Existence of Gibbsian point processes with geometry-dependent interactions

D. Dereudre, R. Drouilhet and H.-O. Georgii

Plan

Motivation

Gibbs framework

- **Classical framework in $\mathbb{R}^d$:** Point process with (pairwise) interaction on the complete graph (ex: Ruelle class of superstable model, Lennard-jones model,...).
- **Classical framework in $\mathbb{Z}^d$:** Lattice field with (pairwise) interaction on the nearest-neighbour graph (Ising model, Potts model,...)
- **New framework in $\mathbb{R}^d$:** Point process with (pairwise) interaction on the nearest-neighbour graph such the Delaunay graph (for example). Introduced by Baddeley-Moller in some bounded domain.
- **Problem:** Existence of such kind of model defined as a stationary point process in $\mathbb{R}^d$.

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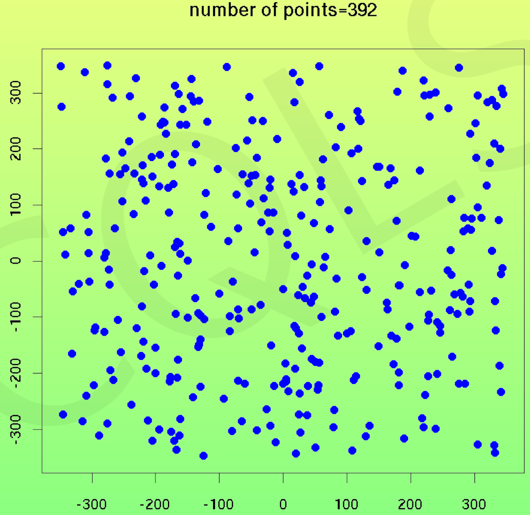
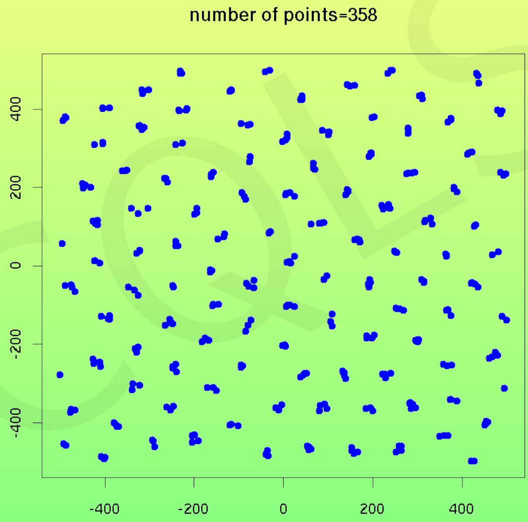
Point process without interaction

Poisson process with intensity 0.0016 (mean=400 points in the domain)



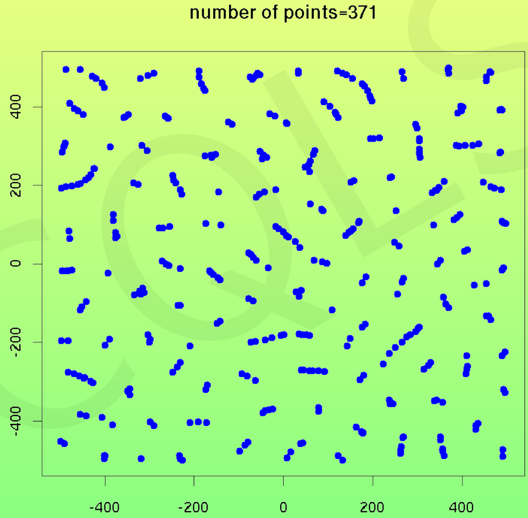
Point process with classical interaction

Multi-Strauss point process



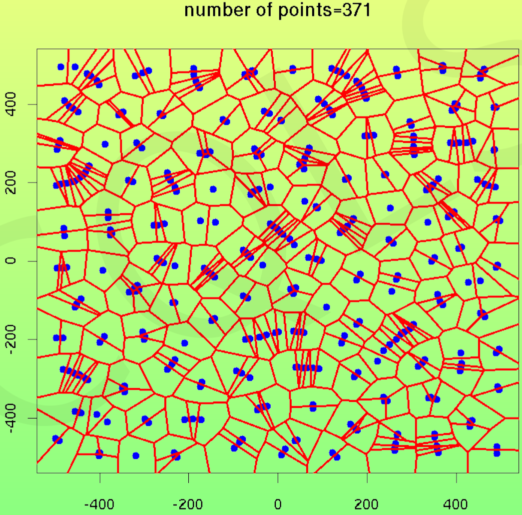
Point process with Delaunay neighbour interaction

Delaunay Multi-Strauss point process



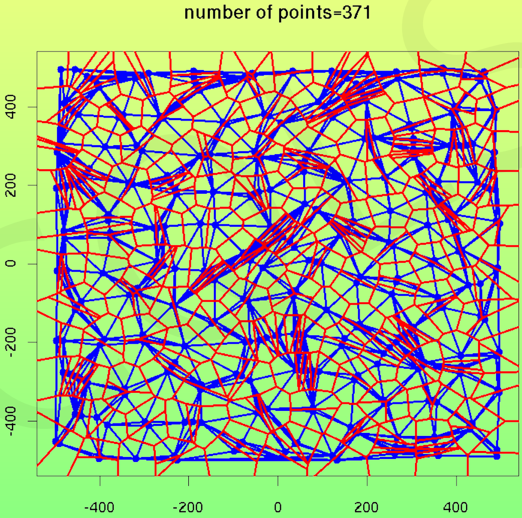
Point process with Delaunay neighbour interaction

The Voronoï diagram



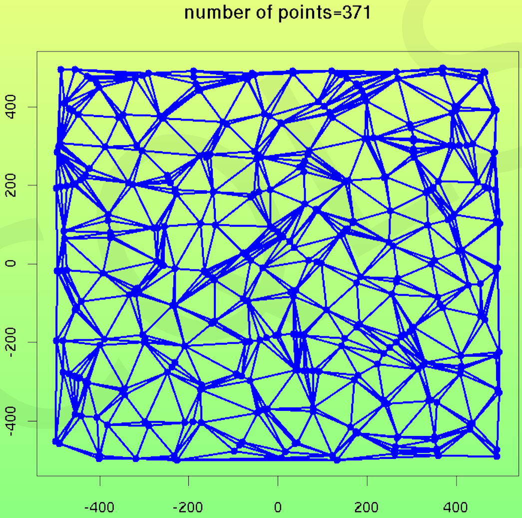
Point process with Delaunay neighbour interaction

The Voronoï diagram and its dual graph



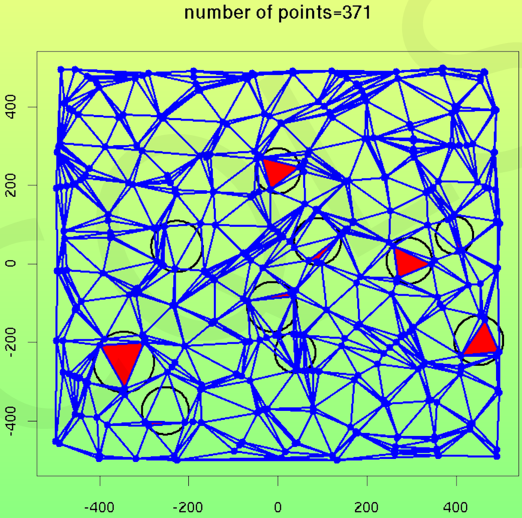
Point process with Delaunay neighbour interaction

This is the Delaunay graph



Point process with Delaunay neighbour interaction

No other points in any circle circumscribing a Delaunay triangle



Point processes: definition and notation

Notation

- $\Delta \Subset \mathbb{R}^d$ and $\Lambda \Subset \mathbb{R}^d$ means Δ and Λ are bounded Borelian sets.
- Let $\Lambda \subset \mathbb{R}^d$ and $\varphi \in \Omega$, $\varphi_\Lambda := \varphi \cap \Lambda \in \Omega_\Lambda$
- **Useful notation:** sum over all configurations φ in Λ

$$\oint_\Lambda d\varphi\, g(\varphi) := \sum_{n=0}^{+\infty} \frac{1}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n\, g\left(\{x_1, \cdots, x_n\}\right)$$

- Poisson measure Π_Λ: $\displaystyle\int_{\Omega_\Lambda} \Pi_\Lambda(d\varphi) g(\varphi) := e^{-|\Lambda|} \oint_\Lambda d\varphi g(\varphi)$

Point process in some domain $\Lambda \subset \mathbb{R}^d$

A point process in Λ is a random variable Φ_Λ with values in Ω_Λ equipped with the smallest σ-field which make measurable all the maps $i_\Delta : \varphi \in \Omega_\Lambda \to |\varphi_\Delta|$ with $\Delta \subset \Lambda \in \mathcal{B}_b$.

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$$\oint_\Lambda^z d\varphi\, g(\varphi) := \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int_\Lambda \cdots \int_\Lambda dx_1 \cdots dx_n\, g\left(\{x_1, \cdots, x_n\}\right)$$

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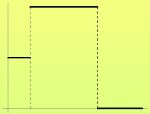
Gibbs Distribution in Λ

$$P_\Lambda(F) = Z_\Lambda^{-1} \oint_\Lambda d\varphi \mathbb{1}_F(\varphi) e^{-V(\varphi)}$$

$$V(\varphi) = \theta_1 |\varphi| + \sum_{\xi \in G_2(\varphi)} g_2(\xi).$$

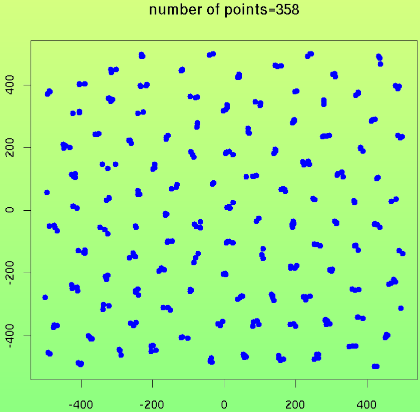
$$g_2(\xi) = \theta_2 \mathbb{1}_{[d_1,d_2[}(\|\xi\|) + \theta_3 \mathbb{1}_{[d_2,d_3[}(\|\xi\|)$$



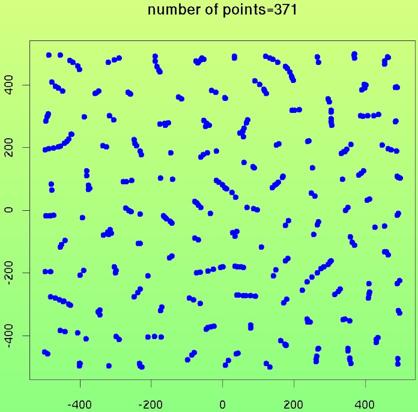
with

$\theta_2 = 2, \theta_3 = 4$

$\mathbf{d} = (0, 20, 80)$



$G_2(\varphi) = \mathcal{P}_2(\varphi)$ and $\theta_1 = -2$



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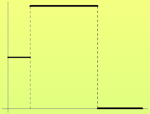

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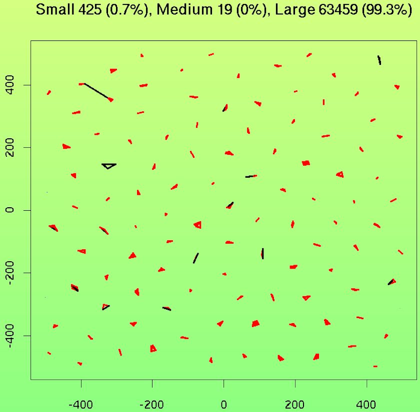

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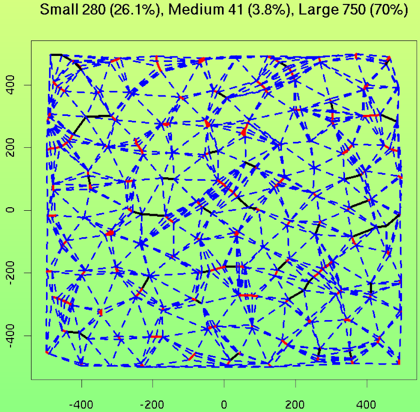


$G_2(\varphi) = Del_2(\varphi)$ and $\theta_1 = 2$

Plan

Gibbs Point Process in bounded domain Λ

Global distribution: $P_\Lambda(F) = \oint_\Lambda d\varphi \mathbf{f}_{\Phi_\Lambda}(\varphi) \mathbb{1}_F(\varphi)$	
Gibbs: $\dfrac{\exp(-\mathbf{V}(\varphi))}{\oint_\Lambda d\varphi \exp(-\mathbf{V}(\varphi))}$	*Poisson:* $\dfrac{1}{\exp(\mathbf{z}\|\Lambda\|)}$
Marginal distribution: $P_{\Lambda,\Delta}(F_\Delta) = \oint_\Delta d\varphi \mathbf{f}_{\Phi_\Delta}(\varphi) \mathbb{1}_{F_\Delta}(\varphi)$ with $\mathbf{f}_{\Phi_\Delta}(\varphi) = \oint_{\Lambda\setminus\Delta} d\varphi^o \mathbf{f}_{\Phi_\Lambda}(\varphi \cup \varphi^o)$	
Gibbs: **generally not explicit**	*Poisson:* $\dfrac{1}{\exp(\mathbf{z}\|\Delta\|)}$
Conditional distribution: $P_{\Lambda,\Delta}(F_\Delta\|\varphi^o) = \oint_\Delta d\varphi \mathbf{f}_{\Phi_\Delta}^{\Phi_{\Lambda\setminus\Delta}=\varphi^o}(\varphi) \mathbb{1}_{F_\Delta}(\varphi)$ with $\mathbf{f}_{\Phi_\Delta}^{\Phi_{\Lambda\setminus\Delta}=\varphi^o}(\varphi) = \frac{\mathbf{f}_{\Phi_\Lambda}(\varphi\cup\varphi^o)}{\mathbf{f}\Phi_{\Lambda\setminus\Delta}(\varphi^o)}$	
Gibbs: $\dfrac{\exp(-\mathbf{V}(\varphi\|\varphi^o))}{\oint_\Delta d\varphi \exp(-\mathbf{V}(\varphi\|\varphi^o))}$	*Poisson:* $\dfrac{1}{\exp(\mathbf{z}\|\Delta\|)}$

where $\mathbf{V}(\varphi|\varphi^o) := V(\varphi \cup \varphi^o) - V(\varphi^o)$ (energy to insert φ in φ^o).

(Stationary) Gibbs Point Process in $\Lambda = \mathbb{R}^d$

Global distribution: $P(F) = \oint_\Lambda d\varphi f_{\Phi_\Lambda}(\varphi) \mathbb{1}_F(\varphi)$	
Gibbs: $\dfrac{\mathbf{\exp(-V(\varphi))}}{\oint_{\boldsymbol{\Lambda}} \mathbf{d\varphi \exp(-V(\varphi))}}$	*Poisson:* $\dfrac{\mathbf{1}}{\mathbf{\exp(z\|\Lambda\|)}}$
Marginal distribution: $P_\Delta(F_\Delta) = \oint_\Delta d\varphi f_{\Phi_\Delta}(\varphi) \mathbb{1}_{F_\Delta}(\varphi)$ with $f_{\Phi_\Delta}(\varphi) = \oint_{\Lambda \setminus \Delta} d\varphi^o f_{\Phi_\Lambda}(\varphi \cup \varphi^o)$	
Gibbs: **generally not explicit**	*Poisson:* $\dfrac{\mathbf{1}}{\mathbf{\exp(z\|\Delta\|)}}$
Conditional distribution: $P_\Delta(F_\Delta \| \varphi^o) = \oint_\Delta d\varphi f_{\Phi_\Delta}^{\Phi_{\Lambda\setminus\Delta} = \varphi^o}(\varphi) \mathbb{1}_{F_\Delta}(\varphi)$ with $f_{\Phi_\Delta}^{\Phi_{\Lambda\setminus\Delta} = \varphi^o}(\varphi) = \dfrac{f_{\Phi_\Lambda}(\varphi \cup \varphi^o)}{f\Phi_{\Lambda\setminus\Delta}(\varphi^o)}$	
Gibbs: $\dfrac{\mathbf{\exp(-V(\varphi\|\varphi^o))}}{\oint_{\boldsymbol{\Delta}} \mathbf{d\varphi \exp(-V(\varphi\|\varphi^o))}}$	*Poisson:* $\dfrac{\mathbf{1}}{\mathbf{\exp(z\|\Delta\|)}}$

where $\mathbf{V(\varphi|\varphi^o)} := \lim_{\Lambda \to \mathbb{R}^d} V(\varphi \cup {\varphi^o}_\Lambda) - V({\varphi^o}_\Lambda)$.

Objective

Stationary Gibbs states

The set $\mathcal{G}_s(V)$ of stationary Gibbs state is nonempty, that is, there exists a translation invariant probability measure P such that:

$$\underbrace{PP_\Delta = P}_{\text{D.L.R. equation}} \quad \Longleftrightarrow \quad \underbrace{P(F|\mathcal{F}_{\Delta^c}) = P_\Delta(F|\cdot)\ P\text{-a.s}}$$

$P =$ distribution of Φ

$P_\Delta(\cdot|\varphi^o) =$ distribution of Φ given $\varphi^o_{\Delta^c}$

General sketch of the proof

- Find $(P_n)_n$ such that $(\mathbf{E_n})$: $P_n P^n_\Delta = P_n$ where $P^n_\Delta \underset{n\to+\infty}{\longrightarrow} P_\Delta$.
- **[GC]** *Gibbs Candidate:* P is an accumulation point of $(P_n)_n$ by relative compactness argument.
- **[GP]** *Gibbs Property:* Prove D.L.R., i.e. $(\mathbf{E_n})$ when $n \to +\infty$.

Plan

Existence of stationary Gibbs models (classical tools)

Restriction to models satisfying:

- **[L] Local property**: $V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) = V(\varphi_\Lambda|\varphi^o_{\widetilde{\Lambda}\setminus\Lambda})$ with $\widetilde{\Lambda} \Subset \mathbb{R}^d$

An interaction function g_2 acting on some graph $\mathcal{G}(\varphi)$ is said to be based on $\mathcal{G}'(\varphi)$$(\subset \mathcal{G}(\varphi))$ if $g_2(\xi) = g_2'(\xi)\mathbf{1}_{\mathcal{G}'(\varphi)}(\xi)$

Assumptions for [L]

- $G_2(\varphi) = \mathcal{P}_2(\varphi)$: **[Range on g_2]** (i.e $g_2(d) = 0$ when $d \geq R$) ($\Leftrightarrow$ [g_2 **based on** $\mathcal{P}^{loc}_{2,R}(\varphi)$] with $\mathcal{P}^{loc}_{2,R}(\varphi)] := \{\xi \in \mathcal{P}_2(\varphi) : ||\xi|| < R\}$)
- $G_2(\varphi) = Del_2(\varphi)$: **[$g_2$ based on $Del^{loc}_{2,R}(\varphi)$]** with
$$Del^{loc}_{2,R}(\varphi) = \bigcup_{\psi \in Del^{loc}_{3,R}(\varphi)} \mathcal{P}_2(\psi)$$
where $R > 0$, $r(\psi)$ the radius of the circumscribed circle of some triangle ψ and $Del^{loc}_{3,R}(\varphi) = \{\psi \in Del_3(\varphi), r(\psi) \leq R\}$.

Existence of stationary Gibbs models (classical tools)

Existence of stationary Gibbs state

1. $\Big(\text{[}\textbf{Superstability}\text{] and [}\textbf{L}\text{]}\Big) \Rightarrow \Big(\mathcal{G}_s(V) \neq \emptyset\Big)$
2. $\Big(\text{([}\textbf{HC}\text{] or [}\textbf{I}\text{]) and [}\textbf{L}\text{]}\Big) \Rightarrow \Big(\text{[}\textbf{LS}\text{] and [}\textbf{L}\text{]}\Big) \Rightarrow \Big(\mathcal{G}_s(V) \neq \emptyset\Big)$

with

- **[LS] Local Stability**: $V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) \geq -K|\varphi_\Lambda|$
- **[HC] Hard-Core**: $V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) = +\infty \Leftarrow (\exists \xi \in \varphi_\Lambda : ||\xi|| < \delta)$
- **[I] Inhibition**: $V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) \geq 0$

Application via **[Superstability]**

- $G_2(\varphi) = \mathcal{P}_2(\varphi)$: tailor-made for this case with g_2 not necessarily nonnegative (but $g_2(0) > 0$)!
- $G_2(\varphi) = Del_2(\varphi)$: **[Superstability]** never true when $d = 2$ (idem when $d > 2$???).

Existence of stationary Gibbs models (classical tools)

Application via **[LS]**

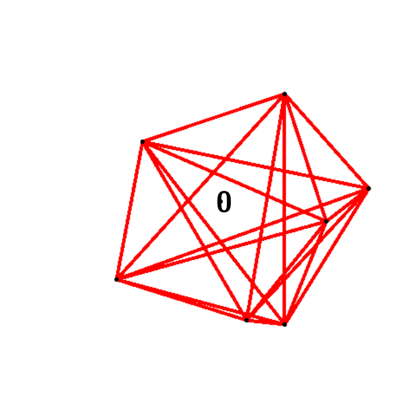
- $G_2(\varphi) = \mathcal{P}_2(\varphi)$:
  1. **[Hard-Core on g_2]** and **[Range on g_2]**
  2. **[Inhibition on g_2 ($g_2 \geq 0$)]**) and **[Range on g_2]**
- $G_2(\varphi) = Del_2(\varphi)$: (Bertin, Billiot, Drouilhet)
  1. **[Hard-Core on g_2]** and **[g_2 based on $Del^{loc}_{2,R}(\varphi)$]**
  2. **[g_2 based on $Del^{\beta_0}_{2,\beta}(\varphi)$]** and **[Range on g_2]** with

$$Del^{\beta_0}_{2,\beta}(\varphi) = \bigcup_{\psi \in Del^{\beta_0}_{3,\beta}(\varphi)} \mathcal{P}_2(\psi)$$

where $\beta_0 \in [0, \pi/3[$, $\beta(\psi)$ the smallest angle of a triangle ψ and $Del^{\beta_0}_{3,\beta}(\varphi) = \{\psi \in Del_3(\varphi), \beta(\psi) > \beta_0\}$.

Local Stability (**complete** versus Delaunay **graph**)

Pointwise local energy ($G_2(\varphi) = \mathcal{P}_2(\varphi)$):
$V(\mathbf{0}|\varphi) := V(\mathbf{0} \cup \varphi) - V(\varphi)$



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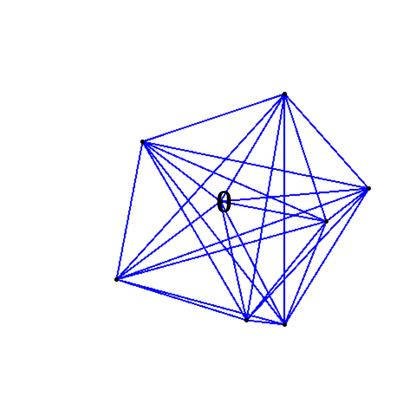

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Pointwise local energy ($G_2(\varphi) = \mathcal{P}_2(\varphi)$):
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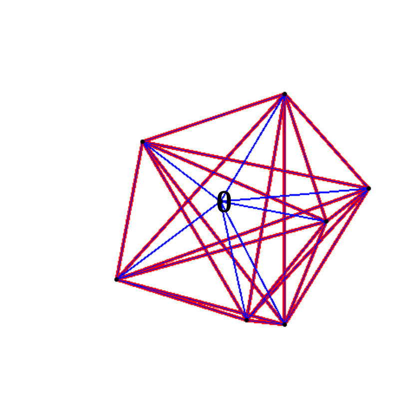

where

$$V^+(\mathbf{0}|\varphi) = \sum_{\substack{\xi^+ \in G_2(\mathbf{0} \cup \varphi) \\ \xi^+ \notin G_2(\varphi)}} g_2\left(\xi^+\right)$$

$$V^-(\mathbf{0}|\varphi) = \sum_{\substack{\xi^- \in G_2(\varphi) \\ \xi^- \notin G_2(\mathbf{0} \cup \varphi)}} g_2\left(\xi^-\right)$$

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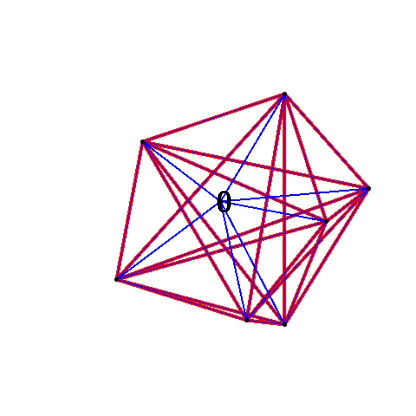

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since $\quad G_2(\varphi) \subset G_2(\mathbf{0} \cup \varphi)$

Local Stability (complete versus **Delaunay graph**)

Pointwise local energy ($G_2(\varphi) = Del_2(\varphi)$):
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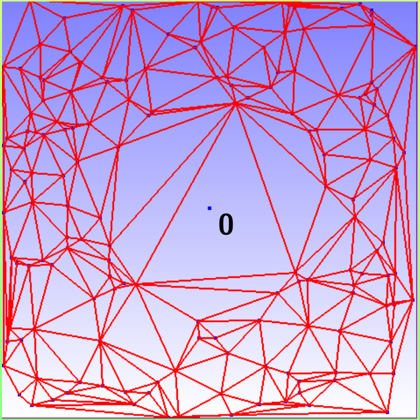

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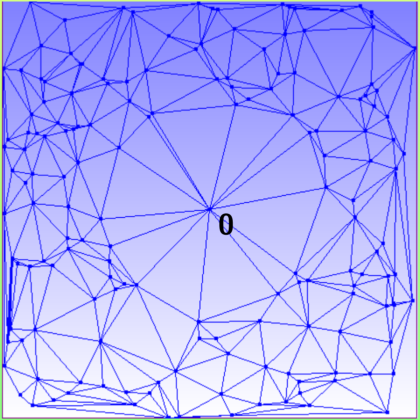

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Pointwise local energy ($G_2(\varphi) = Del_2(\varphi)$):
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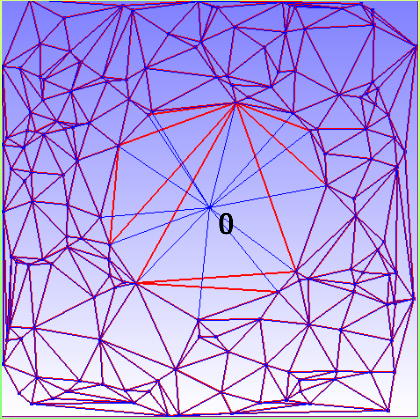

where

$$V^+(\mathbf{0}|\varphi) = \sum_{\substack{\xi^+ \in G_2(\mathbf{0}\cup\varphi) \\ \xi^+ \notin G_2(\varphi)}} g_2\left(\xi^+\right)$$

$$V^-(\mathbf{0}|\varphi) = \sum_{\substack{\xi^- \in G_2(\varphi) \\ \xi^- \notin G_2(\mathbf{0}\cup\varphi)}} g_2\left(\xi^-\right) \neq 0$$

since $\quad G_2(\varphi) \not\subseteq G_2(\mathbf{0} \cup \varphi)$

Existence of stationary Gibbs models (via entropic tools)

Existence of stationary Gibbs state (entropic tools of H.-O. Georgii)

$$\Big(([\mathbf{GC\text{-}GE}] \text{ and } [\mathbf{L}]\Big) \Rightarrow \Big([\mathbf{GC\text{-}IM}] \text{ and } [\mathbf{L}]\Big) \Rightarrow \Big(\mathcal{G}_s(V) \neq \emptyset\Big)$$

with

- **[GC-IM]**: there exists $\varphi^o \in \Omega$ such that
$$I(P_\Lambda(\cdot|\varphi^o)\,;\pi^z_\Lambda) \leq c|\Lambda|$$
where $I(P;Q)$ denotes the relative entropy of P and Q.
- **[GC-GE]** ($\Rightarrow$ **[GC-IM]**): there exists $\varphi^o \in \Omega$ such that
$$V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) > -c_0|\Lambda|, \text{ uniformly on } \varphi_\Lambda \in \Omega_\Lambda$$

Application

- $G_2(\varphi) = \mathcal{P}_2(\varphi)$: (Georgii, Haggström)
[Superstability] ($\Rightarrow$ **[GC-GE]**) and **[Range on g_2]**
- $G_2(\varphi) = Del_2(\varphi)$: (Bertin, Billiot, Drouilhet)
[Inhibition ($g_2 \geq 0$)] and **[g_2 based on $Del^{loc}_{2,R}(\varphi)$]**
(Choosing $\varphi^o = \emptyset$, $V(\varphi_\Lambda|\varphi^o_{\Lambda^c}) = V(\varphi_\Lambda) \geq 0 \Rightarrow$ **[GC-GE]**).

Existence of Gibbs models (local graph and non hereditary)

Gibbs property via local property of the graph (D. Dereudre)

- **Remark**: a nearest-neigbour type graph is local and, for a.s. any φ^o, there exists $\Lambda(\varphi^o) \Subset \mathbb{R}^d$: $V(0|\varphi^o) = V(0|\varphi^o_{\Lambda(\varphi^o)})$ which is clearly less restrictive than **[L]**!
- No longer **[L]** is required and consequently the following models exist:
  1. **[g_2 based on $Del_2(\varphi)$]** and **[Hard-Core on g_2]**
  2. **[g_2 based on $Del_{2,\beta}^{\beta_0}(\varphi)$]**

Non hereditary extension (D. Dereudre)

- **Hereditary property**:

  $V(\varphi) < +\infty \Rightarrow V(\psi) < +\infty$ whenever $\psi \subset \varphi$ is usually required
- Existence of non hereditary Delaunay models is first considered.
- **Example**: **Rigid** models such that $g_2(d) = +\infty$ for $d > D$.

Existence of Gibbs models (local graph and non hereditary)

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- **Remark**: a nearest-neigbour type graph is local and, for a.s. any φ^o, there exists $\Lambda(\varphi^o) \Subset \mathbb{R}^d$: $V(0|\varphi^o) = V(0|\varphi^o_{\Lambda(\varphi^o)})$ which is clearly less restrictive than **[L]**!
- No longer **[L]** is required and consequently the following models exist:

1. **[g_2 based on $Del_2(\varphi)$]** and **[Hard-Core on g_2]**
2. **[g_2 based on $Del_{2,\beta}^{\beta_0}(\varphi)$]**

Non hereditary extension (D. Dereudre)

- **Hereditary property**:

  $V(\varphi) < +\infty \Rightarrow V(\psi) < +\infty$ whenever $\psi \subset \varphi$ is usually required

- Existence of non hereditary Delaunay models is first considered.
- **Example**: **Rigid** models such that $g_2(d) = +\infty$ for $d > D$.

Existence of Gibbs models (local graph and non hereditary)

Gibbs property via local property of the graph (D. Dereudre)

- **Remark**: a nearest-neigbour type graph is local and, for a.s. any φ^o, there exists $\Lambda(\varphi^o) \Subset \mathbb{R}^d$: $V(0|\varphi^o) = V(0|\varphi^o_{\Lambda(\varphi^o)})$ which is clearly less restrictive than **[L]**!
- No longer **[L]** is required and consequently the following models exist:

  1. **[g_2 based on $Del_2(\varphi)$]** and **[Hard-Core on g_2]**
  2. **[g_2 based on $Del_{2,\beta}^{\beta_0}(\varphi)$]**

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- **Hereditary property**:

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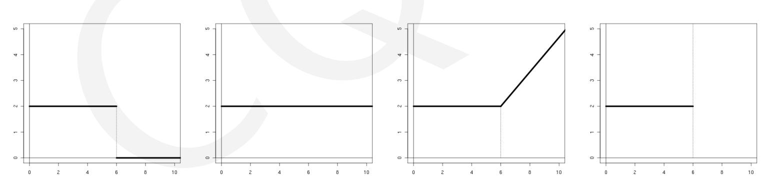
- Existence of non hereditary Delaunay models is first considered.
- **Example**: **Rigid** models such that $g_2(d) = +\infty$ for $d > D$.

Plan

Goals

- Thanks to the entropic tools, replacement of **[Superstability]** or **[LS]** by Stability **[S]**: $V(\varphi_\Lambda) \geq -K|\varphi_\Lambda|$
- Extension to general nearest-neighbour graph (not only the Delaunay graph)
- Locality of the graph instead of the local property.
- Non-hereditary case considered.
- In the Delaunay case, consider the interaction function g_2 of the form:



New contribution (Dereudre, Drouilhet and Georgii)

Definition

- *Hypergraph structure*: measurable subset $\mathcal{E}$ of $\Omega_f \times \Omega$ such that $\eta \subset \omega$ for all $(\eta, \omega) \in \mathcal{E}$.
- *Hyperedge* of ω: $\eta \in \mathcal{E}(\omega) \Leftrightarrow (\eta, \omega) \in \mathcal{E}$.
- *Hyperedge potential*: measurable function φ from $\mathcal{E}$ to $\mathbb{R} \cup \{\infty\}$.
- Hyperedge potential φ is called *shift-invariant* if

$$(\vartheta_x \eta, \vartheta_x \omega) \in \mathcal{E} \text{ and } \varphi(\vartheta_x \eta, \vartheta_x \omega) = \varphi(\eta, \omega), \forall (\eta, \omega) \in \mathcal{E}, \ x \in \mathbb{R}^d.$$

- *Finite horizon property* for φ if for each $(\eta, \omega) \in \mathcal{E}$ there exists some $\Delta \Subset \mathbb{R}^d$ such that

$$(\eta, \tilde{\omega}) \in \mathcal{E} \text{ and } \varphi(\eta, \tilde{\omega}) = \varphi(\eta, \omega) \text{ when } \tilde{\omega} = \omega \text{ on } \Delta. \tag{1}$$

Hamiltonian

$$H_{\Lambda,\omega}(\zeta) := \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \cup \omega_{\Lambda^c})} \varphi(\eta, \zeta \cup \omega_{\Lambda^c}) \quad \text{for } \zeta \in \Omega_\Lambda \tag{2}$$

where

$$\mathcal{E}_\Lambda(\omega) = \left\{\eta \in \mathcal{E}(\omega) : \varphi(\eta, \zeta \cup \omega_{\Lambda^c}) \neq \varphi(\eta, \omega) \text{ for some } \zeta \in \Omega_\Lambda\right\}. \tag{3}$$

which is the set of hyperedges η in a configuration ω for which either η itself or $\varphi(\eta, \omega)$ depends on the points of ω in Λ.

Remark on the conditional density function

$$\frac{\exp(-V(\zeta|\omega_{\Lambda^c}))}{\oint_\Lambda d\zeta \exp(-V(\zeta|\omega_{\Lambda^c}))} = \frac{\exp(-H_{\Lambda,\omega}(\zeta))}{\oint_\Lambda d\zeta \exp(-H_{\Lambda,\omega}(\zeta))}$$

Definition

- $\Omega_{\mathrm{cr}}^{\Lambda}$ consists of the set of configuration $\omega \in \Omega$ which *confines the range of* φ *from* Λ: there exists a set $\partial\Lambda(\omega) \Subset \mathbb{R}^d$ such that $\varphi(\eta, \zeta \cup \tilde{\omega}_{\Lambda^c}) = \varphi(\eta, \zeta \cup \omega_{\Lambda^c})$ whenever $\tilde{\omega} = \omega$ on $\partial\Lambda(\omega)$, $\zeta \in \Omega_\Lambda$ and $\eta \in \mathcal{E}_\Lambda(\zeta \cup \omega_{\Lambda^c})$.
- $\partial\Lambda(\omega) := \Lambda^r \setminus \Lambda$ with Λ^r is the closed r-neighborhood of Λ and $r := r_{\Lambda.\omega}$ is chosen as small as possible.
- $\partial_\Lambda \omega = \omega_{\partial\Lambda(\omega)}$.

For $\omega \in \Omega_{\mathrm{cr}}^{\Lambda}$ we have

$$H_{\Lambda,\omega}(\zeta) := \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \cup \omega_{\Lambda^c})} \varphi(\eta, \zeta \cup \omega_{\Lambda^c}) = \sum_{\eta \in \mathcal{E}_\Lambda(\zeta \cup \partial_\Lambda \omega)} \varphi(\eta, \zeta \cup \partial_\Lambda \omega), \tag{4}$$

and this sum extends over a finite set.

(R) *The range condition*

There exist constants $\ell_R, n_R \in \mathbb{N}$ and $\delta_R < \infty$ such that for all $(\eta, \omega) \in \mathcal{E}$ one can find a horizon Δ as in (1) satisfying the following:
For every $x, y \in \Delta$, there exist ℓ open balls $B_1, \ldots, B_\ell$ (with $\ell \leq \ell_R$) such that

– the set $\cup_{i=1}^{\ell} \bar{B}_i$ is connected and contains x and y, and
– for each i, either $\operatorname{diam} B_i \leq \delta_R$ or $N_{B_i}(\omega) \leq n_R$.

Proposition

Under **(R)**, *for each* $\Lambda \Subset \mathbb{R}^d$ *there exists a set* $\hat{\Omega}_{\mathrm{cr}}^{\Lambda} \in \mathcal{F}_{\Lambda^c}$ *such that* $\hat{\Omega}_{\mathrm{cr}}^{\Lambda} \subset \Omega_{\mathrm{cr}}^{\Lambda}$ *and* $P(\hat{\Omega}_{\mathrm{cr}}^{\Lambda}) = 1$ *for all* $P \in \mathscr{P}_\Theta$ *with* $P(\{\emptyset\}) = 0$.

(S) *Stability.*

The hyperedge potential φ is called *stable* if there exists a constant $c_S \geq 0$ such that

$$H_{\Lambda,\omega}(\zeta) \geq -c_S \; \#(\zeta \cup \partial_\Lambda \omega) \tag{5}$$

for all $\Lambda \Subset \mathbb{R}^d$, $\zeta \in \Omega_\Lambda$ and $\omega \in \Omega_{\mathrm{cr}}^\Lambda$.

- Periodic partition of $\mathbb{R}^d$ into parallelotopes

$$C(k) := \left\{ \mathrm{M}x \in \mathbb{R}^d : x - k \in [-1/2, 1/2[^d \right\}. \tag{6}$$

  with $k \in \mathbb{Z}^d$ and $\mathrm{M} \in \mathbb{R}^{d \times d}$ be an invertible $d \times d$ matrix. For brevity, $C = C(0)$.

- Let Γ be a measurable subset of $\Omega_C \setminus \{\emptyset\}$ and

$$\overline{\Gamma} = \left\{ \omega \in \Omega : \vartheta_{\mathrm{M}k}(\omega_{C(k)}) \in \Gamma \;\text{ for all } k \in \mathbb{Z}^d \right\} \tag{7}$$

  the set of all pseudo-periodic configurations.

(U) *Upper regularity.*

M and Γ can be chosen so that the following holds.

(U1) *Uniform confinement:* $\overline{\Gamma} \subset \Omega_{\mathrm{cr}}^{\Lambda}$ for all $\Lambda \Subset \mathbb{R}^d$, and

$$r_\Gamma := \sup_{\Lambda \Subset \mathbb{R}^d} \sup_{\omega \in \overline{\Gamma}} r_{\Lambda,\omega} < \infty.$$

(U2) *Uniform summability:* $c_\Gamma^+ := \sup_{\omega \in \overline{\Gamma}} \sum_{\eta \in \mathcal{E}(\omega):\, \eta \cap C \neq \emptyset} \frac{\varphi^+(\eta,\omega)}{\#(\hat{\eta})} < \infty,$

where $\hat{\eta} := \{k \in \mathbb{Z}^d : \eta \cap C(k) \neq \emptyset\}$.

(U3) *Strong non-rigidity:* $e^{z|C|}\, \Pi_C^z(\Gamma) > e^{c_\Gamma}$
where c_Γ is defined as in (U2) with φ in place of φ^+.

Theorem

For every hypergraph structure $\mathcal{E}$, hyperedge potential φ and activity $z > 0$ satisfying **(S)**, **(R)** *and* **(U)** *there exists at least one Gibbs measure $P \in \mathscr{G}_\Theta(\varphi, z)$.*

($\hat{\mathbf{U}}$) *Alternative upper regularity.*

M and Γ can be chosen so that the following holds.

($\hat{\text{U}}$1) *Lower density bound:* There exist constants $a, b > 0$ such that $\#(\zeta) \geq a|\Lambda| - b$ whenever $\zeta \in \Omega_f$ is such that $H_{\Lambda,\omega}(\zeta) < \infty$ for some $\zeta \subset \Lambda \Subset \mathbb{R}^d$ and some $\omega \in \overline{\Gamma}$.

($\hat{\text{U}}$2) = (U2) *Uniform summability.*

($\hat{\text{U}}$3) *Weak non-rigidity:* $\Pi^z_C(\Gamma) > 0$.

Theorem

A Gibbs measure $P \in \mathscr{G}_\Theta(\varphi, z)$ exists also under the hypotheses **(S)**, **(R)** *and* **($\hat{\mathbf{U}}$)**.

Simplified upper regularity.

Same as **(U)** and **($\hat{\mathbf{U}}$)** but with Γ chosen as:

$$\Gamma^A = \{\zeta \in \Omega_C : \zeta = \{x\} \text{ for some } x \in A\}.$$

Examples

Polynomially increasing Delaunay edge interactions

Let $d = 2$ and φ be a edge potential on Del_2 which is bounded below such that

$$\phi(\ell) \leq \kappa_0 + \kappa_1 \ell^\alpha \quad \text{for some constants } \kappa_0 \geq 0,\ \kappa_1 \geq 0 \text{ and } \alpha > 0\ .$$

Then there exists at least one Gibbs measure for φ and every activity

$$z > (1+2\varrho_0)e^{3\kappa_0}(3\alpha e^2\kappa_1/2)^{1/\alpha}/(\pi\varrho_0^2).$$

Long Delaunay edge exclusion.

Let $d = 2$ and φ be a pure edge potential on Del_2 which is bounded below and such that there are constants $0 \leq \ell_0 < \ell_1 \leq \ell_2$:

$$\sup_{\ell_0 \leq \ell \leq \ell_1} \phi(\ell) < \infty \quad \text{and} \quad \phi(\ell) = \infty \ \text{ if } \ell > \ell_2.$$

Then there exists at least one Gibbs measure for φ and every $z > 0$.

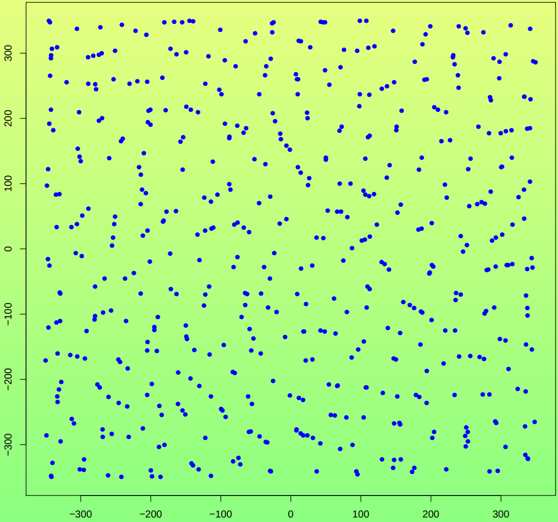
Examples

Many other examples

- Polynomially increasing Delaunay triangle interactions
- Shape-dependent Delaunay triangle interactions
- Many-body interactions of finite range
- Forced-clustering k-nearest neighbor interactions
- Voronoi cell interactions
- Adjacent Voronoi cell interactions

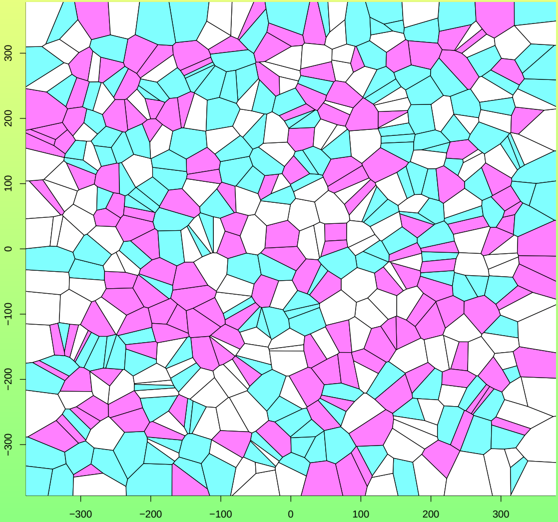
Example 1:

```
gd<-EBGibbs(~2+Del2(l2<1600,theta=2))
run(gd)
```



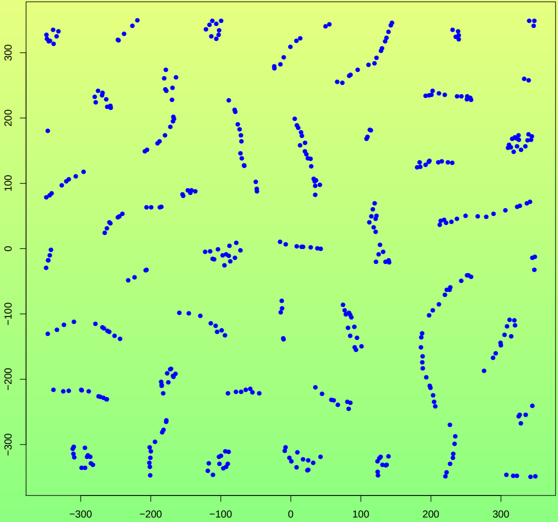
Example 2:

```
gdm<-EBGibbs(~2+Del2(l2<1600,theta=2),mark=EBMark(m=int(1,1:3)))
run(gdm,vcCol=m)
```



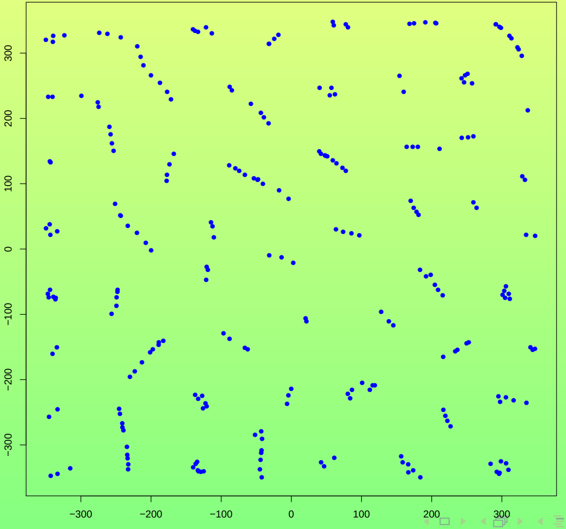
Example 3:

```
gd2<-EBGibbs(~1+Del2(l2<=400,400<l2 & l2<=6400,theta=c(2,4)))
run(gd2)
```



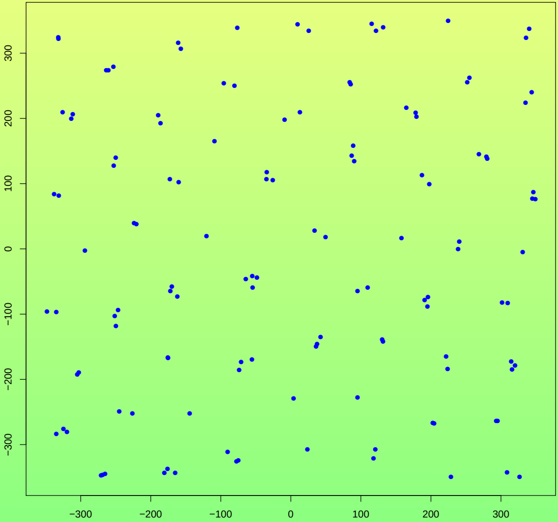
Example 3 (bis):

```
PieceWise<-function(x,b) (b[-length(b)] <= x) & (x < b[-1])
gd2<-EBGibbs(~1+Del2(PieceWise(l,c(0,20,80)),theta=c(2,4)))
run(gd2)
```



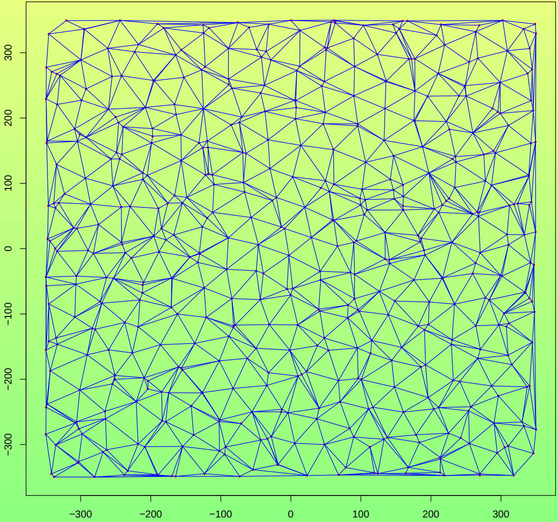
Example 4:

```
ga2<-EBGibbs(~1+All2(l2<=400,400<l2 & l2<=6400,theta=c(2,4)))
run(ga2)
```



Example 5:

```
gd3<-EBGibbs(~2+Del2(l<=40,theta=2)+Del3(sa>=pi/4,theta2=-2))
run(gd3,type=c("dv","de"),dvArgs=list(cex=.5,col="red"))
```



Example 6:

```
gvm<-EBGibbs(~(-50)+Del1((a-aire[v$m])^2,aire=c(100,1000)),
      mark=EBMark(m=int(1,1:2)))
run(gvm,vcCol=m,dvCol=m,dvCex=.5)
```