From hypoellipticity for operators with double characteristics to semi-classical analysis of magnetic Schrödinger operators.

in honor of Johannes Sjöstrand.

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Abstract

In 1972-73, J. Sjöstrand was sitting in the same office as me and completing his paper: Parametrices for pseudodifferential operators with multiple characteristics. Important tools appearing in his paper were Microlocal Analysis and also the introduction of a Grushin's problem (already present in his PHD thesis). During 40 years this technique has been used successfully in many situations. This applies in particular in the analysis of magnetic wells where some of the questions could appear as a rephrasing of questions in hypoellipticity.
We would like to present some of these problems and their solutions and then discuss a few open or solved problems in the subject, including non self-adjoint problems.
These notes are in provisory form and could contain errors.

Many results are presented in the books of Helffer [He1] (1988) (referring to the work with J. Sjöstrand), Dimassi-Sjöstrand, A. Martinez, and Fournais-Helffer [FH2] (2010) (see also a recent course by N. Raymond). The results discussed today were obtained in collaboration with J. Sjöstrand, A. Morame, and for the most recent Y. Kordyukov, X. Pan and Y. Almog.

Other results have been obtained recently by N. Raymond, Dombrowski-Raymond, Popoff, Raymond-Vu-Ngoc, R. Henry.
We mainly look in this talk at the bottom of the (real part of the) spectrum but not only necessarily to the first eigenvalues.

Hypoellipticity questions in the seventies

We focus on operators with double characteristics. At about the same time two papers, one by Johannes Sjöstrand [Sj] and the other by Louis Boutet de Monvel [BdM] (following a first paper of Boutet de Monvel-Trèves) attack and solve the same problem (construct parametrices for these operators implying their hypoellipticity with loss of one derivative in the so-called symplectic case). This was then developed for other cases by A. Grigis (PHD), Boutet-Grigis-Helffer [BGH] and L. Hörmander (see [Ho] and his (4 volumes) book). In the considered case (symplectic), the result is that some subprincipal symbol should avoid some quantity attached to the Hessian of the principal symbol.

As an example, where the theory can be applied, let us look at the operator

$$P := \sum_j (D_{x_j} - A_j(x) D_t)^2 + V_1(x) D_t + V_2(x)\,,$$

on $\mathbb{R}^n \times \mathbb{R}_t$ (or $M \times \mathbb{T}^1$ where M is a compact Riemannian manifold).
The principal symbol is given by

$$(x, t, \xi, \tau) \mapsto |\xi - A\tau|^2\,,$$

and the subprincipal symbol (assuming $\mathrm{div} A = 0$) is

$$(x, t, \xi, \tau) \mapsto \tau V_1\,.$$

The characteristic set in $T^*(\mathbb{R}^{n+1}) \setminus \{(\xi, \tau) = (0,0)\}$ is given by

$$\Sigma := \{\xi = A\tau\,,\, \tau \neq 0\}$$

It has two connected components determined by the the sign of τ and we will concentrate to the component Σ^+ corresponding to $\tau > 0$.

The principal symbol vanishes exactly at order 2 on Σ which is the basic assumption for the theory described above (except Hörmander's result). If we ask for the rank of the symplectic canonical 2-form on Σ, we see that it is immediately related to the rank of the matrix

$$B_{jk} = \partial_k A_j - \partial_j A_k\,.$$

This new object appears in the above context when computing (say for for $\tau = +1$) the Poisson brackets of the functions

$$(x, t, \xi, \tau) \mapsto u_j(x, t, \xi, \tau) = \xi_j - A_j(x)\tau\,,$$

and will be interpreted later as a magnetic field (see also the talk by San Vu Ngoc).

When $n = 2$, the condition that Σ is symplectic simply reads that B_{12} does not vanish.
When $n = 3$, we cannot be in the symplectic situation but can express a condition for constant rank (Grigis case) by writing that $\sum_{j,k} |B_{jk}|^2$ does not vanish. When $n = 4$ we can hope generically for a symplectic situation.
Let us now how the necessary and sufficient condition for hypoellipticity (actually we will analyze microlocal hypoellipticity in $\tau > 0$) with loss of one derivative reads in the case $n = 2$.
We simply get:

$$|B_{12}|(x)(2k+1) + V_1(x) \neq 0\,, \forall k \in \mathbb{N}\,. \tag{1}$$

This in particular implies the hypoellipticity with loss of one derivative that we write in the form

$$||\chi(D_x, D_t)D_t u|| \leq C\,(||P\chi(D_x, D_t)u|| + ||u||)\ ,$$

(χ corresponds to the microlocalization in $\tau > 0$).
For our specific model (independence of t), we get actually (after partial Fourier transform with respect to t)

$$|\tau|||v||_{L^2(M)} \leq C\,(||P_\tau v|| + ||v||)\ , \forall \tau > 0\,, \forall v \in C_0^\infty(\mathbb{R}^n)\,.$$

If we consider the particular case, when

$$V_1(x) = -\lambda\,,$$

we obtain, taking

$$h = \frac{1}{\tau}\,,$$

and dividing by τ^2 the inequality

$$h||v|| \leq C\left(||(hD-A)^2 v - h\lambda|| + h^2||v||\right)\,,$$

if the following condition is satisfied:

$$|B_{12}|(x)(2k+1) - \lambda \neq 0\,,\,\forall k \in \mathbb{N}\,,\,\forall x \in M\,.$$

This can be interpreted as a spectral result for $(hD-A)^2$.

Hypoellipticity with loss of $\frac{3}{2}$ derivatives or more.

We now come back to Condition (1) and assume that for $k = 0$ and some point x_0:

$$|B_{12}|(x_0) + V_1(x_0) = 0\,,$$

Then we can think in the symplectic situation of applying our results on hypoellipticity with loss of $\frac{3}{2}$ derivatives [He0] (actually with σ derivatives with $\frac{3}{2} \leq \sigma < 2$).

Like in the proof of J. Sjöstrand [Sj], we introduce on his suggestion a Grushin's problem permitting microlocally to reduce the problem, in say the case $n = 2$ to the question:
When is the symbol $(y,\eta) \mapsto |B_{12}|(y,\eta) + V_1(y,\eta)$ the symbol of an hypoelliptic operator with (microlocally) loss of $\sigma - 1$ derivatives. This will never be the case when V_1 is real (in particular when $V_1 = \lambda$ but may be we can guess in this way results for non self-adjoint Schrödinger operators !
Hence, when V_1 is real, we can not hope for an hypoellipticity better than with loss of 2 derivatives which will involve the role of V_2. I am not aware of general results giving hypoellipticity with loss of 2 derivatives. If there were any they will lead (taking $V_1(x) = -\lambda_1$ and $V_2(x) = -\lambda_2$) to conditions under which $h\lambda_1 + h^2\lambda_2 + o(h^2)$ cannot belong to the spectrum of $(hD - A)^2$.
Note that this idea was used for establishing a semi-classical Garding-Melin-Hörmander inequality in Helffer-Robert, see also Helffer-Mohamed.

The magnetic Schrödinger Operator

We stop with this approach and will analyze from now on directly the semi-classical problem in a more physical point of view.
Our main object of interest is the Laplacian with magnetic field on a riemannian manifold, but in this talk we will mainly consider, except for specific toy models, a magnetic field

$$\beta = \text{ curl } \mathbf{A}$$

on a regular domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) associated with a magnetic potential $\mathbf{A}$ (vector field on Ω), which (for normalization) satisfies :

$$\text{div } \mathbf{A} = 0 \,.$$

We start from the closed quadratic form Q_h

$$W_0^{1,2}(\Omega) \ni u \mapsto Q_h(u) := \int_\Omega |(-ih\nabla + \mathbf{A})u(x)|^2 \, dx. \tag{2}$$

Let $\mathcal{H}^D(\mathbf{A}, h, \Omega)$ be the self-adjoint operator associated to Q_h and let $\lambda_1^D(\mathbf{A}, h, \Omega)$ be the corresponding groundstate energy. Motivated by various questions we consider the connected problems in the asymptotic $h \to +0$.

- Pb 1 Determine the structure of the bottom of the spectrum : gaps, typically between the first and second eigenvalue.
- Pb2 Find an effective Hamiltonian which through standard semi-classical analysis can explain the complete spectral picture including tunneling.

The case when the magnetic field is constant

The first results are known from Landau at the beginning of the Quantum Mechanics) analysis of models with constant magnetic field β.
In the case in $\mathbb{R}^d$ ($d = 2, 3$), the models are more explicitly

$$h^2 D_x^2 + (hD_y - x)^2 \,,$$

($\beta(x, y) = 1$) and

$$h^2 D_x^2 + (hD_y - x)^2 + h^2 D_z^2 \,,$$

($\beta(x, y, z) = (0, 0, 1)$) and we have:

$$\inf \sigma(\mathcal{H}(\mathbf{A}, h, \mathbb{R}^d)) = h|\beta| \,.$$

Let us now look at perturbations (sometimes strong) of this situation.

The effect of an electric potential

2D with some electric one well potential (Helffer-Sjöstrand (1987)).

First we add an electric potential.

$$h^2 D_x^2 + (hD_y - x)^2 + V(x,y)\,.$$

V creating a well at a minimum of V : $(0,0)$. (V tending to $+\infty$ at ∞).

Harmonic approximation in the non-degenerate case:

$$h^2 D_x^2 + (hD_y - x)^2 + \frac{1}{2} < (x,y)|\mathrm{Hess}\, V(0,0)|(x,y)) \,.$$

$$\lambda_1(h) \sim \alpha h \,.$$

The electric potential plays the dominant role and determines the localization of the ground state. As mentioned to us by E. Lieb, this computation is already done by Fock at the beginning of the quantum mechanics.

2D with some weak electric potential (Helffer-Sjöstrand (1990)).

$$h^2 D_x^2 + (hD_y - x)^2 + h^2 V(x,y)\,.$$

Close to the first Landau level h, the spectrum is given (modulo $\mathcal{O}(h^{\frac{7}{2}})$) by the h-pseudo-differential operator on $L^2(\mathbb{R})$

$$h + h^2 V^w(x, hD_x) + h^3 \left(\mathrm{Tr}\,\mathrm{Hess}\, V\right)^w (x, hD_x)$$

Here, for a given h-dependent symbol p on $\mathbb{R}^2$, $p^w(x, hD_x; h))$ denotes the operator

$$(p^w(x, hD_x)u)(x) = (2\pi h)^{-1} \int e^{\frac{i}{h}(x-y)\cdot\xi} p(\frac{x+y}{2}, \xi; h) u(y) dy d\xi\,.$$

Purely magnetic effects in the case of a variable magnetic field

We introduce

$$b = \inf_{x \in \overline{\Omega}} |\beta(x)| \,, \tag{3}$$

$$b' = \inf_{x \in \partial\Omega} |\beta(x)| \,. \tag{4}$$

Theorem 1 : rough asymptotics for h small

$$\lambda_1^D(\mathbf{A}, h, \Omega) = hb + o(h) \tag{5}$$

The Neumann case is quite important in the case of Superconductivity.

$$\lambda_1^D(\mathbf{A}, h, \Omega) = h \inf(b, \Theta_0 b') + o(h) \tag{6}$$

with $\Theta_0 \in]0,1[$.
This is not discussed further in this talk (see the book Fournais-Helffer).

The consequences are that a ground state is localized as $h \to +0$ for Dirichlet, at the points of $\overline{\Omega}$ where $|\beta(x)|$ is minimum,
All the results of localization are obtained through semi-classical Agmon estimates (as Helffer-Sjöstrand [HS1, HS2] or Simon [Si] have done in the eighties for $-h^2\Delta + V$ or for the Witten Laplacians (Witten, Helffer-Sjöstrand, Helffer-Klein-Nier, Helffer-Nier, Le Peutrec,...) .

There are also Agmon estimates in the magnetic case (Helffer-Sjöstrand, Helffer-Mohamed, Helffer-Raymond, Helffer-Pan, Fournais-Helffer, Bonnaillie, N. Raymond,...). These estimates are not always optimal (see L. Erdös, S. Nakamura, A. Martinez, V. Sordoni).

The case of $\mathbb{R}^n$ or the interior case

2D case (See the talk by San Vu Ngoc)

If

$$b < \inf_{x \in \partial\Omega} |\beta(x)| ,$$

the asymptotics are the same (modulo an exponentially small error) as in the case of $\mathbb{R}^d$: no boundary effect.
In the case of $\mathbb{R}^d$, we assume

$$b < \liminf_{|x| \to +\infty} |\beta(x)| .$$

We assume in addition (generic)

Assumption A

- There exists a unique point $x_{min} \in \Omega$ such that $b = |\beta(x_{min})|$.
- $b > 0$
- This minimum is non degenerate.

We get in $2D$ (Helffer-Morame (2001), Helffer-Kordyukov [HK6] (2009))

Theorem 2

$$\lambda_1^D(\mathbf{A}, h) = bh + \Theta_1 h^2 + o(h^2)\ . \tag{7}$$

where $\Theta_1 = a^2/2b$.

Here

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\,\beta(x_{min})\right)^{1/2}\ .$$

The previous statement can be completed in the following way.

$$\lambda_j^D(\mathbf{A}, h) \sim h \sum_{\ell \geq 0} \alpha_{j,\ell} h^{\frac{\ell}{2}}\,, \tag{8}$$

with

- $\alpha_{j,0} = b$,
- $\alpha_{j,1} = 0$,
- $\alpha_{j,2} = \frac{2d^{1/2}}{b}(j-1) + \frac{a^2}{2b}$,
- $d = \det\left(\frac{1}{2}\mathrm{Hess}\,\beta(x_{min})\right)^{1/2}$.

In particular, we get the control of the splitting $\sim \frac{2d^{1/2}}{b}$.
Note that behind these asymptotics, two harmonic oscillators are present as we see in the sketch. Recent improvments (Helffer-Kordyukov and Raymond–Vu-Ngoc) show that no odd powers of $h^{\frac{1}{2}}$ actually occur.

Interpretation with some effective Hamiltonian

Look at the bottom of the spectrum of

$$h\left(\hat{\beta}^w(x, hD_x) + h\gamma^w(x, hD_x, h^{\frac{1}{2}})\right) .$$

This gives the result modulo $\mathcal{O}(h^2)$, hence it was natural to find a direct proof of this reduction (which is in the physical litterature is called the lowest Landau level approximation).
$\hat{\beta}$ is related to β by an explicit map:

$$\hat{\beta} = \beta \circ \phi \, .$$

Sketch of the initial quasimode proof.

The toy model is

$$h^2 D_x^2 + \left(hD_y - b(x + \frac{1}{3}x^3 + xy^2) \right)^2 .$$

We obtain this toy model by taking a Taylor expansion of the magnetic field centered at the minimum and chosing a suitable gauge.
The second point is to use a blowing up argument $x = h^{\frac{1}{2}} s$, $y = h^{\frac{1}{2}} t$.
Dividing by h this leads (taking $b = 1$) to

$$D_s^2 + (D_t - s + h(\frac{1}{3}s^3 + st^2))^2 .$$

Partial Fourier transform

$$D_s^2 + (\tau - s + h(\frac{1}{3}s^3 + s(D_\tau)^2)^2\,,$$

and translation

$$D_s^2 + \left((-s + h\left(\frac{1}{3}(s+\tau)^3 + (s+\tau)(D_\tau - D_s)^2\right)\right)^2\,.$$

Expand as $\sum_j L_j h^j$, with

- $L_0 = D_s^2 + s^2\,,$
- $L_1 = -\frac{2}{3}s(s+\tau)^3 - s(s+\tau)(D_\tau - D_s)^2 - (s+\tau)(D_\tau - D_s)^2 s\,.$

The second harmonic oscillator appears in the τ variable by considering

$$\phi \mapsto \langle u_0(s), L_1(u_0(s)\phi(\tau))\rangle_{L^2(\mathbb{R}_s)}\,.$$

The recent improvments

In 2013, Helffer-Kordyukov on one side, and Raymond–Vu-Ngoc on the other side reanalyze the problem with two close but different points of view.
The proof of Helffer-Kordyukov is based on

- A change of variable : $(x, y) \mapsto \phi(x, y)$
- Normal form near a point (the minimum of the magnetic field)
- Construction of a Grushin's problem

This approach is local near the point where the intensity of the magnetic field is assumed to be minimum.

A change of variable

After a gauge transform, we assume that $A_1 = 0$ and $A_2 = A$. We just take :

$$x_1 = A(x,y), \quad y_1 = y$$

In these coordinates the magnetic field reads

$$B = dx_1 \wedge dy_1 \,.$$

Normal form through metaplectic transformations

After the change of variables, gauge transformation, partial Fourier transform, and at the end a dilation, we get

$$T^h_{new}(x,y,D_x,D_y;h) = h\sum_{k=0}^{2} h^{k/2}\,\widetilde{T}_k(x,y,D_x,hD_y,h), \tag{9}$$

where:

$$\begin{aligned}\widetilde{T}_0(x,y,D_x,hD_y;h) =&(B^2+A_y^2)(h^{\frac{1}{2}}x+y,hD_y-h^{\frac{1}{2}}D_x)D_x^2\\ &+A_y(h^{\frac{1}{2}}x+y,hD_y-h^{\frac{1}{2}}D_x)D_x x\\ &+xA_y(h^{\frac{1}{2}}x+y,hD_y-h^{\frac{1}{2}}D_x)D_x+x^2\,,\end{aligned}$$

Note that $A_y(0,0)=0$ and $B(0,0)=b_0$.

The Grushin problem

Our Grushin problem takes the form

$$\mathcal{P}_h(z) = \begin{pmatrix} h^{-1} T^h_{new} - b_0 - z & R_- \\ R_+ & 0 \end{pmatrix} \tag{10}$$

where T^h_{new} was introduced above, the operator $R_- : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2)$ is given by

$$R_- f(x,y) = H_0(x) f(y)\,, \tag{11}$$

H_0 being the normalized first eigenfunction of the harmonic oscillator

$$T = b_0^2 D_x^2 + x^2\,,$$

and the operator $R_+ : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R})$ is given by

$$R_+ \phi(y) = \int H_0(x) \phi(x,y)\, dx\,. \tag{12}$$

These "Hermite" operators appear also in [BdM].

One can show that in a suitable sense, this system is invertible:

$$\mathcal{E}(z,h) = \begin{pmatrix} E(z,h) & E_- \\ E_+ & \epsilon_\pm(z,h) \end{pmatrix}$$

Here $\epsilon_\pm(z,h)$ is an h-pseudodifferential operator on $L^2(\mathbb{R}_y)$. At least formally, we have

$$z \in \sigma(T^h_{new}) \text{ if and only if } 0 \in \sigma(\epsilon_\pm(z,h))\,.$$

Although not completely correct, think that

$$\epsilon_\pm(z,h) = \epsilon_\pm(0,h) - z\,.$$

Here we follow Helffer-Sjöstrand (Harper) for the $1D$-problem and Fournais-Helffer ((2D)-Neumann).
Suppose that we have found $z = z(h)$ (possibly admitting an expansion in powers of h) and a corresponding approximate 0-eigenfunction $u_h^{qm} \in C^\infty(\mathbb{R})$ of the operator $\epsilon_\pm(z)$

$$\epsilon_\pm(z) u_h^{qm} = \mathcal{O}(h^\infty)\,,$$

such that the frequency set of u_h^{qm} is non-empty and contained in Ω.

Here we use our right inverse and write:

$$\mathcal{P}_h(z) \circ \mathcal{E}_h(z) \sim I\,, \tag{13}$$

with $\mathcal{E}_h(z)$ as above .
In particular it reads:

$$(T^h_{new} - b_0 - z(h))\epsilon_-(z) + R_-\epsilon_\pm(z) \sim 0\,. \tag{14}$$

The quasimode for our problem is simply $\epsilon_-(z)u_h^{qm}$:

$$(T^h_{new} - h^{-1}\lambda_h)\epsilon_-(z)u_h^{qm} = \mathcal{O}(h^\infty)\,,$$

where $\lambda_h = h(b_0 + z(h))$. The structure of $\epsilon_-(z)$ gives a meaning to this expression.
We recover the previous results on quasi-modes but have extended it to excited states.

The converse

This time we start from the eigenfunction u_h of H^h associated with $\lambda_h \in [hb_0, h(b_0 + \epsilon_0)]$ for $\epsilon_0 > 0$ as above.
The rewriting of H^h leads to an associated eigenfunction u_h of T^h_{new} associated with $h^{-1}\lambda_h$ The aim is to construct an approximate eigenfunction for the operator $\epsilon_\pm(z)$ with $z(h) = \frac{1}{h}(\lambda_h - hb_0)$. Formally, the left inverse of $\mathcal{P}_h(z)$ leads to

$$\mathcal{E}_h(z) \circ \mathcal{P}_h(z) \sim I\,. \tag{15}$$

We extract from this the identity:

$$\epsilon_+(z)(T_{new}h - h^{-1}\lambda_h) + \epsilon_\pm(z)R_+ \sim 0\,. \tag{16}$$

Hence

$$u_h^{qm} = R_+\widetilde{u}_h$$

should be the candidate for an approximate 0-eigenfunction for $\epsilon_\pm(z)$:

$$\epsilon_\pm(z)u_h^{qm} = \mathcal{O}(h^\infty).$$

Birkhoff normal form (see the talk of San Vu Ngoc)

The proof of Raymond–Vu-Ngoc (see also Faure-Raymond-Vu-Ngoc) is reminiscent of Ivrii's approach (see his book in different versions) and uses a Birkhoff normal form. This approach seems to be semi-global but involves more general symplectomorphisms and their quantification.

We consider the h-symbol of the Schrödinger operator with magnetic potential A:

$$H(x, y, \xi, \eta) = |\xi - A_1(x, y)|^2 + |\eta - A_2(x, y)|^2 \,.$$

Theorem (Ivrii—Raymond–Vu-Ngoc)

$\exists$ a symplectic diffeomorphism Φ defined in an open set $\widetilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{C}_{z_2}$ with value in $T^*\mathbb{R}^2$ which sends $z_1 = 0$ into the surface $H = 0$ and such that

$$H \circ \Phi(z_1, z_2) = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty)\,,$$

where f is smooth.
Moreover, the map

$$\Omega \ni (x, y) \mapsto \phi(x, y) := \Phi^{-1}(x, y, \mathbf{A}(x, y)) \in \{\{0\} \times \mathbb{C}_{z_2}\} \cap \widetilde{\Omega}$$

is a local diffeomorphism and

$$f(\phi(x, y), 0) = B(x, y)\,.$$

The statement in Ivrii is Proposition 13.2.11, p. 1218 (in a version of 2012). Unfortunately, there are many misprints.
We have reproduced above the statement as in Raymond–Vu-Ngoc.

Quantum version (after Raymond–Vu-Ngoc)

Theorem: Quantum Normal Form

For h small enough, there exists a global Fourier-Integral operator $\mathcal{U}_h$ (essentially unitary modulo $\mathcal{O}(h^\infty)$) such that

$$\mathcal{U}_h^* \mathcal{H} \mathcal{U}_h = \mathcal{I}_h \, F_h + R_h \,,$$

where

$$I_h = -h^2 \frac{d^2}{dx_1^2} + x_1^2 \,,$$

F_h is a classical h-pseudodifferential operator which commutes with I_h, and R_h is a remainder (with $\mathcal{O}(h^\infty)$ property in the important region).

More precisely, the restriction to the invariant space $H_n \otimes L^2(\mathbb{R}_{x_2})$ (H_n is the n-th eigenfunction) can be seen as a h-pseudodifferential operator in the x_2 variable, whose principal symbol is B. In Ivrii, the relevant statement seems Theorem 13.2.8.

3D case

The problem is partially open (Helffer-Kordyukov [HK8]) in the $3D$ case. What the generic model should be is more delicate. The toy model is

$$h^2 D_x^2 + (hD_y - x)^2 + (hD_z + (\alpha zx - P_2(x,y)))^2$$

with $\alpha \neq 0$, P_2 homogeneous polynomial of degree 2 where we assume that the linear forms $(x,y,z) \mapsto \alpha z - \partial_x P_2$ and $(x,y,z) \mapsto \partial_y P_2$ are linearly independent. We hope to prove :

$$\lambda_1^D(\mathbf{A},h) = bh + \Theta_{\frac{1}{2}} h^{\frac{3}{2}} + \Theta_1 h^2 + o(h^2)\,. \qquad (17)$$

A generic case in $\mathbb{R}^3$

The toy model is

$$h^2 D_x^2 + (hD_y - x)^2 + (hD_z + (\alpha z x - P_2(x,y)))^2$$

with $\alpha \neq 0$, P_2 homogeneous polynomial of degree 2 where we assume that the linear forms $(x,y,z) \mapsto \alpha z - \partial_x P_2$ and $(x,y,z) \mapsto \partial_y P_2$ are linearly independent. We hope to prove :

$$\lambda_1^D(\mathbf{A}, h) = bh + \Theta_{\frac{1}{2}} h^{\frac{3}{2}} + \Theta_1 \, h^2 + o(h^2)\,. \tag{18}$$

More generally, let $M = \mathbb{R}^3$ with coordinates $X = (X_1, X_2, X_3) = (x, y, z)$. and let $\mathbf{A}$ be an 1-form, which is written in the local coordinates as

$$\mathbf{A} = \sum_{j=1}^{3} A_j(X)\, dX_j\,.$$

We are interested in the semi-classical analysis of the Schrödinger operator with magnetic potential $\mathbf{A}$:

$$H^h = \sum_{j=1}^{3} \left(hD_{X_j} - A_j(X)\right)^2\,.$$

The magnetic field β is given by the following formula

$$\beta = B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy\,.$$

We will also use the trace norm of $\beta(x)$:

$$|\beta(X)| = \left[\sum_{j=1}^{3} |B_j(X)|^2\right]^{1/2}\,.$$

Put

$$b = \min\{|\beta(X)|\,:\, X \in \mathbb{R}^3\}$$

and assume that there exist a (connected) compact domain K and a constant $\epsilon_0 > 0$ such that

$$|\beta(X)| \geq b + \epsilon_0, \quad x \notin K\,. \tag{19}$$

Suppose that:

$$b > 0\,, \tag{20}$$

and that there exists a unique minimum $X_0 \in K \subset \mathbb{R}^3$ such that $|B(X_0)| = b_0$, which is non-degenerate:

$$C^{-1}|X - X_0|^2 \leq |\beta(X)| - b \leq C|X - X_0|^2\,. \tag{21}$$

Main statement

Choose an orthonormal coordinate system in $\mathbb{R}^3$ such that the magnetic field at X_0 is $(0,0,b)$. Denote

$$d = \det \mathrm{Hess}\, |\beta(X_0)|\,, \quad a = \frac{1}{2}\frac{\partial^2|\beta|}{\partial z^2}(X_0)\,.$$

Denote by $\lambda_1(H^h) \leq \lambda_2(H^h) \leq \lambda_3(H^h) \leq \dots$ the eigenvalues of the operator H^h in $L^2(\mathbb{R}^3)$ below the essential spectrum.

Theorem (Helffer-Kordyukov) (2011)

Under current assumptions,

$$\lambda_j(H^h) \leq hb + h^{3/2}a^{1/2} + h^2\left[\frac{1}{b}\left(\frac{d}{2a}\right)^{1/2}(j-1) + \nu_2\right] + C_j h^{9/4},$$

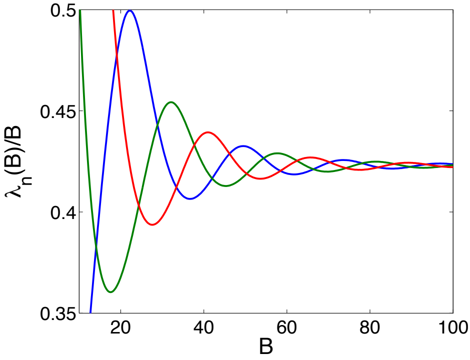
where ν_2 is some specific spectral invariant.

The theorem is based on a construction of quasimodes. The lower bound is open.
One can expect to find an effective hamiltonian using either ideas of Raymond-Vu-Ngoc or normal forms constructed by V. Ivrii.
Interpretation:

$$h^2 D_z^2 + h|\beta|^w(x, hD_x, z) + \dots .$$

Tunneling with magnetic fields

Essentially no results known (except Helffer-Sjöstrand (Pise)) but note that this last result is not a "pure magnetic effect".
There are however a few models where one can "observe" this effect in particular in domains with corners ([BDMV]) (numerics with some theoretical interpretation, see also Fournais-Helffer (book [FH1]) and Bonnaillie (PHD)).



The figure describes the graph of $\frac{\lambda_n(h)}{h}$ as a function of $B = h^{-1}$ for the equilateral triangle and for $n = 1, 2, 3$. Notice that $\Theta_0 \approx 0.59$ is, as expected, larger than $\lim_{h \to +\infty} \frac{\lambda_n(h)}{h}$, which corresponds to the groundstate energy of the Schrödinger operator with constant magnetic field equal to 1 in a sector of aperture $\frac{\pi}{3}$.

Although, one can explain heuristically why these braids appear, mathematical formulas are missing for describing tentatively the main term (and a fortiori proofs).

Other toy models

$$\text{Example 1}: h^2D_x^2 + (hD_y - a(x))^2 + y^2\,.$$

This model is rather artificial (and not purely magnetic) but by Fourier transform, it is unitary equivalent to

$$h^2D_x^2 + (\eta - a(x))^2 + h^2D_\eta^2\,,$$

which can be analyzed because it enters in the category of the miniwells problem treated in Helffer-Sjöstrand [HS1] (the fifth). We have indeed a well given by $\beta = a(x)$ which is unbounded but if we assume a varying curvature $\beta(x)$ (with $\liminf|\beta(x)| > \inf|\beta(x)|$) we will have a miniwell localization. A double well phenomenon can be created by assuming $\beta = a'$ even.

Example 2 : $h^2 D_x^2 + (hD_y - a(x))^2 + y^2 + V(x)\,.$

Here one can measure the explicit effect of the magnetic field by considering, after a partial Fourier transform:

$$h^2 D_x^2 + h^2 D_\eta^2 + (\eta - a(x))^2 + V(x)\,.$$

Example 3:
One can also imagine that in the main (2D)-example, as presented before (see also the talk by San Vu Ngoc), we have a magnetic double well, and that a tunneling effect could be measured using the effective $(1D)$-hamiltonian : $\beta(x, hD_x)$ assuming that b is holomorphic with respect to one of the variables .
(inspired by discussions with V. Bonnaillie-Noël and N. Raymond.)

Example 4:
Also open is the case considered in Fournais-Helffer (Neumann problem with constant magnetic field in say a full ellipse) [FH1].

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