Improvement of Reduced Order Modeling based on Proper Orthogonal Decomposition

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Summary

Context and flow configuration

I - A pressure extended Reduced Order Model based on POD

II - Stabilization of Reduced Order Models
   - Residuals based stabilization method
   - Classical SUPG and VMS methods

III - Improvement of the functional subspace
   - An hybrid DNS/POD ROM method (Database modification)
   - Krylov like method

Conclusions
Context and flow configuration

Context

- Need of Reduced Order Model for Flow Control Purpose
  - To reduce the CPU time
  - To reduce the memory storage during adjoint-based minimization process

- Optimization + POD ROM methods
  - Generalized basis, no POD basis actualization: fast but no "convergence" proof
  - Trust Region POD (TRPOD), POD basis actualization: proof of convergence!

- Drawbacks
  - Need to stabilize POD ROM (lack of dissipation, roundoff errors, pressure term)
  - Basis actualization: DNS → high numerical costs!

- Solutions
  - Efficient ROM & stabilization
  - Low costs functional subspace adaptation during optimization process
Context and flow configuration

Flow Configuration
- 2-D Confined flow past a square cylinder in laminar regime
- Viscous fluid, incompressible and newtonian
- No control

\[ U(y) \]
\[ \Omega \]
\[ D \]
\[ U = 0 \]
\[ L \]
\[ H \]
\[ U = 0 \]

Numerical methods
- Penalization method for the square cylinder
- Multigrids V-cycles method in space
- Gear method in time

C.-H. Bruneau solver

Workshop on POD - 2 april 2008 – p. 4
I - A pressure extended Reduced Order Model

- **Momentum conservation**

Detailed model (exact)

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u}
\]

Temporal discretization

\[
\frac{\mathbf{u}^{n+1}}{\Delta t} + \nabla p^{n+1} - \frac{1}{Re} \Delta \mathbf{u}^{n+1} = \frac{\mathbf{u}^{n}}{\Delta t} - (\mathbf{u}^{n} \cdot \nabla) \mathbf{u}^{n}
\]

Projection onto the pressure extended POD basis (correlations onto \( U = (\mathbf{u}, p)^T \))

\[
\tilde{\mathbf{u}}(\mathbf{x}, t) = \sum_{i=1}^{N} a_i(t) \phi_i(\mathbf{x}) \quad \text{and} \quad \tilde{p}(\mathbf{x}, t) = \sum_{i=1}^{N} a_i(t) \psi_i(\mathbf{x})
\]

\[
\sum_{j=1}^{N} a_j^{(n+1)} \left( \frac{\phi_j}{\Delta t} + \nabla \psi_j - \frac{1}{Re} \Delta \phi_j \right) = \sum_{i=j}^{N} a_i^{(n)} \frac{\phi_j}{\Delta t} + \left( \sum_{j=1}^{N} a_j^{(n)} \phi_j^{(u)} \cdot \nabla \right) \sum_{k=1}^{N} a_k^{(n)} \phi_k^{(u)}
\]
I - A pressure extended Reduced Order Model

After some simplifications

\[
\sum_{j=1}^{N} a_j^{(n+1)} \left( \frac{\phi_i}{\Delta t} + \nabla \psi_i - \frac{1}{Re} \Delta \phi_i \right) = \sum_{j=1}^{N} a_j^{(n)} \phi_j^{(u)} \frac{\Delta t}{\Delta t} + \sum_{j=1}^{N} \sum_{k=1}^{N} a_j^{(n)} \left( \phi_j^{(u)} \cdot \nabla \right) \phi_k^{(u)} a_k^{(n)}
\]

\[
\sum_{j=1}^{N} a_j^{(n+1)} \chi_j = \sum_{j=1}^{N} a_j^{(n)} \xi_j + \sum_{j=1}^{N} \sum_{k=1}^{N} a_j^{(n)} \zeta_{jk} a_k^{(n)}
\]

Least squares

\[
\sum_{j=1}^{N} \chi_i^T \chi_j a_j^{(n+1)} = \sum_{j=1}^{N} \chi_i^T \xi_j a_j^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} a_j^{(n)} \chi_i^T \zeta_{jk} a_k^{(n)}
\]

\[
\sum_{j=1}^{N} L_{ij}^{adm} a_j^{(n+1)} = \sum_{j=1}^{N} B_{ij} a_j^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk} a_j^{(n)} a_k^{(n)}
\]

\[\leftrightarrow\] The ROM does not satisfy a priori the mass conservation

(for non divergence free modes, as NSE-Residual modes)
I - A pressure extended Reduced Order Model

**Mass conservation**

Detailed model

\[ \nabla \cdot u = 0 \]

Projection onto the POD basis

\[ \sum_{j=1}^{N} a_{j}^{(n+1)} \nabla \cdot \phi_{j} = 0 \]

\[ \sum_{j=1}^{N} (\nabla \cdot \phi_{j})^{T} \nabla \cdot \phi_{j} a_{j}^{(n+1)} = 0 \]

\[ \sum_{j=1}^{N} L_{i,j}^{\text{div}} a_{j}^{(n+1)} = 0 \]

Modified ROM

\[ \sum_{j=1}^{N} (\alpha L_{i,j}^{\text{qm}} + \beta L_{i,j}^{\text{div}}) a_{j}^{(n+1)} = \sum_{j=1}^{N} \alpha B_{i,j} a_{j}^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha C_{i,j,k} a_{j}^{(n)} a_{k}^{(n)} \]

→ The ROM has moreover to satisfy the flow rate conservation..
I - A pressure extended Reduced Order Model

Flow rate conservation

For the 2-D confined flow

\[
\int_S u \, dS = \text{Cste}
\]

\[
\sum_{i=1}^{N} a_j(t) \int_S \phi_j^u \, dS = \text{Cste}
\]

\[
\sum_{j=1}^{N} \frac{da_j}{dt} \int_S \phi_j^u \, dS = 0
\]

\[
\sum_{j=1}^{N} L_{ij}^{\text{deb}} a_j^{(n+1)} = \sum_{j=1}^{N} L_{ij}^{\text{deb}} a_j^{(n)}
\]

Finally, the ROM writes

\[
\sum_{j=1}^{N} (\alpha L_{ij}^{qdm} + \beta L_{ij}^{\text{div}} + \gamma L_{ij}^{\text{deb}}) a_j^{(n+1)} = \sum_{j=1}^{N} (\alpha B_{ij} + \gamma L_{ij}^{\text{deb}}) a_j^{(n)} + \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha C_{ijk} a_j^{(n)} a_k^{(n)}
\]
I - A pressure extended Reduced Order Model

**Advantage** no modelisation of the pressure term

\[ Re = 200, \text{ 11 modes } \Rightarrow \text{ convergence towards the exact limit cycles (\(=\) DNS)} \]

**Fig.** : Temporal evolution of the POD ROM coefficients over 25 vortex shedding periods

**Fig.** : Limit cycles of the POD ROM coefficients over 25 vortex shedding periods
I - A pressure extended Reduced Order Model

**Drawbacks** same as usual, *i.e.* lack of dissipation...

\[ Re = 200, \text{ 5 modes } \Rightarrow \text{ convergence towards an erroneous limit cycles (} \neq \text{ DNS) } \]

*Fig.* : Temporal evolution of the POD ROM coefficients over 25 vortex shedding periods

*Fig.* : Limit cycles of the POD ROM coefficients over 25 vortex shedding periods
I - A pressure extended Reduced Order Model

- **Drawbacks** same as usual, *i.e.* lack of dissipation...

\[ Re = 200, \text{ 3 modes} \Rightarrow \text{exponential divergence} \]

**Fig.** : Temporal evolution of the POD ROM coefficients over 25 vortex shedding periods

**Fig.** : Limit cycles of the POD ROM coefficients over 25 vortex shedding periods
II - POD ROM stabilization

► Overview of stabilization methods (non-exhaustive)

- Eddy viscosity
  → Heisenberg viscosity
  → Spectral vanishing viscosity
  → Optimal viscosity

- Penalty method

- Calibration of POD ROM coefficients

► "New" stabilization methods in POD ROM context

- Residuals based stabilization method

- Streamline Upwind Petrov-Galerkin (SUPG) and Variational Multi-scale (VMS) methods
II - POD ROM stabilization

Residuals based stabilization method

⇒ Idea add dominant POD-NSE residual modes to the existing basis

← The POD-NSE residuals are $\mathcal{L}(\tilde{u}(x, t), \tilde{p}(x, t)) = R(x, t)$, where $\tilde{u}$ and $\tilde{p}$ obtained using POD and $\mathcal{L}$ is the NSE operator

Model $A^{[N]}$, unstable POD ROM built with $N$ basis functions $\Phi_i(x)$.

Algorithm

1. Integrate the ROM to obtain $a_i(t)$ and extract $N_s$ snapshots $a_i(t_k)$, $k = 1, \ldots, N_s$.

2. Compute $\tilde{u}(x, t_k) = \sum_{i=1}^{N} a_i(t_k) \phi_i(x)$, $\tilde{p}(x, t_k) = \sum_{i=1}^{N} a_i(t_k) \psi_i(x)$, and $R(x, t_k)$.

3. Compute the POD modes $\Psi(x)$ of the NSE residuals.

4. Add the $K$ firsts residual modes $\Psi(x)$ to the existing POD basis $\Phi_i(x)$ and built a new ROM (here the mass and flow rate constraints are important).

Model $B^{[N;K]}$, PODRES ROM built with $N$ POD basis functions $\Phi_i(x)$ + $K$ RES basis functions $\Psi_i(x)$.
II - POD ROM stabilization

SUPG and VMS methods

⇒ **Idea** approximate the fine scales using the NSE residuals \( \mathbf{R} = (\mathbf{R}_M, \mathbf{R}_C)^T \)

\[
\mathbf{u}'(x, t) = \tau_M \mathbf{R}_M(x, t) \quad \text{and} \quad p'(x, t) = \tau_C \mathbf{R}_C(x, t)
\]

← Class of penalty methods, i.e.

\[
\sum_{j=1}^{N} L_{ij} \frac{da_j}{dt} = \sum_{j=1}^{N} B_{ij}a_j + \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk}a_ja_k + F_i(t)
\]

Model \( C^{[N]} \), SUPG method

\[
F_i^{SUPG}(t) = (\mathbf{\tilde{u}} \cdot \nabla \Phi_i + \nabla \Psi_i, \tau_M \mathbf{R}_M(x, t))_\Omega + (\nabla \cdot \Phi_i, \tau_C \mathbf{R}_C(x, t))_\Omega
\]

Model \( D^{[N]} \), VMS method

\[
F_i^{VMS}(t) = F_i^{SUPG}(t) + (\mathbf{\tilde{u}} \cdot (\nabla \Phi_i)^T, \tau_M \mathbf{R}_M(x, t))_\Omega - (\nabla \Phi_i, \tau_M \mathbf{R}_M(x, t) \otimes \tau_M \mathbf{R}_M(x, t))_\Omega
\]

← Parameters \( \tau_M \) and \( \tau_C \) are determined using adjoint based minimization method
II - POD ROM stabilization

$Re = 200$ and $N = 5$ POD basis function $\rightarrow$ erroneous limit cycles

![Graph showing temporal evolution of the $L_2$ norm of the POD-NSE residuals]

Fig. : *temporal evolution of the $L_2$ norm of the POD-NSE residuals*

![Graphs showing limit cycles of the POD ROM coefficients over 20 vortex shedding periods]

Fig. : *Limit cycles of the POD ROM coefficients over 20 vortex shedding periods*
II - POD ROM stabilization

- $Re = 200$ and $N = 5$ POD basis function $\rightarrow$ erroneous limit cycles

![Graphs showing temporal evolution of the $L_2$ norm of the POD-NSE residuals and limit cycles of the POD ROM coefficients over 20 vortex shedding periods.](image)

**Fig.** : temporal evolution of the $L_2$ norm of the POD-NSE residuals

**Fig.** : Limit cycles of the POD ROM coefficients over 20 vortex shedding periods
II - POD ROM stabilization

- $Re = 200$ and $N = 5$ POD basis function → erroneous limit cycles

Fig. : temporal evolution of the $L_2$ norm of the POD-NSE residuals

Fig. : Limit cycles of the POD ROM coefficients over 20 vortex shedding periods
II - POD ROM stabilization

- $Re = 200$ and $N = 5$ POD basis function $\rightarrow$ erroneous limit cycles

**Fig.**: temporal evolution of the $L_2$ norm of the POD-NSE residuals

**Fig.**: Limit cycles of the POD ROM coefficients over 20 vortex shedding periods
\( \text{Re} = 200 \) and \( N = 3 \) POD basis function → divergence

**Fig.** : temporal evolution of the \( L_2 \) norm of the POD-NSE residuals

**Fig.** : Limit cycle of the POD ROM coefficients over 20 vortex shedding periods
II - POD ROM stabilization

$Re = 200$ and $N = 3$ POD basis function $\rightarrow$ divergence

**Fig.** : temporal evolution of the $L_2$ norm of the POD-NSE residuals

**Fig.** : Limit cycle of the POD ROM coefficients over 20 vortex shedding periods
**II - POD ROM stabilization**

- $Re = 200$ and $N = 3$ POD basis function → divergence

**Fig.** : temporal evolution of the $L_2$ norm of the POD-NSE residuals

**Fig.** : Limit cycle of the POD ROM coefficients over 20 vortex shedding periods
II - POD ROM stabilization

- $Re = 200$ and $N = 3$ POD basis function $\rightarrow$ divergence

Fig.: temporal evolution of the $L_2$ norm of the POD-NSE residuals

Fig.: Limit cycle of the POD ROM coefficients over 20 vortex shedding periods
III - Improvement of the functional subspace

Functional subspace drawbacks, $\Phi_n(x)$: lack of representativity of 3D flows outside the database


- Problems for 3D flow control
- Erroneous turbulence properties (spectrum, etc)
Method 1 : hybrid ROM-DNS method to adapt the functional subspace $\Phi_n(\mathbf{x})$
Goal: determine $\Phi_n(\mathbf{x})$ at $Re_2$ starting from $\Phi_n(\mathbf{x})$ at $Re_1$ for low numerical costs.

- **Database modification**: statistics evolution $\Rightarrow \varphi : \Phi(k) \mapsto \Phi(k+1)$

- Table:

<table>
<thead>
<tr>
<th>ROM</th>
<th>DNS $Re_2$</th>
<th>ROM</th>
<th>DNS $Re_2$</th>
<th>ROM</th>
<th>DNS $Re_2$</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(\mathbf{x}, t_{k-1})$</td>
<td>$U(\mathbf{x}, t_k)$</td>
<td>$U(\mathbf{x}, t_{k+1})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Database modification

$$\tilde{U}[1,\ldots, N_r](\mathbf{x}, t_k) = \sum_{n=1}^{N_r} a_n(t_k) \phi_n(\mathbf{x}),$$

One snapshot modification using few DNS iterations

$$U(\mathbf{x}, t_s) = \tilde{U}[1,\ldots, N_r](\mathbf{x}, t_s) + U_s(\mathbf{x}, t_s).$$

In a general way

$$\tilde{U}(\mathbf{x}, t_k) = \tilde{U}[1,\ldots, N_r](\mathbf{x}, t_k) + \delta_{ks} U_s(\mathbf{x}, t_s),$$
III - Improvement of the functional subspace

2 Modification temporal correlations tensor

\[ C(t_k, t_l) = (U(x, t_k), U(x, t_l))_\Omega \]

\[ = \left( \sum_{i=1}^{N_r} a_i(t_k) \phi_i(x) + U_\perp(x, t_k), \sum_{j=1}^{N_r} a_j(t_l) \phi_j(x) + U_\perp(x, t_l) \right)_\Omega \]

\[ = \sum_{i=1}^{N_r} \sum_{j=1}^{N_r} a_i(t_k) a_j(t_l) (\phi_i(x), \phi_j(x))_\Omega + \left( U_\perp(x, t_k), U_\perp(x, t_l) \right)_\Omega \]

\[ = \delta_{i,j} \]

\[ + \sum_{i=1}^{N_r} a_i(t_k) \left( \phi_i(x), U_\perp(x, t_l) \right)_\Omega + \sum_{j=1}^{N_r} a_j(t_l) \left( U_\perp T(x, t_k), \phi_j(x) \right)_\Omega = 0 \]

Final approximation

\[ C(t_k, t_l) = \sum_{i=1}^{N_r} a_i(t_k) a_i(t_l) + \delta_{k,s} \delta_{l,s} \int_\Omega \sum_{i=1}^{n_c} U_\perp i(x, t_s) U_\perp i(x, t_s) dx. \]
III - Improvement of the functional subspace

\[ \tilde{U}^{[1,\ldots,5]} \]

\[ U \equiv \tilde{U}^{[1,\ldots,40]} \]

\[ U^\perp \text{ contribution} \]

\[ N_r = 5 \]

\[ \lambda_n \]

\[ \text{index of POD modes} \]

\[ N_r = 11 \]

\[ \text{index of POD modes} \]

**Fig.**: Comparison of the temporal correlation tensor eigenvalues evaluated from the exact field, \( U \), and from the \( N_r \)-modes approximated one, \( \tilde{U}^{[1,\ldots,N_r]} \).

\[ \leftrightarrow \] Very good approximation, and very low costs method!
III - Improvement of the functional subspace

3 Functional subspace adaptation

\[
\phi_{k}^{(n+1)}(\mathbf{x}) = \frac{1}{\lambda_{k}^{(n+1)}} \sum_{j=1}^{N} \tilde{U}^{(n)}(\mathbf{x}, t_{j}) a_{k}^{(n+1)}(t_{j})
\]

\[
\phi_{k}^{(n+1)}(\mathbf{x}) = \frac{1}{\lambda_{k}^{(n+1)}} \sum_{i=1}^{N_{r}} \sum_{j=1}^{N} a_{k}^{(n+1)}(t_{j}) a_{i}^{(n)}(t_{j}) \phi_{i}^{(n)}(\mathbf{x}) + \frac{1}{\lambda_{k}} U_{\perp}^{(n)}(\mathbf{x}, t_{s}) a_{k}^{(n+1)}(t_{s}).
\]

\[
\phi_{k}^{(n+1)}(\mathbf{x}) = \sum_{i=1}^{N_{r}} K_{k i}^{(n+1)} \phi_{i}^{(n)}(\mathbf{x}) + S_{k}^{(n+1)}(\mathbf{x}).
\]

Taken \( S^{(n+1)} \) with elements \( S_{i j}^{(n+1)} = S_{j i}^{(n+1)} \), the actualized basis is obtained using the linear application \( \varphi : \mathbb{R}^{n} \times \mathbb{R}^{n} \mapsto \mathbb{R}^{n} \times \mathbb{R}^{n} \) defined as

\[
\varphi : \phi^{(n)} \mapsto \phi^{(n+1)} = \phi^{(n)} K^{(n+1)} + S^{(n+1)}
\]

Incrementation \( n = n + 1 \).
III - Improvement of the functional subspace

Example: we have a POD basis for $Re_1 = 100$, and we want a POD basis for $Re_2 = 200$.

- The POD ROM is evaluated with the current improved POD basis $\Phi_i^{(k)}$.
- The DNS is performed for $Re_2$.

Fig.: Modification of the POD basis functions under the application of the linear transformation $\varphi$.
Streamline representation of the velocity fields. Bifurcation from $Re = 100$ to $Re = 200$.

$\phi_2^{(0)}$  $\phi_2^{(1)} = \varphi \left( \{ \phi_1^{(0)} \}_{i=1}^{N_R}, U^\perp \right)$

$\phi_4^{(0)}$  $\phi_4^{(1)} = \varphi \left( \{ \phi_1^{(0)} \}_{i=1}^{N_R}, U^\perp \right)$

$\leftarrow$ Functional subspace modification
III - Improvement of the functional subspace

Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

$Re_1 = 100$

$Re_2 = 200$
Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

Fig. : temporal evolution of the POD basis
III - Improvement of the functional subspace

Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

Fig. : temporal evolution of the POD basis
III - Improvement of the functional subspace

Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

![Diagram showing temporal evolution of the POD basis](image)

**Fig.**: *temporal evolution of the POD basis*
III - Improvement of the functional subspace

Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

Fig. : *temporal evolution of the POD basis*
III - Improvement of the functional subspace

► Results for a dynamical evolution from $Re_1 = 100$ to $Re_2 = 200$

Fig. : temporal evolution of the POD basis
III - Improvement of the functional subspace

Observations

- Results are very good if a sufficient amount of DNS is performed.

\[ \frac{\text{DNS}}{\text{DNS} + \text{POD ROM}} \text{ greater than } 70\% \]

Possible explanation

\[ \frac{N}{N + P} > 70\% \]

exact "target"

POD ROM

DNS
III - Improvement of the functional subspace

► Observations

- Results are very good if a sufficient amount of DNS is performed
  \( \frac{\text{DNS}}{\text{DNS} + \text{POD ROM}} \) greater than 70%.

► Possible explanation

\[ \frac{N}{N + P} = 70\% \]

- POD ROM

- DNS

- exact "target"
III - Improvement of the functional subspace

Observations

- Results are very good if a sufficient amount of DNS is performed
  \[ \frac{\text{DNS}}{\text{DNS} + \text{PODROM}} \] greater than 70%

Possible explanation

\[ \frac{N}{N + P} \leq 70\% \]

exact "target"

POD ROM

DNS
III - Improvement of the functional subspace

- Method 2: Krylov-like method to improve the functional subspace $\Phi_n(x)$

- Use of the POD-NSE residuals: $\mathcal{L}(\tilde{u}(x, t), \tilde{p}(x, t)) = R(x, t)$, $\tilde{u}$ and $\tilde{p}$ are POD fields, $\mathcal{L}$ is the NSE operator

Algorithm

1. Start with the POD basis to be improved, $\Phi_i$ with $i = 1, \ldots, N$. Let $N_0 = N$.

2. Built and solve the corresponding ROM to obtain $a_i(t)$ and extract $N_s$ snapshots $a_i(t_k)$ with $i = 1, \ldots, N$ and $k = 1, \ldots, N_s$.

3. Compute $\tilde{u}(x, t_k) = \sum_{i=1}^{N} a_i(t_k) \phi_i(x)$, $\tilde{p}(x, t_k) = \sum_{i=1}^{N} a_i(t_k) \psi_i(x)$, and $R(x, t_k)$.

4. Compute the POD modes $\Psi(x)$ of the NSE residuals.

5. Add the $K$ firsts residual modes $\Psi(x)$ to the existing POD basis $\Phi_i(x)$
   - $\Phi \leftarrow \Phi + \Psi$
   - $N \leftarrow N + K$
   - If $N$ is below than a threshold, return to 1. Else, go to 5.

6. Perform a new POD compression with $N = N_0$.
   - If convergence is satisfied, stop. Else, return to 1.
{Φ_i}^{N_0}_{i=1} for Re_1. Let N = N_0

Built and solve the ROM(Φ)
Extract N_s snapshots a_i(t_k)
Compute \( \tilde{u}(x, t_k) \) and \( \tilde{p}(x, t_k) \)
⇒ Coarse scales (Φ)

if N < N_{max}

Compute the residuals \( R(x, t_k) \)
from \( \tilde{u}(x, t_k) \) and \( \tilde{p}(x, t_k) \)
NS operator evaluated for Re_2
⇒ Missing scales (u' = τR)

if N ≥ N_{max}

Perform a new POD
using \( \tilde{u}(x, t_k) \) and \( \tilde{p}(x, t_k) \)
Extract firsts \( N_0 \) modes \( Φ_i \)
Let N = N_0

Add \{Ψ_i\}^K_{i=1} to \{Φ_i\}^N_{i=1}
Let Φ ← Φ + Ψ
Let N ← N + K

Perform a POD from \( R(x, t_k) \)
Extract firsts \( K \) modes \( Ψ_i \)
III - Improvement of the functional subspace

First test case: 1D burgers equation $R_{e1} = 50 \rightarrow R_{e2} = 300$ ($\Phi^{R_{e1}} \cdot \Phi^{R_{e2}} \approx 0.5$)
First test case: 1D burgers equation $Re_1 = 50 \rightarrow Re_2 = 300$ ($\Phi^{Re_1} \cdot \Phi^{Re_2} \approx 0.5$)
III - Improvement of the functional subspace

Second test case: 2D NSE equations $Re_1 = 100 \rightarrow Re_2 = 200$ ($\Phi^{Re_1} \cdot \Phi^{Re_2} \approx 0.5$)
III - Improvement of the functional subspace

Observations

- the decomposition $U'(x, t) = \tau R(x, t)$ is used to stabilize the ROM
  $\leftrightarrow$ Very good results for NSE

- the decomposition $U'(x, t) = \tau R(x, t)$ is used to improve POD basis
  $\leftrightarrow$ Very good results for Burgers, quite bad results for NSE

Possible explanation

- the decomposition $U'(x, t) = \tau R(x, t)$ is only valid for:
  $\leftrightarrow$ small values of $U'(x, t)$ (for instance, non resolved POD modes).
  $\leftrightarrow$ Can we find a good approximation $\tau$ of the elementary Green's function? Not sure...

Future works

- Look for an other decomposition for the missing scales $U'(x, t)$:
  $\leftrightarrow$ $U'(x, t) = M(t)R(x, t)$, where $M \in \mathbb{R}^{3\times3}$
  $\leftrightarrow$ ...??
Conclusions

- **A pressure extended Reduced Order Model**
  - The pressure is naturally included in the ROM ⇒ no modelisation of pressure term...
  - ... but need of modelisation interaction with non resolved modes (dissipation)

- **Stabilization of Reduced Order Models based on POD**
  - Add some residual modes ⇒ Good results
  - SUPG and VMS methods ⇒ Very good results

- **Try to improve the functional subspace**
  - Database modification : an hybrid DNS/ROM method
    - Fast evaluation of temporal correlations tensor
    - Linear actualization of the POD basis
    - DNS must correct ROM ⇒ good results for amount of DNS greater than 70%
  - Improvement using POD-NSE residuals (Krylov like method)
    - Very good for 1D burgers equation but quite poor results for 2D NSE equations
    - Problem with the continuity equation ?
    - Missing scales ≠ "fine scales" ⇒ approximation $U'(x, t) = \tau R(x, t)$ not good!