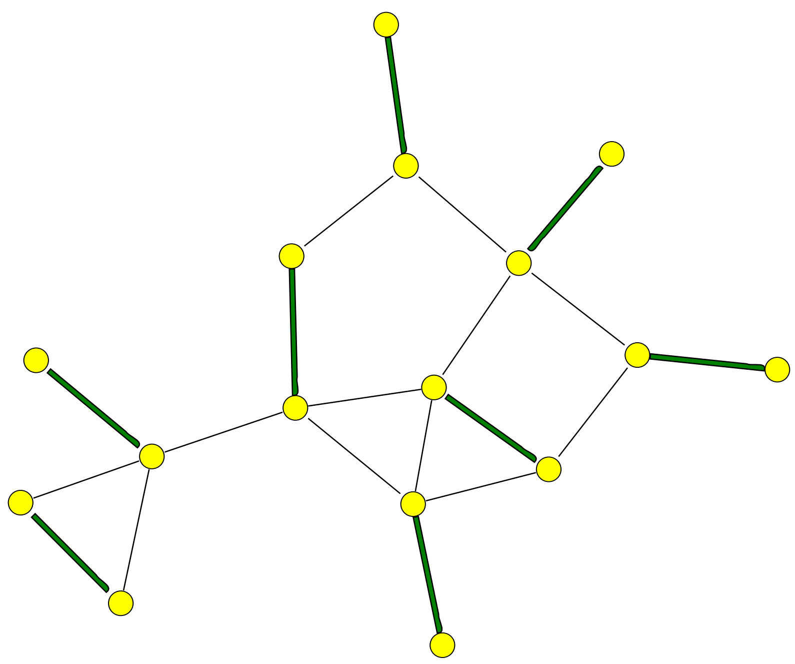
The geometry of polynomials and the validity of the Cavity Method

Justin Salez (ENS & INRIA)



Joint work with Charles Bordenave (CNRS & Uni. Toulouse)
and Marc Lelarge (INRIA & ENS).

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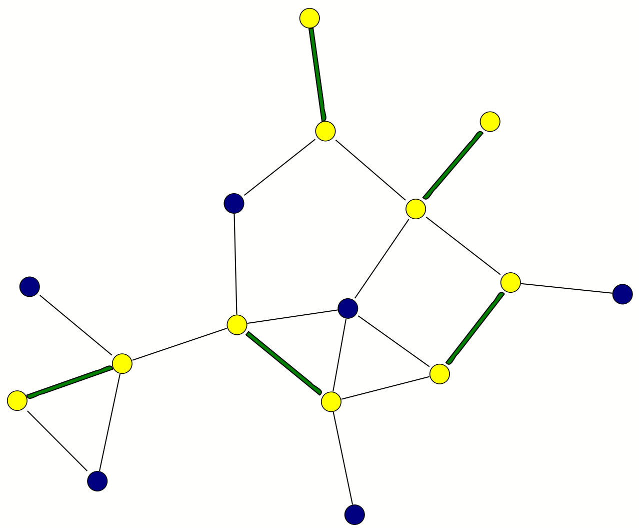

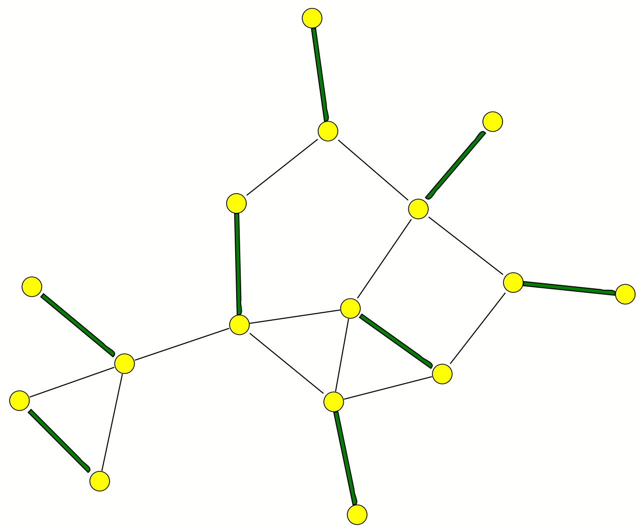

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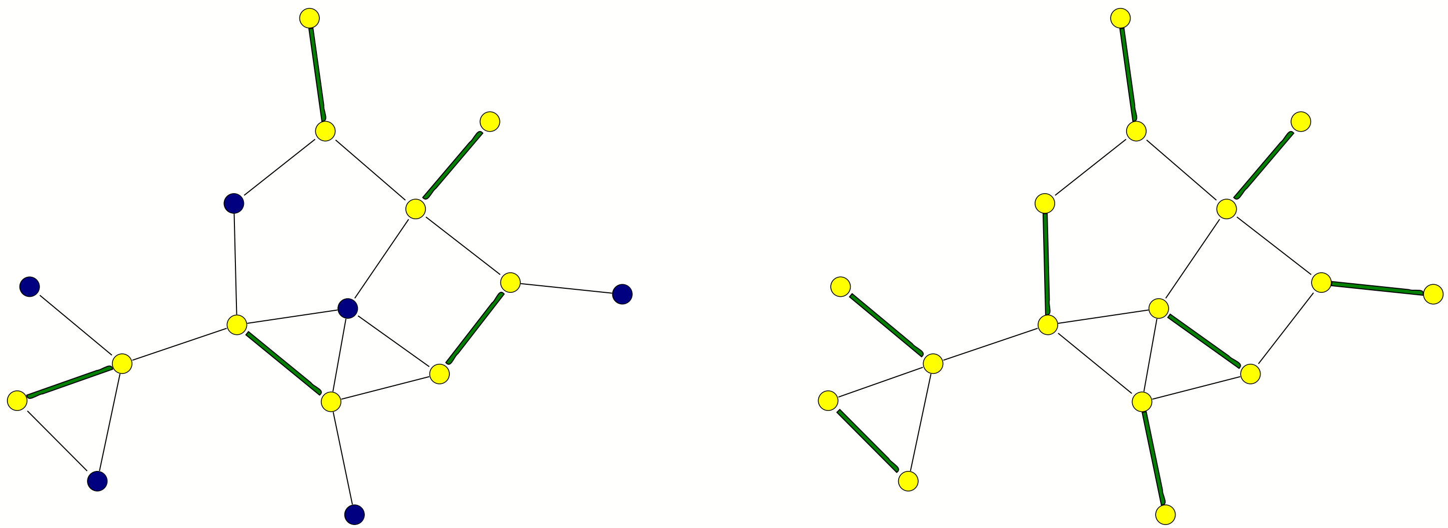

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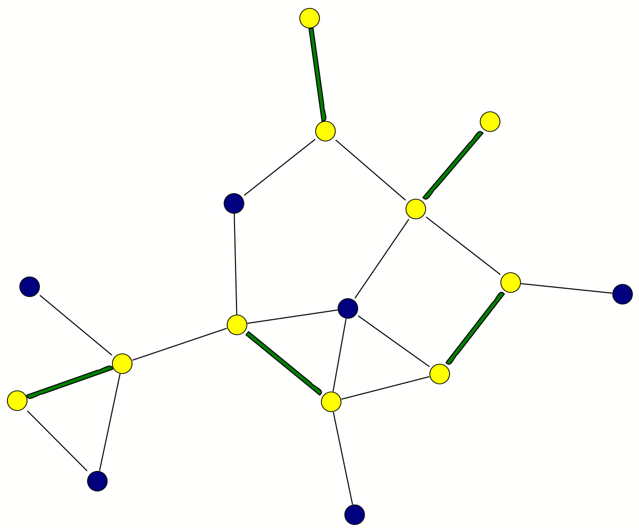 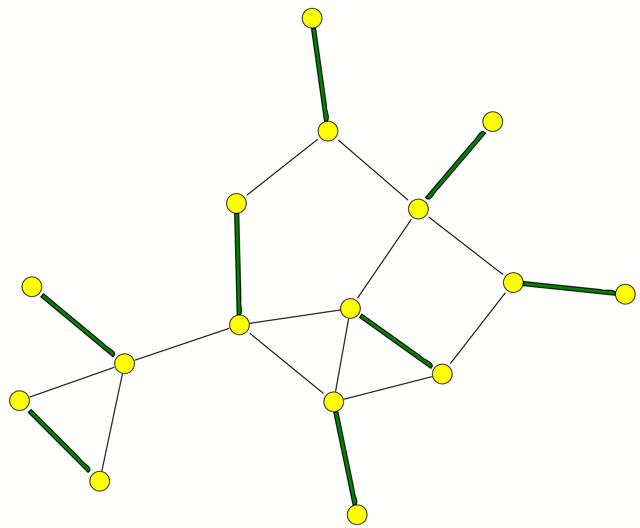

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▷ Typical behavior of $\nu(G)$ when G is a large random diluted graph ?

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Theorem [Karp & Sipser, '82]

$$\frac{\nu(G_n)}{n} \xrightarrow[n\to\infty]{P} 1 - \frac{1}{2}\left(x^* + e^{-cx^*} + cx^* e^{-cx^*}\right),$$

where x^* is the smallest root of $x = e^{-ce^{-cx}}$ in $[0,1]$.

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Theorem For random graphs $G_1, G_2, \ldots$ s.t. $G_n \xrightarrow[n\to\infty]{d} GWT(\phi)$ & $\phi'(1) < \infty$,

$$\frac{\nu(G_n)}{n} \xrightarrow[n\to\infty]{P} \min_{[0,1]} F,$$

where $F = 1 - \frac{1}{2}\left(x\phi'(1-x) + \phi(1-x) + \phi\left(1 - \frac{\phi'(1-x)}{\phi'(1)}\right)\right)$.

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The multi-affine polynomial $1 + x_1 + \ldots + x_d$ is non-vanishing whenever all variables lie in the open right half-plane.

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The rational function $(P^{/e})/(P^{\backslash e})$ is called the influence of $e \in E$ on μ .

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$\triangleright$ even for matchings, computing $P_G(1)$ is known to be # P-complete !

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▷ Cavity Approximation (Mézard & Parisi, 85) : non-rigorous, but really efficient

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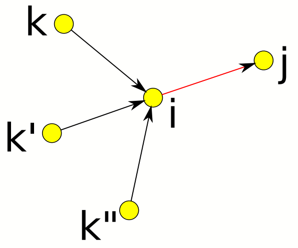

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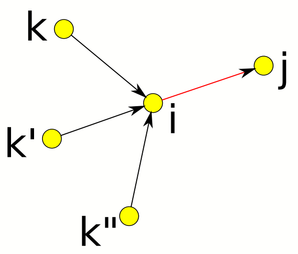

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2. Use the cavity approximation to evaluate the Boltzmann marginals :

$$\mu^z_{G_n}(ij \in \mathcal{F}) \approx \frac{x_{\vec{ij}} x_{\vec{ji}}}{z + x_{\vec{ij}} x_{\vec{ji}}}.$$

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▷ powerful predictions e.g. the assignment problem : $(X_{i,j})_{1\leq i,j\leq n}$ iid uniform on $[0,1]$,

$$\min_{\pi\in\mathfrak{S}_n}\left(\sum_{i=1}^{n} X_{i,\pi(i)}\right) \xrightarrow[n\to\infty]{a.s.} \zeta(2) = \frac{\pi^2}{6}.$$

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But rigorous results remain sparse. Any simple, general conditions for validity ?

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Example : $\mu_i(F) = 1_{\{|F| \leq 1\}}$, and more generally $\mu_i(F) = 1_{\{|F| \leq r\}}$ for $r \in \mathbb{N}$.

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Corollary : When the convergence $G_n \to G$ holds under uniform choice of the root $\circ$,

$$u_{G_n}(z) \to u_G(z) = \frac{1}{2}\mathbb{E}\left[\sum_{i\sim\circ}\mu_G^z\left(i\circ\in\mathcal{F}\right)\right] \text{ and } f_{G_n}(z) \to \int_0^z \frac{u_G(s)}{s}ds.$$

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1. contraction at low activity : if z is sufficiently small, then the cavity operator is contracting, hence there exists a unique, exponentially attractive fixed point.

2. analycity in the activity : the uniform half-plane property guarantees that the cavity operator preserves uniformly bounded analycity in a fixed complex domain containing the positive real line. Hence, the above convergence extends to any $z > 0$.

3. local weak convergence : the cavity operator is “local”, i.e. continuous with respect to local convergence, so we may pass to the limit in the cavity equations. When the limit is a Galton-Watson tree, the cavity equations may be simplified into a recursive distributional equation, which can sometimes be explicitely solved.

CONCLUSION

Gian-Carlo Rota (1932-1999) :

“The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems on location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation.”