An extension of Davis and Lo's contagion model

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Introduction

- Das et al. (2007) : Conditional independence assumption with no contagion effect is rejected by historical default data. The conditional independence assumption is not enough to capture historical default dependency
- Boissay (2006), Jorion and Zhang (2007, 2009) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects
- We present a multi-period extension of Davis and Lo's contagion model

Literature

In the spirit of Davis and Lo's contagion model :

- First models : Davis and Lo (2001) and Jarrow and Yu (2001)
- Extensions : Yu (2007), Egloff, Leippold and Vanini (2007), Rösch, Winterfeldt (2008), Sakata, Hisakado and Mori (2007)

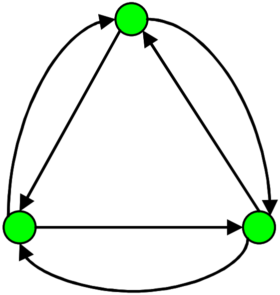
Other contagion models in the credit risk field :

- Copula : Schönbucher and Schubert (2001)
- Interacting particle system : Giesecke and Weber (2004)
- Incomplete information models : Frey and Runggaldier (2008), Fontana and Runggaldier (2009)
- Markov chain models : Schönbucher (2006), Frey and Backhaus (2007), Herbertsson (2007), Laurent, Cousin and Fermanian (2007)

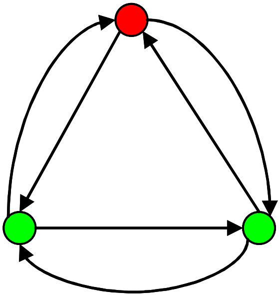
Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

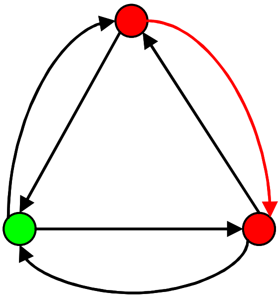
Example : Portfolio with 3 credit references



End of the period : direct default



End of the period : default by contagion (one possibility)



One-period contagion model

- n : number of credit references,
- X_i : direct default indicator of name i (i.e. $X_i = 1$ if i defaults directly, $X_i = 0$ otherwise),
- C_i : indirect default indicator of name i,
- Z_i : default indicator (direct or indirect) such that :

$$Z_i = X_i + (1 - X_i)C_i$$

where :

$$C_i = \mathbb{1}_{\text{at least one } X_j Y_{ji}=1,\ j=1,\dots,n}$$

- $Y_{ji} = 1$ if the contagion link is activated from name j to name i, $Y_{ji} = 0$ otherwise.

Davis and Lo's contagion model

$N = \sum_{i=1}^{n} Z_i$: total number of defaults

Distribution of total number of defaults (Davis and Lo)

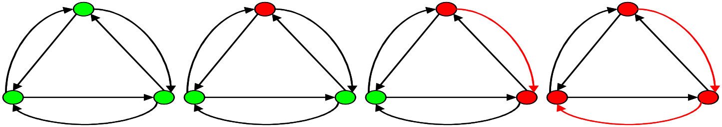
$$\mathrm{P}[N = k] = C_n^k \sum_{i=1}^{k} C_k^i p^i (1-p)^{n-i} (1-(1-q)^i)^{k-i} (1-q)^{i(n-k)}.$$

Under the assumptions :

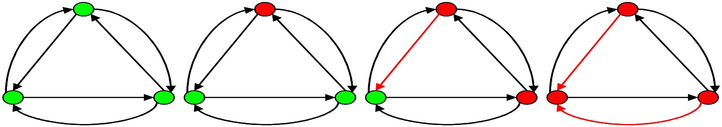
- X_i, $i = 1, \ldots, n$: iid Bernoulli with parameter p
- Y_{ij}, $i, j = 1, \ldots, n$: iid Bernoulli with parameter q
- One default alone may trigger a contamination effect
- A name that has been infected cannot contaminate other names (no chain-reaction effect)

Extension of Davis and Lo's contagion model

Dominos Effect



Two defaults required to trigger a contagion effect



Extension of Davis and Lo's contagion model

Multi-period contagion model : $t = 0, 1, 2, \ldots, T$, in period $[t, t+1]$:

- n : number of credit references,
- X_t^i : direct default indicator of name i,
- C_t^i : indirect default indicator of name i,
- Z_t^i : default indicator (direct or indirect) such that :

$$Z_t^i = Z_{t-1}^i + (1 - Z_{t-1}^i)[X_t^i + (1 - X_t^i)C_t^i]$$

where

$$C_t^i = f\left(\sum_{j \in F_t} Y_t^{ji}\right)$$

- Y_t^{ji}, $i, j = 1, \ldots, n$ are Bernoulli random variables such that $Y_t^{ji} = 1$ if name j may infect name i between t and $t+1$
- F_t is the set of names that are likely to infect other names between t and $t+1$
- f is a contamination trigger function, for example $f = \mathbb{1}_{x \geq 1}$ (Davis and Lo) or $f = \mathbb{1}_{x \geq 2}$

Extension of Davis and Lo's contagion model

$N_t = \sum_{i=1}^n Z_t^i$: total number of defaults at time t

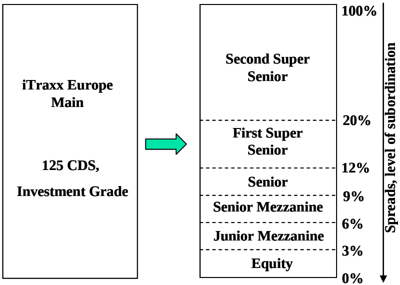
Main result

$$\mathrm{P}\left[N_t = r\right] = \sum_{k=0}^{r} \mathrm{P}\left[N_{t-1} = k\right] C_{n-k}^{r-k} \sum_{\gamma=0}^{r-k} C_{r-k}^{\gamma} \cdot \sum_{\alpha=0}^{n-k-\gamma} C_{n-k-\gamma}^{\alpha} \mu_{\gamma+\alpha,\,t} \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^{j+\alpha} \xi_{j+r-k-\gamma,t}(\gamma).$$

Under the assumptions :

- X_t^i, $i = 1, \ldots, n$ are conditionally independent Bernoulli r.v. with the same marginal distribution $\mu_{k,t}$ and $\mathbf{X}_t = (X_t^1, ..., X_t^n)$, $t = 1, \ldots, T$ are independent vectors
- Y_t^{ji}, $i, j = 1, \ldots, n$ are conditionally independent Bernoulli r.v. with the same marginal distribution and $\mathbf{Y}_t = (Y_t^{ji})_{1 \leq i,j \leq n}$, $t = 1, \ldots, T$ are independent vectors
- $(\mathbf{X}_t)_{t=1,\ldots,T}$ and $(\mathbf{Y}_t)_{t=1,\ldots,T}$ are independent

Calibration on 5-years iTraxx tranche quotes



- Cash-flows of CDO tranches driven by the aggregate loss process

$$L_t = \sum_{i=1}^{n} (1 - R_i) E_i Z_t^i$$

where R_i is the recovery rate associated with name i and E_i is the nominal of i.

We restrict ourselves to the case where for all t :

- $X_t^i \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $\mathrm{E}[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- Y_t^{ij} are iid $Y_t^{ij} \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover

- $n = 125$, $r = 3\%$ (short-term interest rate)
- $R_i = R = 40\%$, $E_i = E = 1$ for any $i = 1, \ldots, n$

$$L_t = (1 - R)N_t$$

- Computation of CDO tranche price only requires marginal loss distributions at several time horizons

Calibration on 5-years iTraxx tranche quotes

Least square calibration procedure : Find $\alpha^* = (p^*, \sigma^*, q^*)$ which minimizes :

$$RMSE(\alpha) = \sqrt{\frac{1}{6}\sum_{i=1}^{6}\left(\frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i}\right)^2}.$$

where

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index
Market prices	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{s}_4$	$\tilde{s}_5$	$\tilde{s}_0$
model prices	$s_1(\alpha)$	$s_2(\alpha)$	$s_3(\alpha)$	$s_4(\alpha)$	$s_5(\alpha)$	$s_0(\alpha)$

Calibration on 5-years iTraxx tranche quotes

Four calibration procedures :

- Calibration 1 : All available market spreads are included in the fitting
- Calibration 2 : The equity [0%-3%] tranche spread is excluded
- Calibration 3 : Both equity [0%-3%] tranche and CDS index spreads are excluded
- Calibration 4 : All tranche spreads are excluded except equity tranche and CDS index spreads.

Two calibration dates before and during the credit crisis :

- 31 August 2005
- 31 March 2008

Calibration on 5-years iTraxx tranche quotes

31 August 2005

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index
Market quotes	24	81	27	15	9	36
Calibration 1	20	114	7	1	1	29
Calibration 2	-	62	32	18	6	8
Calibration 3	-	55	29	18	7	-
Calibration 4	24	-	-	-	-	36

Annual scaled optimal parameters

	p^*	σ^*	q^*
Calibration 1	0.0016	0.0015	0.0626
Calibration 2	0.0007	0.0133	0.0400
Calibration 3	0.0001	0.0025	0.3044
Calibration 4	0.0014	0.002	0.1090

Calibration on 5-years iTraxx tranche quotes

31 March 2008

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index
Market quotes	40	480	309	215	109	123
Calibration 1	28	607	361	228	95	75
Calibration 2	-	505	330	228	112	68
Calibration 3	-	478	309	215	109	-
Calibration 4	40	-	-	-	-	123

Annual scaled optimal parameters

	p^*	σ^*	q^*
Calibration 1	0.0124	0.0886	0
Calibration 2	0.0056	0.0518	0.0400
Calibration 3	0.0012	0.012	0.2688
Calibration 4	0.0081	0.0516	0.0589

Conclusion

We propose a multi-period extension of Davis and Lo's contagion model that accounts for

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex : more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons

- When direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes