Equivariant Toeplitz index

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Asymptotic equivariant index

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In this lecture I wish to describe how the asymptotic equivariant index and how behaves in case of the group SU_2.

I spoke of this some years ago in the case of a torus action, and will first recall that case.

Szegö kernel

The complex sphere $X \subset \mathbb{C}^N$ is endowed with its canonical contact structure coming from its CR structure. The contact form is

$$\lambda = \operatorname{Im} \bar{z} \cdot dz|_X$$

There is a corresponding symplectic cone Σ : the set of positive multiples on λ in T^*X.

This is one half of the real characteristic set of $\bar{\partial}_b$, which carries the microsingularities of functions (or distributions) in the space $\mathbb{H}$ of boundary values of holomorphic functions, or of the Szegö projector S.

The Szegö projector S is the orthogonal projector on the space of boundary functions in L^2(sphere).
It is given by

$$Sf = \frac{1}{v} \int_{sphere} (1 - z \cdot \bar{w})^{-N} f(w)\ d\sigma(w)$$

where $d\sigma(w)$ denotes the standard measure on the sphere, v its total volume

It is quite typically a F.I.O. with complex phase.

Toeplitz operators

Toeplitz operators on the complex sphere X are operators on the space $\mathbb{H}$ of boundary values of holomorphic functions, of the form

$$f \mapsto T_P(f) = S(Pf)$$

where P is a pseudodifferential operator on X and S denotes the Szegö projector.

Toeplitz operators behave exactly like pseudo-differential operators, in particular T_P has a symbol, which is a homogeneous function of degree $\deg P$ on Σ (the restriction to Σ of the symbol of P).

As shown in the work [7] of J. Sjöstrand and myself, Szegö projectors are well behaved on any pseudo-convex complex boundary, and Toeplitz operators can be defined there. More on any compact oriented contact manifold, there is an analogue of the Szegö projector S whose range $\mathbb{H}$ is an analogue of the space of CR functions. However in this more general setting S and H are not canonically defined; if two are constructed $(\mathbb{H}, \mathbb{H}')$ one can only assert that the orthogonal projection from one to the other is a Fredholm operator. So $\mathbb{H}$ is only well defined essentially up to a finite dimensional space. For index computations the topological and contact data cannot suffice.

A useful example is when X is the unit sphere in a holomorphic cone, and S its Szegö projector on the space $\mathbb{H}$ of CR functions.

Group action

Let G be a compact Lie group with a holomorphic linear action on X (more generally a compact Lie group with a contact action: one can then always construct an equivariant generalized Szegö projector)

The infinitesimal generators (vector fields) of the action ($L_v, v \in \mathfrak{g}$) define Toeplitz operators of degree 1. The charateristic set is the set $\operatorname{char} \mathfrak{g} \subset \Sigma$ where the symbols of these generators all vanish.

We will also use its base $Z \subset X$ which is the set where these generating vector fields are all orthogonal to the contact form λ; equivalently the null set of the moment map of the action.

An equivariant Toeplitz operator A (or system of such operators acting on vector bundles) is G-elliptic if it is elliptic on the characteristic set Z (i.e. its symbol is invertible there).

(transversally elliptic in Atiyah's book [2] but in our Toeplitz context there is nothing to be transversal to)

When this is the case, each irreducible representation of G has finite multiplicty in the kernel and cokernel of A, and A has a G-index which is a virtual representation in which all irreducible representation has a finite degree.

This can be represented by a formal series of characters

$$\sum n_\alpha \, \chi_\alpha \quad \in \widehat{R}_G$$

where $\widehat{R}_G$ is the formal completion of the character set R_G (for values $\to \infty$ of the Casimir)

(This in fact always converges in distribution sense to a central distribution on G.)

For contact manifolds where the Toeplitz space $\mathbb{H}$ is only defined up to a Fredholm quasi-isomorphism, the index is not well defined.

However the asymptotic index, i.e. the preceding one mod finite representations, still makes sense

$$\mathrm{AsInd}\,(A) \in \widehat{R}_G / R_G$$

it only depends on the contact structure and not on the choice of generalized Szegö projectors; this was a crucial ingredient in [6].

The asymptotic index is additive and stable by deformation, so it only depends on the K-theoretical element

$$[A] \in K^G(X^Z)$$

defined by its symbol ($K^G(X^Z)$ denotes the equivariant K-theory with compact support in $X - Z$, i.e. the group of stable isotopy class of equivariant bundle homomorphisms $a : E \to F$ on X which are invertible on Z.

Torus action

Let G be a torus $\mathbb{R}^n/\mathbb{Z}^n$ acting linearly on the sphere $X \subset \mathbb{C}^N$. Changing for a suitable orthonormal basis we can suppose that G acts diagonally:

$$g \cdot z = (\chi_k(g) z_k)$$

where $\chi_k = \exp 2i\pi\xi_k$ are characters of G - the infinitesimal character ξ_k is an integral linear form on the Lie algebra $\mathfrak{g} \sim \mathbb{R}^n$.

the symbol of an infinitesimal generator γ is (up to a positive factor)

$$\sum \xi_k(\gamma) z_k \bar{z}_k$$

(same as as its moment)

Thus the characteristic set Z is the pull back of the convex set $\sum \xi_k(\gamma)\lambda_k = 0$ (for all $\gamma \in \mathfrak{g}$) in $\mathbb{R}_+^N$ ($\lambda_k = |z_k|^2,\ \sum \lambda_k = 1$).

An important case is the case where $\mathrm{char}\,\mathfrak{g} = \emptyset$ (elliptic action), i.e. the ξ_k generate a strictly convex cone.
In that case all equivariant homomorphisms are G-elliptic. As an R_G-module the equivariant K-theory $K^G(X)$ is generated by the trivial bundle, isomorphic to

$$R_G/R_G\beta, \quad \text{with} \quad \beta = \prod (1 - \chi_k)$$

provided, as we can always suppose, that there is no fixed point i.e. $\beta \neq 0$. β is the symbol of the Koszul complex, which is used to construct the Bott periodicity homomorphism.

The index of the trivial bundle is the representation of G in the space of holomorphic functions $\mathcal{O}_X$; by the Hilbert-Samuel formula this is

$$\beta^{-1} = \prod (1 - \chi_k)^{-1}$$

being understood that each factor $(1 - \chi_k)^{-1}$ is expanded as a series of positive powers of χ_k.

Also in that case the index map $K^G(X) \to \widehat{R}_G / R_G$ is injective.

In general if $Y \subset X$ is an elliptic coordinate subsphere, there is a transversal Koszul complex $k_{Y^\perp}$ whose cohomology is just $\mathcal{O}_Y$ (in degree 0).
If A is any equivariant Toeplitz homomorphism (or complex) on Y, $k_Y \otimes A$ is a G-elliptic complex on X. The transfer $a \mapsto k_Y \otimes a$ preserves the equivariant index, and the underlying K-theoretical map is the Bott homomorphism.

There is an analogous construction for any equivariant embedding of contact manifolds - but only the asymptotic index is defined and preserved.

A natural conjecture is that in general all G-elliptic complexes come from such embeddings, i.e. $K^G(X - Z)$ is generated by the Bott images of the K-theories of all elliptic subspheres; also that the index map is injective. This is true if G is the circle group (easy), also if it is a 2-torus. It follows from [8] that it is also true if the representation of G in X is symmetric, i.e. the charaters ξ_k can be grouped by opposite pairs. But I still do not have a proof in general.

Anyway a typical index is $\sum_{\xi\in R} P(\xi)\xi$ (mod R_G) where R is a net, set of all $\sum n_j\xi_j$ ($n_j \geq 0$) where the ξ_j are linearly independant (not necessarily a $\mathbb{Z}$-basis) and P a polynomial with integral values on R.

All asymptotic indices are sums translates of such.

SU_2 action

Let G be the group SU_2 acting on a sphere X. The representation ring R_G is a polynomial ring $\mathbb{Z}[V]$ with generator the fundamental representation $V = \mathbb{C}^2$. It is more convenient to use the basis (over $\mathbb{Z}$) formed by the irreducible representations, i.e. the symmetric powers $S^k = S^K(V)$; these are linked by the formal relation $\sum S^k T^k = (1 - V\ T + T^2)^{-1}$.

The sphere X_V of $\mathbb{C}^2$ is obviously elliptic, the corresponding index is

$$\mathrm{AsInd}\,(1_{X_V}) = \sum S^k$$

However the sphere of V is the only elliptic one, the spheres of V^m or S^m, $m \geq 2$ are not. So the constructions for the torus cannot be copied.

Here are examples showing that, intriguingly, asymptotic indices for SU_2 can have the same aspect as in the circle case, i.e. a typical index is

$$\sum P(k) S^{j+mk}$$

where P is a polynomial with integral values (integral linear combination of binomial polynomials).

We will use holomorphic cones (with an SU_2 action) which are not cones - but this is just as good, and one can always embed in some large sphere.

The basic cone is $\mathbb{C}^2 - \{0\}$ which is also the complement of the zero section of the tautological line bundle L over $P^1(\mathbb{C})$. The index was recalled above.

Let now $X_{m,k}$ be the contact sphere of $L^{\otimes m} \otimes \mathbb{C}^k$ with the obvious SU_2 action, SU_2 acting trivially on $\mathbb{C}^k$ (the holomorphic base $X_{m,k}$ identifies with the set of k vectors in $\mathbb{C}^2$ lying all on the same line, not all zero. The action of SU_2 is again obviously elliptic.

The asymptotic index of the trivial line bundle on $X_{m,k}$ is the decomposition in irreducible components of the space of holomorphic functions, and that is the same as the space of sections of the symmetric algebra of the dual bundle $S(L'^{\otimes m} \otimes \mathbb{C}^k)$ over $P^1(\mathbb{C})$: we get

$$\mathrm{AsInd}\,(1_{X_{m,k}}) = \sum \binom{n+k-1}{k-1} S^{mk}$$

(because the space of sections of $L'^{\otimes j}$ is S^j).

We get the translates of this by the following trick: $X_{1,k}$ is a ramified G-covering of $X_{m,k}$ which makes $\mathcal{O}_{X_{1,k}}$ an equivariant coherent $\mathcal{O}_{X_{m,k}}$-module. Inside this z_1^j generates a coherent $\mathcal{O}_{X_{m,k}}$-submodule, with an obvious action of SU_2. This is just as good as a vector bundle because equivariant coherent sheaves have equivariant locally free resolutions.

The asymptotic index is $\sum \binom{n+k-1}{k-1} \; S^{j+mk}$ as announced.

I do not know if there are asymptotic indices other than sums of these.

remarks and examples

1. Asymptotic index from S^2
SU_2 is identified with the group of quadratic polynomials

$$\alpha X^2 + \beta XY + \gamma Y^2.$$

The moment map is obviously equivariant under the action of $SU_2 \times U(1)$ and orbits under this group are parametrized by polynomials $X^2 + aY^2, \; 0 \leq a \leq 1$.

The characteristic set is the orbit of $a = 1$ (polynomials whose roots in P^1 are antipodal). It is elementary to see that the pull back of $(0 < a < 1)$ is a product bundle: the stabilizer of $X^2 - aY^2, 0 < a < 1$ is the constant two-subgroup generated by $(\mathrm{I} \times \{-1\})$. So the complement of Z retracts equivariantly on the orbit of $a = 1$ (polynomial with one double root), which is isomorphic to $X_{2,1}$.

Thus the set of asymptotic indices from S^2 is the same as from $X_{2,1}$ i.e. the R_G module generated by $\sum S^{2k}$

2. The set of asymptotic indices from $X_{2,2}$ is the R_G-module generated by

$$\sum (k+1) S^k$$

This is also the decomposition in irreducible components of $L^2(G)$.

This is not an accident: the contact manifold corresponding to pseudodifferential operators on G is the cotangent sphere $G \times S^2$, and it is not hard to check that this is isomorphic to the sphere of $L \otimes \mathbb{C}^2$.

3. The cone S^3 is identified with the set of third degree polynomials $aX^3 + bX^2Y + cXY^2 + dY^3$, with the standard action of SU_2. It is not elliptic, a typical characteristic element is $X^3 - Y^3$ (the characteristic set is in fact the orbit of this by the group $SU_2 \times U(1)$).

Inside this the polynomials with zero discriminant form an elliptic holomorphic subcone Γ: any such polynomial is conjugate via $SU_2 \times U(1)$ to a polynomial fof the form $P = aX^3 + bX^2Y$ for which we have $\langle P|L_I P\rangle = 3|a|^2 + \frac{1}{3}|b|^2 > 0$.

(because $\|X^3\| = 1, L_I X^3 = 3X^3 \|X^2Y\| = \frac{1}{3}, L_I X^2 Y = X^2 Y$)

Any polynomial $P \in \Gamma$ is of the form $a^2 b$ where a, b are first degree polynomials, depending holomorphically on P (up to scalar factors). It follows that the algebra $\mathcal{O}_\Gamma$ identifies with the subalgebra of polynomials f on $L^{\otimes 2} \times L$ (over $P^1 \times P^1$) such that $f(\lambda a, b) = f(a, \lambda^2 b)$.

The cone $L^{\otimes 2} \times L$ is not elliptic and its algebra $\sum S^{2p} \otimes S^q$ is not of trace class; but the subalgebra $\mathcal{O}_\Gamma$ is ; it is a sum of examples as above :

$$\sum S^{2k} \otimes S^k = \sum([\frac{n}{2}] + 1) S^n$$

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