Logarithmic extension of overconvergent isocrystals and an application

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1.1 Log-∇-modules

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1.1 Log-∇-modules

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O_K: ring of integers of K,

$|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$: valuation, $\Gamma^* := \sqrt{|K|}$.

§1. Log extension of overconvergent isocrystals

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$|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$: valuation, $\Gamma^* := \sqrt{|K|}$.

$K \subseteq L$: a complete field w.r.t. a multiplicative norm extending $|\cdot|$ (denoted also by $|\cdot|$).

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a (log-)∇-module (E, ∇) on $\mathfrak{X}$:=

a locally free module E of finite rank on $\mathfrak{X}$ +

integrable (log) connection $\nabla : E \longrightarrow E \otimes \omega^1_{\mathfrak{X}/L}$.

Assume $\mathfrak{X}$ smooth & log. str. on $\mathfrak{X}$ is defined by

$\mathfrak{D} = \bigcup_{i=1}^{r} \mathfrak{D}_i = \bigcup_{i=1}^{r} \{t_i = 0\}$ an SNCD.

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For a log-∇-module (E, ∇) on $\mathfrak{X}$, the composite

$$E \xrightarrow{\nabla} E \otimes \omega^1_{\mathfrak{X}/L} \twoheadrightarrow E|_{\mathfrak{D}_i} \mathrm{dlog}\, t_i \cong E|_{\mathfrak{D}_i}$$

induces $\mathrm{res}_i \in \mathrm{End}(E|_{\mathfrak{D}_i})$ (residue along $\mathfrak{D}_i$).

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Exponents of (E, ∇) along $\mathfrak{D}_i$:= the roots of minimal monic $P_i(x) \in L[x]$ with $P_i(\mathrm{res}_i) = 0$.

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For an aligned interval I, we put

$$A_L^n(I) := \{x \in \mathbb{A}_L^{n,\mathrm{an}} \mid \forall i, |t_i(x)| \in I\}.$$

$(t_1, \ldots, t_n$: coordinate) with log structure defined by $\bigcup_i \{t_i = 0\}$.

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For $\xi := (\xi_i)_i \in \mathbb{Z}_p^n$, we define the log-$\nabla$-module (M_ξ, ∇_{M_ξ}) on $A_L^n(I)$ by

$$(M_\xi, \nabla_{M_\xi}) := (\mathcal{O}, d + \textstyle\sum_i \xi_i \mathrm{dlog}\, t_i).$$

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(1) A log-∇-module (E, ∇) on $\mathfrak{X} \times A_L^n(I)$ is **Σ-constant** iff

$$(E, \nabla) = \pi_1^*(F, \nabla_F) \otimes \pi_2^*(M_\xi, \nabla_{M_\xi})$$

for some (F, ∇_F) on $\mathfrak{X}$, $\xi \in \Sigma$.

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(2) (E, ∇) is **Σ-semisimple** iff it is a direct sum of Σ-constant ones.

(3) (E, ∇) is **Σ-unipotent** iff it is a succesive extension of Σ-constant ones.

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$j : X \hookrightarrow \overline{X}$: open imm. of smooth k-varieties

s.t. $(\overline{X} \setminus X)_{\mathrm{red}} =: Z = \bigcup_{i=1}^{r} Z_i$ is an SNCD.

($\rightsquigarrow (\overline{X}, Z)$: a log scheme)

Definition

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(1) A charted standard small frame (css frame) enclosing $\overline{U}$ is $(\overline{U}, \overline{\mathcal{X}}, \{t_i\}_{i=1}^r)$, where $\overline{\mathcal{X}}$ is a smooth lift of $\overline{U}$ to a p-adic formal scheme over $\mathrm{Spf}\, O_K$, $\mathcal{Z} = \bigcup_i \{t_i = 0\}$ is an SNCD on $\overline{\mathcal{X}}$ lifting $Z \cap \overline{U}$.

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(2) A charted smooth standard small frame (csss frame) enclosing $\overline{U}$ is $(\overline{U}, \overline{\mathcal{X}}, t)$, where $\overline{\mathcal{X}}$ is a smooth lift of $\overline{U}$ to a p-adic formal scheme over $\mathrm{Spf}\, O_K$, $\mathcal{Z} = \{t = 0\}$ is a conn. smooth divisor on $\overline{\mathcal{X}}$ lifting $Z \cap \overline{U}$. (So $Z \cap \overline{U} = Z_i \cap \overline{U}$ for some i)

(3) A csss frame with generic point enclosing $\overline{U}$ is $(\overline{U}, \overline{\mathcal{X}}, t, L)$, where $(\overline{U}, \overline{\mathcal{X}}, t)$: csss frame, L: a field containing $\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K})$ (where $\mathcal{Z} := \{t = 0\}$) which is complete w.t.r. a multiplicative norm $|\cdot|$ which extends the supremum norm on $\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K})$.

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(1) An overconvergent isocrystal on $(X, \overline{X})$ is a ∇-module on $\mathfrak{U}_\lambda = \{x \in \overline{\mathcal{X}}_K \mid \forall i, |t_i(x)| \geq \lambda\}$ for some $\lambda \in [0, 1) \cap \Gamma^*$ satisfying the 'overconvergent condition'.

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(2) A log convergent isocrystal on $(\overline{X}, Z)$ is a log-∇-module on $(\overline{\mathcal{X}}_K, \mathcal{Z}_K)$ satisfying the 'log-convergent condition'.

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$\mathbf{Isoc}^{\dagger}(X, \overline{X})$: the cat. of overconvergent isocrystals

$\mathbf{Isoc}^{\mathbf{log}}(\overline{X}, Z)$: the cat. of log convergent isocrystals

Recall that $a \in \mathbb{Z}_p$ is p-adically non-Liouville if $\underline{\lim}_{n\to\infty} |n - a|^{1/n} = \underline{\lim}_{n\to\infty} |n + a|^{1/n} = 1.$

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<u>Example</u>: $\forall a \in \mathbb{Z}_{(p)}$ is p-adically non-Liouville.

<u>Definition</u> $\Sigma = \prod_{i=1}^{r} \Sigma_i \subseteq \mathbb{Z}_p^r$ is **(NID)** (resp. **(NLD)**) if $\forall i, \forall a, b \in \Sigma_i$, $a-b$ is not in $\mathbb{Z}_{\neq 0}$ (resp. is p-adically non-Liouville).

$X \hookrightarrow \overline{X}, Z$ as above, $\overline{U} \hookrightarrow \overline{X}$ open affine. $(\overline{U}, \overline{\mathcal{X}}, t, L)$: a csss frame with generic point enclosing $\overline{U}$, $\mathcal{Z} = \{t = 0\}$.

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iff $\forall \overline{U} \hookrightarrow \overline{X}, \forall (\overline{U}, \overline{\mathcal{X}}, t, L)$: a csss frame with generic point, the ∇-module $(E_{\mathcal{E},L}, \nabla_{\mathcal{E},L})$ on $A_L^1[\lambda, 1)$ is Σ-semisimple (Σ-unipotent).

$\mathbf{Isoc}^{\dagger}(\boldsymbol{X}, \overline{\boldsymbol{X}})_{\boldsymbol{\Sigma}\text{-}\mathbf{ss}}$: the cat. of overconv. isocrystals on $(\boldsymbol{X}, \overline{\boldsymbol{X}})$ having $\boldsymbol{\Sigma}$-semisimple monodromy.

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<u>Remark</u>

The above categories depend only on $\overline{\boldsymbol{\Sigma}} := \mathbf{Im}\boldsymbol{\Sigma} \subseteq (\mathbb{Z}_{\boldsymbol{p}}/\mathbb{Z})^{\boldsymbol{r}}$. So we also denote them by $\mathbf{Isoc}^{\dagger}(\boldsymbol{X}, \overline{\boldsymbol{X}})_{\overline{\boldsymbol{\Sigma}}\text{-}\mathbf{ss}}$, $\mathbf{Isoc}^{\dagger}(\boldsymbol{X}, \overline{\boldsymbol{X}})_{\overline{\boldsymbol{\Sigma}}}$.

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iff $\forall \overline{U} \hookrightarrow \overline{X}, \forall (\overline{U}, \overline{\mathcal{X}}, \{t_i\}_{i=1}^{r})$: a css frame, $\forall i$, the exponents along $\mathcal{Z}_{i,K}$ of the log-∇-module $(\widetilde{E}_{\mathcal{E}}, \widetilde{\nabla}_{\mathcal{E}})$ is in Σ_i (and $P_i(\mathrm{res}_i) = 0$ for some $P_i(x) \in K[x]$ without multiple roots).

$\mathbf{Isoc}^{\mathbf{log}}(\overline{\boldsymbol{X}}, \boldsymbol{Z})_{\boldsymbol{\Sigma}\text{-}\mathbf{ss}}$: the cat. of log conv. isocrystals on $(\overline{\boldsymbol{X}}, \boldsymbol{Z})$ having expontents in $\boldsymbol{\Sigma}$ with semisimple residues.

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Theorem 1

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Then $\exists$ equivalences of categories

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<u>Remark</u>

- The case $\Sigma = \{0\}$: due to Kedlaya.

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- The case $\Sigma = \{0\}$: due to Kedlaya.
- Analogue for semi-stable varieties: di Proietto (talk in this conference).

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The case $r > 1$: Take a css frame and extend $(\widetilde{E}_{\mathcal{E}}, \widetilde{\nabla}_{\mathcal{E}})$ to $(\overline{\mathcal{X}}_K, \mathcal{Z}_K)$ 'step by step', by using property slightly stronger than $*$ in some sense.

(called 'overconvergent generization' property)

Prop. 2

<u>Prop. 2</u> $\quad j : X \hookrightarrow \overline{X}, Z = \bigcup_{i=1}^{r} Z_i$ as above.
$\boldsymbol{\Sigma}^i = \prod_{j=1}^{r} \boldsymbol{\Sigma}_j^i \subseteq \mathbb{Z}_p^r$: $(\mathbf{NID}), (\mathbf{NLD})$ sets
for $i = 1, 2$ s.t. $\forall j, \forall \xi_i \in \boldsymbol{\Sigma}_j^i, \xi_1 - \xi_2 \notin \mathbb{Z}_{<0}$.

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Then, for $\boldsymbol{\mathcal{E}_i \in \mathrm{\mathbf{Isoc}}^{\mathbf{log}}(\overline{X}, Z)_{\Sigma^i\text{-}\mathbf{ss}}\ (i = 1, 2)}$,

$$\boldsymbol{\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \xrightarrow{=} \mathrm{Hom}(j^\dagger \mathcal{E}_1, j^\dagger \mathcal{E}_2)}.$$

<u>Prop. 2</u> $j : X \hookrightarrow \overline{X}, Z = \bigcup_{i=1}^{r} Z_i$ as above. $\Sigma^i = \prod_{j=1}^{r} \Sigma_j^i \subseteq \mathbb{Z}_p^r$: $(\mathbf{NID}), (\mathbf{NLD})$ sets for $i = 1, 2$ s.t. $\forall j, \forall \xi_i \in \Sigma_j^i, \xi_1 - \xi_2 \notin \mathbb{Z}_{<0}$. Then, for $\mathcal{E}_i \in \mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma^i\text{-ss}}$ $(i = 1, 2)$, $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \xrightarrow{=} \mathrm{Hom}(j^\dagger \mathcal{E}_1, j^\dagger \mathcal{E}_2)$.

<u>Proof</u>. Reduce to the local calculation on relative polydisc.

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$\mathcal{O}(\sum_i \alpha_i Z_i) :=$ the unique object in

$\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\prod_i \{-\alpha_i\}\text{-ss}}$ sent to $j^\dagger \mathcal{O}$ by $j^\dagger$,

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For $\mathcal{E} \in \mathrm{Isoc}^{\log}(\overline{X}, Z)$, put

$\mathcal{E}(\sum_i \alpha_i Z_i) := \mathcal{E} \otimes \mathcal{O}(\sum_i \alpha_i Z_i)$.

§2. An application: Parabolic log convergent isocrystals

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K, O_K, k: as before, k: perfect,

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$\mathrm{Rep}_{K^\sigma}(\pi_1(X))$: category of finite dimensional continuous representations of $\pi_1(X)$ over K^σ.

$F\text{-}\mathrm{Isoc}(X)^\circ$: category of unit-root convergent F-isocrystals on X.

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For $n \in \mathbb{N}$ prime to p,

$$(\overline{X}, Z)^{1/n} := \overline{X} \times_{[\mathbb{A}_k^r/\mathbb{G}_{m,k}^r],n} [\mathbb{A}_k^r/\mathbb{G}_{m,k}^r]$$

(stack of roots, Cadman, Borne, Iyer-Simpson)

(cf. talk of Vistoli)

Local description: if $\overline{X} = \operatorname{Spec} R$, $Z_i = \{t_i = 0\}$,

$$\overline{X}^{(n)} := \operatorname{Spec} R[s_i]_{1 \le i \le r}/(s_i^n - t_i)_{1 \le i \le r}$$

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We can define

$\mathrm{Isoc}((\overline{X}, Z)^{1/n}) \ \ (F\text{-}\mathrm{Isoc}((\overline{X}, Z)^{1/n})^{\circ})$:

the category of (unit-root) convergent (F-) isocrystals on $(\overline{X}, Z)^{1/n}$.

Prop. 3

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$$\mathbf{Rep}_{K^\sigma}(\pi_1^t(X)) \xrightarrow{=} \varinjlim_{(n,p)=1} F\text{-}\mathbf{Isoc}((\overline{X}, Z)^{\frac{1}{n}})^\circ.$$

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Key ingredients

- the restriction of $\rho \in \mathrm{Rep}_{K^\sigma}(\pi_1^t(X))$ to $\pi_1^t(Y)$ for some tame covering $Y \longrightarrow X$ factors through $\pi_1(\overline{Y}^{\mathrm{sm}})$.

($\overline{Y}$: norm. of $\overline{X}$ in $k(Y)$, $\overline{Y}^{\mathrm{sm}} \subseteq \overline{Y}$ sm. locus.)

- Abhyankar's lemma.

- Crew's equivalence.

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(a) $\forall i, \exists$ isom. $(\mathcal{E}_{\alpha+e_i})_\alpha \cong (\mathcal{E}_\alpha(Z_i))_\alpha$

via which the transition map $(\mathcal{E}_\alpha)_\alpha \longrightarrow (\mathcal{E}_{\alpha+e_i})_\alpha$

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(b) $\exists n \in \mathbb{N}, (n, p) = 1$ s.t. $\mathcal{E}_{[n\alpha]/n} \xrightarrow{=} \mathcal{E}_\alpha$.

Definition

<u>Definition</u> **A unit-root parabolic log convergent F-isocrystal** on $(\overline{X}, Z)$ is $((\mathcal{E}_\alpha)_\alpha, \Psi)$, where

$(\mathcal{E}_\alpha)_\alpha$: a parabolic log conv. isocrystal

$\Psi : \varinjlim (F^*\mathcal{E}_\alpha)_\alpha \longrightarrow \varinjlim (\mathcal{E}_\alpha)_\alpha$: isom as ind-objects s.t. it is unit-root when restricted to X.

Definition

<u>Definition</u> A (unit-root) parabolic log convergent (F-)isocrystal $\mathcal{E}$ on $(\overline{X}, Z)$ is adjusted if, for $\forall \overline{U} \hookrightarrow \overline{X}$ open affine, $\forall(\overline{U}, \overline{\mathcal{X}}, t, L)$ a csss frame with generic point, the ind. system of log-∇-modules $(E_\alpha, \nabla_\alpha)_\alpha$ on $A^1_L[0,1)$ induced by $\mathcal{E}$ is isom. to

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$$(\oplus_{j=1}^{\mu}(\mathcal{O}_{A^1_L[0,1)}, d - \lceil \gamma_j \rceil_{\alpha_i} \mathrm{dlog}\, t))_\alpha$$

for some $\gamma_j \in (-1, 0] \cap \mathbb{Z}_{(p)}$ $(1 \leq j \leq \mu)$, where

i: the unique index with $\overline{U} \cap Z_i \neq \emptyset$,

$\lceil \gamma_j \rceil_{\alpha_i} :=$ the unique elt in $(\alpha_i - 1, \alpha_i] \cap (\gamma_j + \mathbb{Z})$.

(transition: multiplication by some power of t.)

$\mathbf{Par\text{-}Isoc}^{\mathbf{log}}(\overline{\boldsymbol{X}}, \boldsymbol{Z})$: **the cat. of adjusted parabolic log conv. isocrystals on** $(\overline{\boldsymbol{X}}, \boldsymbol{Z})$**.**

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$\mathbf{Par\text{-}}\boldsymbol{F}\mathbf{\text{-}Isoc}^{\mathrm{log}}(\overline{X}, Z)^{\circ}$: the cat. of unit-root adjusted parabolic log conv. F-isocrystals on $(\overline{X}, Z)$.

Theorem 4

Cor. 5

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$$\varinjlim_{(n,p)=1} \mathrm{Isoc}((\overline{X}, Z)^{\frac{1}{n}}) \xrightarrow{=} \text{Par-Isoc}^{\log}(\overline{X}, Z),$$

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<u>Cor. 5</u> $\exists$ an equivalence of categories

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Suffices to prove the equivalences

$$\mathbf{Isoc}((\overline{\boldsymbol{X}},\boldsymbol{Z})^{\frac{1}{n}}) \xrightarrow{=} \mathbf{Isoc}^{\dagger}(\boldsymbol{X},\overline{\boldsymbol{X}})_{(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^{r}\text{-}\mathbf{ss}},$$

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The ess. surj. of the first eq: Take covering

$$\overline{X}^{(n)} \longrightarrow (\overline{X}, Z)^{1/n} = [\overline{X}^{(n)}/\mu_n]$$

and use Theorem 1 for $\Sigma = \{0\}$ on $\overline{X}^{(n)}$.

The ess. surj. of the second eq: For

$\alpha = (\alpha_i)_i \in \mathbb{Z}_{(p)}^r$,

let $\Sigma_\alpha := \prod_{i=1}^r (-((\alpha_i - 1, \alpha_i] \cap \mathbb{Z}_{(p)}))$.

$\mathcal{E}_\alpha :=$ the unique object in $\mathrm{Isoc}^{\log}(\overline{X}, Z)_{\Sigma_\alpha\text{-ss}}$ extending the given object in $\mathrm{Isoc}^\dagger(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r\text{-ss}}$. (Theorem 1)

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extending the given object in

$\mathrm{Isoc}^{\dagger}(X, \overline{X})_{(\mathbb{Z}_{(p)}/\mathbb{Z})^r\text{-ss}}$. (Theorem 1)

By Prop. 2, $(\mathcal{E}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}}$ forms an inductive system,

and we can check that it is indeed adjusted. Done.

Thank you very much!