Amœbas and Coamœbas
in Dimension 2


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Chapter 1

Tropical Mathematics

The purpose of this chapter is to introduce the basic tropical\(^1\) notions which are necessary prerequisites. It is not meant to be a detailed summary of tropical algebraic geometry. For more details on the topic, see [MS09], [RGST03] and [IMS07], for example.

1.1 Tropical “Forests”

In this section we introduce the tropical arithmetic and different settings in which it takes place.

1.1.1 The Tropical Semiring

Let \( \mathcal{T} := \mathbb{R} \cup \{ -\infty \} \) and define \( \forall \ a, b \in \mathcal{T} \)

\[
\begin{align*}
a \oplus b & := \max(a, b) \\
a \otimes b & := a + b
\end{align*}
\]

(\( \mathcal{T}, \oplus, \otimes \)) satisfies the axioms of a *semiring* with identity elements being \(-\infty\) for tropical addition and 0 for tropical multiplication. Indeed, \( \forall \ a \in \mathcal{T} \)

\[
\begin{align*}
a \oplus -\infty & = -\infty \oplus a = \max(-\infty, a) = a \\
a \otimes 0 & = 0 \otimes a = 0 + a = a
\end{align*}
\]

Note that \( \forall \ a \in \mathcal{T} \ a \oplus a = a \) and that if \( a \in \mathbb{R} \), there is no \( b \in \mathcal{T} \) such that \( a \oplus b = -\infty \). In other words, addition is idempotent and only \(-\infty\) has an additive inverse, namely itself. However, every nonzero element in \( \mathcal{T} \) has a multiplicative inverse:

\[
\forall \ a \in \mathcal{T} \setminus \{-\infty\} \ \exists \ a^{-1} := -a \ \text{s.t.} \ a \otimes a^{-1} = a + (-a) = 0
\]

This is why this structure is sometimes also called a *semifield*.

\(^{1}\)The term “tropical” was apparently first used by several French mathematicians to honour Imre Simon, a Brazilian mathematician who was one of the pioneers of tropical algebra ([SS09]).
Tropical semiring \((\mathcal{T}, \boxplus, \boxtimes)\) can be viewed as a limit of a continuous deformation of the semiring \((\mathbb{R}_{\geq 0}, +, \times)\). Namely, for any \(h > 0\) define \(T_h := \mathbb{R} \cup \{-\infty\}\) and
\[
\begin{align*}
a \boxplus_h b &:= h \log(e^{a/h} + e^{b/h}) \\
a \boxtimes_h b &:= a + b
\end{align*}
\] (1.3)
Semirings \(\mathbb{R}_{\geq 0}\) and \(T_h\) are isomorphic via \(\phi : \mathbb{R}_{\geq 0} \rightarrow T_h, t \mapsto h \log t\). As \(h\) approaches 0, \(T_h\) degenerates into \(\mathcal{T}\). This is known as Litvinov-Maslov dequantization\(^2\).

Attempts have been made to resolve the issues of the idempotency and lack of additive inversion. We shall only briefly cover some of them in the following subsections.

1.1.2 The Extended Tropical Semiring

This approach is due to Izhakian (see [Izh08] and [Izh09]). We begin by by gluing two copies \(T := \mathbb{R} \cup \{-\infty\}\) and \(T' := \mathbb{R}' \cup \{-\infty\}\) along \(\{-\infty\}\). We denote the resulting set as \(T^*\). This set is then equipped with a total order, extending the order from \(\mathbb{R}\), in the following way:
\[
\begin{align*}
\forall x &\in T^* \quad -\infty < x \\
\forall a, b &\in \mathbb{R} \quad a \leq b \Rightarrow a < b, a < b', a' < b, a' < b' \\
\forall a &\in \mathbb{R} \quad a < a'
\end{align*}
\] (1.4)
and with tropical operations, extending those from \(\mathcal{T}\), in the following way:
\[
\begin{align*}
\forall x, y &\in T^* \quad x \boxplus y = \begin{cases} 
\max_{<}(x, y) & \text{if } x \neq y \\
\times' & \text{if } x = y \neq -\infty
\end{cases} \\
\forall a, b &\in \mathbb{R} \quad a' \boxtimes b = a \boxtimes b' = a' \boxtimes b' = (a + b)'
\end{align*}
\] (1.5)
\((T^*, \boxplus, \boxtimes)\) is called the extended tropical semiring as it extends \((\mathcal{T}, \boxplus, \boxtimes)\). However, it is not idempotent as \(\forall a \in \mathbb{R} \quad a \boxplus a = a'\).

1.1.3 Tropical Hyperfields

These structures allow us to define a weak notion of a tropical additive inverse (but at a high price).

**Definition 1.1** A set \(X\) with a multivalued operation \((x, y) \mapsto x + y\) is called a \((\text{commutative})\) hypergroup if:

1. the operation \(+\) is associative and commutative;
2. \(\exists 0 \in X\) s.t. \(\forall x \in X \quad 0 + x = x\);
3. \(\forall x \in X \quad \exists! -x \in X\) s.t. \(0 \in x + (-x)\).

\(^2\)See [Viro10] and references therein.
Example 1.1 Define addition on $\mathbb{R}_{\geq 0}$ as

$$a \triangledown b := [|a - b|, a + b]$$

In other words, the sum of $a$ and $b$ is the set of all $c$ such that there exists a triangle with sides $a, b$ and $c$. In the ultrametric case, define addition as

$$a \triangledown b := \begin{cases} 
\max(a, b) & \text{if } a \neq b \\
[0, a] & \text{if } a = b 
\end{cases}$$

Both additions are commutative and associative in the sense that

$$(a \triangledown b) \triangledown c = \bigcup_{\alpha \in a \triangledown b} \{\alpha \triangledown c\} = \bigcup_{\beta \in b \triangledown c} \{a \triangledown \beta\} = a \triangledown (b \triangledown c)$$

Furthermore, $0$ serves as the neutral element and every element is its own unique inverse, i.e.

$$\forall a \in \mathbb{R}_{\geq 0} \quad 0 \in a \triangledown a$$

Thus, $(\mathbb{R}_{\geq 0}, \triangledown)$ is a hypergroup. Along with standard multiplication (which is distributive over $\triangledown$) $\mathbb{R}_{\geq 0}$ obtains a structure that is known as a hyperfield. We can then apply the extended logarithm map $\log : \mathbb{R}_{\geq 0} \to T := \mathbb{R} \cup \{-\infty\}$ to obtain a tropical hyperfield $(T, \bar{\triangledown}, \bar{\star})$.

1.1.4 The Complex Tropical Hyperfield

It is not immediate how one could carry the ideas of subsection 1.1.1 to $\mathbb{C}$. One may attempt to singularize $(\mathbb{C}, +, \times)$ via the Litvinov-Maslov process. Namely, $\forall h > 0$ and $\forall z, w \in \mathbb{C}$ let

$$z \boxplus_h w := S_h^{-1}(S_h(z) + S_h(w))$$

$$z \boxtimes_h w := z + w$$

where $S_h : \mathbb{C} \leftrightarrow \mathbb{C}$ is defined by

$$z \mapsto \begin{cases} 
|z|^\frac{1}{h} \frac{z}{|z|} & \text{if } z \neq 0 \\
0 & \text{if } z = 0 
\end{cases}$$

Let $z \boxplus_0 w := \lim_{h \to 0^+} (z \boxplus_h w)$. This defines a tropical addition on $\mathbb{C}$ and one can check that

$$z \boxplus_0 w = \begin{cases} 
z & \text{if } |z| > |w| \\
w & \text{if } |z| < |w| \\
0 & \text{if } z = -w \\
|z| \frac{z + w}{|z + w|} & \text{if } |z| = |w| \text{ and } z \neq -w 
\end{cases}$$

The problem, however, is that this addition is not continuous or associative. To overcome the issue, we again introduce the “hyper”-structures. Let $z, w \in \mathbb{C}$ and define

$$z \hat{\boxplus} w := \begin{cases} 
\{z\} & \text{if } |z| > |w| \\
\{w\} & \text{if } |w| < |w| \\
\{re^{\theta i} : \theta \in [\alpha, \beta]\} & \text{if } z = re^{\alpha i}, w = re^{\beta i}, \beta - \alpha < \pi \\
\{\zeta \in \mathbb{C} : |\zeta| \leq |z|\} & \text{if } z = -w 
\end{cases}$$
The only part that is not immediately clear in showing that \((\mathbb{C}, \tilde{\oplus})\) is a hypergroup is associativity. We will not reproduce the proof here (see Lemma 2.B. in \([\text{Viro10}]\)). Restricting this operation to the reals makes \((\mathbb{R}, \tilde{\oplus})\) a hypergroup as well. With the standard single-valued tropical multiplication

\[ z \otimes w := z + w, \]

\((\mathbb{C}, \tilde{\oplus}, \otimes)\) and \((\mathbb{R}, \tilde{\oplus}, \otimes)\) become tropical hyperfields.

Tropical algebra and geometry can be studied in these different settings with the hope of generalizing more results from standard algebra and geometry. Certain results such as Bézout’s Theorem (see \([\text{St02}]\)), Degree-Genus Formula, Riemann-Roch Theorem (see \([\text{Mikh06}]\)) and Group Law of Cubics have been established for tropical curves. We shall only be concerned with the tropical semiring \((\mathcal{T}, \tilde{\oplus}, \otimes)\) as defined in subsection 1.1.1.

### 1.2 Tropical Polynomials

A tropical polynomial in \(n\) variables is a formal sum of the form

\[ p := \bigoplus_{\alpha \in \mathbb{N}^n} c_\alpha \otimes x_1^{\alpha_1} \otimes \cdots \otimes x_n^{\alpha_n} \quad (1.6) \]

where \(c_\alpha \in \mathcal{T}\) and all but finitely many equal \(-\infty\) and \(x_i^{\alpha_i} := x_i \otimes \cdots \otimes x_i\). Putting \(x = (x_1, \ldots, x_n)\), we can also use the alternative notation \(c_\alpha \otimes x^{\alpha}\) for monomials.

To each polynomial \(p\) as in (1.6) one associates a tropical polynomial function

\[ p : \mathcal{T}^n \to \mathcal{T}, \quad x \mapsto \bigoplus_{\alpha \in \mathbb{N}^n} c_\alpha \otimes x^{\alpha} = \max_{\alpha \in \mathbb{N}^n} (c_\alpha + \langle \alpha, x \rangle) \quad (1.7) \]

As with the standard definitions, our definitions can be extended to those of Laurent polynomials and Laurent polynomial functions. We can allow \(\alpha \in \mathbb{Z}^n\) provided that we never try to evaluate at \(-\infty\). A tropical Laurent polynomial function is a function of the form

\[ p : (\mathcal{T}\setminus\{-\infty\})^n = \mathbb{R}^n \to \mathcal{T}, \quad x \mapsto \max_{\alpha \in \mathbb{Z}^n} (c_\alpha + \langle \alpha, x \rangle) \quad (1.8) \]

and a tropical Laurent polynomial is a formal sum

\[ p := \bigoplus_{\alpha \in \mathbb{Z}^n} c_\alpha \otimes x^{\alpha}. \quad (1.9) \]

Note that \(p\) is a convex function. Every tropical (Laurent) polynomial function is not representable in a unique way as (1.8). However, it admits a unique representation of this form such that the coefficients \(c_\alpha\) are maximal, i.e. one can attach to it a tropical polynomial. Conversely, given a tropical polynomial, it admits a unique evaluation function as in (1.8), where the coefficients \(c_\alpha\) are maximal.
1.3 Tropical Hypersurfaces

The notion of a zero-locus from standard geometry cannot be generalized to the tropical setting because for any Laurent polynomial function $p$ there is no $x \in \mathbb{R}^n$ such that $p(x) = -\infty$ unless $p$ is identically $-\infty$. We must, therefore, take a different approach.

**Definition 1.2** Let $p$ be as in (1.8). The *tropical hypersurface* $\mathcal{V}_{p}^{\text{trop}}$ associated to $p$ is the set of points at which $p$ is not differentiable, i.e. the subset of $\mathbb{R}^n$ of those points at which the maximum is achieved for at least two distinct affine functions $c_\alpha + \langle \alpha, x \rangle$.

Of course, this definition is the same for all polynomials that have the same “evaluation” so it makes sense to define the hypersurface $\mathcal{V}_{p}^{\text{trop}}$ of a tropical Laurent polynomial $p$ as the hypersurface of its evaluation $p$.

**Note:** If $p$ is a monomial then $\mathcal{V}_{p}^{\text{trop}} = \emptyset$.

Figure 1.1: Graphs of some tropical polynomial functions $p$ and the corresponding curves $\mathcal{V}_{p}^{\text{trop}}$. The coefficients in these examples are symmetric and chosen so that every affine function is represented.
1.4 Legendre-Fenchel Transform

Definition 1.3 Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be an arbitrary function. Its Legendre-Fenchel transform (a.k.a. the convex conjugate) is the following function, defined on the dual space $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ of $\mathbb{R}^n$:

$$\xi \in \mathbb{R}^n \mapsto \tilde{f}(\xi) := \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - f(x)) \in [-\infty, \infty]$$ (1.10)

It is immediate that $f \leq g$ implies $\tilde{f} \geq \tilde{g}$ and therefore $\tilde{\tilde{f}} \leq \tilde{\tilde{g}}$. Furthermore, we have $\tilde{\tilde{f}} \leq f$, with equality holding if and only if one of the following is true:

1. $f$ is a proper\(^3\), lower semi-continuous\(^4\), convex function
2. $f \equiv -\infty$
3. $f \equiv \infty$

In convex analysis, this is known as the Fenchel-Moreau theorem (see [BL06]). Legendre-Fenchel transform plays an important role for us because of the following theorem.

Theorem 1.1 Let $p$ be a tropical Laurent polynomial function. Then $p$ can be represented as

$$p(x) = \max_{\alpha \in \mathbb{Z}^n} (-\tilde{p}(\alpha) + \langle \alpha, x \rangle)$$ (1.11)

Moreover, the coefficients $-\tilde{p}(\alpha)$ are maximal over all possible representations of $p$.

Proof Assume $p \neq -\infty$ and let

$$p(x) = \max_{\alpha \in \mathbb{Z}^n} (c_\alpha + \langle \alpha, x \rangle)$$ (1.12)

be any representation of $p$. Define $f : \mathbb{R}^n \to [-\infty, \infty]$ as

$$f(x) = \begin{cases} -c_\alpha & \text{if } x \in \mathbb{Z}^n \\ \infty & \text{if } x \in \mathbb{R}^n \setminus \mathbb{Z}^n \end{cases}$$

\(^3\)i.e. $f$ is not identically $\infty$ and never takes value $-\infty$.
\(^4\)i.e. \{f > \alpha\} is open for every $\alpha$.\topfraction=0.8\displaystyle
so that $p$ is the Legendre-Fenchel transform of $f$. Since $p$ clearly satisfies condition (1) of the Fenchel-Moreau theorem, we have $p = \hat{p} = \check{f}$ and $\check{p} = \hat{f} = \check{f}$, i.e.

$$\forall x \in \mathbb{R}^n \quad p(x) = \hat{p}(x) := \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \hat{p}(\xi))$$

$$= \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \sup_{\eta \in \mathbb{R}^n} (\langle \xi, \eta \rangle - p(\eta)))$$

$$= \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \sup_{\eta \in \mathbb{R}^n} (\langle \xi, \eta \rangle - \max_{\alpha \in \mathbb{Z}^n} (\langle \alpha, \eta \rangle + c_{\alpha}))))$$

$$= \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \sup_{K_{\alpha}} (\sup_{\eta \in K_{\alpha}} (\langle \xi - \alpha, \eta \rangle - c_{\alpha}))))$$

Here $K_{\alpha}$ denotes the subset of $\mathbb{R}^n$ in which $p(\eta) = \langle \alpha, \eta \rangle + c_{\alpha}$. (We ignore those $\alpha$ for which $c_{\alpha} = -\infty$. If $\xi \neq \alpha \in \mathbb{Z}^n$, there is always a component $K_{\alpha}$ such that $\sup_{\eta \in K_{\alpha}} (\langle \xi - \alpha, \eta \rangle - c_{\alpha})))) = \infty$ (see figure 1.3). This leads to a contradiction as $p(x) = -\infty$ is impossible. We may, therefore, rewrite (1.13) with $\xi \in \mathbb{Z}^n$, i.e. (1.11) holds.

Figure 1.3: The affine plane $\langle \xi - \alpha, \eta \rangle - c_{\alpha}$ in $\mathbb{R}^n \times \mathbb{R}$. If it is not horizontal, i.e. $\xi = \alpha$, the second coordinate is clearly unbounded.

Moreover, we have $\forall \alpha \in \mathbb{Z}^n \quad -\hat{p}(\alpha) = -\check{f}(\alpha) \geq -f(\alpha) = c_{\alpha}$.

This implies that there is a bijective correspondence between tropical Laurent polynomials and tropical Laurent polynomial functions with maximal coefficients.
Chapter 2

Amœbas and Coamœbas

Before we move on to our main objects of interest, we shall introduce some definitions and notations.

\( \mathbb{T}^n \) shall denote the standard complex torus \((\mathbb{C}^*)^n\) and \( \mathbb{P}^{n-1} \) shall denote the standard complex projective space.

**Definition 2.1** A *Laurent polynomial* in \( \mathbb{T}^n \) is an expression of the form

\[
f(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha} z_{\alpha}
\]

where \( c_{\alpha} \in \mathbb{C} \) with \( c_{\alpha} = 0 \) for all but finitely many \( \alpha \) and, as before, \( z_{\alpha} := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \).

**Definition 2.2** The *support* of \( f \) is the set \( A_f := \{ \alpha \in \mathbb{Z}^n : c_{\alpha} \neq 0 \} \).

**Definition 2.3** The *Newton polytope* \( \Delta_f \) of \( f \) is the convex hull of the support of \( f \).

**Remark 2.1** Let \( \text{int} \Delta_f \) denote the interior of \( \Delta_f \subset \mathbb{R}^n \). We will usually assume that \( \text{int} \Delta_f \) is non-empty. If it is not, \( f \) may be reduced to the case of a Laurent polynomial in less than \( n \) variables\(^1\). As with other polytopes, we can view \( \Delta_f \) as an intersection of finitely many halfspaces:

\[
\Delta_f = \bigcap_{k=1}^{N} \{ x \in \mathbb{R}^n : \langle \mu_k, x \rangle \geq \nu_k \} \quad (2.1)
\]

where \( \nu_k \in \mathbb{Z} \) and \( \mu_k \in \mathbb{Z}^n \) are primitive vectors normal to facets\(^2\) of \( \Delta_f \) and facing inwards. If \( \Gamma \) is a face of \( \Delta_f \), we shall denote its relative interior\(^3\) by \( \text{relint}(\Gamma) \) and by \( f_\Gamma \) we shall denote the Laurent polynomial consisting of those monomials of \( f \) that correspond to the face \( \Gamma \), i.e.

\[
f_\Gamma = \sum_{\alpha \in \Gamma \cap A_f} c_{\alpha} z_{\alpha} \quad (2.2)
\]

\(^1\)In this case the powers are linearly dependent so we can write the general term as \( z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} z_n^{\alpha_n} = (z_1 z_2 \cdots z_n)^{\alpha_1} \cdots (z_{n-1} z_n)^{\alpha_n-1} \).

\(^2\)i.e. \((n-1)\)-dimensional faces.

\(^3\)i.e. interior of \( \Gamma \) viewed as a subset of the lowest dimensional hyperplane that contains it.
Defining 2.4 Let $A \subset \mathbb{R}^n$ and $a \in A$. The normal cone of $A$ at $a$ is a convex cone defined as

$$C_a = \{ y \in \mathbb{R}^n : \forall x \in A \langle y, x - a \rangle \leq 0 \} \quad (2.3)$$

Equivalently, we can describe the normal cone by saying that a non-zero vector $y$ belongs to $C_a$ if and only if the following two conditions hold:

1. $y$ is a normal to a hyperplane that supports4 $A$ at $a$;
2. $y$ and $A$ are on the opposite sides of that hyperplane.

Remark 2.2 We will be interested in normal cones of $\Delta_f$. Notice that if $\Gamma$ is a face of $\Delta_f$ and $\xi \in \Gamma$, the cone $C_{\xi}$ is of dimension $n - \dim \Gamma$. In particular, $C_{\xi}$ has a non-empty interior (in $\mathbb{R}^n$) precisely when $\xi$ is a vertex of $\Delta_f$, and it equals $\{0\}$ when $\xi \in \text{int} \Delta_f$. Also notice that $C_{\xi} = \{ x \in \mathbb{R}^n : \langle \xi, x \rangle = \max_{\alpha \in \Delta_f} \langle \alpha, x \rangle \}.

Definition 2.5 Let $r \geq 2$. A set $\{e^{x_1}, \ldots, e^{x_r}\} \subset \mathbb{R}_{>0}$ is called lopsided if for some $j$ we have $e^{x_j} > \sum_{k \neq j} e^{x_k}$. If, particularly, for some $c \geq r - 1$ we have $e^{x_j} > c \max_{k \neq j} e^{x_k}$, the set is said to be $c$-superlopsided. If $c = r - 1$ the set is just said to be superlopsided.

Proposition 2.1 $\{e^{x_1}, \ldots, e^{x_r}\} \subset \mathbb{R}_{>0}$ is not lopsided if and only if there exist $\theta_1, \ldots, \theta_r \in \mathbb{R}$ such that $\sum_{j=1}^r e^{x_j + i\theta_j} = 0$.

Proof
1) Assume $\{e^{x_1}, \ldots, e^{x_r}\}$ is not lopsided. Then $e^{x_r} \leq \sum_{j=1}^{r-1} e^{x_j}$. The triangle inequality implies $|\sum_{j=1}^{r-1} e^{x_j + i\theta_j}| \leq \sum_{j=1}^{r-1} e^{x_j}$, $\forall (\theta_1, \ldots, \theta_{r-1})$. Since $|\sum_{j=1}^{r-1} e^{x_j + i\theta_j}|$ is continuous, it achieves value $e^{x_r}$ for some $(\theta_1, \ldots, \theta_{r-1})$. Hence $\exists \theta_r$ such that $e^{x_r + i\theta_r} = -\sum_{j=1}^{r-1} e^{x_j + i\theta_j}$.
2) Assume $\{e^{x_1}, \ldots, e^{x_r}\}$ is lopsided. Without loss of generality, assume $e^{x_r} > \sum_{j=1}^{r} e^{x_j}$. Then $\sum_{j=1}^{r} e^{x_j + i\theta_j} = 0$ is impossible because by triangle inequality we have $e^{x_r} = |\sum_{j=1}^{r-1} e^{x_j + i\theta_j}| \leq \sum_{j=1}^{r-1} e^{x_j} < e^{x_r}$. \hfill $\square$

Remark 2.3 The definitions of the support, Newton polytope and lopsidedness have an analogous definition in the tropical setting.

2.1 The Hypersurface Case

2.1.1 Amoebas

Definition 2.6 The amoeba $\mathcal{A}_f$ of a Laurent polynomial $f$ is the image of the zero set $\mathcal{V}_f = \{ z \in \mathbb{T}^n : f(z) = 0 \}$ under the map $\text{Log} : \mathbb{T}^n \to \mathbb{R}^n$ given by

$$(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$$

Note that this is a slight abuse of notation as the amoeba depends on the hypersurface $\mathcal{V}_f$, not on $f$. For example, $\mathcal{A}_{f_2} = \mathcal{A}_f$. Amoebas first appeared in [GKZ94] and were so named because of their typical shape (see examples below).

4A hyperplane $H$ is said to support a set $A$ if $A$ lies entirely in one of the two closed half-spaces determined by $H$ and has at least one point on $H$. 

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Remark 2.4 $\mathcal{A}_f$ is a closed set.

Example 2.1 One-dimensional amoebas are discrete sets and, as such, not of great interest to us.

Example 2.2 (Amoeba of a line) Let $f(z, w) = z + w - 1$ and consider $\mathcal{V}_f = \{(z, w) \in \mathbb{T}^2 : f(z, w) = 0\}$. A point $(z, w) \in \mathbb{T}^2$ belongs to $\mathcal{V}_f$ if and only if $\{(1, |z|, |w|)\}$ is not lopsided (cf. Proposition 2.1), i.e.

$$1 \leq |z| + |w|, \quad |z| \leq 1 + |w|, \quad |w| \leq 1 + |z|$$

(2.4)

This implies that $\mathcal{A}_f$ is the image of $\{(u, v) \in \mathbb{R}^2_{>0} : 1 \leq u + v, \quad u \leq 1 + v, \quad v \leq 1 + u\}$ under the map $(u, v) \mapsto (\log u, \log v)$ (see figures 2.1 and 2.2 below).

Recall that the genus of a smooth projective curve of degree $d$ is $(d - 1)(d - 2)/2$ and notice that when we take the Fermat curve $\mathcal{V}_g$ where $g(z, w) = z^d + w^d - 1$, the amoeba $\mathcal{A}_g$ is a dilation of the amoeba $\mathcal{A}_f$ considered above. This shows that some topological information is lost when moving to the world of amoebas.

Example 2.3 Let $f(z, w) = z^3w^4 + z^5 + 40z^3w + z^3 + 80z^2w + 1$. $\Delta_f$ and $\mathcal{A}_f$ are shown in figure 2.3. Note that $\mathcal{A}_f$ has bounded components. We will come back to this example from time to time as it will provide us with relevant illustrations.

Consider a Laurent polynomial $f$. Rational function $1/f$ can be developed as

$$\sum_{\alpha \in \mathbb{Z}^n} c'_{\alpha} z^\alpha$$

where

$$c'_{\alpha} = \frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(x)} \frac{dz_1 \wedge \cdots \wedge dz_n}{f(z)_z^{\alpha_1+1} \cdots z_n^{\alpha_n+1}} = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} c_{\alpha,x+i\theta} f^{-1}(e^{x+i\theta}) d\theta,$$

(2.5)

$x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \mathcal{A}_f$ and $e^z := (e^{x_1}, \ldots, e^{x_n})$. Coefficients $c'_{\alpha}$ do not depend on the connected component of $\mathbb{R}^n \setminus \mathcal{A}_f$ because if $x$ and $y$ are in the same component, $\text{Log}^{-1}(x)$ and $\text{Log}^{-1}(y)$ are homologous. Moreover, we have the following theorem.
Figure 2.3: Newton polytope and amoeba of \( f(z, w) = z^3w^4 + z^5 + 40z^3w^2 + z^3w + 80z^2w + 1 \)

**Theorem 2.1 [GKZ94]** Connected components of \( \mathbb{R}^n \setminus \mathcal{A}_f \) are convex and in bijective correspondence with distinct Laurent expansions of \( 1/f \) (near the origin).

**Proof** This follows from the following facts:

1. \( \mathbb{R}^n \setminus \mathcal{A}_f \) is open and so are its connected components.
2. The domain of convergence of a Laurent series \( f \in \mathbb{C}[z^{\pm 1}, \ldots, z_{n}^{\pm 1}] \) is a logarithmically convex complete Reinhardt domain\(^5\).
3. On any Reinhardt domain there is exactly one convergent Laurent series.

For more on this topic, see [Horm90a], [KK83], [Kr92].

We shall study components of the complement of the amoeba in more detail in section 2.1.4.

### 2.1.2 Coamoebas

**Definition 2.7** The coamoeba \( \mathcal{A}'_f \) of a Laurent polynomial \( f \) is the image of the zero set \( \mathcal{V}_f = \{ z \in \mathbb{T}^n : f(z) = 0 \} \) under the map \( \text{Arg} : \mathbb{T}^n \to (S^1)^n \) given by

\[
(z_1, \ldots, z_n) \mapsto (\arg z_1, \ldots, \arg z_n)
\]

We may also view the coamoeba as the corresponding periodic subset of \( \mathbb{R}^n \).

**Example 2.4** (Coamoeba of a line) Let \( f(z, w) = z + w - 1 \). Writing \( z = r_1e^{i\theta_1} \) and \( w = r_2e^{i\theta_2} \), we can describe the coamoeba \( \mathcal{A}'_f \) as the set of those \((\theta_1, \theta_2)\) in the fundamental polygon \((\pi, \pi)^2\) such that

\[
r_1e^{i\theta_1} + r_2e^{i\theta_2} = 1
\]
has a solution \((r_1, r_2) \in \mathbb{R}^2_{>0}\). We can rewrite (2.6) as
\[
\begin{align*}
    r_1 \cos \theta_1 + r_2 \cos \theta_2 &= 1 \\
    r_1 \sin \theta_1 + r_2 \sin \theta_2 &= 0
\end{align*}
\]  
(2.7)
Clearly there is a solution when \((\theta_1, \theta_2) = (0, 0)\) and this is the only point in \((-\pi, \pi]^2\) for which the second equation in (2.7) is trivial. If \((\theta_1, \theta_2) \neq (0, 0)\), a solution exists if and only if
\[
0 \neq \det \begin{bmatrix}
\cos \theta_1 & \cos \theta_2 \\
\sin \theta_1 & \sin \theta_2
\end{bmatrix} = \cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1
\]
and the solution can be described in terms of \((\theta_1, \theta_2)\) as
\[
\begin{bmatrix}
    r_1 \\
    r_2
\end{bmatrix} = \frac{1}{\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1} \begin{bmatrix}
    \sin \theta_2 \\
    -\sin \theta_1
\end{bmatrix}
\]
For \((\theta_1, \theta_2) \in (-\pi, \pi]^2\), \(\cos \theta_1 \sin \theta_2 - \cos \theta_2 \sin \theta_1 \neq 0\) precisely when \(\theta_1 = \theta_2\) or \(\theta_1 = \theta_2 \pm \pi\). Simple considerations of the signs of trigonometric functions outside of these three lines tell us when \(r_1\) and \(r_2\) are positive and give us the full description of the coamoeba. (cf. figure 2.4a).

![Figure 2.4: Coamoebas of lines, represented in \((-\pi, \pi]^2\)](figure)

We similarly obtain the coamoeba of \(f(z, w) = z + w + 1\) (cf. figure 2.4b). Notice that they differ, unlike the corresponding amoebas. Also, if we let \(\mathcal{A}_g'\) denote the coamoeba of the Fermat curve \(\mathcal{F}_g\) where \(g(z, w) = z^d + w^d - 1\), we notice that \((\theta_1, \theta_2) \in \mathcal{A}_f'\) if and only if \((d\theta_1, d\theta_2) \in \mathcal{A}_g'\).

**Remark 2.5** As the above examples show, coamoebas are not closed in general.

Even though amoebas and coamoebas seem to behave quite differently, it turns out that in the case of coamoebas there is a statement analogous to Theorem 2.1 and we shall establish it in the following subsection (see [NP10]). We shall first lift the coamoeba in \(\mathbb{R}^n\) and, since it need not be closed, take its closure. In other words, we are interested in the components of the open set \(\mathbb{R}^n \setminus \pi^{-1}(\mathcal{A}_f')\) where \(\pi : \mathbb{R}^n \to (\mathbb{S}^1)^n\) denotes the canonical projection.
The Mellin Transform

**Definition 2.8** The (generalized) Mellin transform of a rational function \(1/f\) is given by

\[
M_{1/f}(s) := \int_{\mathbb{R}_0^n} z^s f(z) \, dz = \int_{\mathbb{R}_0^n} e^{s,x} f(e^x) \, dx
\]  

(2.8)

**Definition 2.9** A Laurent polynomial \(f\) is said to be completely non-vanishing on a set \(X\) if for every face \(\Gamma\) of \(\Delta_f\) the Laurent polynomial \(f_\Gamma\) (as defined in (2.2)) does not vanish on \(X\).

**Theorem 2.2** If \(f\) is completely non-vanishing on \(\mathbb{R}_0^n\) then the integral (2.8) converges and defines an analytic function on the tube domain \(\text{int} \Delta_f + i\mathbb{R}_0^n\).

**Proof** Let \(\sigma \in \text{int} \Delta_f\) and \(s \in \sigma + i\mathbb{R}_0^n\). It suffices to show that \(\exists c, k > 0\) such that

\[
|f(e^x)e^{-\langle s,x \rangle}| = |f(e^x)|e^{-\langle \sigma,x \rangle} \geq ce^{|x|}
\]  

(2.9)

when \(x \in \mathbb{R}_0^n\) is far away from the origin. We proceed by induction on \(n\). If \(n = 1\), we have \(\Delta_f = [\alpha, \beta]\) and \(f(z) = c_\alpha z^\alpha + \cdots + c_\beta z^\beta\). Notice that for \(\sigma \in (\alpha, \beta)\) we have

\[
\lim_{x \to \infty} \frac{|f(e^x)e^{-\langle s,x \rangle}|}{c_\beta e^{\langle \beta - \sigma \rangle x}} = \lim_{x \to \infty} \frac{|c_\alpha e^{\langle \alpha - \sigma \rangle x} + \cdots + c_\beta e^{\langle \beta - \sigma \rangle x}|}{c_\beta e^{\langle \beta - \sigma \rangle x}} = \lim_{x \to \infty} \frac{c_\alpha e^{\langle \alpha - \beta \rangle x} + \cdots + 1}{c_\beta} = 1
\]  

(2.10)

Therefore, if \(x\) is sufficiently large and positive, we have

\[
|f(e^x)e^{-\langle s,x \rangle}| \leq \frac{1}{2} |c_\beta| e^{\langle \beta - \sigma \rangle |x|}
\]

and, similarly, if \(x\) is sufficiently large and negative, we have

\[
|f(e^x)e^{-\langle s,x \rangle}| \leq \frac{1}{2} |c_\alpha| e^{\langle \sigma - \alpha \rangle |x|}.
\]

Suppose that (2.9) holds for dimensions \(\leq n - 1\) and consider a Laurent polynomial in \(n\) variables. For every face \(\Gamma\) of \(\Delta_f\) with \(0 \leq \dim \Gamma \leq n - 1\), we can express any \(\sigma \in \text{int} \Delta_f\) as

\[
\sigma = \lambda \sigma_\Gamma + (1 - \lambda) \tau_\Gamma
\]

where \(\sigma_\Gamma \in \text{relint}(\Gamma)^7\) and \(\tau_\Gamma \in \text{relint}(\text{conv}((A_f \setminus \Gamma))^8\). Fix one such point \(\sigma_\Gamma\) for each face \(\Gamma\) and consider the polytope

\[\Delta_\Gamma := \text{conv}((A_f \setminus \Gamma) \cup \sigma_\Gamma)\]

Notice that \(\sigma \in \Delta_\Gamma\) for all faces \(\Gamma\) and that when \(\Gamma\) is a vertex, we have \(\Delta_\Gamma = \Delta_f\).

Let \(\widetilde{C}_\Gamma\) denote the normal cone (recall Definition 2.4) to \(\Delta_\Gamma\) at \(\sigma_\Gamma\) translated by \(\sigma_\Gamma\), i.e.

\[
\widetilde{C}_\Gamma = \{x \in \mathbb{R}_0^n : \forall \xi \in \Delta_f \, \langle \xi - \sigma_\Gamma, x - \sigma_\Gamma \rangle \leq 0\}
\]  

(2.11)

7\text{relint(\_)} denotes the relative interior.
8\text{conv(\_)} denotes the convex hull.
All $\tilde{C}_\Gamma$ are $n$-dimensional because $\sigma_\Gamma$ is a vertex of $\Delta_\Gamma$. Furthermore, these cones cover almost all of $\mathbb{R}^n$ in the sense that $\mathbb{R}^n \setminus (\bigcup \Gamma \tilde{C}_\Gamma)$ is bounded. Let $C_\Gamma$ denote a smaller closed convex cone with vertex $\sigma_\Gamma$ such that $C_\Gamma \setminus \{\sigma_\Gamma\} \subset \text{int}(\tilde{C}_\Gamma)$ and such that $\mathbb{R}^n \setminus (\bigcup \Gamma C_\Gamma)$ is still bounded.

Observe that it suffices to prove (2.9) for $x \in C_\Gamma \setminus B(0, R)$ where $B(0, R)$ is a ball of large radius. Since $f_\Gamma$ depends on less than $n$ variables (recall Remark 2.1) and $\sigma_\Gamma \in \text{relint}(\Delta_{f_\Gamma})$, we deduce from the induction hypothesis that there are constants $c_\Gamma > 0$ such that

$$|f_\Gamma(e^x)e^{-(\sigma_\Gamma,x)}| \geq c_\Gamma$$

Let $g_\Gamma = \sum_{\alpha \in A_x \setminus \Gamma} c_\alpha z^\alpha$ so that $f = f_\Gamma + g_\Gamma$. We may now write

$$f(e^x)e^{-(\sigma_\Gamma,x)} = e^{\langle \sigma_\Gamma, x \rangle} (f_\Gamma(e^x)e^{-(\sigma_\Gamma,x)} + g_\Gamma(e^x)e^{-(\sigma_\Gamma,x)}).$$

Let $x \in C_\Gamma$ and $y = x - \sigma_\Gamma$. We have $e^{\langle \sigma_\Gamma, x \rangle} \geq c_0 e^{k|y|}$ where $c_0 = e^{\langle \sigma_\Gamma - \sigma_\Gamma, y \rangle}$ and

$$k = \min \{ \langle \sigma_\Gamma - \sigma, y \rangle : |y| = 1, \sigma_\Gamma + y \in C_\Gamma \}.$$

We may assume $|x| > |\sigma_\Gamma|$ (since $x \notin B(0, R)$) so that $|x| - |\sigma_\Gamma| \geq |x - \sigma_\Gamma| = y$ and

$$e^{\langle \sigma_\Gamma, x \rangle} \geq c_1 e^{k|x|}$$

where $c_1 = c_0 e^{-k|\sigma_\Gamma|}$.

It now remains to bound the second factor on the right in (2.12) by a positive constant. By induction hypothesis, we have

$$|f_\Gamma(e^x)e^{-(\sigma_\Gamma,x)}| \geq c_\Gamma$$

so it is enough to show that the second term $g_\Gamma(e^x)e^{-(\sigma_\Gamma,x)}$ remains sufficiently small (e.g. $< \frac{c_\Gamma}{2}$). We have

$$g_\Gamma(e^x) = \sum_{\alpha \in A_x \setminus \Gamma} c_\alpha e^{\langle \alpha, x \rangle} \sum_{\alpha \in A_x \setminus \Gamma} \tilde{c}_\alpha e^{\langle \alpha, y \rangle}.$$

Since $\alpha \in \Delta_\Gamma$, we have a positive constant

$$k_\alpha = \min \{ \langle \sigma_\Gamma - \alpha, y \rangle : |y| = 1, \sigma_\Gamma + y \in C_\Gamma \}.$$

Hence

$$|c_\alpha e^{\langle \alpha, x \rangle}| = |\tilde{c}_\alpha e^{\langle \sigma_\Gamma - \alpha, y \rangle}| \leq |c_\alpha| e^{-k_\alpha |y|}.$$

This implies that for a large enough $R'$ we have

$$\forall x \in C_\Gamma |x + \sigma| \geq R' \Rightarrow |g_\Gamma(e^x)e^{-(\sigma_\Gamma,x)}| < \frac{c_\Gamma}{2}$$

and hence

$$|f(e^x)e^{-(\sigma_\Gamma,x)}| \geq \frac{c_\Gamma}{2}.$$

Finally, we conclude that for sufficiently large $R$ we have

$$\forall x \in C_\Gamma \setminus B(0, R) \ |f(e^x)e^{-(\sigma_\Gamma,x)}| \geq c_\Gamma e^{k|x|}$$

with $c = \frac{c_\Gamma e^{k|x|}}{2}$.

It turns out that (2.8) can be extended to a meromorphic function on $\mathbb{C}^n$ in an interesting way. We shall omit the proof of this (see Theorem 2 in [NP10]).
Theorem 2.3 If \( f \) is completely non-vanishing on \( \mathbb{R}^n_{>0} \) and \( \Delta_f \) has a non-empty interior, the Mellin transform \( M_{1/f}(s) \) can be meromorphically extended to \( \mathbb{C}^n \) as

\[
M_{1/f}(s) = \Phi(s) \prod_{k=1}^{n} \Gamma(\langle \mu_k, s \rangle - \nu_k)
\]  

(2.13)

where \( \mu_k \) and \( \nu_k \) are as in (2.1) and \( \Phi(s) \) is an entire function.

Theorem 2.4 For any \( \theta \in \mathbb{R}^n \setminus \pi^{-1}(\omega_f') \), the Laurent polynomial \( f \) is completely non-vanishing on \( \text{Arg}^{-1}(\theta) \).

Proof For a given \( \theta \) we can consider \( f_\theta(z) := f(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \). We have \( \pi^{-1}(\omega_f') + \theta = \pi^{-1}(\omega_f') \) so that \( 0 \in \pi^{-1}(\omega_f') \) if and only if \( \theta \in \pi^{-1}(\omega_f') \) and that \( f_\theta \) is completely non-vanishing on \( \text{Arg}^{-1}(\theta) \) if and only if \( f \) is completely non-vanishing on \( \text{Arg}^{-1}(\theta) \). Therefore it suffices to consider the case \( \theta = 0 \). Note that \( \text{Arg}^{-1}(0) = \mathbb{R}^n_{>0} \).

Assume that \( f \) is not completely non-vanishing on \( \text{Arg}^{-1}(0) \) so that \( 0 \in \pi^{-1}(\omega_f') \) for some face \( \Gamma \), i.e. \( \exists x_0 \in \mathbb{R}^n \) s.t. \( f_\Gamma(e^{ix_0}) = 0 \). We need to show that \( 0 \in \overline{\pi^{-1}(\omega_f')} \).

Case \( \Gamma = \Delta_f \) is trivial, so we will consider faces such that \( \dim \Gamma \leq n-1 \). Let \( \mu \in \mathbb{Z}^n \) and \( m \in \mathbb{Z} \) be such that \( \langle \mu, \alpha \rangle = m \) for \( \alpha \in \Gamma \) and \( \langle \mu, \alpha \rangle < m \) for \( \alpha \in \Delta_f \setminus \Gamma \) and let \( g_\Gamma = f - f_\Gamma \) as before. We have

\[
f_\Gamma(e^{x_0-t\mu}) = \sum_{\alpha \in \Gamma} c_\alpha e^{(x_0,\alpha)-t(\mu,\alpha)} = e^{-tm} \sum_{\alpha \in \Gamma} c_\alpha e^{(x_0,\alpha)} = 0
\]

and

\[
g_\Gamma(e^{x_0-t\mu}) = \sum_{\alpha \in \Delta_f \setminus \Gamma} c_\alpha e^{(x_0,\alpha)-t(\mu,\alpha)} = \sum_{\alpha \in \Delta_f \setminus \Gamma} \hat{c}_\alpha e^{-tk_\alpha}
\]

where \( \hat{c}_\alpha = c_\alpha e^{(x_0,\alpha)} \) and \( k_\alpha = \langle \mu, \alpha \rangle - m > 0 \). Now fix \( \varepsilon > 0 \) and let \( D := D(x_0, \varepsilon) \) be a disk of radius \( \varepsilon \) centered at \( x_0 \) contained in a complex line on which the function \( w \mapsto f_\Gamma(e^w) \) is not identically zero. Since \( f_\Gamma(e^w) \) is not zero on the boundary \( \partial D, \) we have \( |f_\Gamma(e^w)| \geq \delta > 0 \) for \( w \in \partial D \). This implies that \( |f_\Gamma(e^w)| \geq \delta e^{-tm} \) for \( w \in \partial D - t\mu \) for some large positive \( t \). For a sufficiently large \( t \), we can ensure that on \( \partial D - t\mu \) we have \( |g_\Gamma(e^w)| < |f_\Gamma(e^w)| \). By Rouche’s theorem, \( f(e^w) = f_\Gamma(e^w) + g_\Gamma(e^w) \) has a zero \( w_e \) in \( D - t\mu \). Therefore \( e^{w_e} \) belongs to \( \mathcal{W}_\Gamma \). Since \( |\text{Arg}(e^{w_e})| = |\text{Im}(w_e)| < \varepsilon \) and \( \varepsilon \) was arbitrarily chosen, we conclude \( 0 \in \pi^{-1}(\omega_f') \).

Theorems 2.2 and 2.4 allow us to define the \( \theta \)-directional Mellin transform

\[
M^\theta_{1/f}(s) := \frac{1}{(2\pi i)^n} \int_{\pi^{-1}(\theta)} \frac{z^s}{f(z)} \frac{dz}{z} = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} e^{(s,x+i\theta)} \frac{dx}{f(e^{x+i\theta})}.
\]

(2.14)

for any \( \theta \in \mathbb{R}^n \setminus \pi^{-1}(\omega_f') \) and \( s \in \text{int} \Delta_f \).

Lemma 2.5 Integral (2.14) does not depend on the choice of \( \theta \) as long as \( \theta \) remains in the same connected component of \( \mathbb{R}^n \setminus \pi^{-1}(\omega_f') \). We shall denote it by \( M^C_{1/f}(s) \) and call it the \( C \)-directional Mellin transform where \( C \) is the connected component that contains \( \theta \).
Proof We begin by considering the one-dimensional case. By Theorems 2.2 and 2.4, we know that the directional Mellin transform

\[ M^\theta_{1/f}(s) := \int_{\arg^{-1}(\theta)} \frac{z^s}{f(z)} \, dz \]

converges for any \( \theta \in \mathbb{R}^n \setminus \pi^{-1}(\mathcal{A}'_f) \) and \( s \in \text{int} \Delta_f \). In dimension one, the coameba \( \mathcal{A}'_f \) is a discrete set and for any \( \theta \in \mathbb{R}^n \setminus \pi^{-1}(\mathcal{A}'_f) \), \( \arg^{-1}(\theta) \) is the ray \( \{ re^{i\theta} : r \in \mathbb{R}_{>0} \} \).

We claim that for small enough \( |\Delta \theta| \)

\[ \int_{\arg^{-1}(\theta)} \frac{z^s}{f(z)} \, dz = \int_{\arg^{-1}(\theta+\Delta \theta)} \frac{z^s}{f(z)} \, dz. \]

To show this, we integrate along a closed path \( \gamma \) composed of three curves: the line that connects the origin and a point \( Re^{i\theta} \), the arc that connects \( Re^{i\theta} \) and \( Re^{i(\theta+\Delta \theta)} \) and the line that connects \( Re^{i(\theta+\Delta \theta)} \) and the origin (see figure 2.5).

![Figure 2.5: The integration path \( \gamma \)](image)

When \( |\Delta \theta| \) is small enough, \( f \) has no zeroes with argument between \( \theta \) and \( \theta + \Delta \theta \), i.e. \( \gamma \) does not encircle any zeroes. Moreover, as \( R \to \infty \), the integral along the arc approaches zero because the integrand rapidly approaches zero when \( |z| \to \infty \). By Cauchy’s integral theorem, the integrals along the two infinite rays are equal. This implies that the Mellin transform does not change if we change the direction of integration by \( \Delta \theta \) as long as \( \theta \) and \( \theta + \Delta \theta \) are in the same connected component.

To extend this argument to the \( n \)-dimensional case, we must take \( \theta \in \mathbb{R}^n \setminus \pi^{-1}(\mathcal{A}'_f) \) since the coameba need not be closed. We connect \( \theta \) and \( \theta + \Delta \theta \) by a piecewise linear path such that for any segment only one coordinate of \( \theta \) is changed. The claim now follows by successively applying the one-variable argument.

The following theorem grants us a nice analogy to Theorem 2.1.

**Theorem 2.6 (Mellin Inversion Formula)** For every connected component \( C \) of \( \mathbb{R}^n \setminus \pi^{-1}(\mathcal{A}'_f) \), \( 1/f \) may be represented as the following integral

\[ \frac{1}{f(z)} = \int_{\sigma+i\mathbb{R}^n} M^C_{1/f}(s) z^{-s} \, ds \quad (2.15) \]

which converges for all \( z \in \text{Arg}^{-1}(C) \). Here \( \sigma \) denotes an arbitrary point in \( \text{int} \Delta_f \) and \( M^C_{1/f}(s) \) denotes the \( C \)-directional Mellin transform as in (2.14) and Lemma 2.5.
Proof It suffices to show that for all \( s \in \sigma + i\mathbb{R}^n \) such that \( \sigma \in \text{int}\Delta_f \), the function

\[
x \mapsto \frac{e^{(s,x+i\theta)}}{f(e^x+i\theta)}
\]

is in the Schwartz space \( S(\mathbb{R}^n) \) of rapidly decreasing functions\(^9\) and the result will follow from the Fourier inversion formula (see Theorem 7.1.5 in \cite{Horm90b})\(^{10}\).

Assume, without loss of generality, that \( \theta = 0 \). From the proof of Theorem 2.2 we know that

\[
\left| \frac{e^{(s,x)}}{f(e^x)} \right| = \frac{|f(e^x)|}{|f(e^x)|}
\]

is an exponentially decreasing function. We need to show that the same holds for all of its partial derivatives. We have

\[
\frac{\partial}{\partial x_k} \left( \frac{e^{(\sigma,x)}}{f(e^x)} \right) = \frac{\sigma_k e^{(\sigma,x)} f(e^x) - e^{(\sigma,x)} f'(e^x) e^{x_k}}{f(e^x)^2} = \frac{\sigma_k e^{(\sigma,x)} f(e^x)}{f(e^x)^2} - \frac{e^{(\sigma+e_k,x)} f'_k(e^x)}{f(e^x)^2} \tag{2.16}
\]

where \( f'_k \) denotes \( \partial f/\partial x_k \). The first term on the right hand side is a constant multiple of the original function and the second term is of the form

\[
\sum_{\alpha \in A_f} \frac{\alpha_k c_{\alpha} e^{(\sigma+\alpha,x)}}{f(e^x)^2}.
\]

The Newton polytope of \( f^2 \) is \( \Delta_{f^2} = 2\Delta_f \), so \( \forall \alpha \in A_f \sigma + \alpha \in \text{int}\Delta_{f^2} \). This implies that the derivative (2.16) is a sum of finitely many terms, each of which satisfies the conditions of Theorem 2.2. By induction, all derivatives of \( e^{(s,x)}/f(e^x) \) are exponentially decreasing.

\[\square\]

Corollary 2.7 The connected components of \( \mathbb{R}^n \setminus \pi^{-1}(\mathbb{R}^n_f) \) are convex (and in bijective correspondence with distinct representations of \( 1/f \) as in (2.15)).

Proof Let \( C \) be a connected component of \( \mathbb{R}^n \setminus \pi^{-1}(\mathbb{R}^n_f) \). Consider the following two functions defined on the tube domain \( C + i\mathbb{R}^n \):

\[
\theta - i x \mapsto \frac{1}{f(e^{(\theta-x)})} = \frac{1}{f(e^{x+i\theta})}
\]

\[
\theta - i x \mapsto \int_{\sigma+i\mathbb{R}^n} M_{1/f}^C(s) e^{-(x+i\theta,s)} \, ds \tag{2.17}
\]

By Bochner’s theorem (see \cite{Boch38}), they can be analytically extended to the convex hull \( \text{conv}(C) + i\mathbb{R}^n \). By the analytic continuation principle, equality (2.15) extends as well. As \( \text{conv}(C) \) is open and \( C + i\mathbb{R}^n \) is the maximal open set on which we can define \( 1 / f(e^{x+i\theta}) \), it must be that \( \text{conv}(C) \subseteq C \), i.e. \( C \) is convex. The rest of the statement follows from the preceding theorems.

\[\square\]

Remark 2.6 The role of Bochner’s result here is analogous to the role of Abel’s result\(^{11}\) in proving convexity in Theorem 2.1.

\(^9\)\( S(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \} \) where \( \alpha \) and \( \beta \) are multi-indices.

\(^{10}\)The Fourier transform is an automorphism of \( S(\mathbb{R}^n) \) with inverse given by the Fourier inversion formula.

\(^{11}\)If a series \( \sum \alpha c_{\alpha} z^\alpha \) converges near two points \( \zeta \) and \( \eta \) in \( T^n \), it converges on \( \{ z \in T^n : \forall j \ |z_j| \leq |z_j| \leq |\eta_j| \} \).
2.1.3 Amœbas and Lopsidedness

Recall that Proposition 2.1 implies that in the special case of a linear polynomial $f(z) = \sum_{j=0}^n c_j z_j$ where $c_j \neq 0$, $x \in \mathcal{A}_f \Leftrightarrow \{|c_0|, |c_1| e^{x}, \ldots, |c_n| e^{nx} \}$ is not lopsided.

Let $f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$ be any Laurent polynomial. It makes sense to ask whether we have

$$\mathcal{A}_f = \{ x \in \mathbb{R}^n : \{|c_{\alpha}| e^{\alpha(x)} \}_{\alpha \in A_f} \text{ is not lopsided} \}$$

(2.18)

By Proposition 2.1, we always have inclusion from left to right. However, we need not have equality as the following example illustrates.

**Example 2.5** Consider a one-variable polynomial $f(z) = c_0 + c_1 z + \cdots + c_{d-1} z^{d-1} + z^d$.

Suppose $\xi_1, \ldots, \xi_d$ are its roots, indexed so that $|\xi_1| \leq \cdots \leq |\xi_d|$. Let $a_j := \log |\xi_j|$. Let $x \in \mathbb{R}^\alpha \setminus \mathcal{A}_f$. If (2.18) is true, we expect $\{ |c_0|, |c_1| e^{x}, \ldots, |c_{d-1}| e^{(d-1)x}, e^{dx} \}$ to be lopsided. If $x \in (-\infty, a_1)$ is very large and negative, $|c_0|$ will dominate other elements. Similarly, if $x \in (a_d, \infty)$ is very large and positive, $e^{dx}$ will dominate. However, when $x$ is close to an $a_j$ (i.e. the amœba), $\{ |c_0|, |c_1| e^{x}, \ldots, |c_d-1| e^{(d-1)x}, e^{dx} \}$ need not be lopsided. It is an easy exercise to choose suitable coefficients to demonstrate this.

However, it turns out that for any $x \in \mathbb{R}^\alpha \setminus \mathcal{A}_f$, we can find a Laurent polynomial $g(z) = \sum_{\alpha \in A_g} c_{\alpha} z^{\alpha}$ (which depends on $x$) such that $\mathcal{A}_f = \mathcal{A}_g$ and $\{ |c_{\alpha}| e^{\alpha(x)} \}_{\alpha \in A_g}$ is lopsided. We start with the one variable case. The idea is that if $f(z) = \prod_j (z - \xi_j)$ and assuming $|\xi_1| > \cdots > |\xi_d| > 0$, we consider $g(z) = \prod_j (z^k - \xi_j^k)$. When $k$ is large, we have

$$g(z) \approx (z^d)^k - (\xi_1 z^{d-1})^k + \cdots \pm (\xi_1 \ldots \xi_{d-1} z^1)^k \mp (\xi_1 \ldots \xi_d)^k.$$

Suppose $|\xi_{i+1}| < z < |\xi_i|$. When $n \to \infty$, we have $g(z)/(\xi_1 \ldots \xi_{d-1} z^{d-l})^k \to 1$. Therefore one term of $g(z)$ dominates others for sufficiently large $k$. Formally, this is rewritten in the following two technical lemmas whose proofs we omit (see [Purb06] for details).

**Lemma 2.8** Let $0 \leq \beta_0 < \beta_1 \leq \infty$, $\gamma \in [\sqrt{\beta_0/\beta_1}, 1)$ and $k \in \mathbb{Z}_{>0}$. If a polynomial $f(z) = \sum_{j=0}^d c_j z^{kj}$ ($c_j \neq 0$, $d \in \mathbb{Z}_{>0}$) has no roots in $\{ z \in \mathbb{C} : \beta_0 < |z| < \beta_1 \}$ then there is an $l$ such that $\forall \ z_0 \in \{ z \in \mathbb{C} : \gamma^{-1} \beta_0 \leq |z| \leq \gamma \beta_1 \}$ we have

$$\frac{|c_j z_0^{kj}|}{|c_l z_0^{kl}|} < \frac{\sum_{m \geq d-l} |z_0^{m}| (d_\gamma)^m}{2 - e^{d_\gamma}}$$

(2.19)

For a Laurent polynomial $f(z) = \sum_{j=d_1}^{d_2} c_j z^{kj}$ with $d_1 < 0 \leq d_2$ and $c_j \neq 0$ that satisfies the same condition, (2.19) holds with $d = d_2 - d_1$.

We omit the proof but note that $l$ can be determined. Namely, let $f_k(z) = \prod_{j=0}^d (z - \xi_j^k)$ where $|\xi_1| \geq \cdots \geq |\xi_d|$ and let $\xi_0 := \infty$ and $\xi_{d+1} := 0$. As $f(z)$ has no roots in $\{ z : \beta_0 < |z| < \beta_1 \}$, we have $|\xi_{i+1}| \leq \beta_0 < \beta_1 \leq |\xi_i|$ for some $l$ ($0 \leq l \leq d$). This $l$ satisfies (2.19). From this, the following result is obtained.

**Lemma 2.9** Let $0 \leq \beta_0 < \beta_1 \leq \infty$, $\gamma \in [\sqrt{\beta_0/\beta_1}, 1)$, $k \in \mathbb{Z}_{>0}$ and $c_0, D_0, c_1, D_1 \in \mathbb{Z}_{>0}$. Let $\{ f_k(z) \}_{k}$ be a family of polynomials (resp. Laurent polynomials) such that

1. $f_k$ has no roots in $\{ z : \beta_0 < |z| < \beta_1 \}$
(2) all terms of \( f_k \) are of the form \( c_{k,j}z^{kj} \), \( j \in \mathbb{Z} \)

(3) \( \deg(f_k) \leq c_0 k^{D_0+1} \) (resp. \( \maxdeg(f_k) - \mindeg(f_k) \leq c_0 k^{D_0+1} \))

If \( k \) is large enough that

\[ k \log \gamma^{-1} \geq (D_0 + D_1) \log(k) + \log(8/3c_0c_1), \]

then \( \{|c_{k,j}| e^{jx}\}_j \) is \( c_1 k^{D_1} \)-superlopsided for all \( x \in [\log(\gamma^{-1}\beta_0), \log(\gamma/\beta_1)] \).

Theorem 2.10 [Purb06] Let \( f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) and for \( k \in \mathbb{Z}_{>0} \) define

\[ f_k(z) = \prod_{l_1=0}^{k-1} \cdots \prod_{l_n=0}^{k-1} f(e^{2\pi i l_1/k} z_1, \ldots, e^{2\pi i l_n/k} z_n) = \sum_{\alpha \in A_k} c_{k,\alpha} z^{\alpha}. \quad (2.20) \]

Then \( \mathcal{A}_f = \mathcal{A}_{f_k} \) and for any \( x \in \mathbb{R}^{n} \setminus \mathcal{A}_f \) there is a \( k(x) \in \mathbb{Z}_{>0} \) such that for \( k \geq k(x) \) the set \( \{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}\}_{\alpha \in A_{f_k}} \) is lopsided.

The idea behind the proof is reducing the claim to the one variable case by fixing all but one variable of \( f_k \), applying the lemma to find a dominant term and showing that this implies that \( f_k \) has a dominant term as a Laurent polynomial in \( n \) variables. To continue, we will need several important facts.

Proposition 2.2 \( \mathcal{A}_f = \mathcal{A}_{f_k} \)

Proof Since \( f_k \) is a product of terms \( f_{l_1,\ldots,l_n}(z) := f(e^{2\pi i l_1/k} z_1, \ldots, e^{2\pi i l_n/k} z_n) \) and \( |e^{2\pi i l_1/k}| = \cdots = |e^{2\pi i l_n/k}| = 1 \), we have \( \mathcal{A}_{f_{l_1,\ldots,l_n}} = \mathcal{A}_f \) and hence \( \mathcal{A}_{f_k} = \mathcal{A}_{f_{l_1,\ldots,l_n}} = \mathcal{A}_f \).

\( \Box \)

Proposition 2.3 All terms of \( f_k \) are of the form \( cz^{k\alpha} := c z_1^{k\alpha_1} \cdots z_n^{k\alpha_n} \).

Proof \( f_k \) is clearly invariant under action of the cyclic group of roots of \( z^k - 1 \) given by \( (u_1, \ldots, u_n)(z_1, \ldots, z_n) := (u_1 z_1, \ldots, u_n z_n) \) and, therefore, so are all of its monomials. This is only possible if they are of the form \( cz^{k\alpha} \).

\( \Box \)

Definition 2.10 If \( \Delta \) is a polytope, its Ehrhart polynomial \( E_{\Delta}(t) \) is defined as \( \text{card}(t\Delta \cap \mathbb{Z}^n) \).

Its coefficients are generally not known, however, it can be bounded from above in terms of the volume of \( \Delta \). See [BM85].

Proposition 2.4 Suppose \( d = d(\Delta_f) \) is an upper bound for \( \text{card}(\mathbb{Z}^n \cap t\Delta_f)/t^n \). Then \( f_k \) has at most \( dk^{n(n-1)} \) terms.

Proof Note that \( \Delta_{f_k} = k^n \Delta_f \). By Proposition 2.3, the number of terms of \( f_k \) is at most the number of lattice points in \( \frac{1}{k}\Delta_{f_k} = k^{n-1}\Delta_f \) which is bounded by \( dk^{n(n-1)} \).

\( \Box \)

Lemma 2.11 Let \( f(z) = \sum_{\alpha \in A_f} c_{\alpha} z^{\alpha} \) be a Laurent polynomial and let \( x \in \mathbb{R}^{n} \setminus \mathcal{A}_f \). If \( \forall \zeta \in \text{Log}^{-1}(x) \) we have \( |f(\zeta)| < M \) then \( \forall \beta \ |c_{\beta}\zeta^{\beta}| < M \).
Proof Integrating \( M > |f(\zeta)| \) over \( \text{Log}^{-1}(x) \) gives

\[
M \geq \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} \left| \sum_{\alpha} c_{\alpha} e^{(\alpha,x+i\theta)} \right| d\theta_1 \ldots d\theta_n
\]

\[
\geq \frac{1}{(2\pi)^n} \int_{|z_1|=1} \ldots \int_{|z_n|=1} \left| \sum_{\alpha} c_{\alpha} e^{(\alpha,x)} z^{\gamma} \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n} \right|
\]

\[
= |c_\beta e^{(\beta,x)}|
\]

\[
= |c_\beta \zeta^\beta|
\]

\[
\square
\]

Proof (of Theorem 2.10) Let \( x \in \mathbb{R}^n \setminus \mathcal{A}_f = \mathbb{R}^n \setminus \mathcal{A}_{f_k} \) and suppose that its distance from the amoeba is at least \( \varepsilon = \varepsilon(x) > 0 \). Let \( d = d(\Delta_f) > 0 \) be an upper bound for \( E_{\Delta_f}(t)/t^n \) as above and let \( c = \max_j (\max(\pi_j(\Delta_f)) - \min(\pi_j(\Delta_f))) > 0 \) where \( \pi_j : \mathbb{R}^n \to \mathbb{R} \) denotes the projection map \( x \mapsto x_j \). We claim that if

\[
k\varepsilon > (n^2 - 1) \log(k) + \log(\frac{16}{3cd}),
\]

the set \( \{|c_{k,\alpha}|e^{(\alpha,x)}\}_{\alpha \in \mathcal{A}_{f_k}} \) is \( dk^{n(n-1)} \)-superlopsided.

Let \( \zeta \in \text{Log}^{-1}(x) \) and let

\[
f_{j,k,\zeta}(z) := f_k(\zeta_1, \ldots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n) = \sum_{\alpha_j \in \pi_j(A_{f_k})} c_{k,\alpha_j,\zeta} z^{k\alpha_j}.
\]

Note that

\[
\maxdeg(f_{j,k,\zeta}) - \mindeg(f_{j,k,\zeta}) = \max \pi_j(\Delta_{f_k}) - \min \pi_j(\Delta_{f_k})
\]

\[
= k^n (\max \pi_j(\Delta_f) - \min \pi_j(\Delta_f)).
\]

\( f_{j,k,\zeta} \) has no roots in the annulus \( \{z \in \mathbb{C} : e^{\gamma_j - \varepsilon} < |z| < e^{\gamma_j + r} \} \) because otherwise the distance between \( x \) and \( \mathcal{A}_{f_k} \) would be less than \( \varepsilon \). We now apply Lemma 2.9 to \( f_{j,k,\zeta} \) with \( \gamma = e^{\varepsilon}, c_0 = c, D_0 = n - 1, c_1 = 2d \) and \( D_1 = n(n-1) \). If \( k \) satisfies (2.21), it holds that

\[
\{|c_{k,\alpha_j,\zeta}|e^{(k\alpha_j,x)}\}_{\alpha_j \in \pi_j(A_{f_k})} \text{ is } 2dk^{n(n-1)} \text{-superlopsided}.
\]

As \( \text{Log}^{-1}(x) \) is connected, this does not depend on the choice of \( \zeta \). Let \( \nu_j \) denote the index of the dominating term and let \( \nu = (\nu_1, \ldots, \nu_n) \) so that we can write \( f_k \) as

\[
f_k(z) = c_{k,\nu} z^{k\nu} + \sum_{\alpha \in \mathcal{A}_{f_k}, \alpha \neq \nu} c_{k,\alpha} z^{k\alpha}.
\]

Let \( M = |c_{k,\nu} z^{k\nu}| \) and \( \mu = \max_{\alpha \neq \nu} |c_{k,\alpha} z^{k\alpha}| \). Since \( f_k \) has at most \( dk^{n(n-1)} \) terms, we are done if we show that \( M > dk^{n(n-1)} \mu \). For a fixed \( l \), the \( z^{k l} \)-term of \( f_{j,k,\zeta} \) is

\[
\sum_{\alpha \in \mathcal{A}_{f_k}, \alpha_j = l} c_{k,\alpha_j,\zeta} z^{k\alpha_j} = \sum_{\alpha \in \mathcal{A}_{f_k}, \alpha_j = l} c_{k,\alpha_1} z^{k\alpha_1} \ldots z^{k\alpha_{j-1}} z^{k\alpha_{j+1}} \ldots z^{k\alpha_n} z^{k\alpha_j}
\]

(2.24)
For \( l \neq \nu_j \), by (2.23), we have

\[
2dk^{n(n-1)} \left| \sum_{\alpha \in A_{f_k}} c_{k,\alpha} \zeta^{k\alpha} \right| < \sum_{\alpha \in A_{f_k}} \left| c_{k,\alpha} \zeta^{k\alpha} \right| \leq \sum_{\alpha \in A_{f_k}, \alpha_j = \nu_j} \left| c_{k,\alpha} \zeta^{k\alpha} \right| \leq M + \mu dk^{n(n-1)}
\]

Since this does not depend on \( \zeta \), by Lemma 2.11,

\[
2dk^{n(n-1)} |c_{k,\alpha} \zeta^{k\alpha}| < M + \mu dk^{n(n-1)}
\]

holds for all \( \alpha \in A_{f_k} \) such that \( \alpha_j = l \). As this holds for every \( j \), we have

\[
2dk^{n(n-1)} |c_{k,\alpha} \zeta^{k\alpha}| < M + \mu dk^{n(n-1)}
\]

whenever \( \alpha \neq \nu \). In particular, \( \mu = |c_{k,\alpha} \zeta^{k\alpha}| \) for some \( \alpha \neq \nu \). Hence

\[
2dk^{n(n-1)} \mu < M + \mu dk^{n(n-1)}
\]

i.e.

\[
M > 2dk^{n(n-1)} \mu
\]

as required.

\[ \square \]

**Remark 2.7** Note that the lower bound for \( k \) depends only on \( \varepsilon \) and \( \Delta_f \).

**Remark 2.8** If \( d' > d \), we can ensure \( d'k^{n(n-1)}\)-superlopsidedness by the same argument.

**Corollary 2.12** For a Laurent polynomial \( f \), the set of all \( x \) such that \( \{|c_\alpha| e^{\langle \alpha,x \rangle} \}_{\alpha \in A_f} \) is not (super)lopsided converges uniformly to \( \mathcal{A}_f \).

**Proof** This follows from the proof above. If \( x \notin \mathcal{A}_f = \mathcal{A}_{f_k} \), we have

\[
\varepsilon < \left( (n^2 - 1) \log(k) + \log(16/3cd) \right) / k
\]

which converges to 0 as \( k \to \infty \).

\[ \square \]

**2.1.4 Complement Components**

In this section, which is based on [FPT00], we shall deal with the problem of finding components of the complement of a hypersurface amoeba \( \mathcal{A}_f \). Let, as before, \( f(z) = \sum_{\alpha \in A_f} c_\alpha z^\alpha \). The first thing to notice is that slightly changing coefficients \( c_\alpha \) does not suddenly decrease the number of components of \( \mathbb{R}^n \setminus \mathcal{A}_f \). In other words:

**Proposition 2.5** The map \((c_\alpha) \mapsto \text{card}\{\text{components of } \mathbb{R}^n \setminus \mathcal{A}_f \}\) is lower semi-continuous.
Let $C$ be a connected component of amœba $\mathbb{R}^n \setminus \mathcal{A}_f$ and let $x \in C$. Choose $z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \log^{-1}(x)$. For every $1 \leq j \leq n$, fix all arguments $\theta_k$ with $k \neq j$ and consider the following loop

$$[0, 2\pi] \ni \theta_j \mapsto f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}).$$

We know by the classical argument principle that

$$\frac{1}{2\pi i} \int_{\log^{-1}(x)} \frac{f'_j(z)}{f(z)} \, dz_j$$

(2.25)

where $f'_j = \frac{\partial f}{\partial z_j}$, is always an integer. Moreover, it depends continuously on $x$ and $\theta$ which implies that it does not depend on $\theta$ and that it is constant on the connected component $C$. Thus we may define the following.

**Definition 2.11** The order of a connected component $C$ of $\mathbb{R}^n \setminus \mathcal{A}_f$ is defined as $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ where

$$\nu_j := \frac{1}{(2\pi i)^n} \int_{\log^{-1}(x)} \frac{zf'(z)}{f(z)} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}. \quad (2.26)$$

To justify this, simply rewrite (2.26) as

$$\nu_j = \frac{1}{(2\pi i)^n} \int_{[0,2\pi]^n} \frac{e^{x_j+i\theta_j}f'(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})} \, d\theta_1 \cdots d\theta_n$$

$$= \frac{1}{(2\pi i)^n} \int_{[0,2\pi]^n} \frac{e^{x_j+i\theta_j}f'(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})} \, d\theta_j \int_{[0,2\pi]^{n-1}} \prod_{k \neq j} d\theta_k$$

$$= \frac{1}{2\pi} \int_{[0,2\pi]^n} \frac{e^{x_j+i\theta_j}f'(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})}{f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})} \, d\theta_j$$

which is precisely (2.25).

**Remark 2.9** The order depends on the Laurent polynomial $f$, not its amœba $\mathcal{A}_f$. For example, it changes if we replace $f$ by $f^m$ for some $m \geq 2$.

**Lemma 2.13** Let $f$ be a Laurent polynomial, $C$ a connected component of $\mathbb{R}^n \setminus \mathcal{A}_f$, $x \in C$ and $\nu$ the order of $C$. For any $\mu \in \mathbb{Z}^n \setminus \{0\}$, $\langle \mu, \nu \rangle$ equals the number of zero-poles\footnote{i.e. the number of zeroes minus the number of poles counted with multiplicities. Note that when $f$ is a Laurent polynomial, the origin is the only pole and it is of order $-\mindeg f$.} of the one-variable Laurent polynomial

$$w \mapsto f(z_1w^{\mu_1}, \ldots, z_nw^{\mu_n})$$

inside the unit disc $|w| < 1$ where $z = (z_1, \ldots, z_n)$ is an arbitrary point in $\log^{-1}(x)$. 

Proof We know by the classical argument principle that the number of zero-poles of \(f(zw^\mu)\) in the unit disc is given by
\[
\frac{1}{2\pi i} \int_{|w|=1} \frac{df(zw^\mu)}{f(zw^\mu)}.
\] (2.27)

Image of the unit circle \(|w| = 1\) under \(f(zw^\mu)\) is a loop contained in \(\text{Log}^{-1}(x)\) which is homologous to \(\mu_1 \gamma_1 + \cdots + \mu_n \gamma_n\) where \(\gamma_j : [0, 2\pi] \ni t \mapsto (z_1, \ldots, z_{j-1}, z_j e^{it}, z_{j+1}, \ldots, z_n)\) (see [Rud69]). Therefore, we can rewrite (2.27) as
\[
\int_{|w|=1} \frac{df(zw^\mu)}{f(zw^\mu)} = \sum_{j=1}^{n} \mu_j \int_{|\zeta_j| = e^{it}} \frac{f_j'(\zeta)}{f(\zeta)} d\zeta_j
\]
\[
= 2\pi i \sum_{j=1}^{n} \mu_j \nu_j
\]
\[
= 2\pi i \langle \mu, \nu \rangle
\]
which completes the proof.

\[\square\]

**Proposition 2.6** The order of any connected component of \(\mathbb{R}^n \setminus \mathcal{A}_f\) is a point in \(\Delta_f\).

**Proof** It suffices to show that \(\langle \mu, \nu \rangle \leq \max_{\alpha \in \Delta_f} \langle \mu, \alpha \rangle\) for any \(\mu \in \mathbb{Z}^n \setminus \{0\}\). By Lemma 2.13, \(\langle \mu, \nu \rangle\) equals the number of zero-poles of the one-variable Laurent polynomial \(w \mapsto f(zw^\mu)\) in the unit disc. This number is bounded by the top degree of \(f(zw^\mu)\) \(\{\max_{\alpha \in \Delta_f} \langle \alpha, \mu \rangle\}\) which is precisely \(\max_{\alpha \in \Delta_f} \langle \alpha, \mu \rangle\).

\[\square\]

**Proposition 2.7** Two connected components of \(\mathbb{R}^n \setminus \mathcal{A}_f\) cannot have the same order.

**Proof** Let \(C\) and \(C'\) be two distinct connected components of \(\mathbb{R}^n \setminus \mathcal{A}_f\) and \(\nu\) and \(\nu'\) their respective orders. Fix two points \(x \in C \cap \mathbb{Q}^n\) and \(x' \in C' \cap \mathbb{Q}^n\). Note that the segment \([x, x']\) intersects \(\mathcal{A}_f\). Let \(x' - x = t\mu\) for some \(t \in \mathbb{Q}_{>0}\) and \(\mu \in \mathbb{Z}^n\). We will show that \(\langle \mu, \nu' \rangle > \langle \mu, \nu \rangle\). By Lemma 2.13, we know that these two numbers are equal to the number of zero-poles in the unit disc of \(w \mapsto f(z'w^\mu)\) and \(w \mapsto f(zw^\mu)\) respectively, where \(z' \in \text{Log}^{-1}(x')\) and \(z \in \text{Log}^{-1}(x)\). Note that \(z'j/zj = e^{it}\) and hence \(z'w^\mu = z(e^{it}w^\mu)\). This means that we can interpret \(\langle \mu, \nu' \rangle\) as the number of zero-poles of \(f(zw^\mu)\) inside the larger disc \(|w| = e^{it}\). \(f(zw^\mu)\) must have an additional zero in the annulus \(1 < |w| < e^{it}\); otherwise \([x, x']\) would not intersect \(\mathcal{A}_f\).

We are, therefore, justified in indexing the component by its order (and vice versa).

**Proposition 2.8** Let \(C\) be a component of \(\mathbb{R}^n \setminus \mathcal{A}_f\) and \(\nu \in \Delta_f\) its order. Then the normal cone \(\mathcal{C}_\nu\) is the recession cone of \(C\), i.e. \(C + \mathcal{C}_\nu \subset C'\) and no larger affine convex cone is in \(C\).

**Proof** Let \(x \in C\) and \(\mu \in \mathbb{Z}^n \setminus \{0\}\). We need to show that the ray \(x + \mu \mathbb{R}_{>0}\) does not intersect the ameoba \(\mathcal{A}_f\) if and only if \(\langle \mu, \nu \rangle = \max_{\alpha \in \Delta_f} \langle \mu, \alpha \rangle\) (recall Remark 2.2). Lemma 2.13 implies that the ray does not intersect \(\mathcal{A}_f\) precisely when \(w \mapsto f(zw^\mu)\) has all of its zeroes inside the unit disc. The claim now follows from the Fundamental Theorem of Algebra because the top degree of \(f(zw^\mu)\) is \(\max_{\alpha \in \Delta_f} \langle \mu, \alpha \rangle\) and \(\langle \mu, \nu \rangle\) counts zero-poles inside the unit disc.

\[\square\]
Proposition 2.9 Let $\nu \in \Delta_f \cap \mathbb{Z}^n$, $C$ a connected component of $\mathbb{R}^n \setminus \mathcal{A}_f$ and $z \in \Log^{-1}(C)$. If $|c_\nu z^\nu| > \left| \sum_{\alpha \in A_f \setminus \{\nu\}} c_\alpha z^\alpha \right|$ then $\nu$ is the order of $C$.

**Proof** Let $\nu_C$ denote the order of $C$. We show $\nu_C = \nu$ by showing that $\langle \mu, \nu_C \rangle = \langle \mu, \nu \rangle$ holds for every $\mu \in \mathbb{Z}^n \setminus \{0\}$. By Lemma 2.13, $\langle \mu, \nu_C \rangle$ counts zero-poles of $w \mapsto f(zw^\mu)$ inside the unit disc. As $\langle \mu, \nu \rangle$ does the same for $w \mapsto c_\nu z^\nu w^{(\mu, \nu)}$, the claim follows from Rouché’s Theorem.

Suppose $\nu$ is a vertex of $\Delta_f$. We can write

$$f(z) = c_\nu z^\nu \left( 1 + \sum_{\alpha \in A_f \setminus \{\nu\}} c_\alpha c_\nu^{-1} z^{\alpha - \nu} \right) = c_\nu z^\nu (1 + g(z)).$$

(2.28)

and, using the geometric series, construct the Laurent expansion

$$\frac{1}{f(z)} = c_\nu^{-1} z^{-\nu} (1 - g(z) + g^2(z) - \ldots)$$

(2.29)

**Proposition 2.10** There exists $y \in C_\nu$ such that (2.29) converges absolutely for any $z \in \Log^{-1}(y + C_\nu)$. In particular, $f(z) \neq 0$ for such $z$.

**Proof** Series (2.29) will converge absolutely for any $z$ such that $|g(z)| < 1$. We have

$$|g(z)| = \left| \sum_{\alpha \in A_f \setminus \{\nu\}} c_\alpha c_\nu^{-1} z^{\alpha - \nu} \right| \leq \sum_{\alpha \in A_f \setminus \{\nu\}} |c_\alpha c_\nu^{-1}| e^{(\alpha - \nu, x)}$$

where $x = \Log(z)$. If we choose $y \in C_\nu$ such that $\langle y, \alpha - \nu \rangle \ll 0$, we will have $|g(z)| < 1$ whenever $x \in y + C_\nu$.

Figure 2.6: Recession cones for $f(z, w) = z^3 w^4 + z^5 + 40z^3 w^2 + z^3 w + 80z^2 w + 1$

Figure 2.6 illustrates (translated) normal cones of $\Delta_f$ that correspond to components of $\mathbb{R}^n \setminus \mathcal{A}_f$ where $f$ is as in Example 2.3. The two degenerate cones correspond to the two bounded components, while the three full-dimensional cones correspond to the three vertices. The correspondence between certain lattice points in $\Delta_f$ and
components of $\mathbb{R}^n \setminus A_f$ suggests a kind of duality. We shall explore this in the next chapter.

**Theorem 2.14** [FPT00] The number of connected components of $\mathbb{R}^n \setminus A_f$ is at least equal to the number of vertices of $\Delta_f$ and at most equal to $\text{card}(\Delta_f \cap \mathbb{Z}^n)$.

**Proof** The lower bound follows immediately from Propositions 2.9 and 2.10. The upper bound follows immediately from Propositions 2.6 and 2.7.

**Proposition 2.11** [Rull00] For any lattice polytope $\Delta$ and any $\mathcal{N} \subset \Delta_f \cap \mathbb{Z}^n$, it is possible to construct a Laurent polynomial $f$ such that $\Delta_f = \Delta$ and $\mathcal{N}$ is the set of orders of connected components of $\mathbb{R}^n \setminus A_f$.

**Corollary 2.15** Both bounds in Theorem 2.14 can be achieved.

**Definition 2.12** Amœbas achieving the lower bound are called *solid* while the amœbas achieving the upper bound are called *full*.

**Definition 2.13** A Laurent polynomial $f$ is called *maximally sparse* if its support $A_f$ equals the set of vertices of its Newton polytope $\Delta_f$.

**Conjecture 1** (Passare-Rullgård) Maximally sparse polynomials have solid amœbas.

### 2.2 More General Case

It is natural to extend the definitions 2.6 and 2.7 to include more than just hypersurfaces.

**Definition 2.14** Let $I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a proper ideal and $\mathcal{V}_f := \{ z \in \mathbb{T}^n : \forall f \in I \quad f(z) = 0 \}$ its zero set. The amœba $A_f$ (resp. coamœba $A'_f$) of the ideal $I$ is defined as $A_f := \text{Log}(\mathcal{V}_f)$ (resp. $A'_f := \text{Arg}(\mathcal{V}_f)$).

**Remark 2.10** Let $f_1, \ldots, f_k$ be a finite set of generators of $I$. Recall that $\mathcal{V}_f = \cap_{j=1}^k \mathcal{V}_{f_j}$ but note that $A_f \neq \cap_{j=1}^k A_{f_j}$ and $A'_f \neq \cap_{j=1}^k A'_{f_j}$ in general\(^{13}\). Again, it would be more accurate to index the amœba and the coamœba by the algebraic set, however the following proposition justifies our notation.

**Proposition 2.12** With notations as above, the following holds:

$$A_f = \bigcap_{f \in I} A_f, \quad A'_f = \bigcap_{f \in I} A'_f$$  \hspace{1cm} (2.30)

**Proof**

1) As $\forall f \in I \quad \mathcal{V}_f \subset \mathcal{V}_f$, we have $\forall f \in I \quad A_f \subset A_f, \quad A'_f \subset A'_f$ and therefore inclusions

$$A_f \subset \bigcap_{f \in I} A_f, \quad A'_f \subset \bigcap_{f \in I} A'_f$$

\(^{13}\)The dimension need not decrease when we take intersection $A_f \cap A'_g$ but it does decrease when we take $\mathcal{V}_f \cap \mathcal{V}_g$ (and therefore its image under Log or Arg). E.g. consider what happens when $\dim \mathcal{V}_{f_j} < n/2$. 25
2) For a Laurent polynomial \( f(z) = \sum_\alpha c_\alpha z^\alpha \) define \( \widetilde{f}(z) := \sum_\alpha \overline{c_\alpha} z^\alpha \) where \( \overline{c_\alpha} \) denotes the complex conjugate of \( c_\alpha \). For any \( x \in \mathbb{R}^n \setminus \mathcal{A}_f \) define

\[
 f_x(z) := \sum_{j=1}^k \widetilde{f}_j(e^{2\pi i z_{-1}^j}, \ldots, e^{2\pi i z_n^j}) f_j(z_1, \ldots, z_n) \in I \tag{2.31}
\]

If \( z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \text{Log}^{-1}(x) \), we have

\[
 f_x(z) = \sum_{j=1}^k \overline{f}_j(z_1, \ldots, z_n) f_j(z_1, \ldots, z_n)
\]

\[
 = \sum_{j=1}^k \overline{f}_j(z_1, \ldots, z_n) f_j(z_1, \ldots, z_n)
\]

\[
 = \sum_{j=1}^k |f_j(z)|^2 > 0
\]

Therefore \( x \in \mathbb{R}^n \setminus \mathcal{A}_f \) and hence \( \mathbb{R}^n \setminus \mathcal{A}_f \subset \bigcup_{f \in I} (\mathbb{R}^n \setminus \mathcal{A}_f) \). Taking complements gives \( \bigcap_{f \in I} \mathcal{A}_f \subset \mathcal{A}_f \). Analogously, for any \( \theta \in (\mathbb{S}^1)^n \setminus \mathcal{A}_f' \) define

\[
 f_\theta(z) := \sum_{j=1}^k \overline{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n) f_j(z_1, \ldots, z_n) \in I. \tag{2.32}
\]

If \( z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \text{Arg}^{-1}(\theta) \), we have \( f_\theta(z) = \sum_{j=1}^k |f_j(z)|^2 > 0 \) and therefore \( (\mathbb{S}^1)^n \setminus \mathcal{A}_f' \subset \bigcup_{f \in I} (\mathbb{S}^1)^n \setminus \mathcal{A}_f' \). Taking complements gives \( \bigcap_{f \in I} \mathcal{A}_f' \subset \mathcal{A}_f' \).

Remark 2.11 This also holds for ideals of \( \mathbb{C}[Z_1, \ldots, Z_n] \). We can choose suitable monomials \( m_1(z) \) and \( m_2(z) \) such that \( m_1(z) \overline{f}_j(e^{2\pi i z_{-1}^j}, \ldots, e^{2\pi i z_n^j}) \) and \( m_2(z) \overline{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n) \) are polynomials and use

\[
 f_x(z) = \sum_{j=1}^k m_1(z) \overline{f}_j(e^{2\pi i z_{-1}^j}, \ldots, e^{2\pi i z_n^j}) f_j(z_1, \ldots, z_n)
\]

and

\[
 f_\theta(z) = \sum_{j=1}^k m_2(z) \overline{f}_j(e^{-2i\theta_1} z_1, \ldots, e^{-2i\theta_n} z_n) f_j(z_1, \ldots, z_n)
\]

in (2.31) and (2.32) respectively.

Theorem 2.16 Let \( I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) be a proper ideal. Then \( x \in \mathbb{R}^n \) is in \( \mathcal{A}_f \) if and only if \( \{ |c_\alpha| e^{(\alpha,x)} \}_{\alpha \in \mathcal{A}_f} \) is not (super)lopsided for any \( f \in I \).

Proof
1) If \( x \in \mathcal{A}_f \), we cannot have lopsidedness because \( \forall f \in I \quad x \in \mathcal{A}_f \).
2) If \( x \not\in \mathcal{A}_f \) then \( x \not\in \mathcal{A}_f' \). Now apply Theorem 2.10 to \( f_x \).
Chapter 3

From Amœbas to Tropical Geometry

3.1 The Ronkin Function

Definition 3.1 Let \( f \) be a holomorphic function and \( \Omega \subset \mathbb{R}^n \) a connected open set. The Ronkin function (see \cite{Ronk74}) \( R_f : \Omega \to \mathbb{R}^n \) is defined in the following way

\[
R_f(x) := \frac{1}{(2\pi i)^n} \int_{\log^{-1}(x)} \log |f(z_1, \ldots, z_n)| \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}
\]

\[
= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log |f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})| d\theta_1 \cdots d\theta_n
\]

(3.1)

It can be viewed as a generalization of the function that appears in Jensen’s formula\(^1\).

Theorem 3.1 \cite{Ronk01} Let \( f \) be a Laurent polynomial. Then \( R_f \) is convex and affine on connected components of \( \mathbb{R}^n \setminus \mathcal{A}_f \). If \( C \) is a component of order \( \nu_C \), we have

\[
\forall x \in C \ R_f(x) = \langle \nu_C, x \rangle + \tau_C \text{ for some constant } \tau_C \in \mathbb{R}.
\]

Proof That \( R_f \) is convex follows from the fact that \( \log |f| \) is plurisubharmonic (see Theorem 2.12 and Corollary 1 in \cite{Ronk74}).

Let \( C \) be a connected component of \( \mathbb{R}^n \setminus \mathcal{A}_f \), \( x \in C \) and \( z = (e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}) \in \log^{-1}(x) \). For \( j \in \{1, \ldots, n\} \), consider

\[
\frac{\partial}{\partial x_j} \log |f| = \frac{1}{2} \frac{\partial}{\partial x_j} \log |f|^2
\]

\[
= \frac{1}{2} \frac{\partial}{\partial x_j} \log f \bar{f}
\]

\[
= \frac{1}{2} \frac{\partial}{\partial x_j} \left( \log f + \log \bar{f} \right)
\]

\[
= \frac{1}{2} \left( \left( \frac{\partial}{\partial z_j} \log f + \frac{\partial}{\partial z_j} \log f \right) \frac{\partial z_j}{\partial x_j} + \left( \frac{\partial}{\partial \bar{z}_j} \log f + \frac{\partial}{\partial \bar{z}_j} \log f \right) \frac{\partial \bar{z}_j}{\partial x_j} \right)
\]

\[
= \frac{1}{f} \left( \frac{f'_j}{f} \frac{\partial z_j}{\partial x_j} + \frac{\bar{f}_j}{\bar{f}} \frac{\partial \bar{z}_j}{\partial x_j} \right)
\]

(3.2)

\(^1\)Suppose \( f \) is an analytic function in a region of \( \mathbb{C} \) that contains a closed disc \( D(0, r) \), \( \xi_1, \ldots, \xi_k \) are zeroes of \( f \) in the open disc \( D(0, r) \) (counted with multiplicities) and \( f(0) \neq 0 \). Jensen’s formula states that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt = \log |f(0)| + \sum_{j=1}^k \log \frac{1}{r_i}.
\]

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In the last step we used the fact that $f$ is a Laurent polynomial so that $\overline{f(z)} = f(\bar{z})$. Since $z_j = e^{x_j + i\theta_j}$, we have $\partial z_j / \partial x_j = z_j$ and $\partial \overline{z_j} / \partial x_j = \overline{z_j}$. (3.2) now becomes

$$\frac{\partial}{\partial x_j} \log |f| = \frac{1}{2} \left( \frac{f'_j}{f} z_j + \frac{\overline{f'_j}}{f} \overline{z_j} \right)$$

$$= \frac{1}{2} \left( \frac{f'_j}{f} z_j + \left( \frac{f'_j}{f} \overline{z_j} \right) \right)$$

$$= \text{Re} \left( \frac{f'_j}{f} z_j \right)$$

(3.3)

This implies\(^2\) that $\partial R_f / \partial x_j$ is precisely the real part of the integral in (2.26), i.e. equals $\nu_{C,j}$ for any $x \in C$. Hence for $x \in C$ we have $R_f(x) = \langle \nu_C, x \rangle + \tau_C$ where $\tau_C$ is some real constant.

For all components $C$ of $\mathbb{R}^n \setminus \mathcal{A}_f$, let $\tau_C := R_f(x) - \langle \nu_C, x \rangle$. Consider the convex function

$$p_{R_f}(x) = \max_C (\langle \nu_C, x \rangle + \tau_C)$$

(3.4)

and the corresponding tropical Laurent polynomial $p_{R_f} := \bigboxplus_C \tau_C \boxtimes x$. Note that $p_{R_f} \leq R_f$ with equality holding on $\mathbb{R}^n \setminus \mathcal{A}_f$.

**Definition 3.2** The spine $\mathcal{S}_f$ of the amoeba $\mathcal{A}_f$ is the tropical hypersurface $\mathcal{V}_{p_{R_f}}$, i.e. the set of points in $\mathbb{R}^n$ for which $p_{R_f}$ is not differentiable.

The spine turns out to be dual to a certain subdivision of the Newton polytope. We make this precise in what follows (see [PR04]).

### 3.2 Convex Subdivisions

**Definition 3.3** Let $A \subset \mathbb{R}^n$ be a convex set. A collection $\mathcal{I}$ of non-empty closed convex subsets (called cells) of $A$ is called a convex subdivision of $A$ if it satisfies the following conditions.

1. $\bigcup_{\gamma \in \mathcal{I}} \gamma = A$;
2. If $\gamma, \delta \in \mathcal{I}$ are such that $\gamma \cap \delta \neq \emptyset$ then $\gamma \cap \delta \in \mathcal{I}$
3. If $\gamma \in \mathcal{I}$ and $\delta \subset \gamma$ then $\delta \in \mathcal{I}$ if and only if $\delta$ is a face of $\gamma$

By a face of $\gamma$ we mean a set $\{x \in \gamma : \langle \xi, x \rangle = \sup_{y \in \gamma} \langle \xi, y \rangle\}$ for some $\xi \in \mathbb{R}^n$. We say that $\mathcal{I}$ is polytopal if all $\gamma \in \mathcal{I}$ are polytopes.

A face $\delta$ of $\gamma$ has a strictly lower dimension than $\gamma$ which implies that any chain $\gamma_1 \supset \gamma_2 \supset \ldots$ in $\mathcal{I}$ stabilizes and that intersection of any collection of sets in $\mathcal{I}$ is also in $\mathcal{I}$.

To any pair $\delta \subset \gamma$ in $\mathcal{I}$ we shall associate a convex cone, defined as

$$C(\delta, \gamma) = \{ t(x - y) : x \in \gamma, y \in \delta, t \geq 0 \}$$

(3.5)

\(^2\frac{\partial}{\partial x} \int F \, d\eta = \int \frac{\partial}{\partial x} F \, d\eta\) when integrating over a set that does not depend on $x$.\[^2\]
Definition 3.4 Let $A, B \subset \mathbb{R}^n$ be two convex sets and $\mathcal{S}$ and $\mathcal{S}'$ be some convex subdivisions of $A$ and $B$ respectively. $\mathcal{S}$ and $\mathcal{S}'$ are called dual (to each other) if there is a bijection $\mathcal{S} \rightarrow \mathcal{S}'$, $\gamma \mapsto \gamma^*$ such that:

1. $\forall \gamma, \delta \in \mathcal{S} \; \delta \subset \gamma \iff \gamma^* \subset \delta^*$;
2. If $\delta \subset \gamma$ then cones $C(\delta, \gamma)$ and $C(\gamma^*, \delta^*)$ are polar\(^3\) (to each other).

Notice from the definition of our associated cone in (3.5) that $C(\gamma, \gamma)$ is just the linear subspace spanned by $\gamma$ (after translating it to the origin). The second condition above implies that $\gamma$ and $\gamma^*$ are orthogonal and that $\dim \gamma + \dim \gamma^* = n$.

We will show that the function $p_{R_f}$ as defined in (3.4) determines a convex subdivision of $\mathbb{R}^n$ while its Legendre-Fenchel transform $\tilde{p}_{p_{R_f}}(\xi) = \sup_{x \in \mathbb{R}^n}(\langle \xi, x \rangle - p_{R_f}(x))$ (recall Section 1.4) determines a dual subdivision of its Newton polytope. Note that $\Delta_f = \Delta_{p_{R_f}}$ because, as we have seen in the previous chapter, there is a unique component of $\mathbb{R}^n \backslash \mathcal{A}_f$ for each vertex of $\Delta_f$ and $\Delta_f$ is the convex hull of its vertices.

Lemma 3.2 The set of $\xi \in (\mathbb{R}^n)^* \cong \mathbb{R}^n$ for which the function

$$
\mathbb{R}^n \ni x \mapsto p_{R_f}(x) - \langle \xi, x \rangle \tag{3.6}
$$

is bounded from below equals $\Delta_f$.

Proof
1) Suppose $\xi \in \Delta_f$. Then $\xi$ can be written as a convex combination of vertices of $\Delta_f$, i.e. $\xi = \sum_{\sigma \in V_f} t_{\sigma} \sigma$ where $V_f$ denotes the set of vertices and $t_{\sigma} \geq 0$ are constants such that $\sum_{\sigma} t_{\sigma} = 1$. For every $\sigma \in V_f$ we have

$$p_{R_f}(x) - \langle \sigma, x \rangle \geq \tau_{C_\sigma},$$

where $C_\sigma$ denotes the connected component of $\mathbb{R}^n \backslash \mathcal{A}_f$ of order $\sigma$. This implies that

$$p_{R_f}(x) - \langle \xi, x \rangle = p_{R_f}(x) - \sum_{\sigma \in V_f} t_{\sigma} \langle \sigma, x \rangle$$

$$= \sum_{\sigma \in V_f} t_{\sigma} p_{R_f}(x) - \sum_{\sigma \in V_f} t_{\sigma} \langle \sigma, x \rangle$$

$$= \sum_{\sigma \in V_f} t_{\sigma} (p_{R_f}(x) - \langle \sigma, x \rangle)$$

$$\geq \sum_{\sigma \in V_f} t_{\sigma} \tau_{C_\sigma} \tag{3.7}$$

---

\(^3\)If $C$ is a convex cone, its dual cone $C^*$ is defined as $\{ y \in \mathbb{R}^n : \forall x \in C \; \langle y, x \rangle \geq 0 \}$. (This can be defined for any set, not just a convex cone.) When $C$ is closed, we have $C^{**} = C$. The cone $-C^*$ is called the polar cone of $C$. 

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2) Suppose \( \xi \notin \Delta_f \) and let \( \sigma \) be a vertex of \( \Delta_f \) such that \( \xi \notin \sigma - C_\sigma^* \) where \( C_\sigma^* \) denotes the dual cone of \( C_\sigma \). \( \sigma - C_\sigma^* \) is the smallest closed affine convex cone with vertex at \( \sigma \) that contains \( \Delta_f \). Because \( \xi - \sigma \) is strictly outside of the cone \( -C_\sigma^* \), it follows that the hyperplane orthogonal to \( \xi - \sigma \) intersects the interior of \( C_\sigma \) (see figure 3.1). This means that we can find a point \( x' \in C_\sigma \) such that

\[
\langle \xi - \sigma, x' \rangle > 0
\]

If \( x \in C_\sigma \) then \( x + tx' \in C_\sigma \) for any \( t > 0 \) by Proposition 2.8. We therefore have

\[
p_{R_f}(x + tx') - \langle \xi, x + tx' \rangle = -\langle \xi - \sigma, x + tx' \rangle + \tau_{C_\sigma} = -t\langle \xi - \sigma, x' \rangle - \langle \xi - \sigma, x \rangle + \tau_{C_\sigma} \rightarrow -\infty
\]
when \( t \rightarrow \infty \), i.e. (3.6) is not bounded from below.

Figure 3.1: Position of \( \xi \notin \Delta_f \) with respect to \( \Delta_f \)

Hence we may define a function \( H : \Delta_f \times \mathbb{R}^n \rightarrow \mathbb{R} \) in the following way.

\[
H(\xi, x) := p_{R_f}(x) + \bar{p}_{R_f}(\xi) - \langle \xi, x \rangle = \sup_{y \in \mathbb{R}^n} (\langle \xi, y \rangle - p_{R_f}(y)) - (\langle \xi, x \rangle - p_{R_f}(x))
\]

(3.8)

**Remark 3.1** This function is non-negative and convex in each argument when the other argument is fixed. Moreover, for \( x, y \in \mathbb{R}^n \) and \( \xi, \eta \in \Delta_f \) we have

\[
\langle \xi - \eta, x - y \rangle = -H(\xi, x) + H(\xi, y) + H(\eta, x) - H(\eta, y)
\]

(3.9)
Proposition 3.1 For any $x \in \mathbb{R}^n$ there is a point $\xi = \xi_x \in \Delta_f$ such that $H(\xi, x) = 0$.

Proof By Theorem 1.1, we can write $p_{R_f}(x) = \max_C (\langle \nu_C, x \rangle - \tilde{p}_{R_f}(\nu_C))$. If $x \in \mathbb{R}^n$ is fixed, we have $p_{R_f}(x) = \langle \nu_C, x \rangle - \tilde{p}_{R_f}(\nu_C)$ for some $C$. \qed

Lemma 3.3 For any $\xi \in \Delta_f$ the function (3.6) achieves its infimum at some $x \in \mathbb{R}^n$.

Proof If $\xi \in \text{int}\Delta_f$ we know by the argument above that (3.6) is bounded from below. If we let $||x|| \to \infty$, we have $p_{R_f}(x) - \langle \xi, x \rangle \to \infty$. This is because in every direction of $x$, for $||x||$ sufficiently large, there is a $\nu_C \not\in \text{int}\Delta_f$, i.e. $C_\nu$ and $C_{\nu'}$ are unbounded (recall Remark 2.2 and Proposition 2.8), such that $p_{R_f}(x) = \langle \nu_C, x \rangle + \tau_C$ dominates $\langle \xi, x \rangle$. Hence we can find a (sufficiently large) compact set on which the function (3.6) is bounded from below which implies that it achieves its infimum.

On the other hand, if $\xi \in \text{relint}(\Gamma)$ where $\Gamma = \{ \xi \in \Delta_f : \langle \xi, y \rangle = \max_{\alpha \in \Delta_f} (\alpha, y) \}$ is a face of $\Delta_f$ and $y \in \mathbb{R}^n \setminus \{0\}$, we consider the truncated Laurent polynomial $f_\Gamma$ (recall (2.2)). We have

$$p_{R_f}(x) = \max_{C : \nu_C \in \Gamma} (\langle \nu_C, x \rangle + \tau_C)$$

Clearly $p_{R_f} \geq p_{R_f^n}$. By the arguments above, the function $x \mapsto p_{R_f^n}(x) - \langle \xi, x \rangle$ is bounded from below and achieves its infimum at some point $x_0$ and hence at all points $x_0 + ty$ for $t \in \mathbb{R}$. There are at most finitely many $\nu_C \not\in \Gamma$ for which $\langle \nu_C, x_0 \rangle + \tau_C > p_{R_f^n}(x_0)$. For any such $\nu_C$ we have

$$p_{R_f^n}(x_0 + ty) - \langle \nu_C, x_0 + ty \rangle \to \infty$$

as $t \to \infty$. Hence for sufficiently large $t$ we have $p_{R_f^n}(x_0 + ty) = p_{R_f}(x_0 + ty)$. Since $p_{R_f}(x) - \langle \xi, x \rangle \geq p_{R_f^n}(x) - \langle \xi, x \rangle$, this implies that $p_{R_f}(x) - \langle \xi, x \rangle$ achieves its infimum at $x_0 + ty$ for sufficiently large $t$.

If $\xi = \nu_C$ is a vertex of $\Delta_f$, then $p_{R_f}(x) - \langle \xi, x \rangle$ achieves its infimum in the subset of $\mathbb{R}^n$ for which $p_{R_f}(x) = \langle \nu_C, x \rangle + \tau_C$, i.e. $p_{R_f}(x) - \langle \xi, x \rangle = \tau_C$. \qed

Corollary 3.4 For every $\xi \in \Delta_f$ there is a point $x = x_\xi \in \mathbb{R}^n$ such that $H(\xi, x) = 0$.

Lemma 3.5 If $x, y \in \mathbb{R}^n$, there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi + ty)$ is a linear function. If $\xi \in \Delta_f$ and $\eta \in \mathbb{R}^n$ are such that $\xi + t\eta \in \Delta_f$ for small $t > 0$, then there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi + t\eta, x)$ is a linear function.

Proof Let $N$ denote the set of orders of components of $\mathbb{R}^n \setminus \Delta_f$. For $x \in \mathbb{R}^n$, let $A = \{ \nu_C : p_{R_f}(x) = \langle \nu_C, x \rangle + \tau_C \}$, $B = \mathbb{R} \setminus A$, and

$$p_A(x) := \max_{\nu_C \in A} (\langle \nu_C, x \rangle + \tau_C), \quad p_B(x) := \max_{\nu_C \in B} (\langle \nu_C, x \rangle + \tau_C).$$

Clearly $p_{R_f}(x) = p_A(x) > p_B(x)$ and $p_{R_f}(y) = p_A(y)$ when $y$ is in a neighbourhood of $x$. Moreover, we have

$$p_A(x + ty) = \max_{\nu_C \in A} (\langle \nu_C, x + ty \rangle + \tau_C)$$

$$= \max_{\nu_C \in A} (\langle \nu_C, x \rangle + \tau_C + t\langle \nu_C, y \rangle)$$

$$= p_{R_f}(x) + t \max_{\nu_C \in A} \langle \nu_C, y \rangle$$

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To prove the second assertion, suppose $\xi \in \Delta_f$ and $\eta \in \mathbb{R}^n$. Let $x' \in \mathbb{R}^n$ be such that $H(\xi, x') = p_{R_f}(x') + \bar{p}_{R_f}(\xi) - \langle \xi, x' \rangle = 0$ and $\langle \eta, x' \rangle$ as large as possible. Note that

$$\bar{p}_{R_f}(\xi + t\eta) \geq \langle \xi + t\eta, x' \rangle - p_{R_f}(x')$$

$$= \bar{p}_{R_f}(\xi) + t\langle \eta, x' \rangle.$$

Since $p_{R_f}(x) \geq p_A(x)$ and the Legendre-Fenchel transform reverses inequalities, we also have

$$\bar{p}_{R_f}(\xi + t\eta) \leq \bar{p}_A(\xi + t\eta)$$

$$= \sup_{{x \in \mathbb{R}^n}} ((\xi + t\eta, x) - p_A(x))$$

$$= \sup_{{x \in \mathbb{R}^n}} ((\xi, x) - p_A(x) + t\langle \eta, x' \rangle$$

$$= \sup_{{x \in \mathbb{R}^n}} ((\xi, x) - p_{R_f}(x) + t\langle \eta, x' \rangle$$

$$= \bar{p}_{R_f}(\xi) + t\langle \eta, x' \rangle.$$ 

$\Box$

Let $\mathcal{K}$ denote the collection of all sets $K_\xi = \{x \in \mathbb{R}^n : H(\xi, x) = 0\}$. Let $\mathcal{K}'$ denote the collection of all sets $\kappa_x = \{\xi \in \Delta_f : H(\xi, x) = 0\}$.

**Theorem 3.6** $\mathcal{K}$ and $\mathcal{K}'$ as defined above are dual polytopal convex subdivisions of $\mathbb{R}^n$ and $\Delta_f$ respectively, with the correspondence given by

$$K \mapsto K^* = \bigcap_{x \in K} \kappa_x = \{\xi \in \Delta_f : \forall x \in K \ H(\xi, x) = 0\};$$

$$\kappa \mapsto \kappa^* = \bigcap_{\xi \in \kappa} K_\xi = \{x \in \mathbb{R}^n : \forall \xi \in \kappa \ H(\xi, x) = 0\}$$  \hfill (3.10)

**Proof** It follows easily from non-negativity and convexity in each variable of $H(\xi, x)$ that $K_\xi$ and $\kappa_x$ are convex sets. Corollary 3.4 and Proposition 3.1 imply that they are non-empty and that

$$\bigcup_{\xi \in \Delta_f} K_\xi = \mathbb{R}^n, \quad \bigcup_{x \in \mathbb{R}^n} \kappa_x = \Delta_f$$

Hence $\mathcal{K}$ and $\mathcal{K}'$ satisfy condition (1) of Definition 3.3.

If $K_{\xi_1} \cap K_{\xi_2} \neq \emptyset$, we claim that $K_{\xi_1} \cap K_{\xi_2} = K_{(\xi_1 + \xi_2)/2}$. Suppose $x \in K_{(\xi_1 + \xi_2)/2}$. Then $H(\xi_1, x) = H(\xi_2, x) = 0$ and by non-negativity and convexity (in $\xi$), it follows that $H(\xi_1 + \xi_2, x) = 0$, i.e. $x \in K_{(\xi_1 + \xi_2)/2}$. Conversely, suppose $x \in K_{(\xi_1 + \xi_2)/2}$ and let $y \in K_{\xi_1} \cap K_{\xi_2}$. Then $H(\xi_1 + \xi_2, x) = 0$ and, by the preceding argument, $H(\xi_1, y) = H(\xi_2, y) = 0 = H(\xi_1 + \xi_2, y)$. Applying (3.9) to $\xi = \xi_1 + \xi_2$ and $\eta = \xi_1$ (and $\eta = \xi_2$) gives $\frac{1}{2}(\xi_1 - \xi_2, x - y) = -H(\xi_1, x) = H(\xi_2, x)$. $H$ is non-negative and convex so $H(\xi_1, x) = H(\xi_2, x) = 0$, i.e. $x \in K_{\xi_1} \cap K_{\xi_2}$. Hence $\mathcal{K}$ satisfies condition (2) of Definition 3.3. The proof is analogous for $\mathcal{K}'$.

If $K_{\xi_1} \subset K_{\xi_2}$, let $\eta = \xi_1 - \xi_2$ and consider

$$L = \left\{ x \in K_{\xi_2} : \langle \eta, x \rangle = \sup_{y \in K_{\xi_2}} \langle \eta, y \rangle \right\}$$
which is a face of $K_{\xi_2}$. We claim that $K_{\xi_1} = L$. Let $x \in K_{\xi_1}$ and $y \in K_{\xi_2}$. By (3.9), it follows that $\langle \eta, x - y \rangle = H(\xi_1, y) \geq 0$, i.e. $x \in L$. If $x \in L$ and $y \in K_{\xi_1} \subset L$, we have $\langle \eta, x - y \rangle = 0 = -H(\xi_1, x)$, i.e. $x \in K_{\xi_1}$. Hence $K_{\xi_1} = L$.

Conversely, let $\xi_2 \in \Delta_f$ and let $L = \{ x \in K_{\xi_2} : \langle \eta, x \rangle = \sup_{y \in K_{\xi_2}} \langle \eta, y \rangle \}$ be a face of $K_{\xi_2}$. By Lemma 3.5, there is an $\epsilon > 0$ such that $[0, \epsilon] \ni t \mapsto H(\xi_2 + t\eta, x)$ is linear. Let $\xi_1 = \xi_2 + \frac{\epsilon}{2} \eta$. We claim that $L = K_{\xi_1}$.

For $x \notin K_{\xi_2}$, we have $H(\xi_2, x) > 0$. Lemma 3.5 implies
\[
H(\xi_1, x) = H(\xi_2, x) + \frac{\epsilon}{2} \langle \eta, x' - x \rangle > 0
\]
because $x'$ is any point in $L$.

For $x \in K_{\xi_2} \setminus L$ and $y \in L$ we have $H(\xi_2, x) = H(\xi_2, y) = 0$ and, by Lemma 3.5, we also have
\[
H(\xi_1, x) = H(\xi_2, x) + \frac{\epsilon}{2} \langle \eta, y - x \rangle > 0
\]
because $x \notin L$ and $y \in L$.

Finally, if $x, y \in L$, we have $H(\xi_2, x) = H(\xi_2, y) = 0$ and Lemma 3.5 gives
\[
H(\xi_1, x) - H(\xi_1, y) = H(\xi_2, x) - H(\xi_2, y) - \frac{\epsilon}{2} \langle \eta, x - y \rangle = 0
\]
because $x, y \in L$. This implies that $H(\xi_1, x)$ is constant for $x \in L$ and positive for $x \notin L$. By Corollary 3.4, this function attains a zero for some $x$ which means that $H(\xi_1, x) = 0$ for all $x \in L$. Hence $\Sigma$ satisfies condition (3) of Definition 3.3. The proof for $\Sigma'$ is again analogous.

We now show that $\Sigma$ and $\Sigma'$ are dual. Recall the definition of $K \mapsto K^*$ in (3.9). It is clear that
\[
(1) \forall K_1, K_2 \in \Sigma \ K_1 \subset K_2 \Rightarrow K_2^* \supset K_1^*;
(2) \forall \xi \in \Delta_f \ \xi \in (K_\xi)^*
(3) \forall K \in \Sigma \ K \subset K^{**}
\]
and that analogous statements hold for $\Sigma'$. This implies that $K \mapsto K^*$ defines a bijection between $\Sigma$ and $\Sigma'$ that reverses inclusions. Hence condition (1) of Definition 3.4 is satisfied and it only remains to show that whenever $K_1 \subset K_2$, the cones $C(K_1, K_2)$ and $C(K_2^*, K_1^*)$ are polar$^3$.

Let $K_1 \subset K_2$ in $\Sigma$ and let $x \in K_2$, $y \in K_1$, $\xi \in K_2^*$ and $\eta \in K_1^*$. We have
\[
\langle \xi - \eta, x - y \rangle = -H(\xi, x) + H(\xi, y) + H(\eta, x) - H(\eta, y) = H(\eta, x) \geq 0
\]
which implies $C(K_2^*, K_1^*) \subset -C(K_1, K_2)^*$. To prove the opposite inclusion, let $K_2 = K_{\xi}$, $K_1 \subset K_2$ and $\eta \in -C(K_1, K_2)^*$. As in Lemma 3.5, let $\epsilon > 0$ be such that $[0, \epsilon] \ni t \mapsto H(\xi + t\eta, x)$ is linear. We claim that $\xi + \frac{\epsilon}{2} \eta \in K_1^*$. This would imply
\[ \eta = \frac{2}{\epsilon} (\xi + \frac{\epsilon}{2} \eta - \xi) \in C(K_2^*, K_1^*). \] For all \( x \notin K_2 \) we have \( H(\xi, x) > 0 \) and, as before, by Lemma 3.5, we have \( H(\xi + \frac{\epsilon}{2} \eta, x) > 0 \). By Corollary 3.4, \( H(\xi + \frac{\epsilon}{2} \eta, x) = 0 \) for some \( x \). Such \( x \) is in \( K_2 = K_\xi \) by the preceding argument. Let \( y \in K_1 \). Since \( \eta \in -C(K_1, K_2)^* \), we have

\[ 0 \geq \frac{\epsilon}{2} \langle \eta, x - y \rangle = H(\xi + \frac{\epsilon}{2} \eta, y) - H(\xi + \frac{\epsilon}{2} \eta, x) \]

Hence \( H(\xi + \frac{\epsilon}{2} \eta, y) \leq H(\xi + \frac{\epsilon}{2} \eta, x) = 0 \) and, as before, by Lemma 3.5, we have \( H(\xi + \frac{\epsilon}{2} \eta, x) > 0 \). By Corollary 3.4, \( H(\xi + \frac{\epsilon}{2} \eta, x) = 0 \) for some \( x \). Such \( x \) is in \( K_2 = K_\xi \) by the preceding argument. Let \( y \in K_1 \). Since \( \eta \in -C(K_1, K_2)^* \), we have

\[ 0 \geq \frac{\epsilon}{2} \langle \eta, x - y \rangle = H(\xi + \frac{\epsilon}{2} \eta, y) - H(\xi + \frac{\epsilon}{2} \eta, x) \]

Hence \( H(\xi + \frac{\epsilon}{2} \eta, y) \leq H(\xi + \frac{\epsilon}{2} \eta, x) = 0 \) and by non-negativity of \( H \), \( H(\xi + \frac{\epsilon}{2} \eta, y) = 0 \) for all \( y \in K_1 \), i.e. \( \xi + \frac{\epsilon}{2} \eta \in K_1^* \), as wanted. This concludes the proof.

\[ \square \]

**Remark 3.2** There is an interesting way to interpret the subdivision \( \Sigma' \) of \( \Delta_f \). Namely, if \( p_{R_f}(x) = \max_C (\langle \nu_C, x \rangle + \tau_C) \), consider the following subset of \( \mathbb{R}^n \times \mathbb{R} \):

\[ \text{conv}\{ (\nu_C, a) : a \leq -\tilde{p}_{R_f}(\nu_C) \} \]

It is an unbounded polytope contained inside the tube \( \Delta_f \times \mathbb{R} \). Projecting its bounded faces via \((\xi, x) \mapsto \xi\) to \( \Delta_f \) gives rise to a polytopal convex subdivision of \( \Delta_f \) which, in fact, equals \( \Sigma' \). For example, let \( \kappa \) be a full-dimensional face whose set of vertices is \( \{(\alpha_j, -\tilde{p}_{R_f}(\alpha_j))\}_{j} \) where \( \alpha_j \in \{\nu_C\}_C \) and \( j \in \{1, \ldots, k\} \). Let \( (x_\kappa, 1) \) be a normal to \( \kappa \). We have

\[ \kappa = \{ \xi \in \Delta_f : \forall \ j \ - \tilde{p}_{R_f}(\alpha_j) - \langle \xi - \alpha_j, x_\kappa \rangle \geq -\tilde{p}_{R_f}(\nu_C) + \langle \nu_C - \alpha_j, x_\kappa \rangle \} = \{ \xi \in \Delta_f : \forall \ j \ - \tilde{p}_{R_f}(\alpha_j) + \langle \xi, x_\kappa \rangle \geq \sup_C (\langle \nu_C, x_\kappa \rangle - \tilde{p}_{R_f}(\nu_C)) \} = \{ \xi \in \Delta_f : \tilde{p}_{R_f}(\alpha_j) + \tilde{p}_{R_f}(x_\kappa) \leq \langle \xi, x_\kappa \rangle \} = \{ \xi \in \Delta_f : H(\xi, x_\kappa) = 0 \}
\]

Repeating the same argument for all faces leads to the desired conclusion.

![Figure 3.2: Convex subdivision \( \Sigma' \) of \( \Delta_f \) for \( f \) from Example 2.3](image-url)
Theorem 3.7 [PR04] Let $f$ be a Laurent polynomial and $\mathfrak{T}$ and $\mathfrak{T}'$ the dual convex subdivisions of $\mathbb{R}^n$ and $\Delta_f$ respectively, as defined above. Then

1. The spine $\mathcal{S}_f$ is the union of all cells in $\mathfrak{T}$ of dimension less than $n$. Moreover, $\mathcal{S}_f \subset \mathfrak{A}_f$.

2. For any connected component $C$ of $\mathbb{R}^n \setminus \mathfrak{A}_f$, the cell dual to $\{v_C\}$ contains $C$.

3. $\mathcal{S}_f$ is a deformation retract of $\mathfrak{A}_f$.

Proof Let $K_C = \{x : p_{R_f}(x) = \langle v_C, x \rangle + \tau_C\}$. Then $C \subset K_C$. Since the spine is, by definition, the union of the boundaries of the sets $K_C$, it follows that $\mathcal{S}_f \subset \mathfrak{A}_f$. Moreover, it is readily seen that the sets $K_C$ are precisely the full-dimensional cells in $\mathfrak{T}$ and that their dual points are $\{v_C\}$. Thus the lower-dimensional cells are the faces of $K_C$ and their union is, therefore, equal the union of their boundaries, i.e. the spine. This shows (1) and (2).

To prove (3), we construct the deformation retraction. For every connected component $C$ of $\mathbb{R}^n \setminus \mathfrak{A}_f$, take a point $x_C \in C$ and consider the set of all segments from $x_C$ to the boundary of $K_C$. We are done if we show that the union of all such segments contains the amoeba.

Let $x_C$ be one such point and let $y \in \mathbb{R}^n \setminus \{0\}$. Suppose that the half-line $\{x + ty : t \geq 0\}$ does not intersect the boundary of $K_C$. We claim that this implies that it does not intersect the amoeba either. Indeed, if $x + ty \in K_C$ for all $t \geq 0$, this means that

$$\forall C' \langle v_C, x + ty \rangle > \langle v_{C'}, x + ty \rangle.$$ 

For $t > 0$, we may divide by $t$ and let $t \to \infty$, which gives

$$\forall C' \langle v_C, ty \rangle \geq \langle v_{C'}, y \rangle$$

which in turn gives

$$\langle v_C, y \rangle \geq \max_{\xi \in \Delta_f} \langle \xi, y \rangle.$$ 

Let $z_C \in \text{Log}^{-1}(x_C)$ and let $\mu \in \mathbb{Z}^n \setminus \{0\}$ be such that

$$\langle v_C, \mu \rangle \geq \max_{\xi \in \Delta_f} \langle \xi, \mu \rangle. \quad (3.11)$$

By Lemma 2.13, we know that $-\langle v_C, \mu \rangle$ counts the zero-poles of $w \mapsto f(z_C w^{-\mu})$ in the unit disc. Hence the function $w \mapsto w^{\langle v_C, \mu \rangle} f(z_C w^{-\mu})$ is a polynomial and has no zeroes in the unit disc. Applying the maximum principle to its inverse implies that

$$\forall t \geq 0 \frac{1}{\min_{|u|=1} |f(z_C e^{t \mu} w^{-\mu})|} \leq \frac{1}{\min_{|u|=1} |f(z_C w^{-\mu})|},$$

i.e.

$$\forall t \geq 0 \min_{|u|=1} |f(z_C w^{-\mu})| \geq e^{\langle v_C, \mu \rangle \min_{|u|=1} |f(z_C e^{t \mu} w^{-\mu})|}.$$ 

Letting $z_C$ vary over $\text{Log}^{-1}(x_C)$ gives

$$\forall t \geq 0 \min_{z \in \text{Log}^{-1}(x_C + t\mu)} |f(z)| \geq e^{\langle v_C, \mu \rangle} \min_{z \in \text{Log}^{-1}(x_C)} |f(z)| > 0.$$ 

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This remains true for all $\mu \in \mathbb{Q}^n$ that satisfy (3.11) (because it holds for all $t \geq 0$) and, by continuity, extends to $\mu \in \mathbb{R}^n$. In particular, this is true for $\mu = y$ which implies that $x + ty$ does not hit the amoeba.

\[ \square \]

Figures 3.3 and 3.4 illustrate these theorems for $f$ as in Example 2.3.

\[ \text{Figure 3.3: Amœba } \mathcal{A}_f \text{ and its spine } \mathcal{S}_f \]

\[ \text{Figure 3.4: Subdivisions of } \mathbb{R}^2 \text{ and } \Delta_f \text{ superimposed} \]

### 3.3 Spine Approximation

Let $f$ be a Laurent polynomial and recall our definition of the function $f_k$ from Theorem 2.10. Note that

$$\log |f_k(z)| = \sum_{l_1=0}^{k-1} \cdots \sum_{l_n=0}^{k-1} \log |f(e^{2\pi i l_1/k}z_1, \ldots, e^{2\pi i l_n/k}z_n)|.$$ 

and that we may interpret $\frac{1}{k^n} \log |f_k(z)|$ as a Riemann sum for the integral (3.1). This is because we are, in fact, partitioning the segment $[0, 2\pi]$ into $k$ equal parts of length $2\pi/k$ in every of the $n$ variables. This corresponds to partitioning the polycircle $\log^{-1}(x) \ni z$ by multiplying each of the $n$ coordinates by each of the $k$ roots of unity. When $\log |f_k(z)|$ is not bounded, i.e. when $x = \log(z) \in \mathbb{R}^n \setminus \mathcal{A}_f$, we have

$$\lim_{k \to \infty} \frac{1}{k^n} \log |f_k(z)| = R_f(x).$$

Fix $x$ in a connected component $C_\nu$ of $\mathbb{R}^n \setminus \mathcal{A}_f$. By Remark 2.8, for any $\varepsilon$ we can find a large enough $k$ such that $f_k(z) = \sum_{\alpha \in A_{f_k}} c_{k,\alpha} z^\alpha$ satisfies

$$|c_{k,\beta}| e^{\langle \beta, x \rangle} > \frac{1}{\varepsilon} \sum_{\alpha \in A_{f_k} \setminus \{\beta\}} |c_{k,\alpha}| e^{\langle \alpha, x \rangle}$$

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In fact, we have $\beta = k^n\nu_C$, as the following lemma shows.

**Lemma 3.8** Let $f_k(z) = \sum_{\alpha \in \mathcal{A}_k} c_{k,\alpha} z^\alpha$ be as in Theorem 2.10, if $x \in C_\nu \subset \mathbb{R}^n \setminus \mathcal{A}_f$. If $\{ |c_{k,\alpha}| e^{(a,x)} \}$ is lopsided, the dominating term has exponent $k^n\nu_C$.

**Proof** Let $\zeta \in \text{Log}^{-1}(x)$ and $f_{j,k,\zeta}$ as in (2.22). The dominating term of

$$f_{j,k,\zeta}(z) = \prod_{l_1=0}^{k-1} \cdots \prod_{l_n=0}^{k-1} f(e^{2\pi i l_1/k}z_1, \ldots, e^{2\pi i l_{j-1}/k}z_{j-1}, e^{2\pi i l_j/k}z, e^{2\pi i l_{j+1}/k}z_{j+1}, \ldots, e^{2\pi i l_n/k}z_n)$$

(3.12)

is the dominating term of $f_k$ with fixed $z_l = \zeta$ for $l \neq j$. Since (3.12) is a product of $k^n$ terms, each of which has $\nu_{C,j}$ zero-poles in $\{ z : |z| < e^{x_j} \}$, we have that $f_{j,k,\zeta}$ has $k^n\nu_{C,j}$ zero-poles in $\{ z : |z| < e^{x_j} \}$ and this is equal to the exponent of its dominating term. Hence the dominating term of $f_k$ has exponent $k^n\nu_C$.

Thus we have

$$\log |c_{k,\beta} z^\beta| + \log |1 - \varepsilon| \leq \log |f_k(z)| \log |c_{k,\beta} z^\beta| + \log |1 + \varepsilon|.$$  

Letting $k \to \infty$, we deduce that

$$\frac{1}{k^n} \log |c_{k,\beta} z^\beta| = \frac{1}{k^n} \log |c_{k,\beta}| + \langle \nu_C, x \rangle$$

converges to $R_f(x)$ and we may use this to approximate the spine.

**Proposition 3.2** Let $r_k(x) := \max(\log |c_{k,\nu_C}| + \langle k^n\nu_C, x \rangle)$. Then

1. $\mathcal{V}_r \subset \left\{ x : \text{\{c_{k,\alpha}|e^{(\alpha,x)}\}_{\alpha \in \mathcal{A}_k} \text{ is not lopsided}} \right\}$;

2. $\lim_{k \to \infty} \mathcal{V}_r = \mathcal{A}_f$.

**Proof** (1) follows from the fact that the maximum cannot be achieved for two different affine functions when one term dominates.

To show (2), note that $\frac{1}{k^n} \log |c_{k,\beta}| + \langle \nu_C, x \rangle - R_f(x)$ is a constant that approaches zero as $k \to \infty$. Let $C_1$ and $C_2$ be two connected components of $\mathbb{R}^n \setminus \mathcal{A}_f$ and consider the equations

$$\langle \nu_{C_1}, x \rangle + \tau_{C_1} = \langle \nu_{C_2}, x \rangle + \tau_{C_2}$$

and

$$\langle k^n\nu_{C_1}, x \rangle + \log |c_{k,\nu_C}| = \langle k^n\nu_{C_2}, x \rangle + \log |c_{k,\nu_{C_2}}|.$$  

They define two hyperplanes in $\mathbb{R}^n$ that we shall denote by $H$ and $H_k$ respectively. They are parallel and their distance is less than $\varepsilon$ times a constant that depends on $C_1$ and $C_2$. By Theorem 2.14, the number of components is bounded by $\text{card}(\Delta_f \cap \mathbb{Z}^n)$, i.e. there are finitely many choices and we can decrease the distance across all components by taking $k$ sufficiently large.

\[\square\]
Chapter 4

Compactification and Contours

In this chapter we give a brief overview of some additional aspects of amoebas which are of importance.

4.1 Toric Compactification

Let \( f(z) = \sum_{\alpha \in A_f} c_{\alpha} z^\alpha \) be a Laurent polynomial and, as before, let \( \Delta_f \) be its Newton polytope and \( Y_f \subset \mathbb{T}^n \) the corresponding hypersurface. Consider the collection of cones dual to faces of \( \Delta_f \), also known as the dual fan of \( \Delta_f \) and denoted by \( \Sigma_f \). Note that these cones are strongly convex rational polyhedral cones. To \( \Sigma_f \) one associates a toric variety \( X_{\Sigma_f} \). Fan \( \Sigma_f \) can be refined by subdivisions to obtain a new fan \( \Sigma_f' \) which is simplicial\(^1\) and regular\(^2\). Let \( \Sigma_f' \) denote such a refinement and let \( X_{\Sigma_f'} \) denote the toric variety associated to it. \( X_{\Sigma_f'} \) is a resolution of singularities of \( X_{\Sigma_f} \) and there is a proper birational surjective morphism \( \pi: X_{\Sigma_f'} \to X_{\Sigma_f} \). Let \( \overline{V_f} \) denote the Zariski closure of \( V_f \) in \( X_{\Sigma_f'} \). This grants us a compact setting in which the information about the amoeba is preserved. In this setting, the role of the Log map is played by the so-called moment map.

**Definition 4.1** The compactified amoeba \( \widetilde{A}_f \) of \( f \) is the image of \( \overline{V_f} \) under the moment map \( \mu_f : X_{\Sigma_f'} \to \Delta_f \) given by

\[
\mathbb{T}^n \ni z \mapsto \sum_{\alpha \in \Delta_f \cap \mathbb{Z}^n} \frac{|z^\alpha|^\alpha}{\sum_{\alpha \in \Delta_f \cap \mathbb{Z}^n} |z^\alpha|}.
\]

That compactified amoebas behave well is reflected in the following facts (see Theorems 1.11 and 1.12 in chapter 6 of [GKZ94]):

1. \( \mu_f \) is surjective.

---

\(^1\) i.e. a normal algebraic variety that contains \( \mathbb{T}^n \) as an open dense subset, such that the action of \( \mathbb{T}^n \) on itself extends to the whole variety. See [Fult93] and Ch. 5 of [GKZ94].

\(^2\) i.e. for every \( k \), every \( k \)-dimensional cone of the fan is generated by \( k \) linearly independent vectors.

\(^3\) i.e. every \( n \)-dimensional cone of the fan is generated by some \( \eta_1, \ldots, \eta_n \in \mathbb{Z}^n \) which have coprime coordinates and \( \det(\eta_1, \ldots, \eta_n) = \pm 1 \). In other words, every cone of the fan is generated by a subset of a basis of the lattice \( \mathbb{Z}^n \).
(2) For any face $\Gamma$ of $\Delta_f$, $\mu_f^{-1}(\Gamma)$ is the closure of the orbit that corresponds to $\Gamma$. The orbit itself is $\mu_f^{-1}(\text{int}\Gamma)$.

(3) $\mu_f^{-1}(\Gamma)$ is a point if and only if $\Gamma = \{\gamma\}$ is a vertex.

(4) For any vertex $\gamma$, there is a neighbourhood of $\gamma$ that does not intersect $\sim A_f$.

(5) Vertices lie in different connected components of $\Delta_f \setminus \sim A_f$.

(6) If $\Gamma$ is an edge, $\sim A_f \cap \Gamma \neq \emptyset$.

This is illustrated in figure 4.1.

![Figure 4.1: Typical shape of a 2-dimensional compactified amoeba](image)

### 4.2 Logarithmic Gauß Map and Contours

**Definition 4.2** Let $f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial with no multiple factors. The **logarithmic Gauß map** is the rational map $\gamma_f : \mathcal{V}f \to \mathbb{P}^{n-1}$ given by

$$\gamma_f^\text{reg} : z_0 \mapsto [z_0, 1, \partial f / \partial z_1(z_0, \ldots, z_0, n), \ldots, z_0, n, \partial f / \partial z_n(z_0, \ldots, z_0, n)]$$

Geometric interpretation of $\gamma_f$ is as follows. Let $z_0 \in \gamma_f^\text{reg}$ and let $U \subset \mathbb{T}^n$ be a neighbourhood of $z_0$. Choose a branch of $\log_U : U \to \mathbb{C}^n$, given by

$$(z_1, \ldots, z_n) \mapsto (\log(z_1), \ldots, \log(z_n)),$$

and apply the standard Gauß map on $\log(U \cap \gamma_f)$, i.e. associate to $z_0$ a normal to the tangent hyperplane $T_{\log z_0} \log(U \cap \gamma_f)$ of $\log(U \cap \gamma_f)$ at $\log(z_0)$. The direction of the normal corresponds to the point $\gamma_f(z_0)$ and does not depend on the choice of the branch of log.

**Definition 4.3** **Contour** $\mathcal{C}_I$ of an amoeba $\mathcal{A}_I$, where $I \subset \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is a proper ideal, is defined as the set of critical values of $\text{Log}_{|\gamma_f} : \mathbb{T}^n \cap \gamma_f \to \mathbb{R}^n$, i.e. the image of its critical points.
Remark 4.1 We always have \( \log(\mathcal{V}_f^{\text{sing}}) \subset \mathcal{C}_f \).

The contour can give us information about the amœba \( \mathcal{A}_f \) because the boundary of the amœba is contained in the contour, i.e. \( \partial \mathcal{A}_f \subset \mathcal{C}_f \). The contour may also include points from the interior of the amœba.

**Proposition 4.1** [Mikh00] Let \( f \) be a Laurent polynomial with no multiple factors. Then \( \mathcal{C}_f = \log(\gamma_f^{-1}(\mathbb{P}^{n-1} \mathbb{R})) \).

**Proof** Let \( z_0 \in \mathcal{V}_f^{\text{reg}} \), let \( U \) be a neighbourhood of \( z_0 \) and choose a branch of \( \log|U| \). Note that \( \log|U| \) is the composition of \( \log|U \cap \mathcal{V}_f| \) and the projection \( \text{Re} : \mathbb{C}^n \to \mathbb{R}^n \). Hence \( z_0 \) is a critical point if \( d\text{Re} : T_{\log z_0}(\log(U \cap \mathcal{V}_f)) \to \mathbb{R}^n \) is not surjective. The normal direction to the tangent hyperplane \( T_{\log z_0}(\log \mathcal{V}_f) \) can be represented by some vector \( \gamma_f(z_0) \in \mathbb{C}^n \setminus \{0\} \). Hence we can write

\[
T_{\log z_0}(\log(U \cap \mathcal{V}_f)) = \{ x + i\theta \in \mathbb{C}^n : \langle \gamma_f(z_0), x + i\theta \rangle = 0 \}.
\]

If \( \gamma_f(z_0) \) is real, the projection \( d\text{Re} \) is not surjective. If \( \gamma_f(z_0) \) is not real, let \( \gamma_f(z_0) = a + ib \). We can consider \( \langle \gamma_f(z_0), x + i\theta \rangle = 0 \) as a system of linear equations with fixed \( x \), i.e. we can write

\[
\langle a + ib, x + i\theta \rangle = \langle a, x \rangle + i\langle a, \theta \rangle + i\langle b, x \rangle - \langle b, \theta \rangle = 0
\]

which is equivalent to

\[
\langle a, \theta \rangle = -\langle b, x \rangle,
\]

\[
\langle b, \theta \rangle = \langle a, x \rangle.
\]

and solve for \( \theta \). Thus \( z_0 \) is not a critical point of \( \mathcal{V}_f \).

**Corollary 4.1** Let \( f \) be a Laurent polynomial with no multiple factors and with real coefficients. Then

\[
\log(\mathcal{V}_f \cap \mathbb{R}^n) \subset \mathcal{C}_f
\]

**Proof** Points in \( \mathcal{V}_f^{\text{sing}} \) map to \( \mathcal{C}_f \) under \( \log \), so points in \( \mathcal{V}_f^{\text{sing}} \cap \mathbb{R}^n \) are just a special case. Points in \( \mathcal{V}_f^{\text{reg}} \cap \mathbb{R}^n \) map to \( \mathbb{P}^{n-1} \mathbb{R} \) which implies they are critical (by the proof above).

To determine the critical points, we consider the following system of equations:

\[
f(z) = 0;
\]

\[
\left( z_1 \frac{\partial f}{\partial z_1} (z) : \cdots : z_n \frac{\partial f}{\partial z_n} (z) \right) = (\lambda_1 : \cdots : \lambda_n) \in \mathbb{P}^{n-1} \mathbb{R}
\]

(4.1)

By a theorem of Kouchmirenko [Kou76], the number of solutions to a system of \( n \) polynomial equations in \( n \) variables, in which all polynomials have the same Newton polytope \( \Delta_f \), is generically equal to \( n! \text{Vol}(\Delta_f) \). As is stated more precisely in the following proposition, this will typically be the degree of the logarithmic Gauß map. When \( \mathcal{V}_f \) is singular, the degree will be lower because a singular point will satisfy (4.1) for all \( \lambda \in \mathbb{P}^{n-1} \mathbb{R} \).

**Proposition 4.2** [Mikh00] Let \( f \) be a Laurent polynomial with no multiple factors and suppose that \( f \) and all \( z_j \frac{\partial f}{\partial z_j} \) have the same Newton polytope \( \Delta_f \). If \( \mathcal{V}_f \subset \mathcal{X}_{\mathcal{C}_f} \) intersects transversally all orbits corresponding to cones of \( \Sigma_f \), the logarithmic Gauß map \( \gamma_f \) can be extended to a dominant rational map \( \overline{\gamma}_f : \mathcal{V}_f \to \mathbb{P}^{n-1} \) with \( \text{deg} \overline{\gamma}_f = n! \text{Vol}(\Delta_f) \).
The case where we can extend $\gamma_f$ so that $\deg \tilde{\gamma}_f = 1$ is particularly interesting. In this case, the contour $\mathcal{C}_f$ can be parametrized by composing $\tilde{\gamma}_f^{-1} : \mathbb{R}^{n-1} \to \mathcal{V}_f$ with $\text{Log}$. This also leads naturally to $A$-discriminants and hypergeometric functions (see [Kapr91], [GZK99] and [PST04]) which is the context in which the concept of the amoeba originally appeared in [GKZ94].
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