Localized spherical deconvolution

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joint work with
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We observe

\[ Z_i = \varepsilon_i X_i \quad i = 1 \ldots N \]

\( Z_i, X_i \) i.i.d random elements of \( \mathbb{S}^2 \), the unit sphere of \( \mathbb{R}^3 \), \( \varepsilon_i \in SO(3) \) are i.i.d., \( X_i \) and \( \varepsilon_i \) are supposed to be independent.

- Distributions of \( X, Z, \varepsilon \) are absolutely continuous with respect to the uniform probability measure on \( \mathbb{S}^2 \), \( \mathbb{S}^2 \) and the Haar measure on \( SO(3) \) with densities \( f, f_Z \) et \( f_\varepsilon \).
We have the following formula:

\[ f_Z = f_\varepsilon * f \]

For \( f_\varepsilon \in L_2(SO(3)) \), \( f \in L_2(S^2) \), we define as follows the convolution product:

\[ f_\varepsilon * f(\omega) = \int_{SO(3)} f_\varepsilon(u)f(u^{-1}\omega)du \]
Bibliography

- A. van Rooij et F. Ruymgaart (1991). *Regularized deconvolution on the circle and the sphere*
Motivations

- **Astrophysics**: study of the origins of UHECR i.e Ultra High Energy cosmic rays, extreme kinetic energy $10^{20}$ electronvolts.
- **Identify their sources**: Supermassive black holes at the AGN centers (active galactic nuclei), Hypernovae, relic particles from the Big Bang.
- **UHECR arrive with a probability law that we aim at estimating.** We observe the cosmic ray incident points on the Earth. They might be deviated by several phenomenons.
Fourier Analysis on $SO(3)$ and $S^2$

**Definition**

We define the rotational Fourier transform on $SO(3)$

$$f_{mn}^* = \int_{SO(3)} f(g) D_{mn}^l(g) dg, \quad l = 0, 1, 2..., \quad -l \leq m, n \leq l$$

where the $D_{mn}^l$ are the rotational harmonics which form an orthonormal basis of $L_2(SO(3))$

- $f^\star = [f_{m,n}^*]$ is a matrix of dimension $(2l + 1) \times (2l + 1)$ with $l = 0, 1, 2, \ldots$ et $-l \leq m, n \leq l$. 

Fourier Analysis on $SO(3)$ et $S^2$

**Definition**

The Fourier transform on $S^2$ is defined as

$$f_m^l = \int_{S^2} f(g) \overline{Y_m^l}(g) \, dg, \quad l = 0, 1, 2..., \quad -l \leq m \leq l$$

where the $Y_m^l$ are the spherical harmonics which form an orthonormal basis of $L_2(S^2)$.

- $f^l = [f_m^l]$ is an array of size $2l + 1$ with $l = 0, 1, 2, \ldots$ et $-l \leq m \leq l$. 
Classical approach of inverse problems

\[ f_\varepsilon \ast f(\omega) = \int_{SO(3)} f_\varepsilon(u)f(u^{-1}\omega)du \]

**Lemma**

We have for all \(-l \leq m \leq l, l = 0, 1, \ldots, :\)

\[
(f_\varepsilon \ast f)^*_m = \sum_{n=-l}^{l} f^*_{\varepsilon,mn} f^*_n := (f^*_\varepsilon f^*_m)_m. \tag{1}
\]

- We inverse the convolution operator thanks to the Fourier Transform.
Classical approach of inverse problems

- By considering the vectors $f^*l$, $f^*_Z$ and the matrix $f^*_\varepsilon$, for all $l \geq 0$, using (1), we get:

  $$f^*l = (f^*_\varepsilon)^{-1} f^*_Z$$

  $$f^*_m = \sum_{n=-l}^{l} f^*_{\varepsilon-1,mn} f^*_Z, n$$

  where $f^*_{\varepsilon-1,mn} := (f^*_\varepsilon)^{-1}_{mn}$

- We consider the empirical Fourier transform $\hat{f}^*_Z$ of $f^*_Z$

  $$\hat{f}^*_Z, n = 1/N \sum_{j=1}^{N} Y_n^l(Z_j)$$

- We deduce the following estimator $\hat{f}^*_m$

  $$\hat{f}^*_m := \frac{1}{N} \sum_{j=1}^{N} \sum_{n=-l}^{l} f^*_{\varepsilon-1,mn} Y_n^l(Z_j)$$
We get by the inversion formula an estimator of the distribution $f$

$$\hat{f}(\omega) = \sum_{l=0}^{\tilde{N}} \sum_{m=-l}^{l} \hat{f}_m^* Y_{lm}(\omega),$$

with $\tilde{N}$ a parameter depending on the number of observations.

Drawbacks of this method:
The spherical harmonics are not localized on the sphere. This method may be unable to detect irregularities of the target function $f$. 
Spherical harmonic $l = 8 \ m = 2$
Needlet \( j = 3 \) \( \eta = 250 \)
Needlet $j = 5 \ \eta = 5000$
Bibliography about Needlets

Delabrouille, Cardoso, Le Jeune, Betoule, Faï, Guilloux. A full sky, low foreground, high resolution CMB map from WMAP. *Astronomy and Astrophysics*, 2009.


Localization result

\[ \psi_{j\eta}(x) = \sqrt{\lambda_{j\eta}} \sum_{l=2^{j-1}}^{2^{j+1}} b(l/2^j) \sum_{m=-l}^{l} Y_m^l(\xi_{j\eta}) Y_m^l(x). \]

For all \( k \in \mathbb{N} \) there exists a constant \( c_k \) such that for all \( \xi \in S^2 \):

\[ |\psi_{j,\eta}(\xi)| \leq \frac{c_k 2^j}{(1 + 2^j d(\eta, \xi))^k}. \]
Thresholding estimation procedure

\[ f = \sum_j \sum_{\eta \in \mathcal{I}_j} (f, \psi_{j\eta})_{L^2(S^2)} \psi_{j\eta}. \]

By Parseval equality \[ \beta_{j\eta} = (f, \psi_{j\eta})_{L^2(S^2)} = \sum_{lm} f^{*l}_m \psi^{*l}_{j\eta,m} \]
but we already had

\[
\hat{f}^{*l}_m := \frac{1}{N} \sum_{j=1}^{N} \sum_{n=-l}^{l} f^{*l}_{\varepsilon^{-1},mn} \overline{Y_n(Z_j)}
\]

hence an unbiased estimator of \( \beta_{j\eta} \)

\[
\hat{\beta}_{j\eta} = \sum_{lm} \hat{f}^{*l}_m \psi^{*l}_{j\eta,m}.
\] (2)

Finally, an estimator of \( f \) is

\[
\hat{f} = \sum_{j=-1}^{J} \sum_{\eta \in \mathcal{I}_j} t(\hat{\beta}_{j\eta}) \psi_{j\eta}.
\]
where $t$ is a thresholding procedure defined as follows:

$$t(\hat{\beta}_{j\eta}) = \hat{\beta}_{j\eta} I\{ |\hat{\beta}_{j\eta}| \geq \kappa t_N \sigma_j \} \quad \text{with}$$

$$t_N = \sqrt{\frac{\log N}{N}},$$

$$\sigma_j^2 = A \sum_{ln} | \sum_m \psi_{l \eta, m}^* f_{\varepsilon - 1 mn}^* |^2,$$

with $\|f_Z\|_\infty \leq A$. 
**Theorem**

Let $1 \leq p < \infty$, $\nu > 0$, we suppose that

$$\sigma_j^2 := A \sum_{ln} | \sum_m \psi_{j_\eta,m}^* f_{\epsilon^{-1} mn}^*|^2 \leq C 2^{2j\nu}, \quad \forall \ j \geq 0. \quad (3)$$

Take $\kappa^2 \geq \sqrt{3\pi A}$, $\sqrt{3\pi A} \kappa > \max 8p, 2p + 1 \ 2^J = d[t_N]^{\frac{-1}{(\nu+1)}}$ with $t_N = \sqrt{\frac{\log N}{N}}$ et $d > 0$. Then if $\pi \geq 1$, $s > 2/\pi$, $r \geq 1$ (with the restriction $r \leq \pi$ if $s = (\nu + 1)(\frac{p}{\pi} - 1)$), there exists a constant $C$ such that:

$$\sup_{f \in B_{s/\pi,r}(M)} \mathbb{E}\| \hat{f} - f \|^p \leq C(\log(N))^{p-1} [N^{-1/2} \sqrt{\log(N)}]^{\mu p}, \quad (4)$$

where

$$\mu = \begin{cases} \frac{s}{s + \nu + 1}, & \text{if } s \geq (\nu + 1)(\frac{p}{\pi} - 1) \\ \frac{s - 2/\pi + 2/p}{s + \nu - 2/\pi + 1}, & \text{if } \frac{2}{\pi} < s < (\nu + 1)(\frac{p}{\pi} - 1). \end{cases}$$
The case of an unknown noise

\[ \hat{\beta}_{j\eta} = \frac{1}{N} \sqrt{\lambda_{j\eta}} \sum_{l=2j-1}^{2j+1} b(l/2^j) \sum_{m=-l}^{l} \sum_{n=-l}^{l} f_{\varepsilon-1,mn}^* \sum_{u=1}^{N} Y_m^l(\xi_{j\eta}) Y_n^l(Z_u). \]

- We replace the rotational Fourier transform \((f_{\varepsilon}^*)_{mn} := f_{\varepsilon,mn}^*\) by its empirical version.
- \(f_{\varepsilon-1,mn}^*\) denotes the \((m, n)\) element of the matrix \((f_{\varepsilon}^*)^{-1} := f_{\varepsilon-1}^*\) which is the inverse of the \((2l + 1) \times (2l + 1)\) matrix \((f_{\varepsilon}^*)\).
- To get the empirical version \(\hat{f}_{\varepsilon-1,mn}^*\) of \(f_{\varepsilon-1,mn}^*\)
  Compute the empirical matrix \((\hat{f}_{\varepsilon}^*)\) then inverse it to get the matrix \((\hat{f}_{\varepsilon}^*)^{-1} := \hat{f}_{\varepsilon-1}^*\). The \((m, n)\) entry of the matrix \((\hat{f}_{\varepsilon}^*)\) is given by the formula:

\[ \hat{f}_{\varepsilon,mn}^* = \frac{1}{N} \sum_{j=1}^{N} D_{m,n}^l(\varepsilon_j), \]
Simulations: Estimation of the uniform density probability

\[ f = \frac{1}{4\pi} 1_{S^2} \]

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**Table:** Number of non-zero coefficients surviving thresholding

\( \phi \sim U[0, \pi/8] \)

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**Table:** Number of nonzero coefficients surviving thresholding

\( \phi \sim U[0, \pi] \)
Case of an unimodal density probability $f = Ce^{-4|\omega-\omega_1|^2}1_{S^2}$ with $\omega_1 = (0, 1, 0), \omega_1 = (\frac{\pi}{2}, \frac{\pi}{2})$
Observations $\phi \sim U[0, \pi/8]$
Observations $\phi \sim U[0, \pi/8]$
Estimated density $\kappa = 0.5$
Estimated density by the first method
Observations $\phi \sim U[0, \pi/4]$
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