Proper Orthogonal Decomposition in PDE-Constrained Optimization

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jointly with

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Dynamic Programming Principle

$$\begin{cases} \min J(y_0, u) = \int_0^\infty L(y(t), u(t)) e^{-\mu t} dt \\ \dot{y}(t) = f(y(t), u(t)), \quad y(0) = y_0, \; u \in \mathcal{U}_{ad}. \end{cases}$$

value function $v(y_0) = \min_{u \in \mathcal{U}_{ad}} J(y_0, u)$.

(DPP) $\quad v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{ \int_0^T L(y(t; y_0, u), u(t) e^{-\mu t} dt + v(y(T; y_0, u)) e^{-\mu T} \}$

$$(HJB) \quad \mu v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{ (\nabla v(y_0) \, , \, f(y_0, u)) + L(y_0, u) \}$$

replace y_0 by $y^*(t)$.

$$u^*(t) = \mathcal{F}(y^*(t)) \qquad \text{where}$$

$\mathcal{F}(y^*(t)) \in \operatorname{argmin}_{u \in \mathcal{U}_{ad}} \{ (\nabla v(y^*(t)) \, , \, f(y^*(t), u)) + L(y^*(t), u) \}$

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$$J(y,u) = \int_0^\infty \Big(\frac{1}{2}\int_\Omega |y(t,x)|^2\,\mathrm{d}x + \frac{\beta}{2}\,|u(t)|^2\Big) e^{-\mu t}\,\mathrm{d}t,$$

$$\begin{aligned}
y_t - \nu y_{xx} + y y_x &= 0 && \text{in } Q,\\
\nu y_x(\cdot,0) + \sigma_0 y(\cdot,0) &= u && \text{in } (0,\infty)\\
\nu y_x(\cdot,1) + \sigma_1 y(\cdot,1) &= g && \text{in } (0,\infty)\\
y(0,\cdot) &= y_\circ && \text{in } \Omega,
\end{aligned}$$

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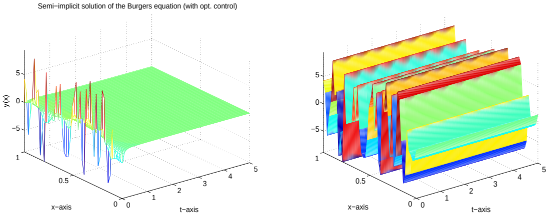


FIGURE 1. Optimal state with random noise (9.0) in the initial condition: feedback design (left) and open-loop design (right).

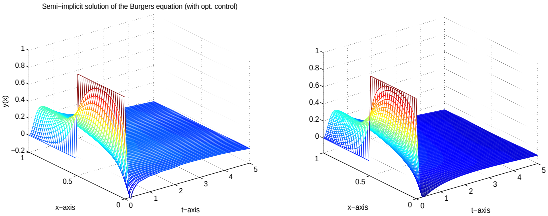


FIGURE 2. Optimal state with random noise (0.25) in the RHS: feedback design (left) and open-loop design (right).

Basics

- Construction of POD–basis

$(X, \langle\cdot,\cdot\rangle_X) \cdots$ HS, $\quad 0 = t_0 < t_1 < \cdots < t_n \leq T$

$\{y_j\}_{j=1}^n \cdots$ snap–shots, e.g. velocity components of fluid

$\mathcal{V} =$ span $\{y_j\}_{j=1}^n, \quad d =$ dim $\mathcal{V}$;

$y_j = \sum_{i=1}^d \langle y_j, \psi_i\rangle_X \psi_i, \quad j = 0, \cdots, n; \quad \{\psi_i\} \cdots$ ONB

POD–basis: for every $\ell \in \{1, \cdots, d\}$

$$\min_{\{\psi_i\}_{i=1}^\ell} \sum_{j=0}^n |y_j - \sum_{i=1}^\ell \langle y_j, \psi_i\rangle_X \psi_j|_X^2$$

subject to $\langle \psi_i, \psi_j\rangle = \delta_{i,j}, 1 \leq i, j \leq \ell$

$$\mathcal{Y}_n : \mathbb{R}^n \to X,$$

$$\mathcal{Y}_n v = \sum_{j=0}^{n} \beta_j \, v_j \, y(t_j),$$

$$\mathcal{Y}_n^* z = \left(\langle z, y(t_j)\rangle_X, .., \langle z, y(t_n)\rangle_X\right)$$

$$\mathcal{R}_n = \mathcal{Y}_n \mathcal{Y}_n^* = \sum_{j=0}^{n} \beta_j y(t_j)\langle y(t_j), \cdot\rangle_X \in \mathcal{L}(X)$$ spatial correlation op.

$$\mathcal{K}_n = \mathcal{Y}_n^* \mathcal{Y}_n = \langle y(t_i), \beta_j y(t_j)\rangle_X \in \mathbb{R}^{n\times n}$$ temporal correlation op.

$\mathcal{R}_n \ldots \{\sigma_j\}, \{\psi_j\}; \quad \mathcal{K}_n \ldots \{\sigma_j\}, \{\varphi_j\}; \qquad \psi_k = \frac{1}{\sqrt{\sigma_k}} \mathcal{Y}_n \varphi_k$

continuous POD

$$\mathcal{Y} : L^2(0, T; \mathbb{R}) \to X, \qquad \mathcal{Y}v = \int_0^T v(t)y(t, \cdot)\, dt$$

$$\mathcal{R} = \mathcal{Y}\mathcal{Y}^* = \int_0^T y(t)\, \langle y(t), \cdot\rangle_X\, dt \quad \in \mathcal{L}(X)$$

$$\mathcal{K} = \mathcal{Y}^*\mathcal{Y} = \int_0^T \bullet \langle y(t, \cdot), y(t, \cdot)\rangle_X\, dt \ \in\ \mathcal{L}(L^2(0, T), L^2(0, T)) \ldots$$

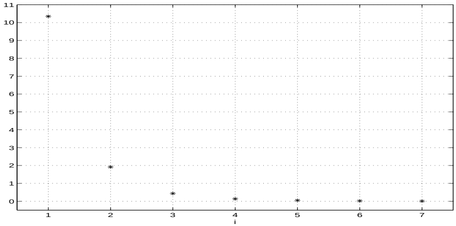
Hilbert-Schmidt

Proposition. Let $y \in L^2(0, t; X)$. Then $\mathcal{K}$ is compact. Except for possibly 0 the eigenvalues of $\mathcal{K}$ and $\mathcal{R}$ coincide. They are positive with identical multiplicities and ψ is *EVE* of $\mathcal{R}$ iff $\frac{1}{\sqrt{2}}\mathcal{Y}^*\psi = \langle y(t, \cdot),\, \psi(\cdot)\rangle_X = \varphi$ is eigenvalue of $\mathcal{K}$.

error formula:

$$\sum_{j=0}^{n} |y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_j \rangle_X \psi_i|_X^2 = \sum_{i=\ell+1}^{d} \lambda_i$$

choice of ℓ: $$I(\ell) = \sum_{k=1}^{\ell} \lambda_k \Big/ \sum_{k=1}^{n} \lambda_k \leq \alpha < 1 \,.$$



Convergence rate for state

$$(*) \quad \begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y = f \text{ in } (0,T]\times\Omega, \\ \text{div } y = 0 \\ y(t) = 0 \text{ on } (0,T]\times\partial\Omega, \quad y(0,\cdot) = y_0 \text{ in } \Omega \end{cases}$$

POD-Galerkin scheme $y^\ell = \sum_{i=1}^{\ell} x_i(t)\psi_i$

$$(\star) \begin{cases} E\dot{x} = Ax + \mathfrak{N}(x) \\ x(0) = x_0, \qquad A\cdots \text{ full} \end{cases}$$

grid = $\{t_i\}_{i=1}^m$, $t_i = i\Delta t$, for snapshots and backwards Euler

$$\mathcal{V} = \text{ span } \{y(t_i), \quad (\Delta t)^{-1}(y(t_i) - y(t_{i-1}))\}_{i=1}^m$$

Convergence rate for state

$$(*) \quad \begin{cases} y_t - \nu \Delta y + (y \cdot \nabla) y = f \text{ in } (0, T] \times \Omega, \\ \text{div } y = 0 \\ y(t) = 0 \text{ on } (0, T] \times \partial\Omega, \quad y(0, \cdot) = y_0 \text{ in } \Omega \end{cases}$$

POD-Galerkin scheme $y^\ell = \sum_{i=1}^{\ell} x_i(t)\psi_i$

$$(\star) \begin{cases} E\dot{x} = A\,x + \mathfrak{N}(x) \\ x(0) = x_0, \qquad A \cdots \text{ full} \end{cases}$$

grid = $\{t_i\}_{i=1}^m$, $t_i = i\Delta t$, for snapshots and backwards Euler

$$\mathcal{V} = \text{ span } \{y(t_i), \quad (\Delta t)^{-1}(y(t_i) - y(t_{i-1}))\}_{i=1}^m$$

fully discrete Galerkin-POD approximation

Theorem

Let y be solution to (*), $\{Y\}_{k=1}^m$ solution to backwards Euler–(POD)–scheme. There exists C independent of ℓ and $\overline{\Delta t}$ (dep. on T) such that

$$\frac{1}{m}\sum_{i=0}^{m}\|Y_i - y(t_i)\|^2_{L^2(\Omega)} \le C\sum_{i=\ell+1}^{\infty}\left(|\langle y_o, \psi_i\rangle_{H^1}|^2 + \lambda_i\right)$$

$$+C(\Delta t)^2|y|_{W^{2,2}(0,T;H^1(\Omega))},$$

for $\Delta t \le \overline{\Delta t}$.

Crank–Nicolson: $(\Delta t)^4$, grid can be decoupled,
without finite differences in snapshot set: $\sim \sum_{i=\ell+1}^{\infty}\frac{1}{(\Delta t)^2}\lambda_i$

semi-continuous Galerkin-POD approximation

$$(E^\ell) \quad \begin{cases} \frac{d}{dt}(y_\ell(t), v)_H + a(y_\ell, v) = \langle f(t), v\rangle_{V^*,V}, \quad t \in (0, T] \\ (y_\ell(0), v)_H = (y_0, v)_H \,, \quad \text{for all } v \in \text{span}\{\psi_i\}_{i=1}^{\ell}. \end{cases}$$

$$\tilde{y}_\ell(t) = \sum_{k=l+1}^{\infty} (y(t), \psi_k)_V \, \psi_k.$$

$\tilde{y}_\ell\,(t) \to 0$ as $\ell \to \infty$ for $t \in [0, T]$.

Theorem

Choose $X = V$*, set* $\rho_\ell := |\frac{d\tilde{y}_\ell}{dt}|^2_{L^2(0,T;H)} + \frac{1}{2}\,|\tilde{y}_\ell\,(0)|^2_H$.
Then for the solution y_ℓ *of* (E^ℓ)

$$|y - y_\ell|^2_{L^2(0,T;V)} \leq \rho_\ell + \sum_{i=\ell+1}^{\infty} \lambda_i \; \underrightarrow{\ell \to \infty} \; 0.$$

a-posteriori estimate for open loop optimal control

for linear quadratic optimal control

$$(P) \qquad \begin{cases} J(y,u) = \frac{1}{2}(\int_0^T |y-z|^2 + \beta|u|^2)\,dt \\ y_t = Ay(t) + Bu(t), \quad y(0) = y_0 \end{cases}$$

$$|u^* - u^\ell|_{L^2(0,T;U)} \le \frac{1}{\beta}|\tilde{J}^{\ell\,\prime}(u^\ell)|$$

where

$$\tilde{J}^{\ell\,\prime}(u^\ell) = \beta u^\ell - p^\ell(u^\ell)$$

Tröltzsch Volkwein

- Galerkin–POD–procedure for optimal control

$$y^\ell = \sum_{i=1}^{\ell} x_i(t)\psi_i, \quad u^\ell = \sum_{i=1}^{\ell} \beta_i(t)\tilde{\psi}_i$$

$$(\star) \begin{cases} E\dot{x} = \tilde{f}(x,\beta) = A\,x + \mathfrak{N}(x) + \beta \\ x(0) = x_0, \qquad A \cdots \text{ full} \end{cases}$$

- low–order optimization problem

$$(P_{POD}) \qquad \min\ \tilde{J}(x,\beta) = \int_0^\infty e^{-\mu t}\tilde{L}(x(t),\ \beta(t)dt \ \text{ subject to } (\star).$$

- solve HJB equation for discretized system

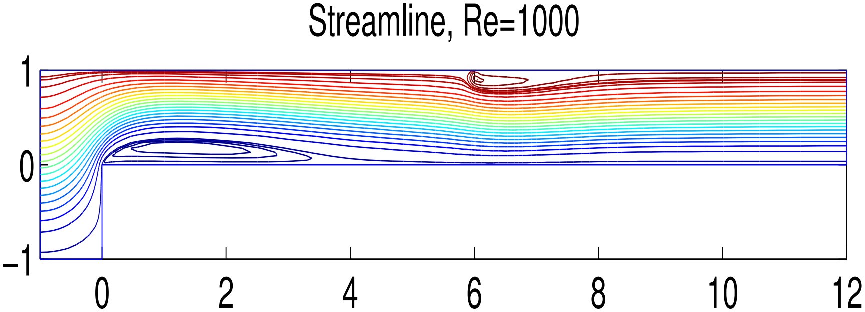
$$\mu V(x) + \min_{\beta \in U} [\tilde{f}(x,\beta)\cdot \nabla V(x) + \tilde{L}(x,\beta)] = 0,$$

$$\beta_{\text{closed loop}}(t) = \beta^*(V(x^*(t)))$$

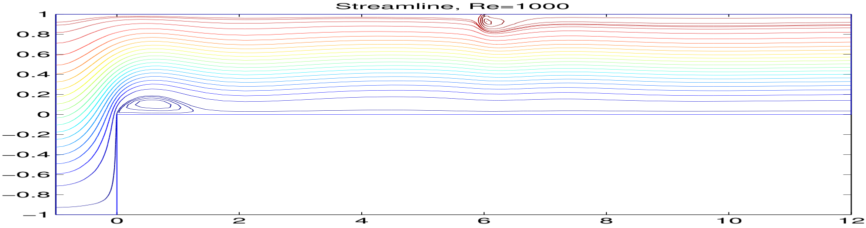
- utilize POD-HJB control in infinite dimensional system

Reynolds number, *Re*	1000
Time horizon, T	10
No. of basis functions	6
No. of controls	2
Bounds on controls	$[-0.5, 0.5]$
Grid system	$12 \times 6 \times 6 \times 4 \times 3 \times 3$
Discount rate, μ	2
Observe region, Ω_o	$[-1, 12] \times [0, 1]$

Table: Parameter Settings



Streamline, Re=1000
1
0
−1
0
2
4
6
8
10
12



Streamline, Re=1000
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
12

unmodelled dynamics 'update POD basis'

where to take snapshots ? (in discrete POD case)

how many basis elements ?

OS-POD optimality-system POD

$$(P) \quad \begin{cases} \min J(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y-z|^2\, dxdt & + \frac{\beta}{2}\int_0^T |u|^2\, dt \\ y_t = \mathcal{A}y + \mathcal{N}(y) & \text{in} \quad (0,T] \times \Omega \\ y(t,\cdot) = \mathcal{B}u = \sum_{k=1}^m u_i(t) b_i(x) & \text{on} \quad (0,T) \times \partial\Omega \\ y(0,\cdot) = y_0 & \text{on} \quad \Omega \end{cases}$$

OS-POD

$$
(P^{\ell}_{OSPOD}) \begin{cases} \min J^{\ell}(x,\psi,u) \text{ over } (x,\psi,u) \in L^2(\mathbb{R}^{\ell}) \times X^{\ell} \times L^2(0,T;\mathbb{R}^m) \\ \text{subject to} \\ E(\psi)\,\dot{x}(t) + A(\psi)x(t) + \mathfrak{N}(x(t),\psi) = B(\psi)u(t),\ t \in (0,T], \\ E(\psi)\,x(0) = x_0(\psi), \\ \frac{d}{dt} y(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = \mathcal{B}u(t), \quad t \in (0,T], \\ y(0) = y_0, \\ \mathcal{R}(y)\psi_i = \lambda_i \psi_i\,, \qquad i = 1,\dots,\ell, \\ \langle \psi_i, \psi_j \rangle = \delta_{ij} \qquad i,j = 1,\dots,\ell. \end{cases}
$$

where $\mathcal{R}(y)\psi_i = \int_0^T \langle y(t), \psi_i \rangle_X \, y(t) dt$.

OS-POD

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(P^{\ell}_{OSPOD}) \begin{cases} \min J^{\ell}(x,\psi,u) \text{ over } (x,\psi,u) \in L^2(\mathbb{R}^{\ell}) \times X^{\ell} \times L^2(0,T;\mathbb{R}^m) \\ \text{subject to} \\ E(\psi)\,\dot{x}(t) + A(\psi)x(t) + \mathfrak{N}(x(t),\psi) = B(\psi)u(t),\ t \in (0,T], \\ E(\psi)\,x(0) = x_0(\psi), \\ \frac{d}{dt}\,y(t) + \mathcal{A}y(t) + \mathcal{N}(y(t)) = \mathcal{B}u(t), \quad t \in (0,T], \\ y(0) = y_0, \\ \mathcal{R}(y)\psi_i = \lambda_i\,\psi_i\,, \qquad i = 1,\dots,\ell, \\ \langle \psi_i,\, \psi_j \rangle = \delta_{ij} \qquad i,j = 1,\dots,\ell. \end{cases}
$$

where $\mathcal{R}(y)\psi_i = \int_0^T \langle y(t), \psi_i\rangle_X\, y(t)dt$.

$$
\begin{aligned}
&\mathcal{L}(z,\xi) \\
&\quad = J^{\ell}(x,\psi,u)+ \\
&\quad \langle E(\psi)\dot{x} + A(\psi)x + \mathfrak{N}(x,\psi) - B(\psi)u, q\rangle_{L^2(0,T,\mathbb{R}^n)} \\
&\quad \langle (y_t + \mathcal{A}y + \mathcal{N}(y) - \mathcal{B}u, p\rangle_{L^2(0,T;L^2(\Omega))} \\
&\quad + \sum_{i=1}^{\ell} \langle (\mathcal{R} - \lambda_i I)\psi_i, \mu_i\rangle_X + \sum_{i=1}^{\ell} (\|\psi_i\|^2 - 1)\eta_i
\end{aligned}
$$

primal variables $z = (x,\psi,u,y)$,
adjoint variables $\xi = (q,p,\mu,\eta)$.

The following optimality system holds:

$$(ADJ\,1)\ \begin{cases} -E(\psi)\,\dot{q}(t) + (A(\psi) + \mathfrak{N}_x^T(x,\psi))\,q(t) = -J_x^\ell(x,\psi,u), \\ q(T) = 0 \end{cases}$$

$$(ADJ\,2)\ \begin{cases} -\dot{p}(t) + \mathcal{A}p(t) + \mathcal{N}'(y(t))^*\,p(t) \\ \qquad = \sum_{i=1}^{\ell} \langle y(t), \mu_i\rangle_X \mathcal{I}^{-1}\psi_i + \langle y(t), \psi_i\rangle_X \mathcal{I}^{-1}\mu_i \\ p(T) = 0 \end{cases}$$

$$\begin{cases} \eta_i = -\frac{1}{2}\,\langle \mathcal{G}_i(x,\psi,u,q), \psi_i\rangle_{X^*,X}, \qquad \mathcal{I} = X^* \to X \\ \mu_i = -(\mathcal{R} - \lambda_i\, I)^{-1}\,[2\,\eta_i\,\psi_i + \mathcal{I}\,\mathcal{G}_i(x,\psi,u,q)], \text{ for } i = 1, \ldots, \end{cases}$$

and

$$\beta u(t) = B^T(\psi)\,q(t) + \mathcal{B}^*\,p(t).$$

$$
\begin{aligned}
\mathcal{G}_i(x,\psi,u,q) = &\int_0^T \left(x_i \Big(\sum_{j=1}^{\ell} x_j \psi_j - z \Big) + q_i \sum_{j=1}^{\ell} \dot{x}_j \, \psi_j + \dot{x}_j \sum_{j=1}^{\ell} q_j \, \psi_j \right) \mathrm{d}t \\
&+ \int_0^T \left(q_i \sum_{j=1}^{\ell} x_j \, \mathcal{A}\psi_j + x_i \sum_{j=1}^{\ell} q_j \, \mathcal{A} \, \psi_j - q_i \sum_{k=1}^{m} b_k \, u_k \right) \mathrm{d}t \\
&+ \sum_{j=1}^{\ell} (x_j(0) \, \psi_j \, q_i(0) + x_i(0) \psi_j \, q_j(0)) - y_0 \, q_i(0) \\
&+ \mathcal{N}\left(\sum_{k=1}^{\ell} x_k \, \psi_k \right) q_i + x_i \sum_{j=1}^{\ell} \mathcal{N}' \left(\sum_{k=1}^{\ell} x_k \, \psi_k \right)^* q_j \psi_j .
\end{aligned}
$$

analogous result for $\mathcal{K}$ formulation.

Algorithm implemented

1. initial (POD) basis $\{\psi_i^0\}_{i=1}^l$
2. solve (P_{POD}^ℓ) inexactly for u^-
3. compute $\mathcal{G}_i, \quad y(u^-), \; p(u^-)$
4. use optimality condition in gradient step to update u^+
5. compute $y(u^+)$ and update POD basis ψ^+

Test examples

$$\min J(y, u_1, u_2) = \frac{1}{2}\int_0^T \int_0^1 |y - z|^2\, dxdt + \frac{\beta}{2}\int_0^T (u_1^2 + u_2^2) dt$$

subject to

$$\begin{aligned}
y_t - \nu y_{xx} + y y_x &= f && \text{in} \quad (0, T] \times (0, 1) \\
\nu y_x(\cdot, 0) + \sigma_0\, y(\cdot, 0) &= u_1 && \text{in} \quad (0, T] \\
\nu y_x(0, 1) + \sigma_1\, y(\cdot, 1) &= u_2 && \text{in} \quad (0, T] \\
y(0, \cdot) &= y_0 && \text{in} \quad (0, 1)
\end{aligned}$$

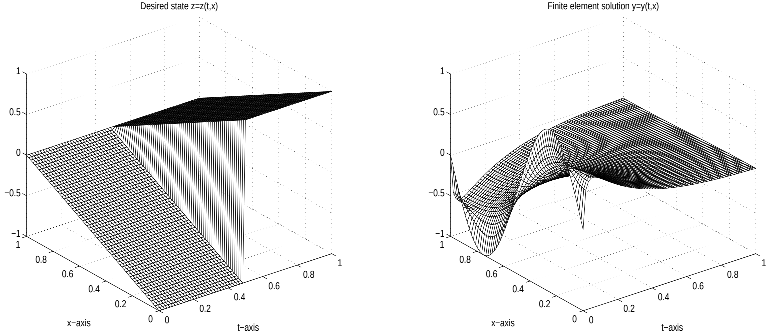


Figure: Desired state (left) and FE solution to the uncontrolled Burgers equation, i.e., $u = v = 0$ (right).

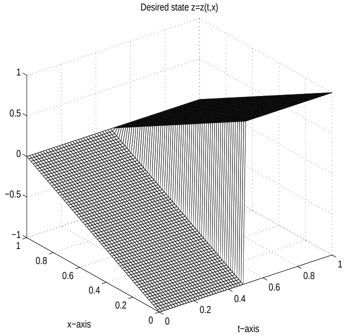
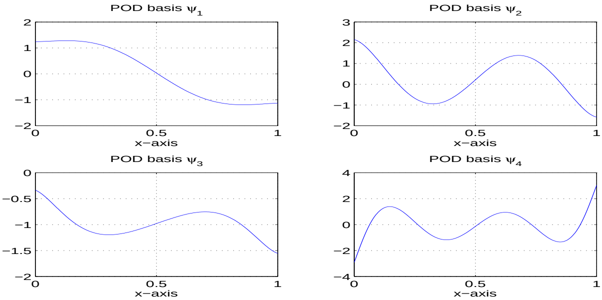
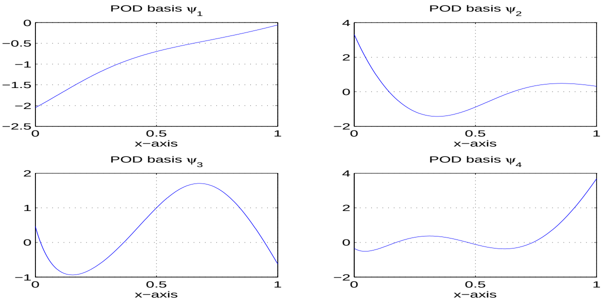
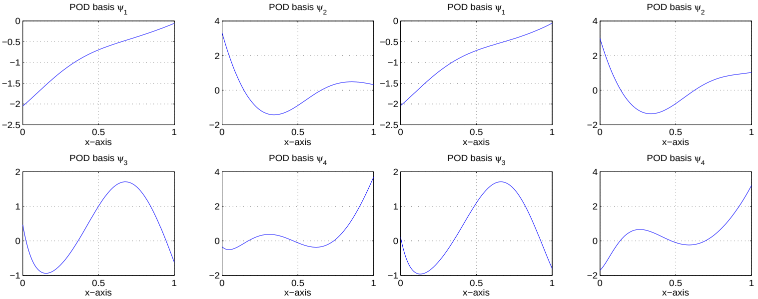


Figure: Desired state :: Optimal POD state for the OS-POD strategy.





POD basis functions: uncontrolled:: final OS-POD update of the controls.



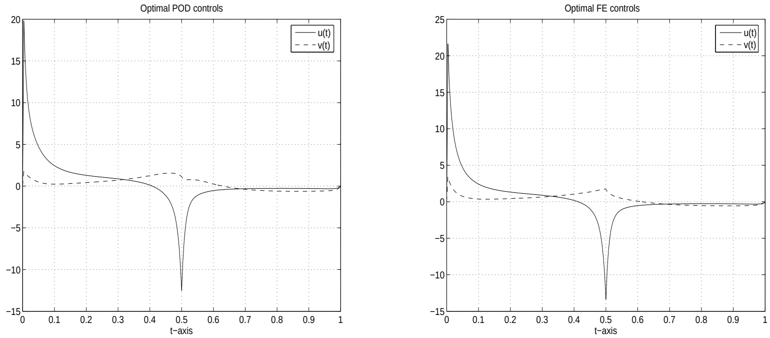
POD basis functions: final OS-POD update of the controls::POD basis functions associated with the optimal FE-SQP controls.

	$n=0$	$n=4$	with FE controls
$\lambda_1/\mathrm{tr}(\mathcal{K}_h)$	0.97187	0.87661	0.88092
$\lambda_2/\mathrm{tr}(\mathcal{K}_h)$	0.02209	0.08051	0.08734
$\lambda_3/\mathrm{tr}(\mathcal{K}_h)$	0.00579	0.02736	0.02744
$\lambda_4/\mathrm{tr}(\mathcal{K}_h)$	0.00025	0.00191	0.00292

Table: Decay of the first four eigenvalues λ_i for the OS-POD strategy for $n=0$, $n=4$ and for the first four eigenvalues associated with the snapshots that are computed by using the optimal FE controls. For $n=0$ we have the decay of the first eigenvalues associated with the uncontrolled solution.

	$J(y,u,v)$
Uncontrolled solution	0.22134
OS-POD	0.03813 (0.03856)
FE-SQP	0.03765

Table: Values of the cost functional for the different approaches (in brackets: value of the cost using POD solution in original system).



Optimal control for OS-POD strategy (left) and optimal FE-SQP controls (right).

partial steps	CPU time
Generate snapshots	0.62 seconds
POD computation	0.09 seconds
Compute ROM	0.03 seconds
SQP solve P^ℓ	14.16 seconds
Compute μ_i's	2.07 seconds
FE dual solve	0.71 seconds
Backtracking in step (4)	0.32 seconds

Table: averaged CPU times for one iteration of OS-POD.

use these (optimal open loop) POD basis elements to solve closed loop problems,
where now, only unresolved dynamics result from noise...
and other systems theoretical tasks.

Arian, Borggard, Gunzburger, Hinze, Ly, Ravindran, Rowley, Sachs, Sirovich, Tran, Willcox,...

Optimal Snapshot Location

where to place additional snapshots ?

$$\dot{y}(t) = Ay(t) + f(t) \text{ for } t \in (0, T]$$
$$y(0) = y_\circ,$$

- fixed time grid $t_j = j\Delta t,\ 0 \le j \le m,\quad$ with $\Delta t = T/m$,
- snapshots $y_j = y(t_j) \in V,\ 0 \le j \le m$,
- new time instances $\bar{t}_k \in [0, T],\quad 1 \le k \le \bar{k}$,
- corresponding snapshots $\bar{y}_k = y(\bar{t}_k),\quad 1 \le k \le \bar{k}$,
- number of POD basis functions $\ell \in \{1, \dots, m\}$.

$$\mathcal{R}(\bar{t})\psi = \sum_{j=0}^{m} \langle y_j, \psi \rangle_H \, y_j + \sum_{k=1}^{\bar{k}} \langle \bar{y}_k, \psi \rangle_H \, \bar{y}_k.$$

The POD basis functions $\{\psi_i\}_{i=1}^{\ell}$ are eigenvectors of $\mathcal{R}(\bar{t})$, i.e.

$$\mathcal{R}(\bar{t})\psi_i = \lambda_i \, \psi_i,$$

POD-Galerkin approximation

$$\begin{aligned} \dot{x}(t) &= A^\ell x(t) + \mathfrak{f}(t) \text{ for } t \in (0, T], \\ x(0) &= x_\circ \end{aligned}$$

$$\min J(x,\bar{t},\psi,\lambda) = \frac{1}{2}\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} x_i(t)\,\psi_i \Big\|_H^2 \,\mathrm{d}t.$$

subject to

$$\begin{aligned}
\dot{x} - A^\ell x - \mathfrak{f}(t) &= 0 && \text{in } L^2(0,T;\mathbb{R}^\ell),\\
x(0) - x_\circ &= 0 && \text{in } \mathbb{R}^\ell,
\end{aligned}$$

$$\begin{pmatrix} \big(\mathcal{R}(\bar{t}) - \lambda_1\big)\psi_1 \\ \vdots \\ \big(\mathcal{R}(\bar{t}) - \lambda_\ell\big)\psi_\ell \end{pmatrix} = 0 \qquad \text{in } H^\ell,$$

$$\begin{pmatrix} 1 - \|\psi_1\|_H^2 \\ \vdots \\ 1 - \|\psi_\ell\|_H^2 \end{pmatrix} = 0 \qquad \text{in } \mathbb{R}^\ell$$

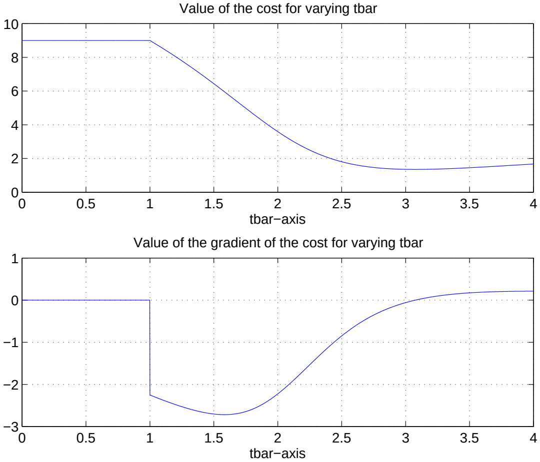
$$\dot{y}(t) = f(t) \text{ for } t \in (0, T] \quad \text{and} \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } t \in [0, T/4) \text{ and } f(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } t \in [T/4, T]$$

Exact solution constant on $(0, T/4)$

POD: Snapshots: $t_0 = 0$ and $\bar{t}$, $\quad \ell = 1$

$$y^\ell(t) = \begin{cases} (\psi^2(\bar{t}))_1 \psi_1(\bar{t}) & \text{for } t \in [0, T/4), \\ \big((t - T/4)(\psi^1(\bar{t}))_1 + (\psi^2(\bar{t}))_1\big)\psi_1(\bar{t}) & \text{for } t \in [T/4, T]. \end{cases}$$


Value of the cost for varying tbar
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
tbar−axis
Value of the gradient of the cost for varying tbar
1
0
−1
−2
−3
0
0.5
1
1.5
2
2.5
3
3.5
4
tbar−axis