An accurate reduced order model for unsteady flows controlled by synthetic jets

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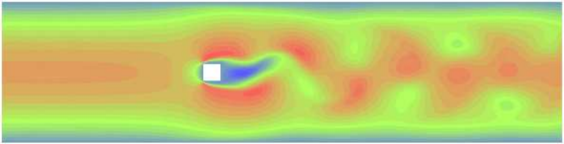
POD ROM of flow past a bluff body

- POD database :
  - Control using actuators placed on the body.
  - Solutions obtained with one control law, or more.

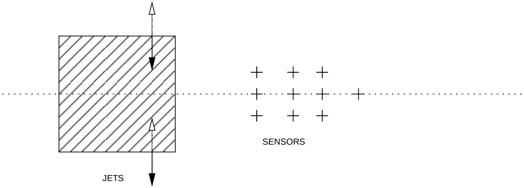
- Build a ROM that :
  - is **Accurate** when integrated with database control law(s)
  - can **predict** solutions for reasonable **changes in control law**
  - can be used for **optimization**

Setup

- Confined square cylinder + incompressible Navier-Stokes



- Simulation of a blowing/suction control using synthetic jets placed on the cylinder :



Reduced Order Model (1)

- Full Navier-Stokes simulation with control law $c(t)$

$$\Rightarrow \text{solutions at } N_t \text{ time instants : } \mathbf{u}(t^k, x), k = 1..N_t$$

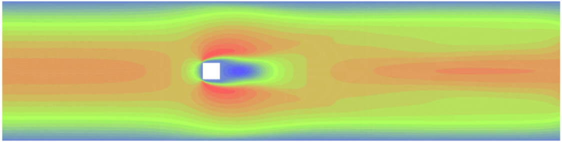
- Definition of snapshots for building a POD basis :

$$\mathbf{w}^k(x) = \mathbf{u}(t^k, x) - \bar{\mathbf{u}}(x) - c(t^k)\mathbf{u}_c(x)$$

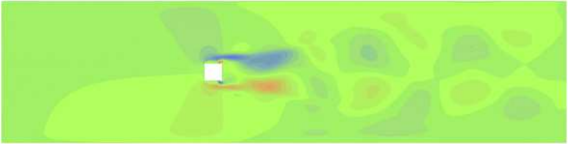
  where functions $\bar{\mathbf{u}}$ and $\mathbf{u}_c$ are chosen such that the snapshots are equal to zero at inflow, outflow, *and jet* boundaries.
- Build POD basis $\Phi_k^{\mathcal{C}}$, and perform Galerkin projection of equations.

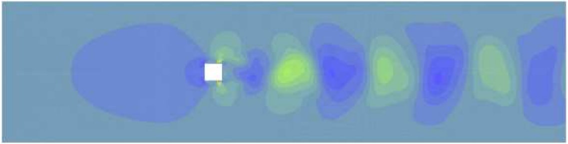
Reduced Order Model (2)

- For $\bar{\mathbf{u}}$: Simulation with $c = 0$ + averaging



- For $\mathbf{u_c}$: Simulation with $c = c^\star$ + averaging $\Rightarrow \bar{\mathbf{u}}'$, $\mathbf{u_c} = (\bar{\mathbf{u}}' - \bar{\mathbf{u}})/c^\star$





Reduced Order Model (3)

- In Navier-Stokes $\mathbf{u}(t, \mathbf{x})$ is replaced by $\bar{\mathbf{u}} + \mathcal{c}(t)\mathbf{u_c} + \sum_{k=1}^{N_r} a_k(t)\Phi_k^{\mathcal{C}}(\mathbf{x})$.
- Projection onto the POD modes leads to a system of ODEs :

$$\left\{ \begin{array}{lcl} \dot{a}_r(t) & = & f_r(\mathbf{a}(t), \mathcal{c}(t), \widehat{\mathbf{X}}) \\ a_r(0) & = & a_r^0 \end{array} \right. \qquad 1 \le r \le N_r$$

$$\text{where : } f_r(\mathbf{a}(t), \mathcal{c}(t), \widehat{X}) \;=\; \begin{array}{l} \widehat{A}_r + \widehat{C}_{kr} a_k(t) - \widehat{B}_{ksr} a_k(t) a_s(t) \\ -\widehat{E}_r \dot{\mathcal{c}}(t) - \widehat{F}_r \mathcal{c}^2(t) + [\widehat{G}_r - \widehat{H}_{kr} a_k(t)]\mathcal{c}(t) \end{array}$$

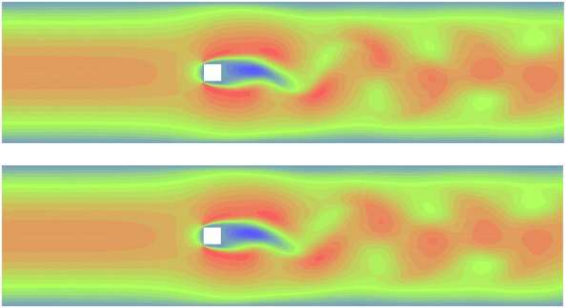
System matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{E}$, $\widehat{F}$, $\widehat{G}$ and $\widehat{H}$ depend only on $\bar{\mathbf{u}}$, $\mathbf{u}_c$ and the modes Φ_k^0.
In the following we call the above model $\widehat{M}^{\mathcal{C}}$.

Model Accuracy (1)

- A small number of modes are needed to capture the energy in a snapshot.

$$\text{If } \hat{a}_k(t) = \left(\mathbf{u}(t,\cdot), \Phi_k^{\mathcal{C}}\right)_2$$

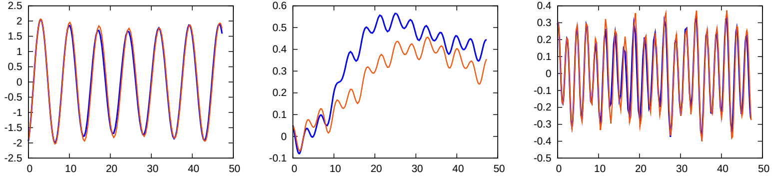
$$\text{Then } \bar{\mathbf{u}}(\mathbf{x}) + c(t)\mathbf{u_c}(\mathbf{x}) + \sum_{k=1}^{N_r} \hat{a}_k(t)\Phi_k^{\mathcal{C}}(\mathbf{x}) \approx \mathbf{u}(t,\cdot)(\mathbf{x}) \text{ for } N_r \text{ small}$$



A solution and its reconstruction with $N_r = 6$

Model Accuracy (2)

- **but** there are differences between, 'a' (solution of $\widehat{M}^{\mathcal{C}}$) and '$\hat{a}$'



a_k vs. $\hat{a}_k$ for $k = 1, k = 3$ and $k = 6$

Calibration (1)

Unresolved modes are modelled as linear combinations of the others and the control law.

$\rightarrow$ Adjust certain system matrices so as to minimize the difference between $\hat{a}_k$ and a_k

- Method 1

$$\min_X \int_0^T \sum_{k=1}^{N_r} \left(\hat{a}_k(t) - a_k(t)\right)^2 dt$$

subject to :

$$\left\{ \begin{array}{rcl} \dot{a}_r(t) & = & f_r(\mathbf{a}(t), c(t), X) \\ a_r(0) & = & a_r^0 \end{array} \right. \qquad 1 \leq r \leq N_r$$

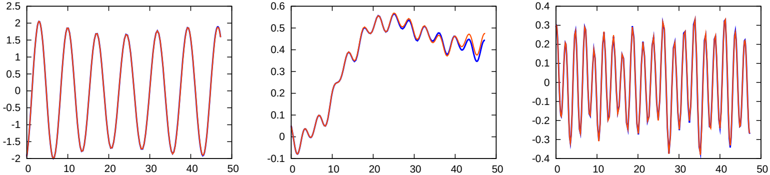
- Method 2

$$\min_X \int_0^T \sum_{k=1}^{N_r} \left(\dot{\hat{a}}_k(t) - f_k(\hat{\mathbf{a}}(t), c(t), X)\right)^2 dt + \alpha \|X - \widehat{X}\|^2$$

with α small.

Calibration (2)

Effect of calibration on time coefficients



a_k vs. $\hat{a}_k$ for $k = 1, k = 3$ and $k = 6$ *after calibration*

Effect of calibration on a cost functional

$$\mathcal{F}(\mathbf{a}) = \sum_{r=1}^{6} a_k^2(t)$$

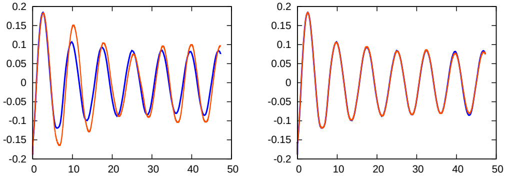
In the example, $\mathcal{F}(\hat{\mathbf{a}}) = 166.98$
Relative error between $\mathcal{F}(\mathbf{a})$ and $\mathcal{F}(\hat{\mathbf{a}})$ before calibration : 4.3 %
Relative error after calibration : 0.35 %

Calibration (3)

Control defined by feedback = extra errors

$$c(t) = \kappa \times v(t, \mathbf{x}_s)$$

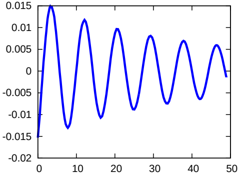
where $\mathbf{x}_s$ is a point in the cylinder wake.



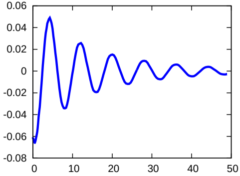
$c(t)$ vs. reconstruction of c during model integration
before and after calibration

Prediction (1)

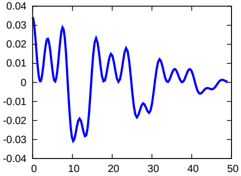
Simulations at $Re = 60$ with $\mathcal{C}^1$,$\mathcal{C}^2$ and $\mathcal{C}^3$.
Models tested with $\mathcal{C}^4$.



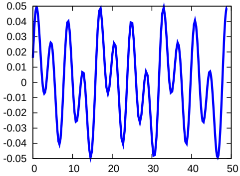
$\mathcal{C}^1$



$\mathcal{C}^2$

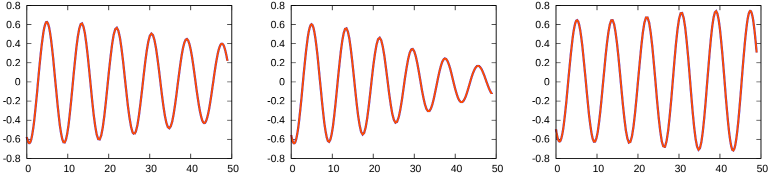


$\mathcal{C}^3$

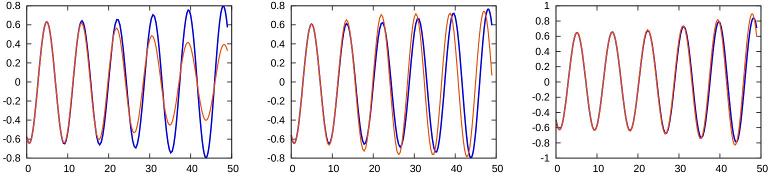


$\mathcal{C}^4$

Prediction (2)



Coefficients a_1 for the three models



Coefficients a_1 for $\mathcal{C}^4$: projections vs. predictions

Errors on $\mathcal{F}(\mathbf{a})$

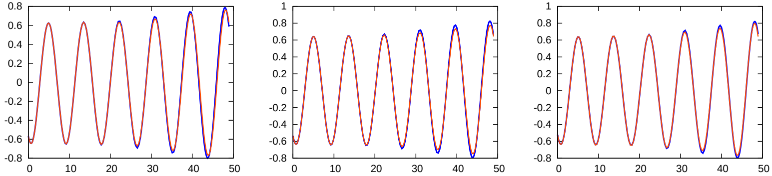
42 %	10.11 %	4.5 %

Model with two control laws

- With N_c different control laws the calibration problem stays the same size if 'Method 2' is used :

$$\min_X \sum_{i=1}^{N_c} \int_0^T \sum_{k=1}^{N_r} \left(\dot{\hat{a}}_k^i(t) - f_k(\hat{\mathbf{a}}^{\mathbf{i}}(t), \mathcal{C}^i(t), X) \right)^2 dt + \alpha \|X - \widehat{X}\|^2$$

- Three 'double control' models

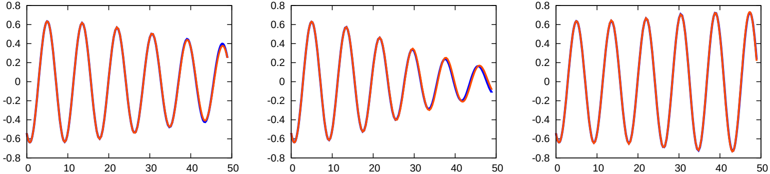


Coefficients a_1 for $\mathcal{C}_4$: projections vs. predictions

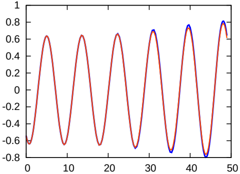
Errors on $\mathcal{F}(\mathbf{a})$

4.9 % 7.0 % 4.0 %

Model with three control laws



Coefficients a_1 for $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$



Coefficients a_1 for $\mathcal{C}_4$: projections vs. predictions

Error on $\mathcal{F}(\mathbf{a})$: 4.6 %

Using the Model for optimization (1)

- Seek c that solves : $\min_c \mathcal{F}(\mathbf{a}) + \beta \int_0^T \dot{c}^2(t)\, dt$

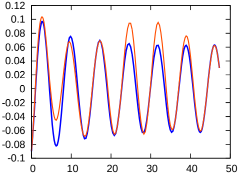
  subject to :

$$\left\{ \begin{array}{lll} \dot{a}_r(t) & = & f_r(\mathbf{a}(t), c(t), X) \\ a_r(0) & = & a_r^0 \end{array} \right. \qquad 1 \leq r \leq N_r$$

- Algorithm :
  - Choose c^0, run N-S with c^0, build POD-ROM model
  - Project c^0 onto a basis of 30 B-Spline functions.
  - $\mathcal{L}(c) =$
  $$\mathcal{F}(\mathbf{a}) + \beta \int_0^T \dot{c}^2(t)\, dt + \int_0^T \sum_r b_r(t) \left(\dot{a}_r(t) - f_r(\mathbf{a}(t), c(t), X) \right) dt$$
  - Calculate gradient of $\mathcal{L}(c^0)$ with respect to the B-Spline control points.
- Example : $\mathrm{Re} = 150, \quad c^0(t) = 0.6 \times v(t, x_s)$
  Initial Value of controlfunctional : $\mathcal{F}(\mathbf{a}) = 178.46$

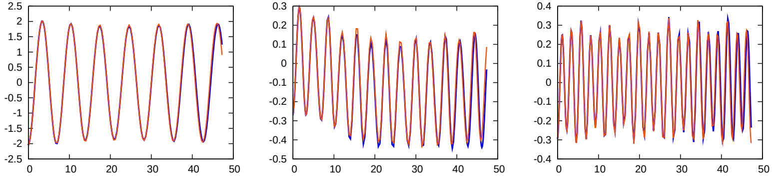
Using the Model for optimization (2)

- Build 'Single control' model.
- Perform one step optimization : $\mathcal{C}^0 \rightarrow \mathcal{C}^1$



- Inject new control law $\mathcal{C}^1$ into N-S
- Project on initial POD base and re-evaluate functional :

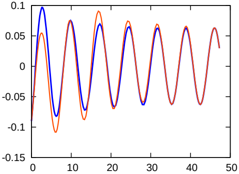
$$\mathcal{F}(\mathbf{a}) = 179.43 > 178.46$$



Evolution of coefficients a_k, $\quad k = 1, k = 3, k = 6$

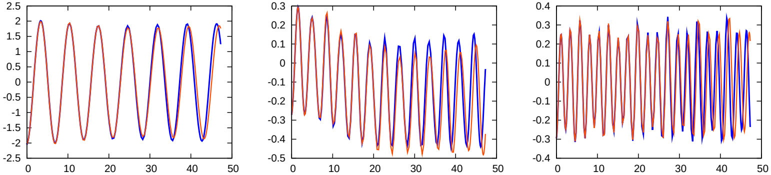
Using the Model for optimization (3)

- Build 'Double control' model using the initial solution and the new one.
- perform one step optimization : $c^0 \rightarrow c^2$
- Inject new control law c^2 into N-S



- Project on initial POD base and re-evaluate functional

$$\mathcal{F}(\mathbf{a}) = 173.6 < 178.46$$



Evolution of coefficients a_k, $k = 1, k = 3, k = 6$