Construction of Hadamard states by pseudo-differential calculus

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joint work with Michał Wrochna

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A quick overview of free Klein-Gordon fields on Minkowski space-time

- Consider on $\mathbb{R}^{1+d}$ the free Klein-Gordon equation:

$$(KG)\ \Box\phi(x) + m^2\phi(x) = 0,\ x = (t, \mathrm{x}),\ \Box = \partial_t^2 - \Delta_{\mathrm{x}}.$$

We are interested in its smooth, real space-compact solutions.

- It admits advanced/retarded Green's functions, with kernels $E_{\pm}(t, \mathrm{x})$ given by

$$\widehat{E}_{\pm}(t, k) = \pm\theta(\pm t)\frac{\sin(\epsilon(k)t)}{\epsilon(k)},\ \epsilon(k) = (k^2 + m^2)^{-\frac{1}{2}}.$$

- the difference $E := E_+ - E_-$ is anti-symmetric, called the Pauli-Jordan function. Clearly

$$E : C_0^\infty(\mathbb{R}^{1+d}) \to \mathrm{Sol}_{\mathrm{sc}}(KG).$$

- Actually $\mathrm{Ran}E = \mathrm{Sol}_{\mathrm{sc}}(KG)$, $\mathrm{Ker}E = (\Box + m^2)C_0^\infty(\mathbb{R}^{1+d})$.

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Free Klein-Gordon fields

- We associate to each (real valued) $u \in C_0^\infty(\mathbb{R}^{1+d})$ a symbol $\phi(u)$ and impose the relations:
- $\phi(u + \lambda v) = \phi(u) + \lambda\phi(v)$, $\lambda \in \mathbb{R}$ ($\mathbb{R}-$linearity),
- $\phi^*(u) = \phi(u)$ (selfadjointness)
- $[\phi(u), \phi(v)] := \mathrm{i}(u|Ev)\mathbb{1}$ (canonical commutation relations).
- taking the quotient of the complex polynomials in the $\phi(\cdot)$ by the above relations, we obtain a $*-$algebra denoted by $\mathcal{A}(\mathbb{R}^{1+d})$ (Borchers algebra).

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The vacuum state on Minkowski

- A quasi-free state ω on $\mathcal{A}(\mathbb{R}^{1+d})$ is a state (positive linear functional) which is uniquely determined by its covariance H defined by:
- $\omega(\phi(\overline{u})\phi(v)) =: (u|Hv) + \mathrm{i}(u|Ev)$.
- Among all quasi-free states, there is a unique state ω_{vac}, the vacuum state such that:
- 1) H_{vac} is invariant under space-time translations, hence is given by convolution with a function $H_{\mathrm{vac}}(x)$,
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What happens for Klein-Gordon fields on a curved space-time ?

- Consider a manifold M with Lorentzian metric g, and the associated Klein-Gordon operator $\Box_g + m^2$.
- $\Box_g + m^2$ should admit unique advanced/retarded Green operators. Answer (Leray): (M, g) should be globally hyperbolic.
- The Borchers algebra $\mathcal{A}(M)$ can then be constructed as before and states on $\mathcal{A}(M)$ can be considered.
- fundamental problem: what is a vacuum state on a curved space-time ?
- Even on Minkowski the notion of vacuum state is observer-dependent (Unruh effect): we singled out the time variable in the positive energy condition.

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The solution: Hadamard states

In the 80's physicists introduced the notion of *Hadamard states*, characterized by the singularity structure of the distributional kernel of their covariances (aka two-point functions).

They share many properties with the vacuum state in Minkowski space: for example the stress-energy tensor can be renormalized w.r.t. a Hadamard state.

In 1996, microlocal analysis entered the scene: Radzikowski showed that Hadamard states can be characterized only in terms of the wave front set of their two-point function.

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Do Hadamard states exist on a globally hyperbolic space-time?

- It is not clear a priori that Hadamard states exist at all !
- Only known construction: [Fulling-Narcovich-Wald 1980]: indirect deformation argument to a static space-time.
- We reconsider the construction of Hadamard states on space-times with metric well-behaved at spatial infinity.
- Working on a fixed Cauchy surface, we can use rather standard pseudo-differential analysis.
- We construct a large class of Hadamard states with pdo covariances, in particular all *pure* Hadamard states.

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Globally hyperbolic space-times

Consider a Lorentzian space-time $(M, g_{\mu\nu}dx^{\mu}dx^{\nu})$, with metric signature $(-, +, \cdots, +)$.

- Using the metric one defines *time-like, causal, space-like* vector fields / curves in M.
- Assume that M is *time-orientable*, i.e. there is a global, continuous time-like vector field on M.
- For $x \in M$, the *future/past causal shadow* of x, $J^{\pm}(x)$ is the set of points reached from x by future/past directed causal curves. For $U \subset M$ $J^{\pm}(U) := \bigcup_{x\in M} J^{\pm}(x)$.
- (M, g) is *globally hyperbolic* if M admits a *Cauchy hypersurface*, i.e. a space-like hypersurface Σ such that each maximal time-like curve in M intersects Σ at exactly one point.

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Globally hyperbolic space-times

- This is equivalent to:
  M is isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta dt^2 + h_t$, where β is a smooth positive function, h_t is a riemannian metric on Σ depending smoothly on $t \in \mathbb{R}$.
- Denote for $x \in M$ by $V_\pm(x) \subset T_x M$ the open *future/past light cones* at x.
- The *dual cones* $V_\pm^*(x) \subset T_x^* M$ are defined as:
  $V_\pm^*(x) = \{\xi \in T_x^* M \ : \ \xi \cdot v > 0, \ \forall v \in V_\pm(x), \ v \neq 0\}$.
- Interpreted as *positive/negative energy* cones.

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Klein-Gordon equations

Consider a globally hyperbolic space-time $(M, g_{\mu\nu}dx^{\mu}dx^{\nu})$.
Standard notations:

$$|g| := \det[g_{\mu\nu}], \quad [g^{\mu\nu}] := [g_{\mu\nu}]^{-1}, \quad dv := |g|^{\frac{1}{2}}dx.$$

We fix a smooth vector potential $A_{\mu}(x)dx^{\mu}$ and a smooth function $\rho : M \to \mathbb{R}$.

- Klein-Gordon operator:

$$P(x, D_x) = |g|^{-\frac{1}{2}}(\partial_{\mu} + \mathrm{i}A_{\mu})|g|^{\frac{1}{2}}g^{\mu\nu}(\partial_{\nu} + \mathrm{i}A_{\nu}) + \rho.$$

Advanced/retarded fundamental solutions

- $P(x, D_x)$ admits unique *advanced/retarded fundamental solutions* $E_\pm$ solving:

$$P(x, D_x) \circ E_\pm = \mathbb{1},$$
$$\mathrm{supp} E_\pm f \subset J^\pm(\mathrm{supp} f), \; f \in C_0^\infty(M),$$

- Moreover $E_- = E_+^*$, for scalar product $(u_1|u_2) = \int_M \overline{u_1} u_2 dv$.

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Symplectic space of solutions

Let $\mathrm{Sol}_{\mathrm{sc}}(P)$ be the space of smooth, *space-compact* solutions of

$$\text{(KG)} \quad P(x, D_x)\phi = 0.$$

- $E = E_+ - E_-$, called the *Pauli-Jordan commutator function*. Note that $E = -E^*$, $PE = EP = 0$.
- One has $\mathrm{Sol}_{\mathrm{sc}}(P) = EC_0^\infty(M)$, $\mathrm{Ker}E = PC_0^\infty(M)$.
- Moreover if we fix a Cauchy hypersurface Σ and set

$$\begin{aligned}\rho: \quad \mathrm{Sol}_{\mathrm{sc}}(P) &\to C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)\\ \phi &\mapsto (\phi_{|\Sigma}, \mathrm{i}^{-1} n^\mu(\nabla_\mu + \mathrm{i}A_\mu)\phi_{|\Sigma}) =: (\rho_0\phi, \rho_1\phi),\end{aligned}$$

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- $E = E_+ - E_-$, called the *Pauli-Jordan commutator function*. Note that $E = -E^*$, $PE = EP = 0$.
- One has $\mathrm{Sol}_{\mathrm{sc}}(P) = EC_0^\infty(M)$, $\mathrm{Ker}\, E = PC_0^\infty(M)$.
- Moreover if we fix a Cauchy hypersurface Σ and set

$$\begin{aligned} \rho : \quad \mathrm{Sol}_{\mathrm{sc}}(P) \quad &\to C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma) \\ \phi \quad &\mapsto (\phi_{|\Sigma}, \mathrm{i}^{-1} n^\mu (\nabla_\mu + \mathrm{i}A_\mu)\phi_{|\Sigma}) =: (\rho_0\phi, \rho_1\phi), \end{aligned}$$

then $\rho : \mathrm{Sol}_{\mathrm{sc}}(P) \to C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$ is bijective.

Symplectic space of solutions

- Denote by σ the canonical symplectic form on $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$:

$$(f|\sigma g) := -\mathrm{i} \int_\Sigma (\overline{f_0} g_1 + \overline{f_1} g_0) ds, \ \ f, g \in C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma),$$

- One has $(u_1|Eu_2) = (\rho \circ Eu_1|\sigma\rho \circ Eu_2)$ for $u_1, u_2 \in C_0^\infty(M)$.
- Hence $(C_0^\infty(M)/PC_0^\infty(M), E)$ is a symplectic space, isomorphic to $(C_0^\infty(\Sigma) \otimes \mathbb{C}^2, \sigma)$ under the map $\rho \circ E$.

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Quantum fields on curved space-times

Since $(C_0^\infty(M), E)$ is a (complex) symplectic space, it is more convenient to generate the Borchers algebra by charged fields:
We associate to each $u \in C_0^\infty(M)$ symbols $\psi(u)$, $\psi^*(u)$ such that:

- $\psi^*(u + \lambda v) = \psi^*(u) + \lambda\psi^*(v)$, $\psi(u + \lambda v) = \psi(u) + \overline{\lambda}\psi(v)$, ($\mathbb{C}-$linearity / anti-linearity).
- $\psi(u)^* = \psi^*(u)$,
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One can again consider the Borchers $*-$algebra $\mathcal{A}(M)$, consisting of polynomials in the fields, quotiented by the above relations.

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Quasi-free states

- the simplest states on $\mathcal{A}(M)$ are *quasi-free states*, defined by the property:

$$\omega(\prod_{i=1}^{n}\psi(u_i)\prod_{j=1}^{m}\psi^*(v_j)) = 0,\ i\neq j$$
$$\omega(\prod_{i=1}^{n}\psi(u_i)\prod_{j=1}^{n}\psi^*(v_j)) = \sum_{\sigma\in S_n}\prod_{i=1}^{n}\omega(\psi(u_i)\psi^*(u_{\sigma(i)})).$$

- The pair of sesquilinear forms

$$(u_1|\Lambda_+ u_2) := \omega(\psi(u_1)\psi^*(u_2)),\ (u_1|\Lambda_- u_2) =: \omega(\psi^*(u_2)\psi(u_1)),$$

are called the *covariances* of the quasi-free state ω.

- A pair of sesquilinear forms $\Lambda_\pm$ are the covariances of a (unique) quasi-free state ω iff

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Hadamard states

The covariances $\Lambda_\pm$ can be viewed as distributions on $M \times M$ (modulo some obvious continuity condition + Schwartz kernel theorem).

- Denote $p(x,\xi) = g^{\mu\nu}(x)\xi_\mu\xi_\nu$ the principal symbol of $P(x, D_x)$,
- $\mathcal{N} = p^{-1}(\{0\})$ *energy surface*, $\mathcal{N}_\pm = \{(x,\xi) \in \mathcal{N} \ : \ \xi \in V^*_\pm(x)\}$, *positive/negative energy surfaces*, $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-$,
- For $X_i = (x_i, \xi_i)$ write $X_1 \sim X_2$ if $X_1, X_2 \in \mathcal{N}$, X_1, X_2 on the same Hamiltonian curve of p.

Definition

ω is a *Hadamard state* if

$$WF(\Lambda_\pm)' \subset \{(X_1, X_2) : X_1 \sim X_2, : X_1 \in \mathcal{N}_\pm\}.$$

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Examples of Hadamard states

- If (M, g) is *stationary* i.e. admits a global time-like Killing vector field, then associated vacuum and thermal states are Hadamard [Sahlmann, Verch '97].
- if (M, g) is asymptotically flat at null infinity, some distinguished states at null infinity are Hadamard [Dappiagi, Moretti, Pinamonti '09]
- the Unruh state on Schwarzschild space-time is Hadamard [Dappiagi, Moretti, Pinamonti '11].
- If (M, g) has a compact Cauchy surface, Hadamard states exist [Junker '97].
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Model Klein-Gordon equation

We consider the following model Klein-Gordon equation:

- $M = \mathbb{R}^{1+d}$, $x = (t, \mathrm{x}) \in \mathbb{R}^{1+d}$

$$a(t, \mathrm{x}, D_{\mathrm{x}}) = -\sum_{j,k=1}^{d} \partial_{\mathrm{x}^j} a^{jk}(x)\partial_{\mathrm{x}^k} + \sum_{j=1}^{d} b^j(x)\partial_{\mathrm{x}^j} - \partial_{\mathrm{x}^j}\overline{b}^j(x) + m(x),$$

- $[a^{jk}]$ uniformly elliptic, a^{jk}, b^j, m uniformly bounded with all derivatives in x, locally in t.
- We consider $P(x, D_x) = \partial_t^2 + a(t, \mathrm{x}, D_{\mathrm{x}})$.
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Pseudo-differential operators

The natural symbol classes of the problem are the classes $S^m(\mathbb{R}^{2d})$, $m \in \mathbb{R}$, consists of functions a such that

$$\partial_x^\alpha \partial_k^\beta a(x,k) \in O(\langle k \rangle^{m-|\beta|}),\ \alpha, \beta \in \mathbb{N}^d.$$

(actually their poly-homogeneous versions).

We denote $\Psi^m(\mathbb{R}^d) := \mathrm{Op}^{\mathrm{w}}(S^m(\mathbb{R}^{2d}))$ the space of pseudodifferential operators of degree m.

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Parametrix for the Cauchy problem

Consider the Cauchy problem for P:

$$(C)\quad \begin{cases} \partial_t^2\phi(t) + a(t, \mathrm{x}, D_{\mathrm{x}})\phi(t) = 0, \\ \phi(0) = f_0, \\ \mathrm{i}^{-1}\partial_t\phi(0) = f_1, \end{cases}$$

essential step to construct Hadamard states for P: characterize solutions with wavefront set in $\mathcal{N}^{\pm}$ in terms of their Cauchy data.

method: construct a sufficiently explicit parametrix for the Cauchy problem (C).

tool: use pseudo-differential calculus (no need for Fourier integral operators, eikonal equations, etc.)

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Step 1: take a square root of $a(t)$: there exists $\epsilon(t, \mathrm{x}, D_{\mathrm{x}}) \in \Psi^1$ s.t. $a(t, \mathrm{x}, D_{\mathrm{x}}) = \epsilon^2(t, \mathrm{x}, D_{\mathrm{x}})$ mod $\Psi^{-\infty}$.

Step 2: construct asymptotic solutions: there exist $b(t) \in \Psi^1$, unique mod. $\Psi^{-\infty}$ with $b(t) = \epsilon(t) + \Psi^0$ such that if $u_+(t) = \mathrm{Texp}(\mathrm{i}\int_0^t b(s)ds)$, $u_-(t) = \mathrm{Texp}(-\mathrm{i}\int_0^t b^*(s)ds)$ one has

$$(\partial_t^2 + \epsilon^2(t))u_\pm(t) = 0 \text{ mod } \Psi^{-\infty}.$$

Step 3: adjust initial conditions: there exists $r \in \Psi^{-1}$, unique mod $\Psi^{-\infty}$ with $r = \epsilon(0)^{-1} + \Psi^{-2}$ and $d_\pm \in \Psi^0$ such that if $r_+ := r$, $r_- := r^*$ and

$$U_\pm(t)f := u_\pm(t)d_\pm(f_0 \pm r_\pm f_1)$$

then $U(t) = U_+(t) + U_-(t)$ is a parametrix for the Cauchy problem (C).

Parametrix for the Cauchy problem

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Step 2: construct asymptotic solutions: there exist $b(t) \in \Psi^1$, unique mod. $\Psi^{-\infty}$ with $b(t) = \epsilon(t) + \Psi^0$ such that if $u_+(t) = \mathrm{Texp}(\mathrm{i}\int_0^t b(s)ds)$, $u_-(t) = \mathrm{Texp}(-\mathrm{i}\int_0^t b^*(s)ds)$ one has

$$(\partial_t^2 + \epsilon^2(t))u_\pm(t) = 0 \text{ mod } \Psi^{-\infty}.$$

Step 3: adjust initial conditions: there exists $r \in \Psi^{-1}$, unique mod $\Psi^{-\infty}$ with $r = \epsilon(0)^{-1} + \Psi^{-2}$ and $d_\pm \in \Psi^0$ such that if $r_+ := r$, $r_- := r^*$ and

$$U_\pm(t)f := u_\pm(t)d_\pm(f_0 \pm r_\pm f_1)$$

then $U(t) = U_+(t) + U_-(t)$ is a parametrix for the Cauchy problem (C).

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Spaces of positive/negative wavefront set solutions

Using Egorov's theorem one gets that $WF(U_{\pm}(t)f) \subset \mathcal{N}_{\pm}$.

First consequence: set

$$\mathrm{Sol}_{\mathcal{E}}(P) := \{\phi \in C^0(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d)) \ : \ P\phi = 0\},$$

(finite energy solutions),

and

$$\mathrm{Sol}^{\pm}_{\mathcal{E}}(P, r) := \{\phi \in \mathrm{Sol}_{\mathcal{E}}(P) \ : \ \phi(0) = \pm r_{\pm} \mathrm{i}^{-1} \partial_t \phi(0))\}.$$

Then:

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1) $\phi \in \mathrm{Sol}^{\pm}_{\mathcal{E}}(P, r) \ \Rightarrow \ WF\phi \subset \mathcal{N}_{\pm}$,

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Hadamard states with pseudo-differential covariances

- Once having fixed r (it is not unique in the construction), we set
$$T(r) := (r + r^*)^{-\frac{1}{2}} \begin{pmatrix} \mathbb{1} & r \\ \mathbb{1} & -r^* \end{pmatrix}.$$
- $T(r)$ diagonalizes the symplectic form:
$$\tilde{\sigma} := (T(r)^{-1})^* \circ \sigma \circ T(r)^{-1} = \begin{pmatrix} -\mathrm{i}\mathbb{1} & 0 \\ 0 & \mathrm{i}\mathbb{1} \end{pmatrix}.$$
- If c is a covariance on $C_0^\infty(\mathbb{R}^d) \otimes \mathbb{C}^2$ (Cauchy data), set
$$\tilde{c} := (T(r)^{-1})^* \circ c \circ T(r)^{-1} =: \begin{pmatrix} \tilde{c}_{++} & \tilde{c}_{+-} \\ \tilde{c}_{-+} & \tilde{c}_{--} \end{pmatrix}.$$

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Hadamard states with pseudo-differential covariances

We can identify a sesquilinear form C on $C_0^\infty(M)$ with a sesquilinear form on $C_0^\infty(\Sigma) \otimes \mathbb{C}^2$ by

$$C = (\rho \circ E)^* \circ c \circ (\rho \circ E).$$

If we fix c we set $C_+ := C$, $C_- := C - \mathrm{i}E$.

Theorem

Assume that c has pdo entries. Then the associate pair $C_\pm$ satisfies the Hadamard condition iff:

$$\mathbb{1} - \tilde{c}_{++}, \tilde{c}_{+-}, \tilde{c}_{-+}, \tilde{c}_{--} \in \Psi^{-\infty}(\mathbb{R}^d).$$

Hadamard states with pseudo-differential covariances

One also has to check the conditions $C_{\pm} \geq 0$, which become

$$\tilde{c} \geq 0, \ \tilde{c} \geq \mathrm{i}\tilde{\sigma}.$$

Theorem

let $a_{-\infty}, b_{-\infty} \in \Psi^{-\infty}$, $a_0 \in \Psi^0$ *with* $\|a_0\| \leq 1$. *Then if*

$$\begin{aligned} \tilde{c}_{++} &= \mathbb{1} + b^*_{-\infty} b_{-\infty}, \ \tilde{c}_{--} = a^*_{-\infty} a_{-\infty}, \\ \tilde{c}_{+-} &= \tilde{c}_{-+} = b^*_{-\infty} a_0 a_{-\infty}, \end{aligned}$$

c is the covariance of a quasi-free Hadamard state.

Pure Hadamard states

One can completely describe *pure* Hadamard states with pdo covariances.

Theorem

c is the covariance of a pure Hadamard state iff

$$\begin{aligned}\tilde{c}_{++} &= \mathbb{1} + a_{-\infty} a_{-\infty}^{*},\\ \tilde{c}_{--} &= a_{-\infty}^{*} a_{-\infty},\\ \tilde{c}_{+-} &= \tilde{c}_{-+}^{*} = a_{-\infty}(\mathbb{1} + a_{-\infty}^{*} a_{-\infty})^{1/2}\end{aligned}$$

for some $a_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$.

Canonical Hadamard state

Choose $a_{-\infty} = 0$ above. The corresponding state has covariance:

$$c(r) = \begin{pmatrix} (r + r^*)^{-1} & (r + r^*)^{-1} r \\ r^*(r + r^*)^{-1} & r^*(r + r^*)^{-1} r \end{pmatrix}.$$

It is called the *canonical Hadamard state* (associated to r).

Theorem

If r, r' are as before, then there exists a symplectic transformation $G \in \mathrm{Sp}(\sigma)$ such that $c(r') = G^* \circ c(r) \circ G$ (covariance under symplectic transformations). Moreover G has pdo entries.

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Basic examples

Consider the *static* case $a(t, \mathrm{x}, D_{\mathrm{x}}) = a(\mathrm{x}, D_{\mathrm{x}})$ independent on t. Then one can define the vacuum (0-temperature) and thermal state.

covariance of the vacuum state: $\tilde{c} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$,

covariance of the thermal state:

$$\tilde{c}_\beta = \begin{pmatrix} (\mathbb{1} - \mathrm{e}^{-\beta\epsilon})^{-1} & 0 \\ 0 & \mathrm{e}^{-\beta\epsilon}(\mathbb{1} - \mathrm{e}^{-\beta\epsilon})^{-1} \end{pmatrix},$$

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Open problems

- Relax condition on the metric at spatial infinity.
- Replace Cauchy surface by a characteristic manifold (backward lightcone).
- Treat fermionic (Dirac) fields.
- What happens for gauge theories (Maxwell, Yang-Mills) ? existence of Hadamard states is still unknown for (linearized) Yang-Mills fields (deformation argument does not work anymore).

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