Overholonomicity of overconvergent F-isocrystals over smooth varieties (work in common with Nobuo Tsuzuki)

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Introduction

Let $\mathcal{V}$ be a complete discrete valuation ring of characteristic 0 with perfect residue field k of characteristic $p > 0$ and field of fractions K.

> Problem from the 60': construction of a p-adic cohomology over k-varieties stable under Grothendieck's six cohomological operations.

This means:

- a category $\mathrm{Coeff}(Y/K)$ of K-objects, for any k-variety Y.
- Grothendieck's operations: the direct image $f_+ : \mathrm{Coeff}(Y'/K) \to \mathrm{Coeff}(Y/K)$ for any morphism $f : Y' \to Y$ of k-varieties, inverse image f^+, extraordinary direct image $f_!$, extraordinary inverse image f^+, tensor product and dual.

Berthelot's Arithmetic $\mathcal{D}$-modules

- Idea: the derived categories of bounded, holonomic complexes of (left by default) $\mathcal{D}$-modules over $\mathbb{C}$-varieties Y are stable under Grothendieck's six operations.

$\rightsquigarrow$ Berthelot's theory of arithmetic $\mathcal{D}$-modules over k-varieties.

Let $\mathcal{E}$ be a coherent arithmetic F-$\mathcal{D}$-module over Y (i.e. endowed with a Frobenius structure).

- Berthelot defined the characteristic variety $\mathrm{Char}(\mathcal{E})$ of $\mathcal{E}$.
- Berthelot-Bernstein's inequality: $\dim \mathrm{Char}(\mathcal{E}) \geq \dim Y$ if $\mathcal{E} \neq 0$. By definition, $\mathcal{E}$ is holonomic if $\dim \mathrm{Char}(\mathcal{E}) \leq \dim Y$ on each irreducible components of Y.
- Holonomicity is stable under external tensor product (Berthelot) and dual functor (Virrion).
- Berthelot's conjectures: the holonomicity is stable under proper direct image and under inverse image.

Contents

1. Definition: arithmetic log-$\mathcal{D}$-modules, overconvergent log-isocrystals and overholonomicity.
2. Comparison theorem between relative log-rigid cohomology and relative rigid cohomology.
3. Overholonomicity of some log-extendable isocrystals.
4. Overholonomicity of overconvergent F-isocrystals on smooth k-varieties (consequence of Kedlaya's semistable theorem and section 2).
5. Consequences:
   - F-overholonomicity equals devissability in overconvergent F-isocrystals,
   - F-overholonomicity is stable under Grothendieck's six operations.

Notation

- Let $g : \mathfrak{X} \to \mathfrak{T}$ be a smooth morphism of smooth formal schemes over $\mathcal{S} = \operatorname{Spf} \mathcal{V}$ of pure relative dimension d (e.g. $\mathfrak{T} = \mathcal{S}$).
- Let $\mathcal{Z}$ be a relatively strict normal crossing divisor of $\mathfrak{X}$ over $\mathfrak{T}$, $\mathcal{Y} := \mathfrak{X} \setminus \mathcal{Z}$ and $\mathfrak{X}^{\#} = (\mathfrak{X}, \mathcal{Z})$ be the logarithmic formal $\mathcal{V}$-scheme with the logarithmic structure associated to $\mathcal{Z}$.
- *Locally will mean:* $\mathfrak{X}$ and $\mathfrak{T}$ are affine and there exist relatively local coordinates of $\mathfrak{X}$ over $\mathfrak{T}$, say $z_1, z_2, \cdots, z_d$, such that the irreducible component $\mathcal{Z}_i$ of $\mathcal{Z} = \cup_{i=1}^{s} \mathcal{Z}_i$ is defined by $z_i = 0$.
- For any multi-index $\underline{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d$, we pose:

$$\underline{\partial}_{\#}^{[\underline{n}]} = \frac{1}{n_1! \cdots n_d!} \left(\prod_{i=1}^{s} \prod_{j=0}^{n_i - 1} (\partial_{\# i} - j) \right) \partial_{s+1}^{n_{s+1}} \cdots \partial_d^{n_d},$$

where $\partial_{\# i} = z_i \frac{\partial}{\partial z_i}$ for $1 \leq i \leq s$, $\partial_i = \frac{\partial}{\partial z_i}$ for $s+1 \leq i \leq d$.

Logarithmic arithmetic $\mathcal{D}$-modules

- We denote by $\mathcal{D}_{\mathfrak{X}^\#}$ the *usual* ring of differential operators on $\mathfrak{X}^\#$. This is a locally free $\mathcal{O}_{\mathfrak{X}}$-module and $\underline{\partial}_{\#}^{[\underline{n}]}$ with $\underline{n} \in \mathbb{N}^d$ form a basis as such an $\mathcal{O}_{\mathfrak{X}}$-module.
- Berthelot, and Montagnon for the logarithmic situation, constructed the sheaf of differential operators of finite level on $\mathfrak{X}^\#$ denoted by $\mathcal{D}^\dagger_{\mathfrak{X}^\#}$.
- *Locally*, $P \in \Gamma(\mathfrak{X}^\#, \mathcal{D}^\dagger_{\mathfrak{X}^\#})$ iff $\exists c,\ \eta \in \mathbb{R}$ with $\eta < 1$ such that

$$P = \sum_{\underline{n} \in \mathbb{N}^d} a_{\underline{n}} \underline{\partial}_{\#}^{[\underline{n}]} | a_{\underline{n}} \in \mathcal{O}_{\mathfrak{X}}, \text{ and } \| a_{\underline{n}} \| \leq c\eta^{n_1 + \cdots + n_d}.$$

- Noot-Huyghe's interpretation: $\mathcal{D}^\dagger_{\mathfrak{X}^\#}$ is the weak p-adic completion as $\mathcal{O}_{\mathfrak{X}}$-algebra of $\mathcal{D}_{\mathfrak{X}^\#}$.

Let D (e.g. $D = Z$) be a divisor of X (the special fiber of $\mathfrak{X}$), $\mathfrak{U} := \mathfrak{X} \setminus D$, and $\mathfrak{U}^{\#}$ the restriction of $\mathfrak{X}^{\#}$ on $\mathfrak{U}$.

• We denote by $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)$ Berthelot's sheaf of section on $\mathfrak{X}$ with overconvergent singularities along D.

> Local description: if D is defined by $f = 0$ in X for $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, then $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}} = \mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^{\dagger} \otimes_{\mathcal{V}} K$ (where $\mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]^{\dagger}$ is the weak completion as $\mathcal{O}_{\mathfrak{X}}$-algebra of $\mathcal{O}_{\mathfrak{X}}[\frac{1}{f}]$).

• We denote by $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)$ the ring of differential operators on $\mathfrak{X}^{\#}$ with overconvergent singularities along D. Noot-Huyghe's interpretation: $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}} = \left(\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}_{\mathfrak{X}^{\#}}\right)^{\dagger} \otimes_{\mathcal{V}} K$, where "$\dagger$" means the weak completion as $\mathcal{O}_{\mathfrak{X}}$-algebra.

• When $D = \emptyset$, we note $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}$ instead of $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}\emptyset)$. When $D = Z$, $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}Z) = \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)$.

• We get the functor $({}^{\dagger}D) := \mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}} -$.

Link with rigid analytic K-spaces

- Let $\mathfrak{X}_K$ be the Raynaud generic fiber of $\mathfrak{X}$ which is a rigid analytic K-space and $g_K : \mathfrak{X}_K \to \mathfrak{T}_K$ the induced morphism.
- For any sheaf $\mathcal{H}$ on $\mathfrak{X}_K$, the sheaf of sections of $\mathcal{H}$ overconvergent along D is

$$j_U^\dagger \mathcal{H} := \varinjlim_V \alpha_{V*}(\mathcal{H}|_V),$$

where V runs over all strict neighborhoods of $\mathfrak{U}_K$ in $\mathfrak{X}_K$ (i.e., admissible open subsets $\mathfrak{U}_K \subset V \subset \mathfrak{X}_K$ such that $V \cup (\mathfrak{X}_K \setminus \mathfrak{U}_K)$ is an admissible cover of $\mathfrak{X}_K$) and where $\alpha_V : V \to \mathfrak{X}_K$ is the canonical inclusion.
- We have $\mathrm{sp}_*(j_U^\dagger \mathcal{O}_{\mathfrak{X}_K}) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}}({}^\dagger D)_{\mathbb{Q}}$, where $\mathrm{sp} : \mathfrak{X}_K \to \mathfrak{X}$ is the specialization morphism.

Overconvergent log-isocrystals

Let $\mathfrak{X}_K^{\#} = (\mathfrak{X}_K, \mathcal{Z}_K)$ be the rigid analytic space endowed with the logarithmic structure associated to $\mathcal{Z}_K$. Let E be a locally free $j_U^{\dagger}\mathcal{O}_{\mathfrak{X}_K}$-modules of finite type with an integrable logarithmic connection $\nabla : E \to j_U^{\dagger}\Omega^1_{\mathfrak{X}_K^{\#}/\mathcal{T}_K} \otimes_{j_U^{\dagger}\mathcal{O}_{\mathfrak{X}_K}} E$. The pair (E, ∇) (or simply E) is called a log-isocrystal on $U^{\#}/\mathcal{T}_K$ overconvergent along D if *locally*:

- there exist a strict neighborhood V of $\mathfrak{U}_K$ in $\mathfrak{X}_K$ and a locally free $\mathcal{O}_V$-module $\mathcal{E}$ and an integrable $\nabla : \mathcal{E} \to (\Omega^1_{\mathfrak{X}_K^{\#}/\mathcal{T}_K}|_V) \otimes_{\mathcal{O}_V} \mathcal{E}$ such that $j_U^{\dagger}(\mathcal{E}, \nabla) = (E, \nabla)$
- and the *overconvergent condition holds:* $\forall \xi \in]0,1[$, $\exists$ an affinoid strict neighborhood $W \subset V$ of $\mathfrak{U}_K$ in $\mathfrak{X}_K$ such that, $\forall e \in \Gamma(W, \mathcal{E})$,

$$||\underline{\partial}_{\#}^{[\underline{n}]}(e)||\xi^{|\underline{n}|} \to 0 \ \ (\text{as } |\underline{n}| \to \infty).$$

• A convergent log-isocrystal on $X^{\#}/\mathcal{T}_K$ is by definition a log-isocrystal on $X^{\#}/\mathcal{T}_K$ overconvergent along $\emptyset$. When $\mathcal{T} = \mathcal{S}$, we say *log-isocrystal on $U^{\#}$ overconvergent along D* or *convergent log-isocrystal on $X^{\#}$*.

Theorem

The functors sp^* and sp_* induce quasi-inverse equivalences between the category of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#}}({}^{\dagger}D)_{\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X}}({}^{\dagger}D)_{\mathbb{Q}}$ and the category of log-isocrystals on $U^{\#}$ overconvergent along D.

• Remark: Shiho's category of convergent log-isocrystals on $X^{\#}$ over $\mathcal{S}$ is equivalent to that of coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$-modules, locally projective of finite type over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$.

Overholonomicity

Definition

Let $r \geq -1$ be an integer and $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.

- The complex $\mathcal{E}$ is (-1)-overholonomic if $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.
- The complex $\mathcal{E}$ is 0-overholonomic if $\forall$ smooth morphism $f : \mathcal{P}' \to \mathcal{P}$, $\forall$ divisor T' of P', $({}^{\dagger}T') \circ f^{*}(\mathcal{E})$ is (-1)-overholonomic.
- If $r \geq 1$, $\mathcal{E}$ is r-overholonomic if $\mathcal{E}$ is $r-1$-overholonomic and if $\forall$ smooth morphism $f : \mathcal{P}' \to \mathcal{P}$, $\forall$ divisor T' of P', $\mathbb{D} \circ ({}^{\dagger}T') \circ f^{*}(\mathcal{E})$ is $(r-1)$-overholonomic, where $\mathbb{D}$ is the dual functor: $\mathbb{D}(-) = \mathbb{R}\mathcal{H}om_{\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}}(-, \mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^{-1})[\dim X]$.
- The $(F\text{-})$complex $\mathcal{E}$ is overholonomic if it is r-overholonomic for every r. We denote by $(F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ the derived category of overholonomic $(F\text{-})$complexes of $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-modules.

Stability of the overholonomicity

Theorem

Let $f : \mathfrak{X}' \to \mathfrak{X}$ be a morphism, $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$, $\mathcal{E}' \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}})$. Then:

- *If $\mathcal{E}$ has a Frobenius structure, then $\mathcal{E}$ is holonomic.*
- *The complex $f^{!}(\mathcal{E}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X}',\mathbb{Q}})$.*
- *If f is proper, $f_{+}(\mathcal{E}') \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.*
- *For any closed subscheme Z of X, $\mathbb{D}(\mathcal{E}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$ and $\mathbb{R}\underline{\Gamma}^{\dagger}_{Z}(\mathcal{E}) \in (F\text{-})D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}})$.*
- *The stability under $\otimes$ of the overholonomicity <u>with Frobenius structure</u> is obtained at the end of this work.*

Until the end and unless otherwise specified, E will be a convergent log-isocrystal on $X^{\#}/\mathcal{T}_K$. Also, $\mathcal{E} := \mathrm{sp}_*(E)$ will be the corresponding coherent $\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}$-module which is locally projective of finite type as $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}$-module.

Definition

- An exponent of E along $\mathcal{Z}_i$ (an irreducible component of $\mathcal{Z}$) is a eigenvalue of the residue of ∇ along $\mathcal{Z}_{iK}$.
- A p-adic integer α is a p-adic Liouville number if the radius of convergence of formal power series, either $\sum_{n\in\mathbb{Z}_{\geq 0}, n\neq\alpha} x^n/(n-\alpha)$ or $\sum_{n\in\mathbb{Z}_{\geq 0}, n\neq-\alpha} x^n/(n+\alpha)$, is less than 1.

- For any integer m, let $\mathcal{I}_{\mathcal{Z}}$ be the sheaf of ideals corresponding to $\mathcal{Z} \hookrightarrow \mathfrak{X}$ and pose:

$$\mathcal{E}(m\mathcal{Z}) := \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{I}_{\mathcal{Z}}^{\otimes -m}.$$

- Let $u : \mathfrak{X}^{\#} \to \mathfrak{X}$ be the canonical morphism. We compute the isomorphism:

$$u_{+}(\mathcal{E}) \xrightarrow{\sim} \left(\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathcal{Z})\right) \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}} \mathcal{E}.$$

- The canonical inclusion $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(\mathcal{Z}) \subset \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}}$ induces the morphism

$$\rho : u_{+}(\mathcal{E}) \to \mathcal{D}^{\dagger}_{\mathfrak{X}}({}^{\dagger}Z)_{\mathbb{Q}} \otimes_{\mathcal{D}^{\dagger}_{\mathfrak{X}^{\#},\mathbb{Q}}} \mathcal{E} =: \mathcal{E}({}^{\dagger}Z) = \mathrm{sp}_{*}(j^{\dagger}_{Y} E).$$

- The *exact triangle of localization* of $u_{+}(\mathcal{E})$ with respect to Z is:

$$\mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) \to u_{+}(\mathcal{E}) \xrightarrow{\rho} \mathcal{E}({}^{\dagger}Z) \to +1.$$

The comparison theorem.

Theorem (Comparison theorem between relative log-rigid cohomology and relative rigid cohomology)

Suppose that, along each irreducible component of $\mathcal{Z}$, E satisfies the following (a), (b) and (c) conditions:

(a) *none of differences of exponents is a p-adic Liouville number,*

(b) *none of exponents is a p-adic Liouville number,*

(c) *none of exponents is a positive integer.*

Then the canonical morphism

$$\mathbb{R}g_{K*}(\Omega^{\bullet}_{\mathfrak{X}^{\#}_{K}/\mathcal{T}_K} \otimes_{\mathcal{O}_{\mathfrak{X}_K}} E) \to \mathbb{R}g_{K*}(\Omega^{\bullet}_{\mathfrak{X}_K/\mathcal{T}_K} \otimes_{\mathcal{O}_{\mathfrak{X}_K}} j^{\dagger}_{Y} E)$$

is an isomorphism.

Remark

- In the heart of the proof, we need the so called Christol's transfert theorem.
- The comparison theorem is not any more true if we remove the condition (*c*).
  Counter-example: The convergent log isocrystal $\mathrm{sp}^*\mathcal{O}_{\mathfrak{X}}(-\mathcal{Z})_{\mathbb{Q}}$ satisfies the conditions (*a*) and (*b*) but not the comparison theorem.

In fact, we have the strongest fact (consequence of the above comparison theorem):

Theorem

If $\mathcal{E}$ satisfies conditions (*a*), (*b*) *and* (*c*), *then the canonical morphism* $u_+(\mathcal{E}) \to \mathcal{E}(^\dagger Z)$ *is an isomorphim.*

Theorem

Suppose that $\mathcal{E}$ satisfies the following condition:

(NL) none of elements of $\mathrm{Exp}(\mathcal{E})^{\mathrm{gr}}$, the group generated by the exponents of $\mathcal{E}$, is a p-adic Liouville number.

Then $u_+(\mathcal{E})$ is overholonomic.
In particular, since $u_+(\mathcal{E})(^\dagger Z) \xrightarrow{\sim} \mathcal{E}(^\dagger Z)$, so is the overconvergent isocrystal $\mathcal{E}(^\dagger Z)$ on U associated to $\mathcal{E}$.

Let $r \geq -1$, $n \geq 0$ be two integers. When the dimension of smooth formal schemes are $\leq n$, let us consider the next properties

($P_{n,r}$) The module $u_+(\mathcal{E})$ is r-overholonomic ;

($Q_{n,r}$) The complex $\mathbb{R}\underline{\Gamma}^\dagger_Z u_+(\mathcal{E})$ is r-overholonomic ;

($R_{n,r}$) The module $\mathcal{E}(^\dagger Z)$ is r-overholonomic.

I.1) *Reduction to the affine case.*
Since $(Q_{n,r})$ is local in $\mathfrak{X}$, from Kedlaya's theorem, we can suppose that there exist a finite étale morphism $h : \mathfrak{X} \to \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$. Denote by $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ the coordinate hyperplanes of $\mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}$, $\mathfrak{H} := \mathfrak{H}_1 \cup \cdots \cup \mathfrak{H}_r$, $\mathfrak{Z}' := h^{-1}(\mathfrak{H})$. More precisely, we get from Kedlaya's theorem the canonical commutative diagram:

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, equal] & \mathfrak{X} \arrow[r, "h"] & \mathfrak{A} = \mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\} \\
(\mathfrak{X}, \mathfrak{Z}) \arrow[u, "u"] & (\mathfrak{X}, \mathfrak{Z}') \arrow[l] \arrow[u, "v"] \arrow[r, "h^{\#}"] & \mathfrak{A}^{\#} = (\mathrm{Spf}\,\mathcal{V}\{t_1, \ldots, t_n\}, \mathfrak{H}), \arrow[u, "w"]
\end{tikzcd}$$

where the right square is cartesian.
We reduce to the case where $u = v$ (technical) and then $u = w$ (easier).

We get the following canonical diagram

$$\begin{tikzcd}
\mathfrak{H}_1 \arrow[r, hook, "i_1"] & \mathfrak{A} \arrow[r, "g_1"] & \mathfrak{H}_1 \\
\mathfrak{H}_1^{\#} \arrow[u, "w_1"] \arrow[r, hook, "i_1"] & \mathfrak{A}^{\#} \arrow[u, "w"] \arrow[r, "g_1"] & \mathfrak{H}_1^{\#}, \arrow[u, "w_1"]
\end{tikzcd}$$

where $\mathfrak{H}_1' = \cup_{i=2}^{s} \mathfrak{H}_1 \cap \mathfrak{H}_i$ and $\mathfrak{H}_1^{\#} := (\mathfrak{H}_1, \mathfrak{H}_1')$.

Theorem (Description of the cone of the comparison morphism)

Suppose furthermore that $\mathcal{E}$ satisfies the (NL) conditions. Then the complex

$$\mathrm{Cone}\left(g_{1+}^{\#}(\mathcal{E}) \to g_{1+}(\mathcal{E}({}^{\dagger}H_1))\right)$$

is isomorphic to a complex of coherent $\mathcal{D}^{\dagger}_{\mathfrak{H}_1^{\#}}({}^{\dagger}H_1')_{\mathbb{Q}}$-modules, locally projective of finite type as $\mathcal{O}_{\mathfrak{H}_1}({}^{\dagger}H_1')_{\mathbb{Q}}$-modules and satisfying the (NL) conditions.

Sketch of the proof:

- For m large enough, $\mathcal{E}(m\mathfrak{H}_1)$ satisfies conditions (a), (b), (c). Then: $g_{1+}^{\#}\mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}(m\mathfrak{H}_1)) = 0$.
- We notice $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}(\mathfrak{H}_1)/\mathcal{E}) = \mathcal{E}(\mathfrak{H}_1)/\mathcal{E}$. Then: $g_{1+}^{\#}\mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}(\mathfrak{H}_1)/\mathcal{E}) = g_{1+}^{\#}(\mathcal{E}(\mathfrak{H}_1)/\mathcal{E})$.
- We apply the functor $g_{1+}^{\#}\mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}$ to the exact sequence $0 \to \mathcal{E} \to \mathcal{E}(\mathfrak{H}_1) \to \mathcal{E}(\mathfrak{H}_1)/\mathcal{E} \to 0$ and so on.

I.3) End of the step

From I.2), $g_{1+}^{\#} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E})$ is a complex of coherent $\mathcal{D}^{\dagger}_{\mathfrak{H}_1^{\#},\mathbb{Q}}$-modules which are locally projective $\mathcal{O}_{\mathfrak{H}_1,\mathbb{Q}}$-modules of finite type satisfying the (NL) condition. Then, by $(P_{n-1,r})$, $w_{1+} \circ g_{1+}^{\#} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E})$ is r-overholonomic. Moreover, since $i_{1+} i_1^! w_+(\mathcal{E}) \xrightarrow{\sim} \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1} w_+(\mathcal{E})$,

$$w_{1,+} \circ g_{1+}^{\#} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}) \xrightarrow{\sim} g_{1+} \circ w_+ \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1}(\mathcal{E}) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} g_{1+} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{H_1} \circ w_+(\mathcal{E}) \xrightarrow{\sim} i_1^! w_+(\mathcal{E}).$$

Hence $i_1^! w_+(\mathcal{E})$ is r-overholonomic and then so is $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_1} w_+(\mathcal{E})$. Symmetrically, we obtain for any $i = 1, \ldots, r$ that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H_i} w_+(\mathcal{E})$ is r-overholonomic. This implies that $\mathbb{R}\underline{\Gamma}^{\dagger}_{H} w_+(\mathcal{E})$ is r-overholonomic (using Mayer-Vietoris exact triangles and the stability of r-overholonomicity by local cohomological functors).

Since the case $r = 0$ is similar, we can suppose $r \geq 1$. It is sufficient to prove:

$$\forall D \subset X \text{ divisor, } \mathbb{D}(\mathcal{E}(^\dagger Z \cup D)) \text{ is } (r-1)\text{-overholonomic.}$$

1) We come down to the case where $D = \emptyset$: by using de Jong's desingularization theorem, there exist a commutative diagram

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "f"] & \mathfrak{X} \\ X' \arrow[u, hook] \arrow[r, "a_0"] & X \arrow[u, hook] \end{tikzcd},$$

such that f is a proper smooth morphism, a_0 is proper, surjective, generically finite and étale, X' is smooth, $Z' := a_0^{-1}(Z \cup D)$ is a strict NCD of X'. We pose $a_0^\sharp : (X', Z') \to (X, Z)$.

Then $\mathcal{E}' := \mathbb{R}\underline{\Gamma}^{\dagger}_{X'} f^{!}(\mathcal{E}({}^{\dagger}Z \cup D))$ "is associated to $j^{\dagger}_{Z'} a_0^{\sharp *}(E)$" (this has locally a meaning) as an isocrystal on $X' \setminus Z'$ overconvergent Z'. Moreover, $\mathcal{E}({}^{\dagger}Z \cup D)$ is a direct factor of $f_{+}(\mathcal{E}')$. Since the $(r-1)$-overholonomicity is stable by f_{+}, we reduce to the case where $D = \emptyset$.
2) By applying $\mathbb{D}$ to the exact triangle of localization of $u_{+}(\mathcal{E})$ with respect to Z, we get:

$$\mathbb{D}(\mathcal{E}({}^{\dagger}Z)) \to \mathbb{D} \circ u_{+}(\mathcal{E}) \to \mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E}) \to +1.$$

- We have the *log-relative duality isomorphism*:

  $$\mathbb{D} \circ u_{+}(\mathcal{E}) \xrightarrow{\sim} u_{+}(\mathcal{E}^{\vee}(-\mathcal{Z})),$$

  where $\mathcal{E}^{\vee} = \mathcal{H}om_{\mathcal{O}_{\mathfrak{X},\mathbb{Q}}}(\mathcal{E}, \mathcal{O}_{\mathfrak{X},\mathbb{Q}})$. Since $\mathcal{E}^{\vee}(-\mathcal{Z})$ satisfies the (NL) condition, by $(P_{n,r-1})$ this implies that $\mathbb{D} \circ u_{+}(\mathcal{E})$ is $(r-1)$-overholonomic.
- By $(Q_{n,r})$, $\mathbb{D} \circ \mathbb{R}\underline{\Gamma}^{\dagger}_{Z} \circ u_{+}(\mathcal{E})$ is $(r-1)$-overholonomic.

Step (III): Conclusion.

a) We have checked:

- $(P_{n,r-1}) + (Q_{n,r}) \Rightarrow (R_{n,r})$ for any $n \geq 0$, $r \geq 0$.
- $\forall\ n \geq 1$, $r \geq -1$, $(P_{n-1,r}) \Rightarrow (Q_{n,r})$.

This implies $(P_{n,r-1}) + (P_{n-1,r}) \Rightarrow (Q_{n,r}) + (R_{n,r})$. Using the exact triangle of localisation, we get: $(Q_{n,r}) + (R_{n,r}) \Rightarrow (P_{n,r})$. Then:

$$(P_{n,r-1}) + (P_{n-1,r}) \Rightarrow (P_{n,r}).$$

b) Since $(P_{0,r})$ and $(P_{n,-1})$ are satisfied, we have finished the proof.

Theorem (Kedlaya's semistable reduction theorem)

Let Y be a smooth irreducible k-variety, X be a partial compactification of Y, $Z := X \setminus Y$, E be an F-isocrystal on Y overconvergent along Z. Then E admits semistable reduction, i.e., there exist

1. *a proper, surjective, generically étale morphism $f : X' \to X$,*
2. *an open immersion $X' \hookrightarrow \overline{X}'$ into a smooth projective variety over k such that $Z' := f^{-1}(Z) \cup (\overline{X}' \setminus X')$ is a strict normal crossing divisor of $\overline{X}'$*

such that the isocrystal $f^*(E)$ on $Y' := f^{-1}(Y)$ overconvergent along $Z' \cap X'$ is log-extendable on X' (i.e. there exists a log-isocrystal with nilpotent residues convergent on the log-scheme $(X', Z' \cap X')$ whose induced overconvergent isocrystal on Y' is $f^*(E)$).

Theorem

- *We have a equivalence of categories of the form*

$$\mathrm{sp}_{(Y,X),+} : \mathrm{Isoc}^{\dagger}(Y, X/K) \cong \mathrm{Isoc}^{\dagger\dagger}(Y, X/K),$$

  where $\mathrm{Isoc}^{\dagger\dagger}(Y, X/K)$ *is a category of some arithmetic* $\mathcal{D}$*-modules.*
- *The objects of* $F\text{-}\mathrm{Isoc}^{\dagger\dagger}(Y, X/K)$ *are overholonomic.*

Proof of the second point.

Using Kedlaya's semistable reduction theorem and the stability of overholonomicity under direct image and under extraordinary inverse image we reduce to the case of log-extendable F-isocrystals, which was already checked. $\square$

Devissage in overconvergent isocrystals

Theorem

Let $\mathcal{P}$ be a separated smooth formal scheme over $\mathcal{V}$, T a divisor of P, $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\dagger}_{\mathcal{P}}({}^{\dagger}T)_{\mathbb{Q}})$. Consider the following assertions:

1. *The complex $\mathcal{E}$ is overholonomic;*
2. *The complex $\mathcal{E}$ is devissable in overconvergent isocrystals.*

We have always $1 \Rightarrow 2$. The converse is true with Frobenius structures.

Corollary

The overholonomicity with Frobenius structure is stable under tensor products.