particular, for these reducible casesthe larger Hodge-Tate weight appears on the subobject exactly as in the crystalline reducible non-semisimple cases in Proposition 4.18 (but now the gap between weights is necessarily 1). Hence, the unique unramified quotient character ψ determines the 2-dimensional representation space (though not its non-split crystalline extension structure) up to isomorphism.

Applying the unramified twist by ψ^{-1} brings us to the case $\lambda = 1$ because $\mathrm{D}_{\mathrm{st}}^*$ is tensor-compatible. Now, since $\mathrm{D}_{\mathrm{st}}^*(\mathbb{Q}_p) = \mathrm{D}_{\mathrm{cris}}^*(\mathbb{Q}_p) \simeq K_0[0]$, we see that up to unramified twisting, the 2-dimensional reducible non-crystalline semistable representations of $G_{\mathbb{Q}_p}$ are parametrized by a single parameter $c \in \mathbb{Q}_p$. Note that to choose a basis of the line $\mathrm{D}_{\mathrm{cris}}^*(\mathbb{Q}_p)$ amounts to making a choice of $\mathbb{Q}_p$-basis of the canonical line $\mathbb{Q}_p(1) = \mathbb{Q}_p \cdot t \subset B_{\mathrm{cris}} \subset B_{\mathrm{dR}}$.

Focusing on the case $\psi = 1$, we have described all of the lines in the space

$$\mathrm{H}^1_{\mathrm{st}}(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1)) := \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \subset \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathrm{H}^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$$

of extension classes with underlying semistable representation. There is a distinguished line whose non-zero elements are the non-split crystalline extenison classes of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ (all of which are mutually isomorphic as representation spaces, forgetting the extension structure). The set of other lines is naturally parametrized by a parameter c as above. By Kummer theory, $\mathrm{H}^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$ is 2-dimensional when $p > 2$. Hence, the proved existence of a line of crystalline classes and a line whose non-zero elements are semistable classes shows (via the preservation of semistability under subrepresentations, quotients, and direct sums) that when $p > 2$ all elements in $\mathrm{H}^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$ correspond to semistable representations, and that there is adistinguished line consisting of the crystalline classes. We have the following generalization,

Proposition 4.22. *For any p-adic field K, each element in $\mathrm{H}^1(G_K, \mathbb{Q}_p(1))$ corresponds to a semistable G_K-representation and there is a $\mathbb{Q}_p$-line consisting of the crystalline classes.*

Proof. This is a restatement of Proposition 3.27. ∎

Chapter 5

p-adic Galois representations from elliptic curves over $\mathbb{Q}_p$

In this chapter we work with primes $p > 5$. Fix $K = \mathbb{Q}_p$ and its algebraic closure as $\overline{K} = \overline{\mathbb{Q}_p}$ with the absolute Galois group $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Let $E/\mathbb{Q}_p$ be an elliptic curve and let $T_p(E)$ be its p-adic Tate module which is a free $\mathbb{Z}_p$-module of rank 2. We set $V_p(E) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$, a $\mathbb{Q}_p$-vector space of dimension 2. There is a natural action of the absolute Galois group $G_{\mathbb{Q}_p}$ on $T_p(E)$ and $V_p(E)$. Therefore, we get representations

$$\rho_p : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T_p(E)) \quad \text{and} \quad \rho_p : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V_p(E)).$$

For more details on elliptic curves please refer to [Sil13, Chap. III, V and VII]

The goal of next few sections is to study the p-adic Galois representations coming from elliptic curves over $\mathbb{Q}_p$ and give a classification of such representations based on reduction type of elliptic curves.

5.1 Introduction

In [Vol00], we find an explicit description of the p-adic Galois representations coming from elliptic curves over $\mathbb{Q}_p$. We mention the key result below.

Theorem 5.1. *Let V_p be a 2-dimensional p-adic representation of $G_{\mathbb{Q}_p}$. Then the following assertions are equivalent.*

(i) *There exists and elliptic curve E over $\mathbb{Q}_p$ such that $V_p(E)$ is isomorphic to V_p.*

(ii) *The representation V_p is potentially semistable (i.e., semistable over a p-adic field K) and the associated filtered $(\varphi, N, G_{K/\mathbb{Q}_p})$-module is isomorphic to an object of the list D^* (cf. 5.2.1).*

Using certain necessary conditions Volkov describes a list D^* of isomorphism classes of objects in $\mathrm{MF}_K(\varphi, N)$ arising from elliptic curves over $\mathbb{Q}_p$. This is the content of our Subsection 5.2.1, where we explicitly describe these modules along with respective Frobenius- and monodromy-actions. Next, given and elliptic curve $E/\mathbb{Q}_p$, based on its j_E-invariant and reduction type we describe the object on D^* which is isomorphic to $V_p(E)$. In other words, after application of the functor $\mathrm{D}_{\mathrm{pst}}$, which we shall describe below, we get $\mathrm{D}_{\mathrm{pst}}(V_p(E))$ and so we compare with the objects in the list D^*. We do all the computations and draw some parallels from the results proven in Chapter 3 as well. The theorem also claims the converse problem of constructing examples of elliptic curves out of any given object in the list D^*. We do not prove the converse and instead give several (though not all) explicit examples of elliptic curves $E/\mathbb{Q}_p$ via Weierstrass equations such that $V_p(E)$ is on the list D^*.

Before we start, we discuss some notations and formalism first.

5.1.1 Notations

Let $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ be an unramified extension of degree 2. Let $\pi_{12} \in \overline{\mathbb{Q}_p}$ such that $\pi_{12}^{12} + p = 0$. Let $\pi_6 = \pi_{12}^2$, $\pi_4 = \pi_{12}^3$, $\pi_3 = \pi_{12}^4$, $\pi_2 = \pi_{12}^6$, $\pi_1 = -p$. Also, let ζ_{12} be a primitive 12-th root of unity and let $\zeta_6 = \zeta_{12}^2$, $\zeta_4 = \zeta_{12}^3$, $\zeta_3 = \zeta_{12}^4$, $\zeta_2 = \zeta_{12}^6$.

For $e \in \{1, 2, 3, 4, 6\}$ consider $\mathbb{Q}_p(\pi_e)$: it is a totally ramified extension of $\mathbb{Q}_p$ of degree e. Since $p > 5$, the ramification index e is prime to p and therefore $\mathbb{Q}_p(\pi_e)/\mathbb{Q}_p$ is tamely ramified. Let K_e be the Galois closure of $\mathbb{Q}_p(\pi_e)$ in $\overline{\mathbb{Q}_p}$ and $G_{K_e/\mathbb{Q}_p} = \mathrm{Gal}(K_e/\mathbb{Q}_p)$ the Galois group, let I_e the inertia subgroup of the extension $\overline{\mathbb{Q}_p}/K_e$. Since $(\mathbb{Z}/e\mathbb{Z})^\times$ has order 1 or 2, therefore $p \equiv 1 \mod e$ or $p \equiv -1 \mod e$.

Based on the discussion above we would encounter the following possibilities:

(1) $K_1 = \mathbb{Q}_p$ and $G_{K_1/\mathbb{Q}_p} = 1$.

(2) $K_2 = \mathbb{Q}_p(\pi_2)$ and $G_{K_2/\mathbb{Q}_p} = \langle \tau_2 \rangle$ where τ_2 is defined such that $\tau_2(\pi_2) = -\pi_2$.

(3) If $e \in \{3, 4, 6\}$ and $e \mid (p-1)$, then $K_e = \mathbb{Q}_p(\pi_e)$ and $G_{K_e/\mathbb{Q}_p} = \langle \tau_e \rangle$ where τ_e is defined such that $\tau_e(\pi_e) = \zeta_e \pi_e$.

(4) If $e \in \{3, 4, 6\}$ and $e \mid (p+1)$ then $K_e = \mathbb{Q}_{p^2}(\pi_e) = \mathbb{Q}_p(\pi_e, \zeta_e)$. $G_{K_e/\mathbb{Q}_p} = \langle \tau_e \rangle \rtimes \langle \omega \rangle$ where τ_e is defined such that $\tau_e(\pi_e) = \zeta_e \pi_e$, $\tau_e(\zeta_e) = \zeta_e$ and ω is the list of the absolute Frobenius which fixes π_e and $\omega(\zeta_e) = \zeta_e^{-1}$. It is easy to observe that $\omega\tau_e = \tau_e^{-1}\omega$.

If K'_e is another Galois extension of $\mathbb{Q}_p$ of ramification index e then there exists a finite unramified extension M of $\mathbb{Q}_p$ such that $MK_e = MK'_e$.

Now we take a look at quadratic extensions of $\mathbb{Q}_p$. The group $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^4$ has order 4 and there are exactly 3 quadratic extensions of $\mathbb{Q}_p$, one unramified and two totally ramified. We write these extensions as $M_1 = \mathbb{Q}_{p^2}$, $M_2 = \mathbb{Q}_p(\pi_2)$ and M_3 (for example if $4 \mid (p+1)$ then $M_3 = \mathbb{Q}_p(\zeta_4\pi_2)$).

Let $\mathcal{N}_p = \{a \in \mathbb{Z}$ such that $|a| \leq 2\sqrt{p}\}$ and $\mathcal{N}_p^\times$ its set of non-zero elements. Of course, $\mathcal{N}_p^\times = 2\lfloor 2\sqrt{p} \rfloor$. Let $\Phi_e \in \mathbb{Q}[X]$ be the e-th cyclotomic polynomial. Set $\gamma_e = \zeta_e + \zeta_e^{-1} = \mathrm{Tr}(\Phi_e)$ and we have

$$\mathrm{Tr}(\Phi_e) = \begin{cases} -1 & \text{if } e = 3 \\ 0 & \text{if } e = 4 \\ 1 & \text{if } e = 6. \end{cases}$$

For $e \in \{3, 4, 6\}$ and $e \mid (p-1)$, let $\mathcal{N}_{p,e}^\times \subset \mathcal{N}_p^\times$ be the set if all $a \in \mathbb{Z}$ such that $(\gamma_e^2 - 4)(a^2 - 4p)$ is a square in $\mathbb{Q}$.

Lemma 5.2. *Let $p \geq 5$. If $4 \mid p-1$ then the set $\mathcal{N}_{p,4}^\times = \{a \in \mathbb{Z} \mid a^2 - 4p \equiv -1 \mod (\mathbb{Q}^\times)^2\}$ is in bijection with $\mu_4(\overline{\mathbb{Q}})$ and if $3 \mid p-1$ then the sets $\mathcal{N}_{p,3}^\times = \mathcal{N}_{p,6}^\times = \{a \in \mathbb{Z} \mid a^2 - 4p \equiv -3 \mod (\mathbb{Q}^\times)^2\}$ are in bijection with $\mu_6(\overline{\mathbb{Q}})$. Moreover, if $12 \mid p-1$ then these sets ($\mathcal{N}_{p,3}^\times$ and $\mathcal{N}_{p,4}^\times$) are disjoint.*

Proof. If $p \equiv 1 \mod 4$ then $\mathcal{N}_{p,4}^\times = \{a \in \mathbb{Z} \mid -(a^2 - 4p) \in (\mathbb{Q}^\times)^2\} = \{a \in \mathbb{Z} \mid 4p = a^2 + b^2, b \in \mathbb{Z}\}$ is in bijection with $\{a \in \mathbb{Z} \mid p = a^2 + b^2,\ b \in \mathbb{Z}\}$. Let σ_4 be the conjugation in $\mathbb{Q}(\zeta_4) = \mathbb{Q}(\sqrt{-1})$ (i.e., the generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q})$ and $N_{\mathbb{Q}(\zeta_4)/\mathbb{Q}}(x) = x\sigma_4(x)$ for $x \in \mathbb{Q}(\zeta_4)$ is the norm). The ring of integers of $\mathbb{Q}(\zeta_4)$ is $\mathbb{Z}[\zeta_4] = \{a + \zeta_4 b,\ a, b \in \mathbb{Z}\}$. Let $N_{p,4} = \{x \in \mathbb{Z}[\zeta_4] \mid N_{\mathbb{Q}(\zeta_4)/\mathbb{Q}}(x) = p\}$, it is a non-empty set since $\mathbb{Z}[\zeta_4]$ is a principal domain and $p \equiv 1 \mod 4$ (which is to say that -1 is a quadratic residue modulo p). Any element of $N_{p,4}$ provides an element in the set $\{a \in \mathbb{Z} \mid p = a^2 + b^2,\ b \in \mathbb{Z}\}$. Hence, the map from $N_{p,4}$ to $\mathcal{N}_{p,4}^\times$ is clearly surjective and any two elements x_1, x_2 have the same image if and only if $x_2 = \sigma_4(x_1)$. So we have a bijection $N_{p,4}/\langle \sigma_4 \rangle \xrightarrow{\sim} \mathcal{N}_{p,4}^\times$. Now if $x_0 \in N_{p,4}$, the set $N_{p,4}$ consists of $x_0, \sigma_4(x_0)$ as well as their product with elements of norm 1 i.e., the units $(\mathbb{Z}[\zeta_4])^\times = \langle \zeta_4 \rangle$, so we deduce a bijection $N_{p,4}/\langle \sigma_4 \rangle \xrightarrow{\sim} \langle \zeta_4 \rangle$ and hence we have the first claim.

If $p \equiv 1 \mod 3$, let $\zeta_3 = \zeta_6^2$, a primitive third root of unity. Note as σ_3 the conjugation and $N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}$ as the norm for the quadratic extension $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ over $\mathbb{Q}$; the ring of integers in $\mathbb{Z}[\zeta_3] = \{(a + \sqrt{-3}b) \mid a, b \in \mathbb{Z},\ a \equiv b \mod 2\mathbb{Z}\}$ which is a principal domain and its units are elements of norm 1. The set $N_{p,3} = \{x \in \mathbb{Z}[\zeta_3] \mid N_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(x) = p\}$ is non-empty because $\mathbb{Z}[\zeta_3]$ is principal and $p \equiv 1 \mod 3$ (which is to say that -3 is a quadratic residue modulo p). Now the proof is quite similar to the previous case: the set $\mathcal{N}_{p,3}^{\times} = \{a \in \mathbb{Z} \mid 12p = 3a^2 + b^2, b \in \mathbb{Z}\}$ is in bijection with $N_{p,3}/\langle\sigma_3\rangle$, which is in bijection with $(\mathbb{Z}[\zeta_3])^{\times} = \langle\zeta_6\rangle$. ■

Now we mention some facts about elliptic curves that will be used later. Let E be an elliptic curve over $\mathbb{Q}_p$. Let us assume that $\upsilon_p(j_E) < 0$, i.e. E has potentially multiplicative reduction at prime p. Then upto twisting E by a quadratic character, we may assume that $E = E_q$ [MP06, pg. 300], where E_q is the Tate curve with $q \in \mathbb{Q}_p^{\times}$, $\upsilon_p(q) \geq 1$ and q is only determined by the modular invariant j_E (refer Theorem 3.26). In Section 3.4 we discussed the p-adic representation V_pE_q. Recall that by the choice of p-power compatible roots of unity, from (3.5) we have an exact sequence of $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-modules,

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T_pE_q \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

On tensoring with $\mathbb{Q}_p$, we obtain a short exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules,

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V_pE_q \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{5.1}$$

Next, we assume that $\upsilon_p(j_E) \geq 0$. In this case E has potentially good reduction, i.e., it acquires good reduction over a finite extension of $\mathbb{Q}_p$. *défaut de semistabilite* (*dst*) is defined as the minimal ramification index e, for which E has good reduction. This is given as $dst(E) = 12/\gcd(12, \upsilon_p(\Delta_E))$ where Δ_E is the discriminant of E. For a minimal Weierstrass model of E, $0 \leq \upsilon_p(\Delta_E) < 12$ and $\upsilon_p(j_E) \geq 0$ implies $\upsilon_p(\Delta_E)$ is coprime to 12. Therefore

$$dst(E) = e = \begin{cases} 1 & \text{if } \upsilon_p(\Delta_E) = 0 \\ 2 & \text{if } \upsilon_p(\Delta_E) = 6 \\ 3 & \text{if } \upsilon_p(\Delta_E) \in \{4, 8\} \\ 4 & \text{if } \upsilon_p(\Delta_E) \in \{3, 9\} \\ 6 & \text{if } \upsilon_p(\Delta_E) \in \{2, 10\} \end{cases}$$

Notice that $\phi(n) \in \{1, 2\}$ where ϕ is the Euler's totient function.

$e = dst(E)$ is coprime to p. E acquires good reduction over a totally ramified extension of $\mathbb{Q}_p$ of degree e; if L is such an extension, let $\tilde{E}_L = (E \times_{\mathbb{Q}_p} L) \times_L \mathbb{F}_p$ be the reduced curve over $\mathbb{F}_p$ and $a_p(E) = a_p(\tilde{E}_L)$, the trace of the characteristic polynomial of the Frobeniu acting on $V_l(\tilde{E}_L)$ for $l \neq p$. $a_p(\tilde{E}_L)$ is a rational integer independent of $l \neq p$ and $a_p(\tilde{E}_L) = p + 1 - \#\tilde{E}_L(\mathbb{F}_p)$. $\tilde{E}/\mathbb{F}_p$ is *ordinary* if $p \nmid a_p(\tilde{E}_L)$ and *supersingular* if $p \mid a_p(\tilde{E}_L)$. If E has good reduction over L and the reduced curve is ordinary, then the connected part $\mathrm{E}_L(p)^{\circ}$ of the p-divisble group $E_L(p)$ has height 1 and we have the following exact sequence

$$0 \longrightarrow E_L(p)^{\circ} \longrightarrow E_L(p) \longrightarrow \tilde{E}_L(p) \longrightarrow 0$$

which induces a short exact sequence [Mum74, Pg. 147]

$$0 \longrightarrow T_p(E_L(p)^{\circ}) \longrightarrow T_p(E) \longrightarrow T_p(\tilde{E}_L) \longrightarrow 0. \tag{5.2}$$

Tensoring it with $\mathbb{Q}_p$ gives an exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules.

5.1.2 Some Formalism

We recall that $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ is the category of p-adic representations of G i.e., $\mathbb{Q}_p$-vector spaces of finite dimension with a continuous and linear action of $G_{\mathbb{Q}_p}$. Also $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_{\mathbb{Q}_p})$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(G_{\mathbb{Q}_p})$ are respectively the full subcategories of p-adic crystalline and semistable representations of $G_{\mathbb{Q}_p}$. Let K be a finite Galois extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}_p}$. We can similarly define the full subcategories $\mathrm{Rep}_K^{\mathrm{cris}}(G_{\mathbb{Q}_p})$ and $\mathrm{Rep}_K^{\mathrm{st}}(G_{\mathbb{Q}_p})$ respectively, of crystalline and semistable K-representations of $G_{\mathbb{Q}_p}$. Moreover, we define $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pcris}}(G_{\mathbb{Q}_p})$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{pst}}(G)$ respectively, as the full subcategories of potentially crystalline and potentially semistable representations of G.

Let $G_{K/\mathbb{Q}_p} = \mathrm{Gal}(K/\mathbb{Q}_p)$, and K_0 the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Let σ be the absolute Frobenius on K_0. We define the category of filtered $(\varphi, N, G_{K/\mathbb{Q}_p})$-modules below, similar to the definition 2.10 of filtered (φ, N)-modules.

Definition 5.3. A *filtered* $(\varphi, N, G_{K/\mathbb{Q}_p})$-module is a K_0-vector space D equipped with

(a) A σ semi-linear action of $G_{K/\mathbb{Q}_p}$ (the inertia subgroup acts linearly).

(b) A σ semi-linear, $G_{K/\mathbb{Q}_p}$-equivariant and injective Frobenius map $\varphi : D \longrightarrow D$.

(c) A K_0-linear and $G_{K/\mathbb{Q}_p}$-equivariant endomorphism $N : D \longrightarrow D$ such that $N\varphi = p\varphi N$.

(d) A decreasing, separated and exhaustive filtration on $D_K = K \otimes_{K_0} D$ given by K-vector subspaces $\{\mathrm{Fil}^i D_K, i \in \mathbb{Z}\}$. It must be stable under the action of $G_{K/\mathbb{Q}_p}$ which extends semi-linearly to D_K.

Morphism: A morphism $f : D_1 \longrightarrow D_2$ between two filtered $(\varphi, N, G_{K/\mathbb{Q}_p})$-modules is a K_0-linear map commuting with the action of $G_{K/\mathbb{Q}_p}$, φ and N such that if we write f_K for the K-linear map obtained from f by scalar extension, then $f_K(\mathrm{Fil}^i D_{1,K}) \subset \mathrm{Fil}^i D_{2,K}$ for every $i \in \mathbb{Z}$.

We denote by $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$ the full subcategory of filtered $(\varphi, N, G_{K/\mathbb{Q}_p})$-modules of finite dimension (i.e., the action of $G_{K/\mathbb{Q}_p}$ on D is discrete). Similarly, denote $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$ the full subcategory of objects in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$ such that $N = 0$. For $K = \mathbb{Q}_p$ we observe that the definition above coincides with definition 2.10 and so we write unambiguously the categories $\mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ and $\mathrm{MF}_{\mathbb{Q}_p}(\varphi)$. The Hodge-Tate weight of an object D of dimension 2 in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$ is a pair (r, s) such that $\mathrm{Fil}^i D_K = D_K$ if $i \leq r$ and $\mathrm{Fil}^i D_K = 0$ if $i > s$. For an object $D \in \mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$ of dimension d we have the definitions of *Newton number* and *Hodge Number* respectively, from definitions 2.14 and 2.24. Therefore we can also impose the admissibility condition of definition 2.28 on D.

Now we give the functors connecting the categories of Galois representations and the semi-linear algebra objects we discussed above. From (3.1) we have the following functors,

$$\mathrm{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{\mathbb{Q}_p}(\varphi)$$
$$\mathrm{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N).$$

We write the contravariant functors as

$$\mathrm{D}^*_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{\mathbb{Q}_p}(\varphi)$$
$$\mathrm{D}^*_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N).$$

where $\mathrm{D}^*_{\mathrm{cris}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, B_{\mathrm{cris}})$ and $\mathrm{D}^*_{\mathrm{st}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, B_{\mathrm{st}})$. Moreover, in similar fashion we can also define the functors

$$\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p} : \mathrm{Rep}^{\mathrm{cris}}_{K}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{K}(\varphi)$$
$$\mathrm{D}^*_{\mathrm{st},K/\mathbb{Q}_p} : \mathrm{Rep}^{\mathrm{st}}_{K}(G_{\mathbb{Q}_p}) \longrightarrow \mathrm{MF}_{K}(\varphi, N).$$

with $\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{cris}})$ and $\mathrm{D}^*_{\mathrm{st},K/\mathbb{Q}_p}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{st}})$. Here $G_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$. These are full, faithful and exact tensor functors which establish the anti-equivalence of the categories $\mathrm{Rep}^{\mathrm{cris}}_K(G_{\mathbb{Q}_p})$ and $\mathrm{Rep}^{\mathrm{st}}_K(G_{\mathbb{Q}_p})$ with their respective essential image.

Now we set $\mathrm{D}^*_{\mathrm{pcris}}$ and $\mathrm{D}^*_{\mathrm{pst}}$ respectively, as the functors obtained by taking the direct limit of $\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}$ and $\mathrm{D}^*_{\mathrm{st},K/\mathbb{Q}_p}$ where K varies over all finite Galois extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}_p}$. These are contravariant functors from $\mathrm{Rep}^{\mathrm{pcris}}_{\mathbb{Q}_p}(G)$ and $\mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(G)$ to the direct limit of $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$ and $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$ respectively.

In the Chapter 2 we saw the classification of admissible filtered (φ, N)-modules in dimension 1 and 2 when the base field is $\mathbb{Q}_p$. In this chapter we turn our attention towards p-adic representations $V_p(E)$ coming from elliptic curves defined over $\mathbb{Q}_p$. Upon application of the appropriate contravariant functor, we obtain certain semi-linear algebraic objects in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi, N)$. Our first objective is to classify all such "interesting" objects. These objects are mentioned explicitely in the list we call D^*. Next, we consider elliptic curves and show that for any elliptic curve $E/\mathbb{Q}_p$ there exists an object $D \in \mathrm{D}^*$ such that $\mathrm{D}_{\mathrm{pst}}(V_p(E)) \simeq D$. In fact, the converse is also true, i.e., for any object on the list D^* there exists an elliptic curve $E/\mathbb{Q}_p$ such that $\mathrm{D}_{\mathrm{pst}}(V_p(E))$ is isomorphic to the object we started with. We mention some explicit examples for this case, however we do not discuss the proof of the converse statement. A detailed account and proof of these statements can be found in the paper [Vol00].

5.2 Some objects in the category $\mathbf{MF}_K(\varphi, N)$

5.2.1 A list of objects from $\mathbf{MF}_K(\varphi, N)$ of Hodge-Tate weight (0,1)

In this section we describe the objects of $\mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ that are of interest to us i.e., $\mathbb{Q}_p$-vector spaces with some extra structure. We describe it with a list which makes the parametrization of objects easier. Objects in each separate case are related by certain "quadratic twists" which we describe later.

Case 1. $(\mathrm{D}^*_{\mathrm{m}})$: Two-dimensional objects $D \in \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ of Hodge-Tate weight (0,1) such that there exists $V \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ with either $D \simeq \mathrm{D}^*_{\mathrm{st}}(V)$ or $D \simeq \mathrm{D}^*_{\mathrm{st}}(V')$ where $V' \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ is a twist of V by a quadratic character. Moreover, it must be that $V, V^{'} \notin \mathrm{Rep}^{\mathrm{pcris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$. These objects are parametrized as $\mathrm{D}^*_{\mathrm{m}}(e, \lambda, \alpha)$ where $e \in \{1, 2\}$, $\lambda \in \{\pm 1\}$ and $\alpha \in \mathbb{Q}_p$.

(a) $\mathbf{e = 1}$. In this case, we have $K = K_1 = \mathbb{Q}_p$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = \lambda e_1$, $\varphi(e_2) = \lambda p e_2$ and the monodromy-action as $N(e_1) = 0$, $N(e_2) = e_1$ i.e.,

$$[\varphi] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda p \end{pmatrix} \quad \text{and} \quad [N] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha e_1 + e_2)\mathbb{Q}_p, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

(b) $\mathbf{e = 2}$. In this case, we have $K = K_2 = \mathbb{Q}_p(\pi_2)$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = \lambda e_1$, $\varphi(e_2) = \lambda p e_2$ and the monodromy-action as $N(e_1) = 0$, $N(e_2) = e_1$ i.e.,

$$[\varphi] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda p \end{pmatrix} \quad \text{and} \quad [N] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha e_1 + e_2)\mathbb{Q}_p(\pi_2), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Moreover, the action of $G_{K/\mathbb{Q}_p} = \langle \tau_2 \rangle$ is described as $\tau_2(e_1) = -e_1$ and $\tau_2(e_2) = -e_2$.

Case 2. $(\mathrm{D}^*_{\mathrm{c}})$: Two-dimensional objects $D \in \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ such that there exists $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ with either $D \simeq \mathrm{D}^*_{\mathrm{cris}}(V)$ or $D \simeq \mathrm{D}^*_{\mathrm{cris}}(V')$ where $V' \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ is a twist of V by a quadratic character. These objects are parametrized in two different ways. The different cases arise owing to whether or not the characteristic polynomial of the Frobenius-action, φ is irreducible over $\mathbb{Q}_p$.

First we mention the case when the charateristic polynomial of φ is reducible over $\mathbb{Q}_p$. The $\mathbb{Q}_p$-vector spaces are parametrized as $\mathrm{D}^*_{\mathrm{c}}(e, a_p, \alpha)$ where $e \in \{1, 2\}, a_p \in \{0, 1\}$ and $\alpha \in \mathbb{Q}_p$. Let $u \in \mathbb{Z}_p^\times$ be the unique element satisfying $u + u^{-1}p = a_p$. Such a u exists because $a_p \in \mathbb{Z}_p^\times$.

(a) $\mathbf{e=1}$. In this case, we have $K = K_1 = \mathbb{Q}_p$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = ue_1$, $\varphi(e_2) = u^{-1}pe_2$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} u & 0 \\ 0 & u^{-1}p \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha e_1 + e_2)\mathbb{Q}_p, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

(b) $\mathbf{e=2}$. In this case, we have $K = K_2 = \mathbb{Q}_p(\pi_2)$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = ue_1$, $\varphi(e_2) = u^{-1}pe_2$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} u & 0 \\ 0 & u^{-1}p \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha e_1 \otimes 1 + e_2 \otimes 1)\mathbb{Q}_p(\pi_2), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Moreover, the action of $G_{K/\mathbb{Q}_p} = \langle \tau_2 \rangle$ is given as $\tau_2(e_1) = -e_1$ and $\tau_2(e_2) = -e_2$.

Next we deal with the case when the characteristic polynomial of φ is irreducible in $\mathbb{Q}_p[X]$. In this case the $\mathbb{Q}_p$-vector spaces are parametrized as $\mathrm{D}_{\mathrm{c}}^*(e, 0)$ with $e \in \{0, 1\}$.

(a) $\mathbf{e=1}$. In this case, we have $K = K_1 = \mathbb{Q}_p$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = e_2$, $\varphi(e_2) = -pe_1$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ \mathbb{Q}_p e_1, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

(b) $\mathbf{e=2}$. In this case, we have $K = K_2 = \mathbb{Q}_p(\pi_2)$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = e_2$, $\varphi(e_2) = -pe_1$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (e_1 \otimes 1)\mathbb{Q}_p(\pi_2), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Moreover, the action of $G_{K/\mathbb{Q}_p} = \langle \tau_2 \rangle$ is given as $\tau_2(e_1) = -e_1$ and $\tau_2(e_2) = -e_2$.

Case 3. $(\mathrm{D}^*_{\mathrm{pc}})$: Two-dimensional objects $D \in \mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ such that there exists $V \in \mathrm{Rep}^{\mathrm{pcris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ with $D \simeq \mathrm{D}^*_{\mathrm{pcris}}(V)$. Moreover, it must be that $D \not\simeq \mathrm{D}^*_{\mathrm{cris}}(V)$ for some $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ or V a twist by quadratic character of a crystalline representation. As in the previous case, these objects are again parametrized in two different ways. The different cases arise owing to whether or not the characteristic polynomial of the Frobenius-action, φ is irreducible over $\mathbb{Q}_p$.

First we mention the case when the characteristic polynomial of φ is reducible over $\mathbb{Q}_p$. The $\mathbb{Q}_p$-vector spaces are parametrized as $\mathrm{D}^*_{\mathrm{pc}}(e, a_p, \varepsilon, \alpha)$ where $e \in \{3, 4, 6\}$ and $e \mid p-1$, $a_p \in \mathcal{N}^{\times}_{p,e}$, $\varepsilon \in \{\pm 1\}$ and $\alpha \in \{0, 1\}$. Let $u \in \mathbb{Z}_p^{\times}$ be the unique element satisfying $u + u^{-1}p = a_p$. Again, such a u exists because $a_p \in \mathbb{Z}_p^{\times}$.
We have $K = K_e = \mathbb{Q}_p(\pi_e)$ and $D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$ with the Frobenius-action given as $\varphi(e_1) = ue_1$, $\varphi(e_2) = u^{-1}pe_2$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} u & 0 \\ 0 & u^{-1}p \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \le 0 \\ (\alpha e_1 \otimes \pi_e^{-\varepsilon} + e_2 \otimes \pi_e^{\varepsilon})\mathbb{Q}_p(\pi_e), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Moreover, the action of $G_{K/\mathbb{Q}_p} = \langle \tau_e \rangle$ is given as $\tau_e(e_1) = \zeta_e^{\varepsilon} e_1$ and $\tau_e(e_2) = \zeta_e^{-\varepsilon} e_2$.

Next we deal with the case when the characteristic polynomial of φ is irreducible. The $\mathbb{Q}_p$-vector spaces are parametrized as $\mathrm{D}^*_{\mathrm{pc}}(e, 0, \alpha)$ where $e \in \{3, 4, 6\}$ and $e \mid p+1$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$.

We have $K = K_e = \mathbb{Q}_{p^2}(\pi_e)$ and $D = \mathbb{Q}_{p^2} e_1 \oplus \mathbb{Q}_{p^2} e_2$ with the Frobenius-action given as $\varphi(e_1) = e_2$, $\varphi(e_2) = -pe_1$ and the monodromy-action as $N(e_1) = N(e_2) = 0$ i.e.,

$$[\varphi] = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad [N] = 0$$

The filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \le 0 \\ (\alpha e_1 \otimes \pi_e^{-1} + e_2 \otimes \pi_e)\mathbb{Q}_{p^2}(\pi_e), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Moreover, the action of $G_{K/\mathbb{Q}_p} = \langle \tau_e \rangle \rtimes \langle \omega \rangle$ is given as $\omega(e_1) = e_1$, $\omega(e_2) = e_2$ and $\tau_e(e_1) = \zeta_e e_1$, $\tau_e(e_2) = \zeta_e^{-1} e_2$.

5.2.2 Description of quadratic twists

Let D_0 be an object from the list D^* above. Let $V_0 \in \mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(G)$ such that $D_0 \simeq \mathrm{D}^*_{\mathrm{pst}}(V_0)$. We say that D_1 in the list D^* is a quadratic twist of D_0 if there exists $V_1 \in \mathrm{Rep}^{\mathrm{pst}}_{\mathbb{Q}_p}(G)$ such that $D_1 \simeq \mathrm{D}^*_{\mathrm{pst}}(V_0)$ and V_1 is a twist by a quadratic character of V_0 corresponding to the quadratic extension M_1, M_2 or M_3 of $\mathbb{Q}_p$.

In what follows we specify four objects from each case in the list, D_0, D_1, D_2, D_3 where (D_0, D_1) and (D_2, D_3) are related via twist by an unramified quadratic character, whereas (D_0, D_2), (D_0, D_3), (D_1, D_2) and (D_1, D_3) are related via twist by a ramified character. We have the following relations,

D_0	D_1	D_2	D_3
$\mathrm{D}_{\mathrm{m}}^*(1, b, \alpha)$	$\mathrm{D}_{\mathrm{m}}^*(1, -b, \alpha)$	$\mathrm{D}_{\mathrm{m}}^*(2, b, \alpha)$	$\mathrm{D}_{\mathrm{m}}^*(2, -b, \alpha)$
$\mathrm{D}_{\mathrm{c}}^*(1, a_p, \alpha)$	$\mathrm{D}_{\mathrm{c}}^*(1, -a_p, \alpha)$	$\mathrm{D}_{\mathrm{c}}^*(2, a_p, \alpha)$	$\mathrm{D}_{\mathrm{c}}^*(2, -a_p, \alpha)$
$\mathrm{D}_{\mathrm{c}}^*(1, 0)$	$\mathrm{D}_{\mathrm{c}}^*(1, 0)$	$\mathrm{D}_{\mathrm{c}}^*(2, 0)$	$\mathrm{D}_{\mathrm{c}}^*(2, 0)$
$\mathrm{D}_{\mathrm{pc}}^*(4, a_p, \varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(4, -a_p, \varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(4, a_p, -\varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(4, -a_p, -\varepsilon, \alpha)$
$\mathrm{D}_{\mathrm{pc}}^*(3, a_p, \varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(3, -a_p, \varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(6, a_p, -\varepsilon, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(6, -a_p, -\varepsilon, \alpha)$
$\mathrm{D}_{\mathrm{pc}}^*(4, 0, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(4, 0, -\alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(4, 0, p^2\alpha^{-1})$	$\mathrm{D}_{\mathrm{pc}}^*(4, 0, -p^2\alpha^{-1})$
$\mathrm{D}_{\mathrm{pc}}^*(3, 0, \alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(3, 0, -\alpha)$	$\mathrm{D}_{\mathrm{pc}}^*(6, 0, p^2\alpha^{-1})$	$\mathrm{D}_{\mathrm{pc}}^*(6, 0, -p^2\alpha^{-1})$

Remark 5.4. (i) In the third case $D_0 = D_1$ and $D_2 = D_3$ i.e., the unramified twists give isomorphic representations.

(ii) In the sixth and seventh case, for $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$, $\alpha = -\alpha$ if and only if $\alpha \in \{0, +\infty\}$ in which case the ramified twists give isomorphic representations.

(iii) If an object D of the list D^* comes from an elliptic curve over $\mathbb{Q}_p$ i.e., there exists $E/\mathbb{Q}_p$ such that $D \simeq \mathrm{D}_{\mathrm{pst}}^*(V_p(E))$ then the objects D_i with $i \in \{1, 2, 3\}$ come from elliptic curve E_i obtained by twisting E by a quadratic character corresponding to the quadratic extension M_i as in [Sil13, Examp. X.2.4].

5.3 Classification of $\mathbb{Q}_p[G]$-modules V_pE

In this section we look at $\mathbb{Q}_p$-vector spaces $V_pE = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pE$ where T_pE is the usual p-adic Tate module for some elliptic curve $E/\mathbb{Q}_p$. This examination is divided into 3 cases just as in our list D^*. For the first case, we look at the elliptic curves having potentially multiplicative reduction over $\mathbb{Q}_p$. In the second case, we deal with the elliptic curves acquiring good reduction over $\mathbb{Q}_p$ or a quadratic extension of $\mathbb{Q}_p$. Finally, we examine the case where an elliptic curve $E/\mathbb{Q}_p$ has potentially good reduction where it turns out that E acquires good reduction over an extension of $\mathbb{Q}_p$ of degree either 3, 4 or 6.

5.3.1 Case 1: Potentially multiplicative reduction

From (5.1) we have the exact sequence,

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V_pE_q \longrightarrow \mathbb{Q}_p \longrightarrow 0.$$

We know that any extension V of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ is semistable from Proposition 3.27. Recall that we write $V_pE_q = \mathbb{Q}_pe \oplus \mathbb{Q}_pf$ for a choice of basis elements (e, f). Note that we are working with the contravariant functor $\mathrm{D}_{\mathrm{st}}^* = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(\cdot, B_{\mathrm{st}})$. So we define $\mathbb{Q}_p$-linear $G_{\mathbb{Q}_p}$-equivariant morphisms w and z from V_pE_q into B_{st}^+ by setting

$$\begin{aligned} w(e) &= 0, \quad w(f) = 1; \\ z(e) &= t, \quad z(f) = \frac{1}{m}\log[\tilde{q}] \end{aligned}$$

where $m \in \mathbb{N}_{\geq 1}$ such that $q = u_qp^m$ for $u_q \in \mathbb{Z}_p^\times$ and $\tilde{q}$ as defined in section 3.4. z is injective and its image does not depend on the choice of the element $\tilde{q}$. Since we have $D := \mathrm{D}_{\mathrm{st}}^*(V_pE_q) = \mathrm{Hom}_{\mathbb{Q}_p[G]}(V_pE_q, B_{\mathrm{st}})$ is a $\mathbb{Q}_p$-vector space of dimenison 2. Also, w and z are clearly linearly independent over $\mathbb{Q}_p$, therefore $D = \mathbb{Q}_pw \oplus \mathbb{Q}_pz$. In addition, from the Frobenius map φ on B_{st} we have $\varphi(t) = pt$ and $\varphi(\log[\tilde{q}]) = p\log[\tilde{q}]$ since $\tilde{q} \in R(\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p})$; $N(t) = 0$ and $N(\log[\tilde{q}]) = m$. From this it is easily deduced that $\varphi(w) = w$, $\varphi(z) = pz$ and $N(w) = 0$, $N(z) = w$ i.e.,

$$[\varphi] = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad [N] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This gives D a structure of (φ, N)-module over $\mathbb{Q}_p$. It could be seen that $\mathbb{Q}_p w$ is a subobject of D while $\mathbb{Q}_p z$ is not, owing to the fact that $N(z) = w$. Therefore the exact sequence 3.5 does not split.

Notice that we are in the situation $N \neq 0$ of Section 2.8 with $\lambda_1 = 1$. Therefore, we can immediately conclude from Lemma 2.43 that $D \simeq D_{\{1,\alpha_q\}}$ for some unique $\alpha_q \in \mathbb{Q}_p$. The filtration on D is given as,

$$\mathrm{Fil}^i D = \begin{cases} D, & \text{if } i \leq 0 \\ (\alpha_q w + z)\mathbb{Q}_p, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

To determine α_q we observe that by definition we have $\mathrm{Fil}^1 D = \mathrm{Hom}_{\mathbb{Q}_p[G]}(V_pE_q, tB_{\mathrm{dR}}^+)$ i.e., $\theta(\alpha_q w(a) + z(a)) = 0$ for any $a \in V_pE_q = \mathbb{Q}_p e \oplus \mathbb{Q}_p f$. Since $w(e) = 0$ and $z(e) = t \in ker\ \theta$, we see that the condition above could be re-written as $\theta(\alpha_q w(f) + z(f)) = \theta(\alpha_q + \log[\tilde{q}]/m) = 0$. Since $\log[\tilde{q}] \equiv \log_p(u_q)$ mod $ker\ \theta$, we get that $\theta(m\alpha_q + \log_p(u_q)) = 0$. But $m\alpha_q + \log_p(u_q) \in \mathbb{Q}_p$, therefore $m\alpha_q + \log_p(u_q) = 0$. So we have $\alpha_q = -\log_p(u_q)/\upsilon_p(q)$ where $q = u_q p^{\upsilon_p(q)}$ and $\upsilon_p(q) \geq 1$.

Remark 5.5. (i) The map $p\mathbb{Z}_p \setminus \{0\} \longrightarrow \mathbb{Q}_p$ where $q \mapsto \alpha_q$ is surjective. However, it is not injective. For $q_1, q_2 \in p\mathbb{Z}_p$ we have that $\alpha_{q_1} = \alpha_{q_2}$ if and only if $\log_p(u_{q_1}^{\upsilon_p(q_2)}) = \log_p(u_{q_2}^{\upsilon_p(q_1)})$ i.e., $u_{q_1}^{\upsilon_p(q_2)(p-1)} = u_{q_2}^{\upsilon_p(q_1)(p-1)}$ which is equivalent to $q_1^{\upsilon_p(q_2)(p-1)} = q_2^{\upsilon_p(q_1)(p-1)}$.

(ii) From Lemma 2.43 we conclude that if D_1, D_2 are two (φ, N)-modules, as above, corresponding to $q_1, q_2 \in p\mathbb{Z}_p \setminus \{0\}$ such that $D_1 \simeq D_2$ then $\alpha_{q_1} = \alpha_{q_2}$.

(iii) In the category $\mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ it can immediately be observed that $\mathrm{D}_{\mathrm{st}}^*(V_pE_q)$ is isomorphic to $\mathrm{D}_{\mathrm{m}}^*(1, 1, \alpha)$ with $\alpha = \alpha_q \in \mathbb{Q}_p$.

(iv) From Proposition 2.45, we also see that $\alpha_q \in \mathbb{Q}_p$ parametrizes all non-isomorphic objects $D_{\{1,\alpha_q\}}$ of Hodge Tate weight $(0, 1)$ and they all come from Tate's curve over $\mathbb{Q}_p$ because the map $q \mapsto \alpha_q$ is surjective. In addition, we see that for two Tate curves E_{q_1} and E_{q_2} we have that $V_pE_{q_1} \simeq V_pE_{q_2}$ if and only if $q_1^{\upsilon_p(q_2)(p-1)} = q_2^{\upsilon_p(q_1)(p-1)}$ i.e., E_{q_1} must be isogenous to E_{q_2} over $\mathbb{Q}_p$.

(v) By twisting Tate's curve $E_q/\mathbb{Q}_p$ with three possible quadratic characters corresponding to the quadratic extensions M_1, M_2 and M_3, we obtain all the objects $\mathrm{D}_{\mathrm{m}}^*(e, b, \alpha)$ where $e \in \{1, 2\}$, $b \in \{\pm 1\}$ and $\alpha \in \mathbb{Q}_p$, of the list D^*.

5.3.2 Case 2: Good reduction, $e \in \{1, 2\}$

Let E be an elliptic curve over $\mathbb{Q}_p$ with $\upsilon_p(j_E) \geq 0$ and $e = dst(E) \in \{1, 2\}$. Up to a twist of E by a ramified quadratic character correpsonding to the extension $\mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$, we may assume that E has good reduction over $\mathbb{Q}_p$ (i.e., $e = 1$). We write E, also for the scheme over $\mathbb{Z}_p$ from which we get the elliptic curve over $\mathbb{Q}_p$; $E(p)$ for the associated p-divisible group; $\tilde{E} = E \times_{\mathbb{Z}_p} \mathbb{F}_p$ the special fiber; $a_p = a_p(\tilde{E})$. We have $V_pE = V_p(E(p))$ and the determinant is $\mathbb{Q}_p(1)$.

We know that every object $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ coming from a p-divisible group (or Barsotti-Tate group) is crystalline [Fon82b, §6], and the Hodge-Tate weight of $D = \mathrm{D}_{\mathrm{cris}}^*(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, B_{\mathrm{cris}})$ is $(0, 1)$. Therefore the object $D = \mathrm{D}_{\mathrm{cris}}^*(V_pE)$ of $\mathrm{MF}_{\mathbb{Q}_p}^{\mathrm{ad}}(\varphi)$ is a 2 dimensional $\mathbb{Q}_p$-vector space, equipped with a $\mathbb{Q}_p$-linear Frobenius map $\varphi : D \longrightarrow D$ verifying $\varphi^2 - a_p\varphi + p = 0$. Its Hodge-Tate weight is $(0, 1)$ and so $\mathrm{Fil}^1 D$ is a $\mathbb{Q}_p$-line. Moreover, these two data on D must satisfy the admissibility conditions of Definition 2.28.

We deal with the cases of $\tilde{E}$ being supersingular or ordinary separately.

1. First of all we suppose $a_p = 0$ i.e., $\tilde{E}$ is *supersingular* (the connected part $E(p)^\circ$ of $E(p)$ is of height 2 and $V_p(E) = V_p(E(p)^\circ)$). So the characteristic polynomial of φ, $P_\varphi(X) = X^2 + p$ is irreducible in $\mathbb{Q}_p[X]$ and therefore, no $\mathbb{Q}_p$-line of D is stable under φ. Let $e_1 \in D$ non-zero and $e_2 = \varphi(e_1)$ then (e_1, e_2) form a basis of D in which the matrix of φ is given as

$$[\varphi] = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix}.$$

The admissibility condition of Definition 2.28 is also satisfied, since $t_H(D) = 1 = \det[\varphi] = t_N(D)$ and there are no proper sub-objects of D in $\mathrm{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$. Now we observe that we are in the setting of Proposition 2.48 and therefore $D \simeq D_{\{0,p\}}$. The filtration on D is given as,

$$\mathrm{Fil}^i D = \begin{cases} D, & \text{if } i \leq 0 \\ \mathbb{Q}_p e_1, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Remark 5.6. (i) In the category $\mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ it can immediately be concluded that $\mathrm{D}^*_{\mathrm{cris}}(V_pE)$ is isomorphic to $\mathrm{D}^*_{\mathrm{c}}(1,0)$ from the list D^*.

(ii) Twisting by a quadratic character corresponding to extension $\mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$, we get the object $\mathrm{D}^*_{\mathrm{c}}(2,0)$ in $\mathrm{MF}_{\mathbb{Q}_p(\pi_2)}(\varphi)$ of the list D^*.

2. Next we suppose that $a_p \neq 0$ i.e., $a_p \in \mathcal{N}_p^{\times}$ and $\tilde{E}$ is *ordinary*. In the exact sequence of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$0 \longrightarrow V_p(E(p)^{\circ}) \longrightarrow V_pE \longrightarrow V_p\tilde{E} \longrightarrow 0 \tag{5.3}$$

the action of inertia subgroup $I_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ on the $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-module of rank one, $V_p\tilde{E}$ is trivial. By application of the contravariant functor $\mathrm{D}^*_{\mathrm{cris}}$, we obtain another short exact sequence in $\mathrm{MF}^{\mathrm{ad}}_{\mathbb{Q}_p}(\varphi)$

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0 \tag{5.4}$$

where $D = \mathrm{D}^*_{\mathrm{cris}}(V_pE)$, $D_1 = \mathrm{D}^*_{\mathrm{cris}}(V_p\tilde{E})$ and $D_2 = \mathrm{D}^*_{\mathrm{cris}}(V_p(E(p)^{\circ}))$. It is obvious that the exact sequence 5.3 splits in $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ if and only if the exact sequence 5.4 splits in $\mathrm{MF}_{\mathbb{Q}_p}(\varphi)$.

Since $a_p \not\equiv 0 \mod p\mathbb{Z}_p$, by Hensel's lemma [Mil08, Th. 7.33] the polynomial $P_{\varphi}(X) = X^2 - a_pX + p$ has two distinct linear factors in $\mathbb{Q}_p[X]$ i.e., two distinct roots in $\mathbb{Q}_p$. Let $u = u(a_p)$ be the unique element of $\mathbb{Z}_p^{\times}$ such that $u + u^{-1}p = a_p$. So we have $P_{\varphi}(X) = (X - u)(X - u^{-1}p)$ and φ is diagonalizable over D. Let (e_1, e_2) be $\mathbb{Q}_p$-basis of diagonalization of φ in D such that $\varphi(e_1) = ue_1$ and $\varphi(e_2) = u^{-1}pe_2$. The admissibility criterion is satisfied for D since $t_H(D) = 1 = \det[\varphi] = t_N(D)$. Now if we assume that D is not a direct sum of two admissible (φ, N)-modules of dimension 1 then we are in the setting of Proposition 2.48. In this case the filtration on D by Remark 2.49 (i) is given as

$$\mathrm{Fil}^i D = \begin{cases} D, & \text{if } i \leq 0 \\ (e_1 + e_2)\mathbb{Q}_p, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Otherwise, we may assume that D is indeed a direct sum of two admissible (φ, N)-modules of dimension 1 which is equivalent to saying that the exact sequence 5.3 splits. In this case we are in the setting of Remark 2.49 (iii) and therefore the filtration on D is given as,

$$\mathrm{Fil}^i D = \begin{cases} D, & \text{if } i \leq 0 \\ \mathbb{Q}_p e_2, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

Remark 5.7. (i) In the category $\mathrm{MF}_{\mathbb{Q}_p}(\varphi, N)$ it can immediately be concluded that $\mathrm{D}^*_{\mathrm{cris}}(V_pE)$ is isomorphic to $\mathrm{D}^*_{\mathrm{c}}(1, a_p, \alpha)$ from the list D^* with $a_p \in \mathcal{N}_p^{\times}$ and $\alpha \in 0, 1$. Moreover, the exact sequence 5.3 splits if and only if $\alpha = 0$.

(ii) Twisting by a quadratic character corresponding to extension $\mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$, we get the object $\mathrm{D}^*_{\mathrm{c}}(2, a_p, \alpha)$ in $\mathrm{MF}_{\mathbb{Q}_p(\pi_2)}(\varphi)$ of the list D^*.

5.3.3 Case 3: Potentially good reduction, $e \in \{3, 4, 6\}$

Let E be an elliptic curve over $\mathbb{Q}_p$ such that $\upsilon_p(j_E) \geq 0$ and its *défaut de stabilite*, $\mathrm{dst}(E) = e \in \{3, 4, 6\}$. Then E acquires good reduction over a totally ramified extension $L = L_e = \mathbb{Q}_p(\pi_e)/\mathbb{Q}_p$ of degree e and it does not have good reduction over any smaller extension i.e., an extension of $\mathbb{Q}_p$ with ramification index striclty smaller than e. We take K as the Galois closure of L inside $\overline{\mathbb{Q}_p}$. We denote by K_0 the maximal unramified extension of $\mathbb{Q}_p$ inside K. So we have $K_0 = \mathbb{Q}_p$ if $p \equiv 1 \mod e\mathbb{Z}$ and $K_0 = \mathbb{Q}_{p^2}$ if $p \equiv -1 \mod e\mathbb{Z}$. We denote by E_L and E_K respectively the $\mathcal{O}_L = \mathbb{Z}_p[\pi_e]$ and $\mathcal{O}_K = \mathbb{Z}_{p^2}[\pi_e]$ schemes which give us the elliptic curves $E \times_{\mathbb{Q}_p} L$ and $E \times_{\mathbb{Q}_p} K$; $E_L(p)$ and $E_K(p) = E_L(p) \times_{\mathcal{O}_L} \mathcal{O}_K$ respectively, the associated p-divisible groups; $\tilde{E_L} = E_L \times_{\mathcal{O}_L} \mathbb{F}_p$ and $\tilde{E_K} = E_K \times_{\mathcal{O}_K} \mathbb{F}_{p^2} = \tilde{E_L} \times_{\mathbb{F}_p} \mathbb{F}_{p^2}$ their special fibers and $a_p = a_p(\tilde{E})$. We have $V_pE_L = V_p(E_L(p))$ as $\mathbb{Q}_p[G_L]$-modules and $\Lambda^2 V_pE = \mathbb{Q}_p(1)$.

We know that for any object V of $\mathrm{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ which is potentially Barsotti-Tate is potentially crystalline and the Hodge-Tate weight of $D = \mathrm{D}^*_{\mathrm{cris}}(V)$ is $(0,1)$. The object $D = \mathrm{D}^*_{\mathrm{cris}}(V_pE) = \mathrm{D}_{\mathrm{cris},K/\mathbb{Q}_p}(V_pE)$ is in $\mathrm{MF}^{\mathrm{ad}}_{K/\mathbb{Q}_p}(\varphi)$: it is a K_0-vector space of dimension 2, equipped with a σ-linear Frobenius map $\varphi : D \longrightarrow D$, a semi-linear action of $G_{K/\mathbb{Q}_p} = \mathrm{Gal}(K/\mathbb{Q}_p)$ on D which commutes with φ. There is also a decreasing filtration on $D_K = K \otimes_{K_0} D$ which is stable by the action of $G_{K/\mathbb{Q}_p}$ when extended semi-linearly to D_K. The filtration is given such that $\mathrm{Fil}^0 D_K = D_K$, $\mathrm{Fil}^1 D_K = K$-line and $\mathrm{Fil}^2 = 0$. D must also satisfy the admissibility criteria of Definition 2.28. Recall that an object of $\mathrm{MF}_{k/\mathbb{Q}_p}(\varphi)$ is admissible if and only if the object in $\mathrm{MF}_K(\varphi)$ obtained by forgetting the action of $G_{K/\mathbb{Q}_p}$ is admissible. Moreover, we have $\Lambda^2 D = \mathbb{Q}_p\{1\}$, i.e., $\Lambda^2 D = \mathbb{Q}_p$ on which φ acts with multiplication by p, $G_{K/\mathbb{Q}_p}$ acts trivially and $\mathrm{Fil}^1 \Lambda^2 D = \mathbb{Q}_p$, $\mathrm{Fil}^2 \Lambda^2 D = 0$.

The inertia subgroup of $G_{K/\mathbb{Q}_p}$, $I(K/\mathbb{Q}_p) = \langle \tau_e \rangle$ acts K_0-linearly over D and we get a morphism $\nu : \langle \tau_e \rangle \longrightarrow \mathrm{Aut}_{K_0}(D)$ such that ν is injective. Indeed, if $H = ker\ \nu \subset I(K/\mathbb{Q}_p)$ is non-trivial, then $D = \mathrm{D}^*_{\mathrm{cris}}(V_pE)$ is be an object of $\mathrm{MF}_{K^H}(\varphi)$. This means V_pE is crystalline over K^H and so E acquires good reduction over the field K^H whose remification index is lower that $e = \mathrm{dst}(E)$. But this is not possible, and hence ν is injective. We identify τ_e with its image under ν, and it is therefore an element of order $e \in \{3,4,6\}$ in $\mathrm{Aut}_{K_0}(D)$. Its determinant is 1 since $\Lambda^2 D = \mathbb{Q}_p\{1\}$. Finally, we deduce that the characteristic polynomial of τ_e is $P_\varphi(X) = (X - \zeta_e)(X - \zeta_e^{-1}) \in \mathbb{Z}[X]$ since $(\mathbb{Z}/e\mathbb{Z})^\times = \{\pm 1\}$. In particular, the $K_0 = \mathbb{Q}_p(\zeta_e)$-linear automorphism is diagonalisable with distinct eigenvalues in D.

The Frobenius map $\varphi : D \longrightarrow D$ is σ-semi-linear. Since $\mathrm{D}^*_{\mathrm{cris}}(V_pE)$ is an object of $\mathrm{MF}_L(\varphi)$ (forgetting the action of $G_{K/\mathbb{Q}_p}$ on D) and E acquires good reduction over L (with $\mathbb{F}_p$ as its residue field), therefore φ satisfies the polynomial $X^2 - a_pX + p$. More precisely, the filtered φ-module over L, $\mathrm{D}^*_{\mathrm{cris},L}(V_pE) = \mathrm{Hom}_{\mathbb{Q}_p[G_L]}(V_pE, B_{\mathrm{cris}})$ gives, upon tensoring with K_0, the filtered $(\varphi, G_{K/L})$-module $\mathrm{D}^*_{\mathrm{cris,\ K/L}}(V_pE)$.

We deal with the two cases of $e \mid (p-1)$ and $e \mid (p+1)$ separately.

1. $\mathbf{e \mid (p-1)}$. In this case φ is $\mathbb{Q}_p$-linear and the relation $\varphi\tau_e = \tau_e\varphi$ implies that φ is diagonalizable in the basis of eigenvectors of τ_e. In particular, the characteristic polynomial of φ, $X^2 - a_pX + p$ splits in $\mathbb{Q}_p[X]$. This is equivalent to $a_p \not\equiv 0 \mod p$, i.e., $a_p \neq 0$ and $\tilde{E_L}$ is ordinary. Let $u = u(a_p)$ be the unique element of $\mathbb{Z}_p^\times$ such that $u + u^{-1} = a_p$. There exists a $\mathbb{Q}_p$-basis (e_1, e_2) of D in which the respective matrices of φ and τ_e are given as,

$$[\varphi] = \begin{pmatrix} u & 0 \\ 0 & u^{-1}p \end{pmatrix} \quad \text{and} \quad [\tau_e] = \begin{pmatrix} \zeta_e^\varepsilon & 0 \\ 0 & \zeta_e^{-\varepsilon} \end{pmatrix}$$

with $\varepsilon \in (\mathbb{Z}/e\mathbb{Z})^\times = \{\pm 1\}$. This gives the structure of $(\varphi, G_{K/\mathbb{Q}_p})$-module of $D = \mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}(V_pE)$. Moreover, $a_p \in \mathcal{N}^\times_{p,e}$ and the $(\varphi, G_{K/\mathbb{Q}_p})$-module defined for $\varepsilon = 1$ is not isomorphic to the one defined for $\varepsilon = -1$ (see [Vol00, Pg. 111]).

Because of the same reasons as in Proposition 2.48, the admissibility condition of Definition 2.28 implies that $\mathrm{Fil}^1 D_K \neq (e_1 \otimes 1)K$. So $\mathrm{Fil}^1 D_K = (e_1 \otimes \beta + e_2 \otimes 1)K$ with $\beta \in K = \mathbb{Q}_p(\pi_e)$. The admissibility condition is then satisfied for D_K. $\mathrm{Fil}^1 D_K$ is stable under the action of $G_{K/\mathbb{Q}_p}$ if and only if $\tau_e(e_1 \otimes \beta + e_2 \otimes 1) = \zeta_e^\varepsilon e_1 \otimes \tau_e(\beta) + \zeta_e^{-\varepsilon} e_2 \otimes 1 \in (e_1 \otimes \beta + e_2 \otimes 1)\mathbb{Q}_p(\pi_e)$ i.e., $\zeta_e^{2\varepsilon}\tau_e(\beta) = \beta$. This is equivalent to saying that $\tau_e(\pi_e^{2\varepsilon}\beta) = \zeta_e^{2\varepsilon}\pi_e^{2\varepsilon}\tau_e(\beta) = \pi_e^{2\varepsilon}\beta$ i.e., $\pi_e^{2\varepsilon}\beta \in \mathbb{Q}_p$. By writing

$\beta = \alpha\pi_e^{-2\varepsilon}$ with $\alpha \in \mathbb{Q}_p$ we get that the filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha \cdot e_1 \otimes \pi_e^{-\varepsilon} + e_2 \otimes \pi_e^{\varepsilon})\mathbb{Q}_p(\pi_e), & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases}$$

If $\alpha = 0$, we get that $\mathrm{Fil}^1 D_K = (e_2 \otimes \pi_e^{\varepsilon})\mathbb{Q}_p(\pi_e)$. Otherwise, for $\alpha \neq 0$, similar to Remark 2.49 (i) we can take $\mathrm{Fil}^1 D_K = (e_1 \otimes \pi_e^{-\varepsilon} + e_2 \otimes \pi_e^{\varepsilon})\mathbb{Q}_p(\pi_e)$.

Remark 5.8. (i) In the category $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$, it can immediately be concluded that $\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}(V_pE)$ is isomorphic to $\mathrm{D}^*_{\mathrm{pc}}(e, a_p, \varepsilon, \alpha)$ of the list D^* given in Case 3 of section 5.2.1.

(ii) The quadruplet $(e, a_p, \varepsilon, \alpha)$ varies over the set $\{n = 3, 4 \text{ or } 6 \text{ such that } n \mid (p-1)\} \times \mathcal{N}^\times_{p,e} \times \{\pm 1\} \times \{0, 1\}$ and parametrizes all of the objects $\mathrm{D}^*_{\mathrm{pc}}(e, a_p, \varepsilon, \alpha)$. These objects are not isomorphic to each other in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$.

(iii) From the short exact sequence of $\mathbb{Q}_p[G]$-modules

$$0 \longrightarrow V_p(E_k(p)^\circ) \longrightarrow V_pE \longrightarrow V_p(\tilde{E_K}) \longrightarrow 0$$

upon application of the functor $\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}$ we get a short exact sequence in $\mathrm{MF}^{\mathrm{ad}}_{K/\mathbb{Q}_p}(\varphi)$

$$0 \longrightarrow D_1 \longrightarrow D \longrightarrow D_2 \longrightarrow 0$$

where $D_i = \mathbb{Q}_p e_i$ for $i = 1, 2$. This exact sequence splits if and only if $\alpha = 0$.

2. $\mathbf{e \mid (p+1)}$. In this case φ is σ semi-linear, $\det\varphi = p$ and $\sigma(\zeta_e) = \zeta_e^{-1}$. We write $D_0 := D^{\langle\omega\rangle} = \{x \in D \mid \omega x = x\}$, and so we have $D_0 = \mathrm{D}^*_{\mathrm{cris},L}(V_pE)$ and $\mathbb{Q}_{p^2} \otimes_{\mathbb{Q}_p} D_0 = D$. The relation $\omega\varphi = \varphi\omega$ implies that $\varphi D_0 \subset D_0$ and the restriction of φ to D_0 is $\mathbb{Q}_p$-linear. Let (e_1, e_2) be a $\mathbb{Q}_p$-basis which diagonalizes τ_e. Upto reordering the basis (e_1, e_2) to (e_2, e_1), one can assume that $\tau_e e_1 = \zeta_e e_1$ and $\tau_2 e_2 = \zeta_e^{-1} e_2$ (i.e., $\varepsilon = 1$). The relation $\tau_e\omega = \omega\tau_e^{-1}$ gives $\tau_e(\omega e_1) = \zeta_w(\omega e_1)$ and $\tau_e(\omega e_2) = \zeta_e^{-1}(\omega e_2)$, so $\omega e_i \in \mathbb{Q}_{p^2} e_i$ for $i = 1, 2$. Now the σ semi-linearity of ω implies that $D_0 \cap \mathbb{Q}_{p^2} e_i \neq 0$ for $i = 1, 2$. We deduce that there exists a $\mathbb{Q}_p$-basis of D_0 which we again write as (e_1, e_2) such that $\tau_e e_1 = \zeta_e e_1$ and $\tau_e e_2 = \zeta_e^{-1} e_2$ in $D = \mathbb{Q}_{p^2} \otimes_{\mathbb{Q}_p} D_0$. Finally the relation $\tau_e\varphi = \varphi\tau_e$ gives that $\tau_e(\varphi e_1) = \zeta_e^{-1}\varphi e_1$ and $\tau_e(\varphi e_2) = \zeta_e(\varphi e_2)$, whence $\varphi e_1 \in \mathbb{Q}_{p^2} e_2$ and $\varphi e_2 \in \mathbb{Q}_{p^2} e_1$. But since $\varphi D_0 \subset D_0$, we have that $\varphi e_1 \in \mathbb{Q}_p e_2$ and $\varphi e_2 \in \mathbb{Q}_p e_1$. Since $\det\varphi = p$, we therefore have $\varphi e_1 = a e_2$, $\varphi e_2 = -pa^{-1} e_1$ where $a \in \mathbb{Q}_p^\times$. Then upto changing (e_1, e_2) to $(e_1, a e_2)$, we conclude that there exists a $\mathbb{Q}_p$-basis (e_1, e_2) of D such that

$$[\varphi] = \begin{pmatrix} 0 & -p \\ 1 & 0 \end{pmatrix} \quad , \quad [\tau_e] = \begin{pmatrix} \zeta_e & 0 \\ 0 & \zeta_e^{-1} \end{pmatrix} \quad \text{and} \quad [\omega] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, we see that $\varphi^2 + p = 0$ and therefore $a_p = 0$ i.e., $\tilde{E_L}$ is supersingular. This gives the structure of $(\varphi, G_{K/\mathbb{Q}_p})$-module of $D = \mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}(V_pE)$.

Let us determine the K-line $\mathrm{Fil}^1 D_K$. Since there are no proper K_0-vector subspaces of D stable under φ (see Lemma 2.46) the admissibility condition is trivially satisfied. Now $\mathrm{Fil}^1 D_K$ must be stable under the action of $G_K/\mathbb{Q}_p = \langle\tau_e\rangle \rtimes \langle\omega\rangle$ which extends semi-linearly to D_K. This is true for the case when $\mathrm{Fil}^1 D_K = (e_1 \otimes 1)K$. Otherwise, let $\mathrm{Fil}^1 D_K = (e_1 \otimes \beta + e_2 \otimes 1)K$ with $\beta \in K = \mathbb{Q}_p(\pi_e)$. Then $\omega(e_1 \otimes \beta + e_2 \otimes 1) = e_1 \otimes \omega(\beta) + e_2 \otimes 1 \in \mathrm{Fil}^1 D_K$ if and only if $\omega(p) = \beta$ i.e., $\beta \in \mathbb{Q}_p(\pi_e) = L$ (this is fine since D_0 is an object of $\mathrm{MF}_L(\varphi)$ and $\mathbb{Q}_{p^2} \otimes_{\mathbb{Q}_p} D_0$: the filtration over D_K comes from the filtration over $(D_0)_L$). Now, $\tau_e(e_1 \otimes \beta + e_2 \otimes 1) = \zeta_e e_1 \otimes \tau_e(\beta) + \zeta_e^{-1} e_2 \otimes 1 \in \mathrm{Fil}^1 D_K$ if and only if $\zeta_e^2 \tau_e(\beta) = \beta$ which is equivalent to saying that $\pi_e^2\beta \in \mathbb{Q}_{p^2}$. Therefore $\pi_e^2\beta \in \mathbb{Q}_{p^2} \cap \mathbb{Q}_p(\pi_e) = \mathbb{Q}_p$ and we have that $\beta = \alpha\pi_e^{-2}$ with $\alpha \in \mathbb{Q}_p$. Finally, we deduce that the filtration on D_K is given as

$$\mathrm{Fil}^i D_K = \begin{cases} D_K, & \text{if } i \leq 0 \\ (\alpha \cdot e_1 \otimes \pi_e^{-1} + e_2 \otimes \pi_e)\mathbb{Q}_{p^2}(\pi_e), & \text{if } i = 1 \\ 0, & \text{if } i > 1 \end{cases}$$

where $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$. It is clear that $\mathrm{Fil}^1 D_K = (e_1 \otimes 1)K$ if $\alpha = +\infty$.

Let $D' \in \mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$ be another object such that $D' = \mathbb{Q}_p e'_1 \oplus \mathbb{Q}_p e'_2$; $\varphi(e'_1) = e'_1$, $\varphi(e'_2) = -pe'_2$; $\omega(e'_1) = e'_1$, $\omega(e'_2) = e'_2$; $\tau_e(e'_1) = \zeta_e e'_1$, $\tau_e(e'_2) = \zeta_e^{-1} e'_2$ and $\mathrm{Fil}^1 D_K = (\alpha' \cdot e'_1 \otimes \pi_e^{-1} + e'_2 \otimes \pi_e)K$ with $\alpha' \in \mathbb{P}^1(\mathbb{Q}_p)$. Let $\psi : D \longrightarrow D'$ be a non-trivial morphism in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$. Then $\psi\omega = \omega\psi$ implies that $\psi(D_0) \subset D'_0 = \mathbb{Q}_p e'_1 \oplus \mathbb{Q}_p e'_2$ and $\psi\tau_e = \tau_2\psi$ implies that $\psi(e_i) \in \mathbb{Q}_{p^2} e'_i$ for $i = 1, 2$. Therefore $\psi(e_1) = ae_1$ and $\psi(e_2) = be_2$ with $a, b \in \mathbb{Q}_p$. Now $\psi\varphi = \varphi\psi$ gives us that $a = b$. Finally, we see that $\psi_K(\mathrm{Fil}^1 D_K) \subset \mathrm{Fil}^1 D'_K$ if and only if $\alpha = \alpha'$.

Remark 5.9. (i) In the category $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$, it can immediately be concluded that $\mathrm{D}^*_{\mathrm{cris},K/\mathbb{Q}_p}(V_pE)$ is isomorphic to $\mathrm{D}^*_{\mathrm{pc}}(e, 0, \alpha)$ of the list D^* given in Case 3 of section 5.2.1.

(ii) The pair (e, α) varies over the set $\{n = 3, 4 \text{ or } 6 \text{ such that } n \mid (p+1)\} \times \mathbb{P}^1(\mathbb{Q}_p)$ and parametrizes all of the objects $\mathrm{D}^*_{\mathrm{pc}}(e, 0, \alpha)$. These objects are not isomorphic to each other in $\mathrm{MF}_{K/\mathbb{Q}_p}(\varphi)$.

(iii) We see that for $E/\mathbb{Q}_p$ such that $\upsilon_p(j_E) \geq 0$ and $\mathrm{dst}(E) = e \geq 3$, we have

$$\begin{cases} e \mid (p-1), \implies \tilde{E}_L \text{ is ordinary} \\ e \mid (p+1), \implies \tilde{E}_L \text{ is supersingular.} \end{cases}$$

5.4 Examples: Potentially good reduction case

In the previous section we studied the objects $\mathrm{D}_{\mathrm{pst}}(V_pE)$ for E and elliptic curve definedover $\mathbb{Q}_p$. In case E has potentially multiplicative reduction over $\mathbb{Q}_p$, we saw that E can be taken as Tate's curve or its quadratic twist. However, in case E has potentially good reduction over $\mathbb{Q}_p$, we did not provide any examples. The goal of this section is to construct some examples in the missing cases. We do this in two parts depending on fact if the reduced curve (over an extension of $\mathbb{F}_p$) is either ordinary or supersingular.

5.4.1 Ordinary curves

Case 1. $(\mathbf{4} \mid \mathbf{p-1})$: For each $a_{p,k} \in \mathcal{N}^{\times}_{p,4}$, $1 \leq k \leq 4$ from lemma (?), let $u_k \in \mathbb{F}_p^{\times}$ be an element such that for

$$\tilde{E}_k : y^2 = x^3 + u_k x$$

the trace of the Frobenius of is $a_{p,k}$. $\{u_k, 1 \leq k \leq 4\}$ form a system of representative of $\mathbb{F}_p^{\times}/(\mathbb{F}_p^{\times})^4$. These curves are ordinary since $4 \mid p-1$ [Sil13, Thm. 4.1(a)] and the j-invariant of each of these curves is 1728.

Now, $\{[u_k](-p)^i, 1 \leq k \leq 4, 0 \leq i \leq 3\}$ is a system of representative of $\mathbb{Q}_p^{\times}/(\mathbb{Q}_p^{\times})^4$, where $[u_k] \in \mathbb{Z}_p^{\times}$ is the Teichmüller representative of u_k. Set

$$E_{ik} : y^2 = x^3 + [u_k](-p)^i x$$

for $1 \leq k \leq 4$ and $0 \leq i \leq 3$. E_{ik} are elliptic curves over $\mathbb{Q}_p$ with j-invariant equal to 1728.

For a fixed k, E_{ik} are isomorphic over $\mathbb{Q}_p(\pi_4)$, each has potentially good reduction and the reduced curve of E_{ik} over $\mathbb{Q}_p(\pi_4)$ is $\tilde{E}_k$. The curve $E_{i+2,k}$ is a qudratic twist, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$, of E_{ik} for $i \in \{0, 1\}$.

So for each $k \in \{1, 2, 3, 4\}$, we have

$$\begin{aligned} \mathrm{D}^*_{\mathrm{pcris}}(V_pE_{0,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, a_{p,k}, 0) \\ \mathrm{D}^*_{\mathrm{pcris}}(V_pE_{1,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, a_{p,k}, 1, 0) \\ \mathrm{D}^*_{\mathrm{pcris}}(V_pE_{2,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, a_{p,k}, 0) \\ \mathrm{D}^*_{\mathrm{pcris}}(V_pE_{3,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, a_{p,k}, -1, 0). \end{aligned}$$

Let

$$E'_{i,k} : y^2 = x^3 + [u_k](-p)^i x + (-p)^{n(i)}$$

for $1 \le k \le 4$, $0 \le i \le 3$ and $n(i) = 1, 2, 4, 5$ for $i = 0, 1, 2, 3, 4$, respectively. These are elliptic curves with j-invariant equal to $1728 \mod p\mathbb{Z}_p$ but different from 1728. All these curves have potentially good reduction with $dst(E'_{ik}) \in \{1, 2, 4\}$. For any i, k the reduced curve of E'_{ik} over $\mathbb{Q}_p(\pi_4)$ is $\tilde{E}_k$. The curve $E'_{i+2,k}$ is a quadratic twist of $E'_{i,k}$, $i \in \{0, 1\}$, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$.

So for each $k \in \{1, 2, 3, 4\}$, we have

$$\begin{aligned}
\mathrm{D}^*_{\mathrm{pcris}}(V_p E'_{0,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, a_{p,k}, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E'_{1,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, a_{p,k}, 1, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E'_{2,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, a_{p,k}, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E'_{3,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, a_{p,k}, -1, 1).
\end{aligned}$$

Case 2. $(\mathbf{3} \mid \mathbf{p-1})$: From lemma 5.2 for each $a_{p,k} \in \mathcal{N}^{\times}_{p,3}$, $1 \le k \le 6$, let $v_k \in \mathbb{F}_p^{\times}$ be an element such that for

$$\tilde{\mathcal{E}}_k : y^2 = x^3 + v_k$$

the trace of the Frobenius of is $a_{p,k}$. $\{v_k, 1 \le k \le 6\}$ form a system of representatives of $\mathbb{F}_p^{\times}/(\mathbb{F}_p^{\times})^6$. These curves are ordinary since $3 \mid p-1$ [Sil13, Thm. 4.1(a)] and the j-invariant of each of these curves is 0.

Now, $\{[v_k](-p)^i, 1 \le k \le 6, 0 \le i \le 5\}$ is a system of representative of $\mathbb{Q}_p^{\times}/(\mathbb{Q}_p^{\times})^6$, where $[v_k] \in \mathbb{Z}_p^{\times}$ is the Teichmüller representative of v_k. Set

$$\mathcal{E}_{ik} : y^2 = x^3 + [v_k](-p)^i$$

for $1 \le k \le 6$ and $0 \le i \le 5$. $\mathcal{E}_{ik}$ are elliptic curves over $\mathbb{Q}_p$ with j-invariant equal to 0.

For a fixed k, $\mathcal{E}_{ik}$ are isomorphic over $\mathbb{Q}_p(\pi_6)$, each has potentially good reduction and the reduced curve of $\mathcal{E}_{ik}$ over $\mathbb{Q}_p(\pi_6)$ is $\tilde{\mathcal{E}}_k$. The curve $\mathcal{E}_{i+3,k}$ is a qudratic twist, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$, of $\mathcal{E}_{ik}$ for $i \in \{0, 1, 2\}$.

So for each $k \in \{1, 2, 3, 4\}$, we have

$$\begin{aligned}
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{0,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, a_{p,k}, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{1,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, a_{p,k}, 1, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{2,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, a_{p,k}, 1, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{3,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, a_{p,k}, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{4,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, a_{p,k}, -1, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_{5,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, a_{p,k}, -1, 0).
\end{aligned}$$

Let

$$\mathcal{E}'_{i,k} : y^2 = x^3 + (-p)^{m(i)}x + [v_k](-p)^i$$

for $1 \le k \le 4$, $0 \le i \le 3$ and $m(i) = 1, 1, 2, 3, 3, 4$ for $i = 0, 1, 2, 3, 4, 5$, respectively. These are elliptic curves with j-invariant equal to $0 \mod p\mathbb{Z}_p$ but not zero itself. All these curves have potentially good reduction with $dst(\mathcal{E}'_{ik}) \in \{1, 2, 3, 6\}$. For any i, k the reduced curve of $\mathcal{E}'_{ik}$ over $\mathbb{Q}_p(\pi_6)$ is $\tilde{\mathcal{E}}_k$. The curve $\mathcal{E}'_{i+3,k}$ is a quadratic twist of $\mathcal{E}'_{i,k}$, $i \in \{0, 1, 2\}$, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$.

So for each $k \in \{1, 2, 3, 4, 5, 6\}$, we have

$$\begin{aligned}
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{0,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, a_{p,k}, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{1,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, a_{p,k}, 1, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{2,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, a_{p,k}, 1, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{3,k}) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, a_{p,k}, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{4,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, a_{p,k}, -1, 1) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}'_{5,k}) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, a_{p,k}, -1, 1).
\end{aligned}$$

5.4.2 Supersingular curves

Case 1. $(\mathbf{4} \mid \mathbf{p+1})$: Let

$$E_i : y^2 = x^3 + (-p)^i x$$

over $\mathbb{Q}_p$ where $i \in \{0, 1, 2, 3\}$. The j-invariant of these elliptic curves is 1728 and all of them are isomorphic over $\mathbb{Q}_p(\pi_4)$. All these curves have potentially good reduction and the reduced curve over $\mathbb{F}_p$, given by $y^2 = x^3 + x$ is of supersingular type. For $i \in \{0, 1\}$, E_{i+2} is a quadratic twist of E_i, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$. So we have

$$\begin{aligned}
\mathrm{D}^*_{\mathrm{pcris}}(V_p E_0) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E_1) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, 0, +\infty) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E_2) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p E_3) &\simeq \mathrm{D}^*_{\mathrm{pc}}(4, 0, 0).
\end{aligned}$$

Case 2. $(\mathbf{3} \mid \mathbf{p+1})$: Let

$$\mathcal{E}_i : y^2 = x^3 + (-p)^i x$$

over $\mathbb{Q}_p$ where $i \in \{0, 1, 2, 3, 4, 5\}$. The j-invariant of these elliptic curves is 0 and all of them are isomorphic over $\mathbb{Q}_p(\pi_6)$. All these curves have potentially good reduction and the reduced curve over $\mathbb{F}_p$, given by $y^2 = x^3 + 1$ is of supersingular type. For $i \in \{0, 1, 2\}$, $\mathcal{E}_{i+3}$ is a quadratic twist of $\mathcal{E}_i$, by a ramified character corresponding to extension $M_2 = \mathbb{Q}_p(\pi_2)/\mathbb{Q}_p$. So we have

$$\begin{aligned}
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_0) &\simeq \mathrm{D}^*_{\mathrm{c}}(1, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_1) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, 0, +\infty) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_2) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, 0, +\infty) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_3) &\simeq \mathrm{D}^*_{\mathrm{c}}(2, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_4) &\simeq \mathrm{D}^*_{\mathrm{pc}}(3, 0, 0) \\
\mathrm{D}^*_{\mathrm{pcris}}(V_p \mathcal{E}_5) &\simeq \mathrm{D}^*_{\mathrm{pc}}(6, 0, 0).
\end{aligned}$$

This concludes our list of examples.

Appendix A

Hodge-Tate representations

Definition A.1. A p-adic field is a field K of characteristic 0 that is complete with respect to a fixed discrete valuation that has a perfect residue field k of characteristic $p > 0$.

Let K be a p-adic field with a fixed algebraic closure $\overline{K}/K$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ is written as G_K and $\mathbb{C}_K := \widehat{\overline{K}}$, the completion of $\overline{K}$ endowed with its unique absolute value extending the given absolute value on K.

The first class of "good" p-adic representations of G_K were those of Hodge-Tate type; this class was discovered by Serre and Tate in there study of p-adic representations arising from abelian varieties with good reduction over p-adic fields, and in this section we will examine this class of representations. This chapter is intended to be a quick recollection of facts on Hodge-Tate representations. We mention important results but without attempting to prove any of the statements. All the statements have been taken from [BC09, Chap.2].

The most basic ingredient in this study is the p-adic cyclotomic character which we define below and note some remarks about it.

Definition A.2. Let K be a field with fixed separable closure K^s/K and let p be a prime distinct from char K. Let $\mu_{p^n} = \mu_{p^n}(K^s)$ denote the group of p^n-th roots of unity in $(K^s)^\times$, and let μ_{p^∞} denote the rising union of these subgroups. The action of G_K on μ_{p^∞} is given by $g(\zeta) = \zeta^{\chi(g)}$ for a unique $\chi(g) \in \mathbb{Z}_p^\times$: for $\zeta \in \mu_{p^n}$ the exponent $\chi(g)$ only matters modulo p^n, and $\chi(g) \mod p^n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ describes the action of g on the finite cyclic group μ_{p^n} of order p^n. Thus, $\chi \mod p^n$ has open kernel (corresponding to the finite etension $K(\mu_{p^n})/K$) and χ is continuous. χ is called the *p-adic cyclotomic character* of K.

Remark A.3. The p-adic Tate module $\varprojlim_n \mu_{p^n}(\overline{K})$ of the group GL_1 over K is a free $\mathbb{Z}_p$-module of rank 1 and we denote its as $\mathbb{Z}_p(1)$. This does not have a canonical basis, and a choice of basis amounts to a choice of compatible system $(\zeta_{p^n})_{n\geq 1}$ of primitive p-power roots of unity (satisfying $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$). The natural action of G_K on $\mathbb{Z}_p(1)$ is given by the $\mathbb{Z}_p^\times$-valued p-adic cyclotomic character χ and sometimes it is convenient to fix a choice of basis of $\mathbb{Z}_p(1)$ and to thereby view $\mathbb{Z}_p(1)$ as $\mathbb{Z}_p$ endowed with a G_K-action by χ.

For any $r \geq 0$, define $\mathbb{Z}_p(r) = \mathbb{Z}_p(1)^{\otimes r}$ and $\mathbb{Z}_p(-r) = \mathbb{Z}_p(r)^\vee$ (linear dual: $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}_p)$ for any finite free $\mathbb{Z}_p$-module M) with the naturally associated G_K-actions (from functoriality of tensor products and duality), so upon fixing a basis of $\mathbb{Z}_p(1)$ we identify $\mathbb{Z}_p(r)$ with the $\mathbb{Z}_p$-module $\mathbb{Z}_p$ endowed with the G_K-action χ^r for all $r \in \mathbb{Z}$. If M is an arbitrary $\mathbb{Z}_p[G_K]$-module, we let $M(r) = \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} M$ with its natural G_K-action, so upon fixing a basis of $\mathbb{Z}_p(1)$ this is simply M with the modified G_K-action $g \cdot m = \chi(g)^r g(m)$ for $g \in G_K$ and $m \in M$. We also have isomorphisms $(M(r))(r') \simeq M(r + r')$ for $r, r' \in \mathbb{Z}$ and $(M(r))^\vee \simeq M^\vee(-r)$ for $r \in \mathbb{Z}$ and M finite free over $\mathbb{Z}_p$ or over a p-adic field.

Following is a fundamental fact about $\mathbb{C}_K$.

Proposition A.4. *The field $\mathbb{C}_K$ is algebraically closed.*

A.1 Theorems of Tate-Sen and Faltings

Definition A.5. A $\mathbb{C}_K$-representation of G_K is a finite-dimensional $\mathbb{C}_K$-vector space W equipped with a continuous G_K-action map $G_K \times W \to W$ that is semilinear (i.e., $g(cw) = g(c)g(w)$ for any $c \in \mathbb{C}_k$ and $w \in W$). The category of such objects (using $\mathbb{C}_K$-linear G_K-equivariant morphisms) is denoted $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Example A.6. If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ then $W := \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is an object in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Remark A.7. The category $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is an abelian category with evident notions of tensor product, direct sum and exact sequence. We can also define the dual object: for any $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, the dual $W^\vee = \mathrm{Hom}(W, \mathbb{C}_K)$ on which G_K acts as $(g \cdot l)(w) = g(l(g^{-1}(w)))$. Also, we have isomorphisms $W \simeq (W^\vee)^\vee$ and $W_1^\vee \otimes W_2^\vee \simeq (W_1 \otimes W_2)^\vee$ as well as evaluation morphism $W \otimes W^\vee \to \mathbb{C}_K$ in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Following is a deep result by Faltings.

Theorem A.8. *Let K be a p-adic field. For smooth proper K-schemes X, there is a canonical isomorphism*

$$\mathbb{C}_K \otimes_K \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \simeq \oplus_p (\mathbb{C}_K(-q) \otimes_K \mathrm{H}^{n-q}(X, \Omega^q_{X/K})$$

in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$, where the G_K-action on the rights side is defined through the action on each $\mathbb{C}_K(-q) = \mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-q)$. In particular, non-canonically

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \simeq \oplus_q \mathbb{C}_K(-q)^{h^{n-q,q}}$$

in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ with $h^{p,q} = \dim_K \mathrm{H}^p(X, \Omega^q_{X/K})$.

Remark A.9. (i) This isomorphism enables us to recover the K-vector spaces $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ from $\mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ by means of operations that make sense on all objects in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

(ii) We cannot recover the p-adic representation space $\mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ from the Hodge cohomologies $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$. In general, the operation $V \rightsquigarrow \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ loses a lot of information about V.

Example A.10. Let E be an elliptic curve over K with split multiplicative reduction, and consider the representation space $V_pE = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pE \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. The theory of Tate curves in Section 3.4 provides and exact sequence

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V_pE \longrightarrow \mathbb{Q}_p \longrightarrow 0 \tag{A.1}$$

that is non-split in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ for all finite extensions K'/K inside of $\overline{K}$.

If we apply $\overline{K} \otimes_{\mathbb{Q}_p} (\cdot)$ to (A.1) then we get an exact sequence

$$0 \longrightarrow \overline{K}(1) \longrightarrow \overline{K} \otimes_{\mathbb{Q}_p} V_pE \longrightarrow \overline{K} \longrightarrow 0$$

in the category $\mathrm{Rep}_{\overline{K}}(G_K)$ of semilinear representations of G_K on $\overline{K}$-vector spaces. This sequence cannot split in $\mathrm{Rep}_{\overline{K}}(G_K)$. Indeed, assume on the contrary that it splits. Since $\overline{K}$ is a directed union of finite subextensions K'/K, there would then exist such a K' over which the splitting occurs. That is, applying $K' \otimes_{\mathbb{Q}_p} (\cdot)$, to (A.1) would give an exact sequence admitting a G_K-equivariant K'-linear splitting. Viewing this as a split sequence of $K'[G_{K'}]$-modules, we could apply a $\mathbb{Q}_p$-linear projection $K' \to \mathbb{Q}_p$ that restricts to identity on $\mathbb{Q}_p \subset K'$ so as to recover (A.1) equipped with a $\mathbb{Q}_p[G_{K'}]$-linear splitting. But it does not split in $\mathrm{Rep}_{\mathbb{Q}_p}(G_{K'})$, so we have a contradiction. Hence, applying $\overline{K} \otimes_{\mathbb{Q}_p} (\cdot)$ to (A.1) gives a non-split exact sequence in $\mathrm{Rep}_{\overline{K}}(G_K)$, as claimed.

But if we instead apply $\mathbb{C}_K \otimes_{\mathbb{Q}_p} (\cdot)$ to (A.1) then the resulting sequence in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ does (uniquely) split.

Definition A.11. Let G be a topological group and M a topological G-module. The continuous cohomology group $\mathrm{H}^1_{\mathrm{cont}}(G, M)$ (or $\mathrm{H}^1(G, M)$) is defined using continuous 1-cochains.

Example A.12. Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character. We identify $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbb{C}_K(\eta))$ with the set of isomorphism classes of extensions

$$0 \longrightarrow \mathbb{C}_K(\eta) \longrightarrow W \longrightarrow \mathbb{C}_K \longrightarrow 0 \tag{A.2}$$

in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ as follows: using the matrix description

$$\begin{pmatrix} \eta & * \\ 0 & 1 \end{pmatrix}$$

of such a W, the homomorphism property for the G_K-action on W says that the upper right entry function is a 1-cocycle on G_K with values in $\mathbb{C}_K(\eta)$, and changing the choice of basis for $\mathbb{C}_K$-linear splitting changes this function by a 1-coboundary. The continuity of the 1-cocycle says exactly that the G_K-action on W is continuous. Changing the choice of $\mathbb{C}_K$-basis of W that is compatible with the filtration in (A.2) changes the 1-cocycle by a 1-coboundary. In this way we get a well-defined continuous cohomology class, and the procedure can be reversed (up to isomorphism of extension structure).

Theorem A.13 (Tate-Sen). *For any p-adic field K we have $K = \mathbb{C}_K^{G_K}$ and $\mathbb{C}_K(r)^{G_K} = 0$ for $r \neq 0$ (i.e., if $x \in \mathbb{C}_K$ and $g(x) = \chi(g)^{-r}x$ for every $g \in G_K$ and some $r \neq 0$ then $x = 0$). Also, $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbb{C}_K(r)) = 0$ if $r \neq 0$ and $\mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbb{C}_K)$ is 1-dimensional over K.*

More generally, if $\eta : G_K \to \mathcal{O}_K^\times$ is a continuous character such that $\eta(G_K)$ is a commutative p-adic Lie group of dimension at most 1 (i.e., $\eta(G_K)$ is finite or contains $\mathbb{Z}_p$ as an open subgroup) and if $\mathbb{C}_K(\eta)$ denotes $\mathbb{C}_K$ with the twisted G_K-action $g \cdot c = \eta(g)g(c)$ then $\mathrm{H}^i_{\mathrm{cont}}(G_K, \mathbb{C}_K(\eta)) = 0$ for $i = 0, 1$ when $\eta(I_K)$ is infinite and these cohomologies are 1-dimesnional over K when $\eta(I_K)$ is finite (i.e., when the splitting field of η over K is finitely ramified).

Remark A.14. This result implies that all exact sequence (as in example above) are split when $\eta(I_K)$ is infinite. Moreover, in such cases splitting is unique.

A.2 Hodge-Tate decomposition

The companion to the theorem of Tate-Sen is a lemma of Serre and Tate that we now state. For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and $q \in \mathbb{Z}$, consider the K-vector space

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W \mid g(w) = \chi(g)^{-q}w \text{ for all } g \in G_K\},$$

where the isomorphism rests on a choice of basis of $\mathbb{Z}_p(1)$. We have a natural G_K-equivariant K-linear multiplication map $K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \simeq W$. So by extending scalars $K \to \mathbb{C}_K$, defines a map,

$$\mathbb{C}_K(-q) \otimes_K W\{q\} \longrightarrow W$$

in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ for all $q \in \mathbb{Z}$.

Lemma A.15 (Serre-Tate). *For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, the natural $\mathbb{C}_K$-linear G_K-equivariant map*

$$\xi_W : \oplus_q(\mathbb{C}_K(-q) \otimes_K W\{q\}) \longrightarrow W \tag{A.3}$$

is injective. In particular, $W\{q\} = 0$ for all but finitely many q and $\dim_K W\{q\} < +\infty$ for all q, with $\sum_q \dim_K W\{q\} \leq \dim_{\mathbb{C}_K} W$; equality holds here if and only if ξ_W is an isomorphism.

Remark A.16. In the special case $W = \mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathrm{H}^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ for a smooth proper scheme X over K. Falting's theorem (from before) says that ξ_W is an isomorphism and $W\{q\}$ is canonically K-isomorphic to $\mathrm{H}^{n-q}(X, \Omega^q_{X/K})$ for all $q \in \mathbb{Z}$.

Example A.17. Let $W = \mathbb{C}_K(\eta)$ for a continuous character $\eta : G_K \to \mathbb{Z}_p^\times$. By the Tate-Sen Theorem A.13, $W\{q\} = \mathbb{C}_K(\eta\chi^{-q})^{G_K}$ is 1-dimensional over K if $\eta\chi^{-q}|_{I_K}$ has finite order (equivalently, if $\eta = \chi^q\psi$ for a finitely ramified character $\psi : G_K \to \mathbb{Z}_p^\times$) and $W\{q\}$ vanishes otherwise. In particular, there is at most one q for which $W\{q\}$ can be nonzero, since if $W\{q_1\}$, $W\{q_2\} \neq 0$ with $q_1 \neq q_2$ then $\eta = \chi^{q_1}\psi_1$ and $\eta = \chi^{q_2}\psi_2$ with finitely ramified $\psi_1, \psi_2 : G_K \rightrightarrows \mathbb{Z}_p^\times$, so $\chi^r|_{I_K}$ has finite image for $r = q_1 - q_2 \neq 0$, which is not possible.

Definition A.18. A representation W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate if ξ_W is an isomorphism. We say that V in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is Hodge-Tate if $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge-Tate.

Example A.19. If W is Hodge-Tate then because ξ_W is an isomorphism we have a non-canonical isomorphism $W \simeq \mathbb{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ with $h_q = \dim_K W\{q\}$. Conversely, consider an object $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ admitting a finite direct sum decomposition $W \simeq \oplus\mathbb{C}_K(-q)^{h_q}$ in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ with $h_q \geq 0$ for all q and $h_q = 0$ for all but finitely may q. The Tate-Sen Theorem A.13 gives that $W\{q\}$ has dimension h_q for all q, so $\sum_q \dim_K W\{q\} = \sum_q h_q = \dim_{\mathbb{C}_K} W$ and hence W is Hodge-Tate. In other words, the intrinsic property of being Hodge-Tate is equivalent to the concrete propert of being isomorphic to a finite direct sum of various objects $\mathbb{C}_K(r_i)$.

Definition A.20. For any Hodge-Tate object W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ we define the *Hodge-Tate weights* of W to be those $q \in \mathbb{Z}$ such that $W\{q\} := (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W)^{G_K}$ is non-zero and then we call $h_q := \dim_K W\{q\} \geq 1$ the multiplicity of q as a Hodge-Tate weight of W.

Remark A.21. If W is Hodge-Tate then so is $W^\vee$, with negated Hodge-Tate weights (compatible with multiplicities).

A.3 Formalism of Hodge-Tate representations

Definition A.22. A ($\mathbb{Z}-$) *graded* vector space over a field K is a K-vector space D equipped with direct sum decomposition $\oplus_{q\in\mathbb{Z}} D_q$ for K-subspaces $D_q \subset D$ (and we define the q-th graded piece of D to be $\mathrm{gr}^q(D) := D_q$). Morphisms $f : D_1 \to D_2$ between graded K-vector spaces are K-linear maps that respect the grading (i.e., $f(D'_q) \subset D_q$ for all q). The category of these objects is denoted as Gr_K; we let $\mathrm{Gr}_{K,f}$ denote the full subcategory such that any for object $D \in \mathrm{Gr}_{K,f}$ we have that $\dim_K D < +\infty$.

Remark A.23.
- (i) For any field K, Gr_K is an abelian category with the evident notions of kernel, cokernel, and exact sequence.
- (ii) Write $K\langle r\rangle$ for $r \in \mathbb{Z}$ to denote the K-vector space K endowed with the grading for whih the unique non-vanishing graded piece is in degree r.
- (iii) For $D, D' \in \mathrm{Gr}_K$, the tensor product $D \otimes D'$ is defined to have $D \otimes_K D'$ as the underlying K-vector space and the q-th graded piece is given by $\oplus_{i+j=q}(D_i \otimes_K D'_j)$.
- (iv) If $D \in \mathrm{Gr}_{K,f}$ then the dual $D^\vee$ has underlying K-vector space given by the K-linear dual and its q-th graded piece is $D^\vee_{-q}$.
- (v) Note, $K\langle r_1\rangle \otimes K\langle r_2\rangle = K\langle r_1 + r_2\rangle$, $K\langle r\rangle^\vee = K\langle -r\rangle$, the natural valuation map $D \otimes D^\vee \to K\langle 0\rangle$ and the double duality isomorphism $D \simeq (D^\vee)^\vee$ in $\mathrm{Gr}_{K,f}$ are morphisms in Gr_K.
- (vi) A map in Gr_K is an isomorphism if and only if it is a linear isomorphism in each separate degree.

Definition A.24. The covariant functor $\underline{\mathrm{D}} = \underline{\mathrm{D}}_K : \mathrm{Rep}_{\mathbb{C}_K}(G_K) \to \mathrm{Gr}_K$ is defined as

$$\underline{\mathrm{D}}(W) = \bigoplus_q W\{q\} = \bigoplus_q (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W)^{G_K}. \tag{A.4}$$

$\underline{\mathrm{D}}$ is obviously left-exact.

Remark A.25. The Serre-Tate Lemma A.15 says that $\underline{D}$ takes values in $\mathrm{Gr}_{K,f}$ and more specifically that $\dim_K \underline{\mathrm{D}}(W) \leq \dim_{\mathbb{C}_K} W$ with equality if and only if W is *Hodge-Tate*. As a simple example, the Tate-Sen Theorem A.13 gives that $\underline{D}(\mathbb{C}_K(r)) = K\langle -r\rangle$ for every $r \in \mathbb{Z}$.

Proposition A.26. *If*

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0$$

is a short exact sequence in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ *and* W *is Hodge-Tate then so are* W' *and* W''*, in which case the sequence*

$$0 \longrightarrow \underline{\mathrm{D}}(W') \longrightarrow \underline{\mathrm{D}}(W) \longrightarrow \underline{\mathrm{D}}(W'') \longrightarrow 0$$

in $\mathrm{Gr}_{K,f}$ *is short exact.*

Remark A.27. The proposition says that any subrepresentation or quotient representation of a Hodge-tate representation is again Hodge-Tate. The converse is false in the sense that if W' and W'' are Hodge-Tate then W can fail to be Hodge-Tate.

Theorem A.28. *For any* $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ *the natural map* $K' \otimes_K \underline{\mathrm{D}}_K(W) \to \underline{\mathrm{D}}_{K'}(W)$ *in* $\mathrm{Gr}_{K',f}$ *is an isomorphism for all finite extensions* K'/K *contained in* $K \subset \mathbb{C}_K$*. Likewise, the natural map* $\widehat{K^{\mathrm{un}}} \otimes_K \underline{\mathrm{D}}_K(W) \to \underline{D}_{\widehat{K^{\mathrm{un}}}}(W)$ *in* $\mathrm{Gr}_{\widehat{K^{\mathrm{un}}},f}$ *is an isomorphism. In particular, for any finite extension* K'/K *inside of* $\overline{K}$*, an object* W *in* $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ *is Hodge-Tate if and only if it is Hodge-Tate when viewed in* $\mathrm{Rep}_{\mathbb{C}_K}(G_{K'})$*, and similarly* W *is Hodge-Tate in* $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ *if and only if it is Hodge-Tate when viewed in* $\mathrm{Rep}_{\mathbb{C}_K}(G_{\widehat{K^{\mathrm{un}}}}) = \mathrm{Rep}_{\mathbb{C}_K}(I_K)$*.*

Remark A.29. The Hodge-Tate property is insensitive to replacing K with a finite extension or restricting to the inertia group. The insensitivity to inertial restriction is a "good" feature but insensitivity to finite (possibly ramified) extensions is a "bad" feature, indicating that the Hodge-Tate property is not sufficiently find, for example, to distinguish between good and potentially good reduction for elliptic curves.

Definition A.30. The *Hodge-Tate ring* of K is the $\mathbb{C}_K$-algebra $B_{\mathrm{HT}} = \oplus_{q\in\mathbb{Z}}\mathbb{C}_K(q)$ in which multiplication is defined via the natural map $\mathbb{C}_K(q_1) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q_2) \simeq \mathbb{C}_K(q_1+q_2)$.

B_{HT} is a graded $\mathbb{C}_K$-algebra i.e., its graded pieces are $\mathbb{C}_K$-subspaces with respect to which multiplication is additive in the degrees, and the natural G_K-action respects the gradings and the ring structure. Concretely, if we choose a basis t of $\mathbb{Z}_p(1)$ then we can identify B_{HT} with the Laurent polynomial ring $\mathbb{C}_K[t, t^{-1}]$ with the obvious grading (by monomial in t) and G_K-action (via $g(t^i) = \chi(g)^i t^i$ for $i \in \mathbb{Z}$ and $g \in G_K$).

By Tate-Sen Theorem A.13, we have $B_{\mathrm{HT}}^{G_K} = K$. For any $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, we have,

$$\underline{\mathrm{D}}(W) = \bigoplus_q (\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} W)^{G_K} = (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W)^{G_K}$$

in Gr_K, where the grading is induced from the one on B_{HT}.

In the reverse direction, let $D \in \mathrm{Gr}_{K,f}$, so $B_{\mathrm{HT}} \otimes_K D$ is a graded $\mathbb{C}_K$-vector space with typically infinite $\mathbb{C}_K$-dimension:

$$\mathrm{gr}^n(B_{\mathrm{HT}} \otimes_K D) = \bigoplus_q \mathrm{gr}^q B_{\mathrm{HT}} \otimes_K D_{n-q} = \bigoplus_q \mathbb{C}_K(q) \otimes_K D_{n-q}.$$

Moreover, the G_K-action on $B_{\mathrm{HT}} \otimes_K D$ arising from that on B_{HT} respects the grading since such compatibility holds in B_{HT}, so we get

$$\underline{\mathrm{V}}(D) := \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_K D) = \bigoplus_K \mathbb{C}_K(-q) \otimes_K D_q \in \mathrm{Rep}_{\mathbb{C}_K}(G_K) \tag{A.5}$$

since D_q vanishes for all but finitely many q and is finite-dimensional over K for all q (as $D \in \mathrm{Gr}_{K,f}$). By inspection $\underline{\mathrm{D}}$ is a Hodge-Tate representation and $\underline{\mathrm{V}} : \mathrm{Gr}_{K,f} \to \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is a covariant exact functor.

Example A.31. For each $r \in \mathbb{Z}$, recall that $K\langle r\rangle$ denotes the 1-dimensional K-vector space K endowed with unique non-trivial graded piece in degree r. One can check that $\underline{\mathrm{V}}(K\langle r\rangle) = \mathbb{C}_K(-r)$. In particular, $\underline{\mathrm{V}}(K\langle 0\rangle) = \mathbb{C}_K$.

For any W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$, the multiplicative structure on B_{HT} defines a natural B_{HT}-linear composite comparison morphism

$$\gamma_W : B_{\mathrm{HT}} \otimes \underline{\mathrm{D}}(W) \hookrightarrow B_{\mathrm{HT}} \otimes_K (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) \longrightarrow B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W$$

that respects the G_K-actions and the gradings.

The Serre-Tate Lemma A.15 has the following reformulation:

Lemma A.32. *For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$, the comparison morphism γ_W is injective. It is an isomorphism if and only if W is Hodge-Tate, in which case there is a natural isomorphism*

$$\underline{\mathrm{V}}(\underline{\mathrm{D}}(W)) = \mathrm{gr}^0(B_{\mathrm{HT}} \otimes \underline{\mathrm{D}}(W)) \simeq \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) = \mathrm{gr}^0(B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) = W$$

in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

Theorem A.33. *The covariant functors $\underline{\mathrm{D}}$ and $\underline{\mathrm{V}}$ between the categories of Hodge-Tate representations in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and finite-dimensional objects in Gr_K are quasi-inverse equivalences.*

For any $W, W' \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ the natural map $\underline{\mathrm{D}}(W) \otimes \underline{\mathrm{D}}(W') \to \underline{\mathrm{D}}(W \otimes W')$ in Gr_K induced by the G_K-equivariant map

$$(B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W) \otimes_{\mathbb{C}_K} (B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} W') \longrightarrow B_{\mathrm{HT}} \otimes_{\mathbb{C}_K} (W \otimes_{\mathbb{C}_K} W')$$

defined by multiplication in B_{HT} is an isomorphism when W and W' are Hodge-Tate.

Likewise, if W is Hodge-Tate then the natural map

$$\underline{\mathrm{D}}(W) \otimes_K \underline{\mathrm{D}}(W^\vee) \longrightarrow \underline{\mathrm{D}}(W \otimes W^\vee) \longrightarrow \underline{\mathrm{D}}(W \otimes W^\vee) \longrightarrow \underline{\mathrm{D}}(\mathbb{C}_K) = K\langle 0\rangle$$

in Gr_K is a perfect duality (between $W\{q\}$ and $W^\vee\{-q\}$ for all q), so the induced map $\underline{\mathrm{D}}(W^\vee) \to \underline{\mathrm{D}}(W)^\vee$ is an isomorphism in $\mathrm{Gr}_{K,f}$. In other words, $\underline{\mathrm{D}}$ is compatible with tensor products and duality on Hodge-Tate objects. Similar compatibilities hold for $\underline{\mathrm{V}}$ with respect to tensor products and duality.

Definition A.34. Let $\mathrm{Rep}_{\mathrm{HT}}(G_K) \subset \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ be the full subcategory of objects V that are Hodge-Tate (i.e., $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is Hodge-Tate in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$), define the functor $\mathrm{D}_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Gr}_{K,f}$ by

$$\mathrm{D}_{\mathrm{HT}}(V) = \underline{\mathrm{D}}_K(\mathbb{C}_K \otimes_{\mathbb{Q}_p} V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

with grading induced by that on B_{HT}.

Remark A.35. (i) $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ is stable under tensor products, duality, subrepresentations and quotients (but not extensions) in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. The formation of D_{HT} naturally commutes with finite extension on K as well as with scalar extension to $\widehat{K^{\mathrm{un}}}$.

(ii) The functor D_{HT} is exact and is compatible with tensor products and duality.

(iii) The comparison morphism $\gamma_v : B_{\mathrm{HT}} \otimes_K \mathrm{D}_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$ for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is an isomorphism precisely when V is Hodge-Tate and hence $\mathrm{D}_{\mathrm{HT}} : \mathrm{Rep}_{\mathrm{HT}}(G_K) \to \mathrm{Gr}_{K,f}$ is a faithful functor.

Whereas $\underline{\mathrm{D}}$ on the category $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is fully faithful into $\mathrm{Gr}_{K,f}$, D_{HT} on the catgeory $\mathrm{Rep}_{\mathrm{HT}}(G_K)$ is not fully faithful. For example, if $\eta : G_K \to \mathbb{Z}_p^\times$ has finite order then $\mathrm{D}_{\mathrm{HT}}(\mathbb{Q}_p(\eta)) \simeq K\langle 0\rangle = \mathrm{D}_{\mathrm{HT}}(\mathbb{Q}_p)$ by the Tate-Sen Theorem A.13, $\mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p$ have no nonzero maps between them when $\eta \neq 1$.

To improve on D_{HT} so as to get a fully faithful functor from a nice category of p-adic representations of G_K into a category of semilinear algebra objects, following two things are done: refinement of B_{HT} to a ring with more structure (going beyond a mere grading with compatible G_K-action) and introduction of a target semilinear algebra category which is richer than $\mathrm{Gr}_{K,f}$.

Appendix B

Kähler differentials

In this section we recall some of the definitions needed in our computations. For a general treatment of the theory of Kähler differentials one could take a look at [Liu06, Ch. 6].

Definition B.1. Let A be a commutative ring with unity. Let B be an A-algebra and M a B-module. An A-derivation of B into M is an A-linear map $d : B \to M$ such that the Leibniz rule

$$d(b_1 b_2) = b_1 db_2 + b_2 db_1, \ \ b_i \in B$$

is verified, and that $da = 0$ for every $a \in A$.

We denote the set of these derivations as $\mathrm{Der}_A(B, M)$.

Definition B.2. Let B be an A-algebra. The module of *relative differentials* or module of *Kähler differetials* of B over A is a B-module $\Omega_{B/A}$ endowed with an A-derivation $d : B \to \Omega_{B/A}$ satisfying the universal property: For any B-module M and for any A-derivation $d' : B \to M$, there exists a unique homomoprhism of B-modules $\psi : \Omega_{B/A} \to M$ such that $d' = \psi \circ d$.

$$\begin{tikzcd}
B \arrow[r, "d'"] \arrow[d, "d"] & M \\
\Omega_{B/A} \arrow[ur, dashed, "\psi"] &
\end{tikzcd}$$

Proposition B.3. *The module of relative differential forms* $(\Omega_{B/A}, d)$ *exists and is unique up to unique isomorphism.*

Proof. The uniqueness follows from the definition, as for any solution to a universal problem. Therefore, we only need to show existence. Let F be the free B-module generated by the symbols db, $b \in B$. Let E be the submodule of F generated by the element of the form da, $a \in A$; $d(b_1 + b_2) - db_1 - db_2$ and $d(b_1 b_2) - b_1 db_2 - b_2 db_1$ with $b_i \in B$. Set $\Omega_{B/A} = F/E$ and $d : B \to \Omega_{B/A}$ which sends b to the image of db in $\Omega_{B/A}$. Now it is clear that $(\Omega_{B/A}, d)$ has the required properties. ■

Proposition B.4. *For any B-module M there is a canonical isomorphism*

$$\begin{aligned} \mathrm{Hom}_B(\Omega_{B/A}, M) &\xrightarrow{\simeq} \mathrm{Der}_A(B, M) \\ \psi &\longmapsto \psi \circ d. \end{aligned}$$

Proof. Let $\partial \in \mathrm{Der}_A(B, M)$, then we define $\psi(dx) := \partial(x)$ and surjectiveness follows. Moreover, if $\psi \circ d$ is the trivial derivation then $\psi(dx) = 0$ for all x, hence $\psi = 0$ and the map is injective. ■

Example B.5. Let A be a ring and let B be the polynomial ring $A[T_1, T_2, \dots, T_n]$. We will show that $\Omega_{B/A}$ is the free B-module generated by dT_i. Let $F \in B$, and let $d' : B \to M$ be an A-derivation into a B-module M. Using the definition of derivation, we immediately obtain that $d'F = \sum_i (\partial F / \partial T_i) d' T_i$, where $(\partial F/\partial T_i)$ is the partial derivative in the usual sense. Therefore, d' is entirely determined by the images of the T_i. Let Ω be the free B-module generated by the symbols dT_i, $1 \leq i \leq n$. Let $d : B \to \Omega$ be the map defined by $d(F) = \sum_i (\partial F/\partial T_i) dT_i$. Now it is easily seen that (Ω, d) fulfills the conditions of the universal property of the module $\Omega_{B/A}$. Hence $\Omega_{B/A} \simeq \Omega$.

Let $\rho : B \to C$ be a homomorphism of A-algebras. Then it follows from the universal property that there exist canonical homomorphisms of C-modules.

$$\alpha : \Omega_{B/A} \otimes_A C \longrightarrow \Omega_{C/A}, \quad \beta : \Omega_{C/A} \longrightarrow \Omega_{C/B}.$$

By definition $\alpha(db \otimes c) = cd\rho(b)$.

Following are some of the properties of the modules of differentials.

Proposition B.6. *Let b be an algebra over a ring A.*

(i) *(Base Change) For any A-algebra A', let us set $B' = A \otimes_A A'$. There exists a canonical isomorphism of B'-modules $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$.*

(ii) *Let $B \to C$ be a homomorphism of A-algebras. Let α, β be as above, then*

$$\Omega_{B/A} \otimes C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

is an exact sequence.

Proof. (i) The canonical derivation $d : B \to \Omega_{B/A}$ induces an A'-derivation $d' = d \otimes Id_{A'} : B' \to \Omega_{B/A} \otimes_A A' = \Omega_{B/A} \otimes_B B'$. Now it easily follows that $(\Omega_{B/A} \otimes_B B', d')$ verifies the universal property of $\Omega_{B'/A'}$.

(ii) It suffices to show that for any C-module N, the dual sequence

$$0 \longrightarrow \mathrm{Hom}_C(\Omega_{C/B}, N) \longrightarrow \mathrm{Hom}_C(\Omega_{B/A}, N) \longrightarrow \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N)$$

is exact. We have $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N) = \mathrm{Hom}_B(\Omega_{B/A}, N)$. By Proposition B.4 this sequence is canonically isomorphic to the sequence

$$0 \longrightarrow \mathrm{Der}_B(C, N) \longrightarrow \mathrm{Der}_A(C, N) \longrightarrow \mathrm{Der}_A(B, N),$$

the last homomorphism being the composition with $B \to C$. It follows from the definition of a derivation that this sequence is exact.

■

Next, we recall basic definitions on de Rham cohomology on affine schemes. As we will see that only this special case is needed in our case. One could take a look at the notes of [Maz17] for a general discussion on algebraic de Rham cohomology.

Definition B.7. Let A be a commutative ring with unity and let B be an A-algebra of finite type. We define the module of *Kähler differential p-forms* as

$$\Omega^p_{B/A} := \wedge^p \Omega_{B/A}.$$

If $\omega_p \in \Omega^p_{B/A}$ and $\omega_q \in \Omega^q_{B/A}$ we have $\omega_p \wedge \omega_q = (-1)^{p+q} \omega_q \wedge \omega_p$ so that $\oplus_p \Omega^p_{B/A}$ is a graded commutative B-algebra. Note by definition $\Omega^0_{B/A} = B$.

The following proposition is easy to see.

Proposition B.8. *There exists a unique map $d : \oplus_p \Omega^p_{B/A} \to \oplus_p \Omega^p_{B/A}$ of degree 1 such that*

(i) *$d \circ d = 0$;*

(ii) *in degree zero it is the canonical mpa* $d : B \to \Omega_{B/A}$*;*

(iii) $d(\omega_p \wedge \omega_q) = (d\omega_p) \wedge \omega_q + (-1)^p \omega_p \wedge d\omega_q$

Definition B.9. (de Rham cohomology). With the above notations let $\Omega^\bullet_{B/A}$ be the complex (starting in degree 0)

$$B \xrightarrow{d} \Omega^1_{B/A} \xrightarrow{d} \Omega^2_{B/A} \longrightarrow \cdots \longrightarrow \Omega^i_{B/A} \longrightarrow \cdots$$

and we define the n-th de Rham cohomology group as

$$\mathrm{H}^n_{\mathrm{dR}}(B/A) := \mathrm{H}^n(\Omega^\bullet_{B/A}) = \frac{ker(\Omega^n_{B/A} \xrightarrow{d} \Omega^{n+1}_{B/A})}{im(\Omega^{n-1}_{B/A} \xrightarrow{d} \Omega^n_{B/A})}.$$

Since the de Rham complex is functorial (covariant) in B, so is de Rham cohomology.

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