Jean-Michel Bony

# Two or Three Things that I Know About Johannes

“Microlocal Analysis and Spectral Theory”
“En l’honneur de Johannes Sjöstrand”

C .I. R. M., September 23 , 2013

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

17, RUE DESCARTES - 75230 PARIS CEDEX 05

# SUR UNE CLASSE D'OPERATEURS DE TYPE SOUS-ELLIPTIQUE

par J. J. DUISTERMAAT et J. SJOSTRAND

exposé par J. SJOSTRAND

Exposé N° VII

15 Novembre 1972

$P$ : ΨDO of order $m$ on $X$ s.t. $\frac{1}{i}\{p,\overline{p}\} \neq 0$ on $\Sigma = \mathrm{Char}(P)$.

$$\Sigma^{\pm} = \left\{(x,\xi) \,\middle|\, p = 0\,;\ \pm\frac{1}{i}\{p,\overline{p}\} > 0\right\}$$

Microlocal model : $D_n + ix_n D_{n-1}$ (Mizohata)

Then $\exists\ F\,, F^{+}, F^{-}$ of a precise form s.t.

$$FP \equiv I - F^{+}\ \ ;\ PF \equiv I - F^{-}\ \ ;\ \left(F^{\pm}\right)^2 \equiv F^{\pm}$$

$$F : H^s \to H^{s+m-1/2}\ \ ;\quad F^{\pm} : H^s \to H^s$$

$$\mathrm{WF}'(F) = \mathrm{diag}\,(T^*X \setminus 0)\ \ ;\ \mathrm{WF}'(F^{\pm}) = \mathrm{diag}\,(\Sigma^{\pm})$$

Fourier Intégraux à phase complexe (10 p.)

with A. Melin — January 1974

**Precept N°1**

Go into the complex if you can.

(M. Riesz?)

Integrals $\int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta$

Phase $\varphi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^N)$ ; $\mathrm{Im}(\varphi) \geq 0$

Homogeneous, no critical point, non degenerate (as in FIO1)

- almost analytic extension $\tilde{\varphi}$
- Morse lemma, Stationary phase (saddle point)
- (jet of) positive Lagrangean submanifold $\Lambda$, image of

$$\left\{(x,\theta) \in \mathbf{C}^n \times \mathbf{C}^N \middle| \, d_\theta \tilde{\varphi} = 0\right\} \mapsto (x, d_x \tilde{\varphi}) \in \mathbf{C}^n \times \mathbf{C}^n$$

## Applications

- Oblique derivative problem (with A. Melin)

- Singularities of Bergman and Szegö kernels
  (with L. Boutet de Monvel)

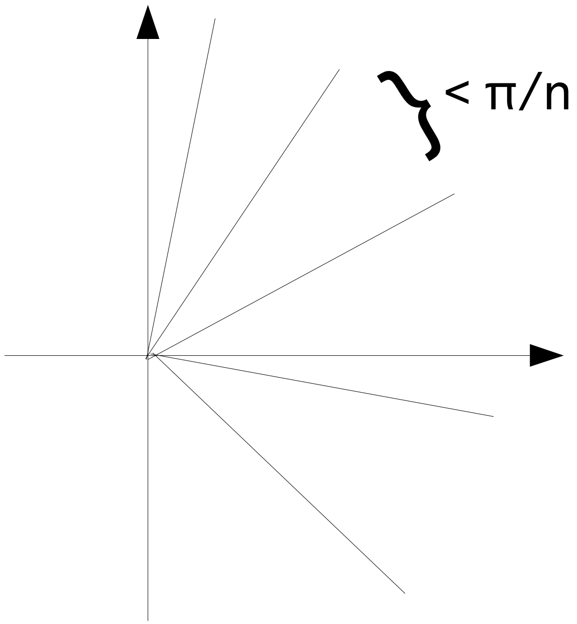
- . . .

Elliptic operators singular in one point (with M. S. Baouendi)

$P = \sum a_\alpha(x) D_x^\alpha$ ; $a_\alpha$ hom. pol. of d° $|\alpha|$

Assumptions : $P$ elliptic outside 0 + condition (C)

- $P$ is not hypoelliptic
- Any $C^\infty$ solution is analytic



Precept N°2

Nonselfadjoint is beautiful

On the eigenvalues of a class of hypoelliptic operators

I
II
III
IV

(I–III: with A. Menikoff)

| Precept N°3 |
|---|
| Write long series of long papers. |

## Hypoelliptic operators with loss of one derivative

$P$ pseudo, selfadjoint, symbol $p_m + p_{m-1} + \cdots$, .

$p_m \geq 0$, $\Sigma = \mathrm{Char}(P) = p_m^{-1}(0)$ (not necessarily smooth)

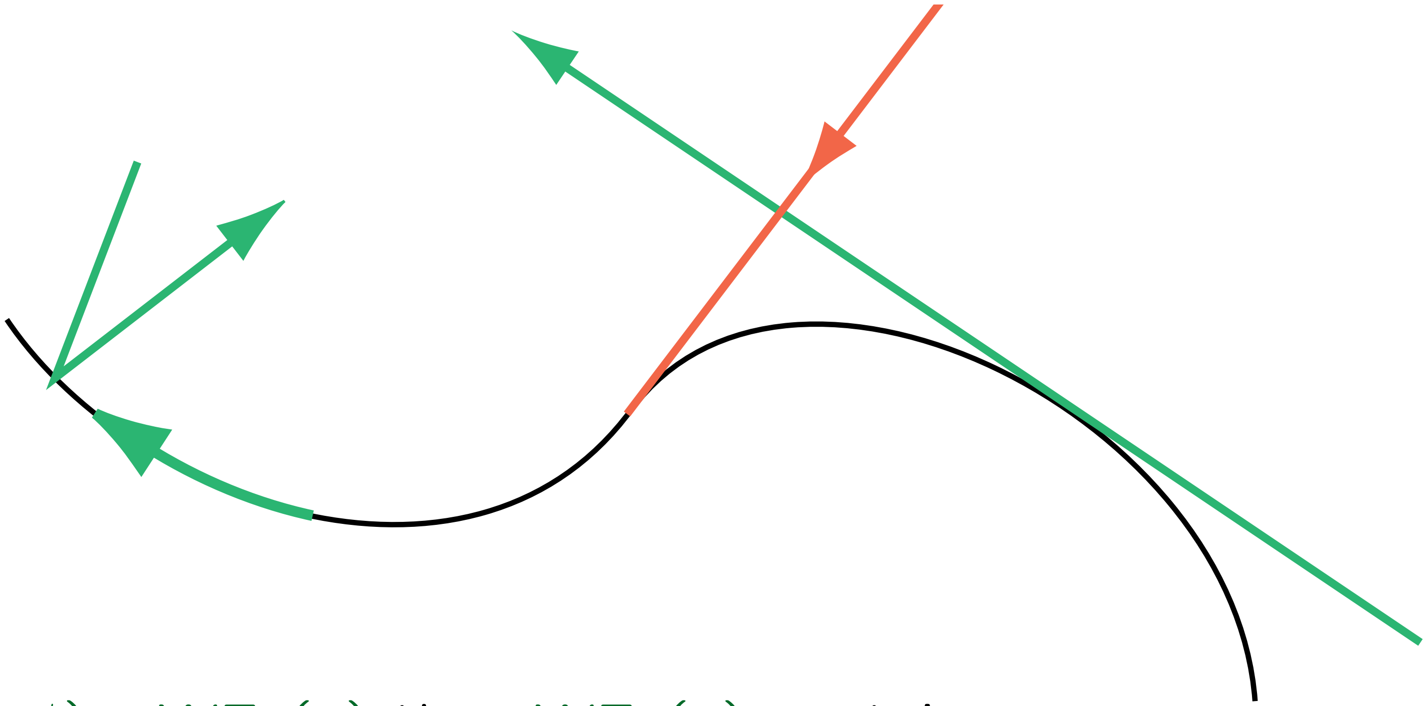
Assumption relating the subprincipal symbol and the eigenvalues of the fundamental matrix on $\Sigma$.

$N(\lambda) =$ number of eigenvalues $\leq \lambda$ (or $N^{\pm}(\lambda)$)

$e^{-tP}$ constructed as a FIO with (quasihomogeneous) complex phase.

asymptotic in terms of measure of level sets in the phase space

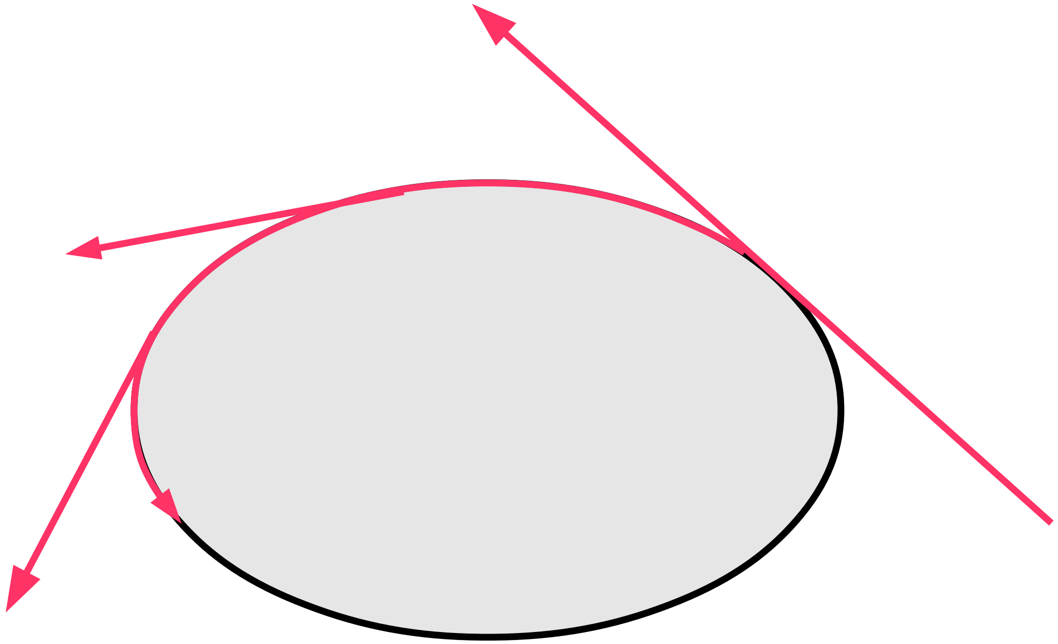
Singularities of Boundary Value Problems I and II (with R. Melrose)



TH. If $(x, \xi) \in \mathrm{WF_b}(u)$ then $\mathrm{WF_b}(u)$ contains

$$\left\{ \begin{array}{c} \text{the} \\ \text{at least one} \end{array} \right\} \text{ maximal ray starting from } (x, \xi).$$

Propagation of analytic singularities for second order Dirichlet problem I, II, III + microhyperbolic BVPb + ...



Same results for $\mathrm{WF}_{\mathrm{a}}(u)$ but propagation along analytic rays.

+ results on impossible configurations.

Sato-Kawai-Kashiwara :

$$\delta(x) = \int_{\mathbf{S}^{n-1}} \frac{h(x,\eta)}{\big(\langle x\,,\,\eta\rangle + i\,|x|^2 + i0\big)^n}\,d\eta = \int \Phi(x,\eta)\,d\eta$$

Fourier $\longrightarrow$ idem with $i\,|x|^2$ replaced by 0.

$(x_0,\eta_0) \notin \mathrm{WF_a}(u) \Longleftrightarrow \int \Phi(x-y,\eta)u(y)\,dy$ analytic near $(x_0,\eta_0)$.

## Bros-Iagolnitzer

$$Tu(x,\xi,\xi_0) = \int e^{-i\langle x\,,\xi\rangle - \xi_0|x-y|^2}\, u(y)\, dy \qquad \xi_0 > 0.$$

Actually, a closed $n$-form $Tu\, d\xi_1 \ldots \wedge d\xi_n + \sum T_j u\, d\xi_0 \wedge \widehat{d\xi_j}$

$(\overline{x},\overline{\xi}) \notin \mathrm{WF_a}(u) \Longleftrightarrow |Tu(x,\xi,\xi_0)| \le C(1+|\xi|)^N e^{-\alpha\xi_0}$

for $\xi$ in a conic neighbourhood of $\overline{\xi}$ and $0 < \xi_0 < \gamma\,|\xi|$.

Inversion formula. Deformation of the contour of integration.

## Resolution of identity

Operators $A = (A_\lambda)$ with Schwartz kernel:

$$A(x,y) = \lambda^{3n/2} \int e^{i\lambda\Phi(x,y,\alpha)} a(x,y,\alpha,\lambda)\, d\alpha$$

$\Phi(x,y,\alpha_x,\alpha_\xi)$ holomorphic in a nbhd of $(x_0,x_0,x_0,\xi_0) \in \mathbf{R}^{4n}$

such as $\Phi(x,y,\alpha) = \left\langle x-y\,,\,\alpha_\xi \right\rangle + \frac{i}{2}\left((x-\alpha_x)^2 + (y-\alpha_x)^2\right)$

For $\pi_\alpha = \lambda^{3n/2} e^{i\lambda\Phi(x,y,\alpha)} a(x,y,\alpha,\lambda)$, one has (so to speak)

$\mathrm{WF}'_{\mathrm{a}}(\pi_\alpha)$ "=" $(\alpha_x,\alpha_x,\alpha_\xi,\alpha_\xi)$ as $\lambda \to \infty$.

For $P(x,\xi,\lambda) = e^{-i\lambda\langle x\,,\,\xi\rangle} A\left(e^{i\lambda\langle\cdot\,,\,\xi\rangle}\right)$, one has $A = P(x, \frac{D_x}{\lambda}, \lambda)$.

Particular case : $\mathrm{Id} = \int \pi_\alpha\, d\alpha$

astérisque
95

1982

# singularités analytiques microlocales

J. SJÖSTRAND

# équation de Schrödinger et propagation des singularités...

B. LASCAR - J. SJÖSTRAND

société mathématique de france

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**S**pace $H_\varphi(\Omega)$. ($\varphi$ real psh in $\Omega \subset \mathbf{C}^n$) :

$u(z,\lambda)$ holomorphic in $z$ s.t. $\forall\varepsilon, \forall K,\ |u(z,\lambda)| \leq C_{K,\varepsilon} e^{\lambda(\varphi(z)+\varepsilon)}$.

$u \sim v$ if $\forall K, \exists\varepsilon,\ |u-v| \leq C_K e^{\lambda(\varphi(z)-\varepsilon)}$

Geometry : $\Lambda = \left\{(z, \frac{2}{i}\frac{\partial\varphi}{\partial z})\right\} \subset \mathbf{C}^{2n}$.

For $\varphi$ strictly psh, it is I-lagrangean and R-symplectic.

- Fourier transformation For $\varphi$ psh s.t. $\varphi''(x_0)$ non deg. signature 0.

For $\xi \in \mathbf{C}^n$ near $\xi_0 = \frac{2}{i}\varphi'(x_0)$ set $\varphi^*(\xi) = \text{crit.val.}_x(\varphi + \operatorname{Im} x\cdot\xi)$

$$H_{\varphi,x_0} \ni u(x,\lambda) \longmapsto \mathcal{F}u(\xi,\lambda) = \int_{\Gamma_\xi} e^{-i\lambda x\cdot\xi}\, u(x,\lambda)\, dx \in H_{\varphi^*,\xi_0}$$

↖ "good contour"

- ΨDO : $H_{\varphi,x_0} \to H_{\varphi,x_0}$

- FIO : $H_{\varphi,x_0} \to H_{\psi,y_0}$

$$Au(y,\lambda) = \int_{\Gamma_y} a(x,y,\theta,\lambda)u(x,\lambda)\, dx\, d\theta\ ;\ a \in H_{\Phi,(x_0,y_0,\theta_0)}$$

Transformation of FBI. Let $\varphi(x,y)$ holomorphic near $(x_0, y_0) \in \mathbf{C}^n \times \mathbf{R}^n$

At that point, assume $\frac{\partial\varphi}{\partial y} = -\eta_0 \in \mathbf{R}^n$; $\mathrm{Im}\,\frac{\partial^2\varphi}{\partial y^2} >> 0$; $\det \frac{\partial^2\varphi}{\partial x \partial y} \neq 0$.

Set $y(x) =$ the point where $\mathrm{Im}\,\varphi(x,\cdot)$ is minimum, $\eta(x) = -\frac{\partial\varphi}{\partial y}(x, y(x))$.

Then $x \longmapsto (y(x), \eta(x))$ is a diffeo (near $x_0$) $\mathbf{C}^n \to T^*\mathbf{R}^n$.

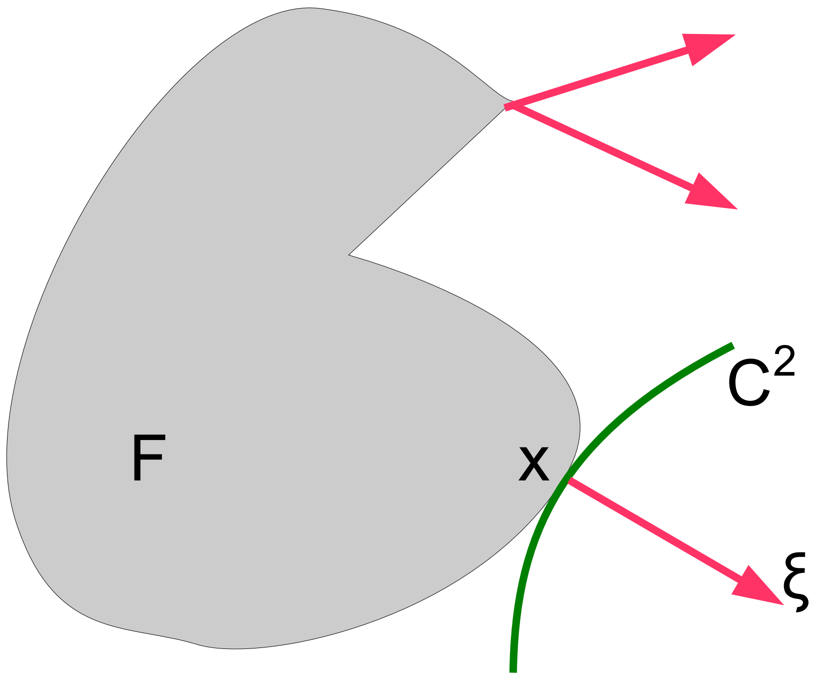
For $a$ classical elliptic symbol of order 0, the transformation

$$Tu(x,\lambda) = \int e^{i\lambda\varphi(x,y)} a(x,y,\lambda)\chi(y)u(y)\,dy$$

maps $\mathcal{D}(\mathbf{R}^n) \to H_\Phi$ with $\Phi(x) = -\varphi(x, y(x))$.

TH. $(y(x_1), \eta(x_1)) \notin \mathrm{WF_a}(u) \Longleftrightarrow Tu(x,\lambda) \equiv 0$ in $H_{\Phi,x_1}$.

## Direct infinitesimal geometry




Conormal "fiber bundle" of F

$N^*F \subset T^*\mathbf{R}^n \setminus 0$ (not closed)

- (JMB) if $p(x,\xi)$ and $q(x,\xi)$ vanish on $N^*F$, then $\{p,q\}$ also.

- (JS) If $p(x,\xi)$ vanishes on $N^*F$

$(x(t),\xi(t))$ : bicharacteristic of $p$ starting from $(x_0,\xi_0) \in N^*F$, then it remains in $N^*F$ for $|t| < \varepsilon$.

End of the first part :

"Ten years of the life of Johannes"

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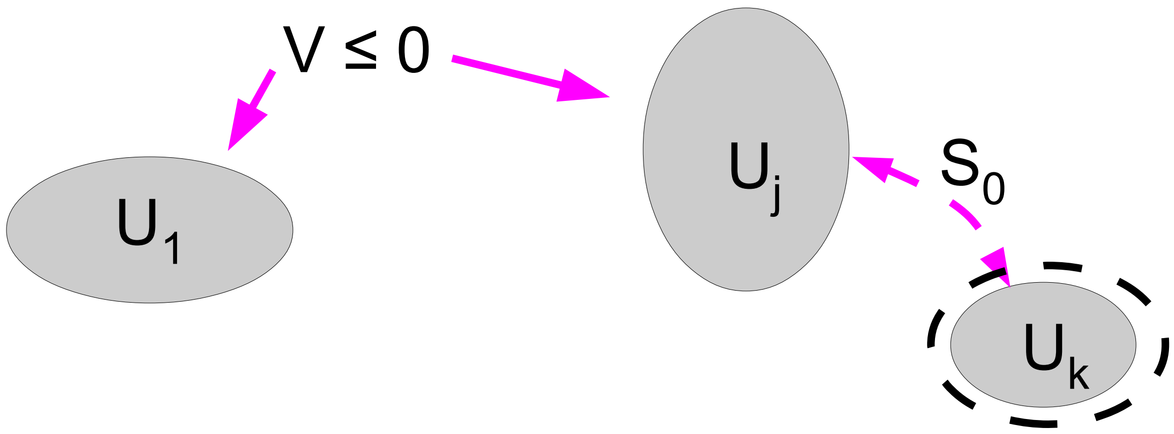
## Multiple wells in the semiclassical limit I to VI + ... (with B. Helffer)

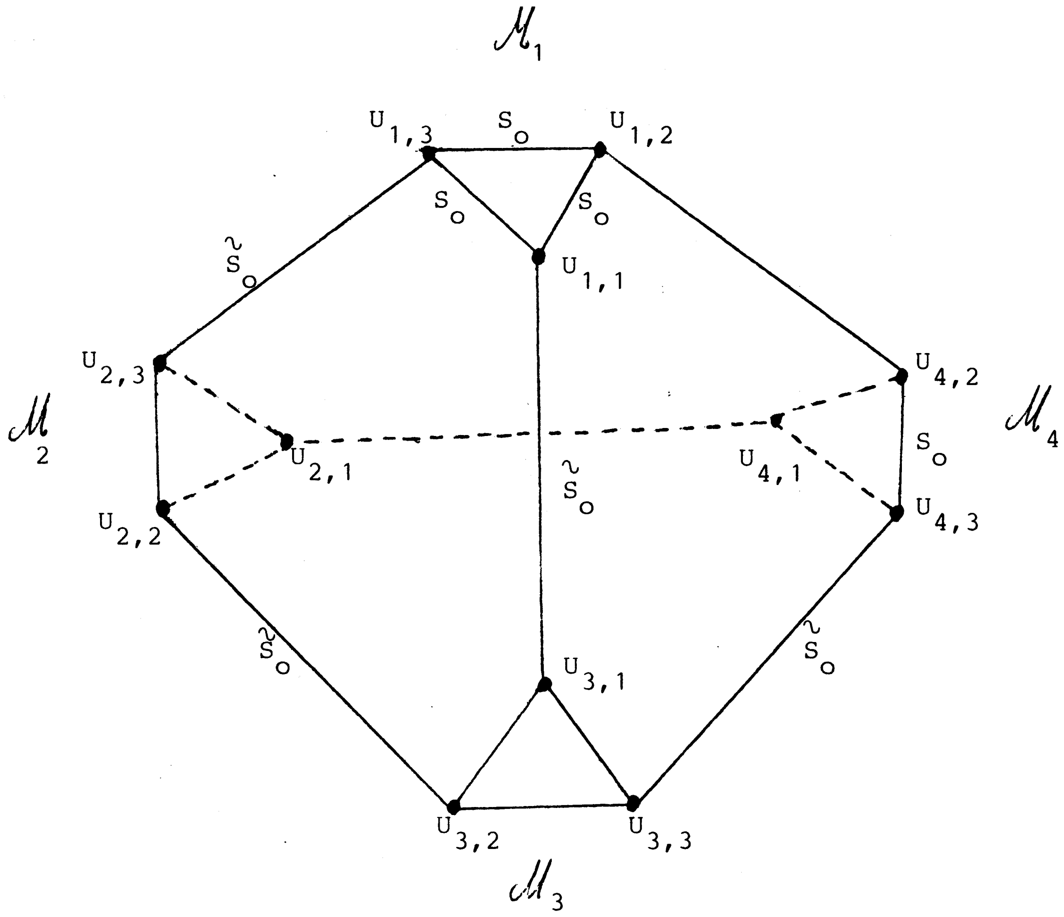
Eigenvalues near 0 of $P = -h^2\Delta + V(x)$ ($\underline{\lim}\, V(x) > 0$ for $x \to \infty$)




$\Lambda(k) =$ eigenvalues of Dirichlet Pb. in $\complement\widetilde{U_k}$

$$\Lambda(P) = \left\{\bigcup_k \Lambda(k)\right\} + \left\{\text{interaction } O(e^{-S_0/h})\right\} + \left\{\text{error } O(e^{-2S_0/h})\right\}$$

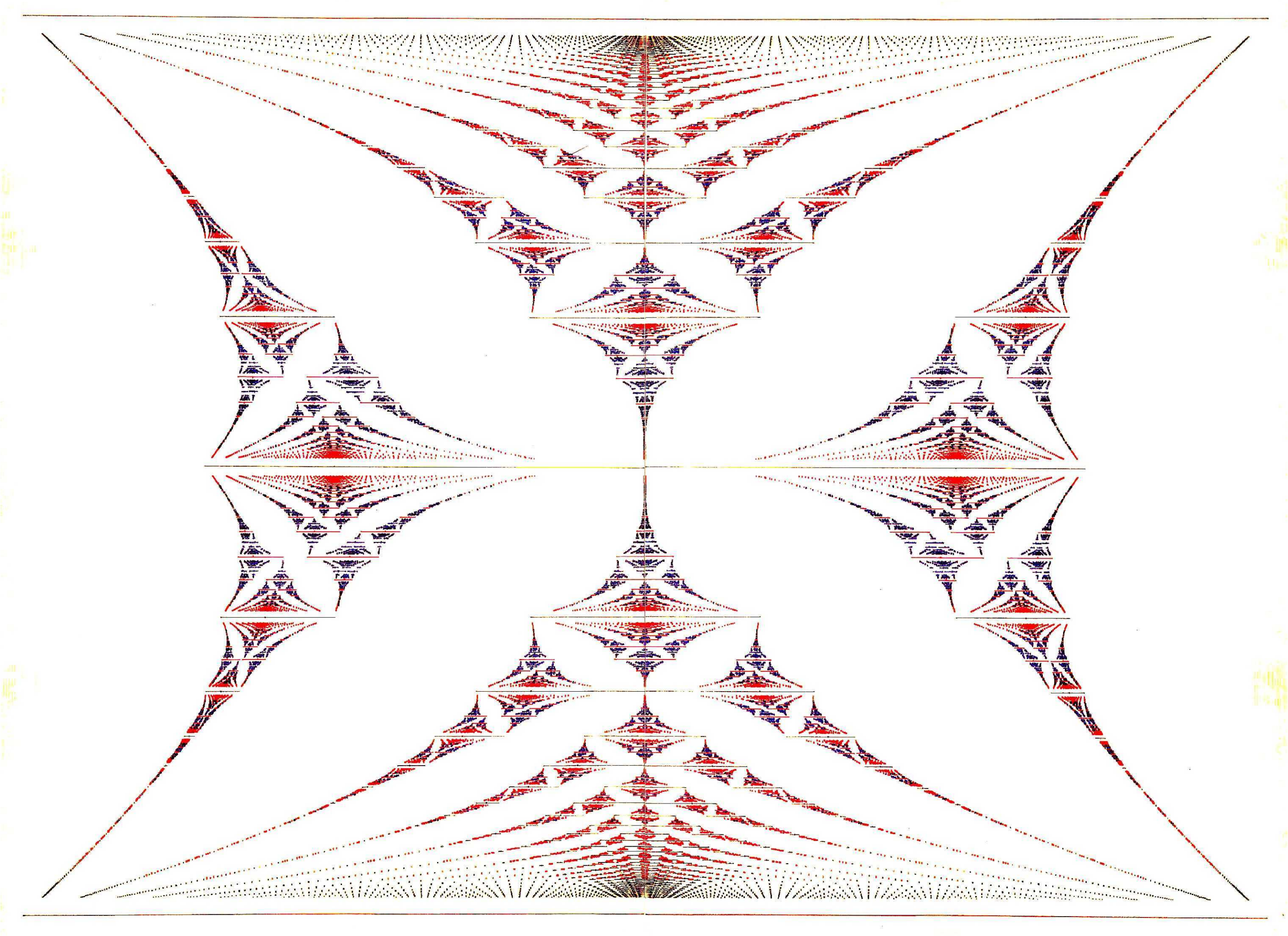
• Molecule




• Witten complex. Computation of the cohomology of a Riemannian manifold via the complex $d_f = e^{-f/h} h d\, e^{f/h}$ ($f$ Morse function)

Harper equation I, II, III + magnetic Schrödinger + (with B. Helffer)

$$P = cos(hD_x) + cosx \quad \text{in } L^2(\mathbf{R})$$



## RESONANCES

$28 + 6 = 34$ articles

- Invasion of the subject by microlocal analysis.
- The well in the island.
- Microlocal version of the complex scaling. Upper bounds.
- The black box.
- Trace formulas. Lower bounds.

## Nonselfadjoint Operators

Pseudospectrum : $z \in \sigma_\varepsilon(P) \iff \left\|(z-P)^{-1}\right\| \geq 1/\varepsilon$ (or $z \in \sigma(P)$)

• For $P = p(x, hD)$, the set $\Sigma = \overline{p(\mathbf{R}^{2n})}$ is a "good approximation" :

— inside, if $\frac{1}{i}\{p, \overline{p}\} > 0$, $\left\|(z-P)^{-1}\right\| \geq 1/h^N$

— for $z \in \partial\Sigma$, under a "non trapping assumption" , the disk $D(z_0, h \log(1/h))$ contains no eigenvalue.

• Non s.a. perturbations of s.a. operators in dimension 2 (I, II, IIIa with Hitrik): precise estimates of $\#(\sigma(P) \cap \text{rectangle})$.

• Random perturbations of non s.a. operators : the spectrum of $P + \delta(h)Q_\omega$ satisfy a Weyl law with probability $\to 1$.
Perturbation by a potential.
Almost sure Weyl law.