Linear Differential Operators of Infinite Order —— Another interaction of microlocal analysis and exact WKB analysis

Takahiro KAWAI
RIMS, Kyoto University
(joint work with S. Kamimoto and T. Koike)

1. Connection formula on the Borel plane

(1.1) $\left[\dfrac{d^2}{dx^2} - \eta^2\left(\dfrac{1}{x} + \dfrac{\eta^{-2}}{x^2}\beta\right)\right]\psi = 0$

(1.2) $S^2 + \dfrac{dS}{dx} = \eta^2\dfrac{1}{x} + \dfrac{\beta}{x^2}$

(1.3) $S_j = c_j\, x^{-1-j/2}$

(1.4) $$S_{odd} = [S^{(+)} - S^{(-)}]/2$$

$$= \sum_{j \geq 0} S_{2j-1} \eta^{1-2j}$$

(1.5) $$\psi_{\pm} = (S_{odd})^{-1/2} \exp\left(\pm \int_0^x S_{odd}\, dx\right)$$

$$= \exp\left(\pm \eta \int_0^x \frac{dx}{\sqrt{x}}\right)\left(\sum_{n \geq 0} \psi_{\pm, n} \sqrt{x}^{\,-n+1/2} \eta^{-n-1/2}\right.$$

(1.6) $$\psi_{\pm 0} = 1$$

$$\psi_{+,n} = \frac{(-1)^n}{n!} \prod_{j=0}^{n-1} \left(\beta - \frac{1}{4}(j-1)\left(j+\frac{3}{2}\right)\right)$$

$$\psi_{-,n} = (-1)^n \psi_{+,n}$$

(1.7) $$\psi_{\pm, B}(x, y; \beta)$$

$$= \sum_{n \geq 0} \frac{\psi_{\pm, n}(\beta)}{\Gamma(n+1/2)} \left(\frac{y}{\sqrt{x}} \pm 2\right)^{n-1/2}$$

(1.8) $\psi_{+,B}(x, y ; \beta) = f_+(s ; \beta)$

$$s = \frac{1}{4}\left(\frac{y}{\sqrt{x}} + 2\right) \quad (= (*1))$$

(1.9) $$0 = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2} - \frac{\beta}{x^2}\right)\psi_{+,B}$$

$$= -\frac{1}{4x^2}\left[s(1-s)\frac{d^2}{dx^2} + \left(\frac{3}{2} - 3s\right)\frac{d}{ds} + 4\beta\right] f_+$$

$c = \frac{3}{2}$, $a+b+1 = 3$, $ab = -4\beta$ $[b \mapsto a']$

(1.10) $u = s^{1-c} F(a+1-c, a'+1-c, 2-c; s)$

(1.11) $\psi_{+,B}(x, y ; \beta)$

$$= \frac{1}{\sqrt{4\pi}} s^{-1/2} F\left(a - \frac{1}{2}, a' - \frac{1}{2}; \frac{1}{2}; s\right)\Big|_{s=(*)}$$

(1.12) $a + a' = 2$, $aa' = -4\beta$

(1.13) singular points of $\psi_{+,B}$:

(i) $s = 0$, i.e., $y = -2\sqrt{x}$

(ii) $s = 1$, i.e., $y = 2\sqrt{x}$; not immediately seen in (1.3)

The effect of (ii) on the Borel sum $\psi_+(x,\eta)$

(1.14) $\psi_+(x, \eta)$

$$= \int_{-2\sqrt{x}}^{\infty} \exp(-\eta y)\, \psi_{+,B}(x, y ; \beta)\, dy$$

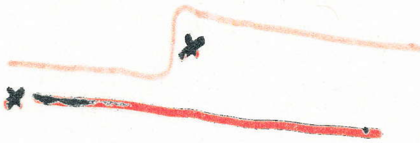
$x = 1 + i\varepsilon$ $\qquad \varepsilon < 0$

$(\,|\varepsilon| \ll 1\,)$ $\qquad -2\sqrt{x}$

$\times\, 2\sqrt{x}$

Figure: $\qquad \varepsilon = 0$

$\varepsilon > 0$



$\Longrightarrow$

sing $\psi_{+,B}$ near $s = 1$? $\quad$ Gauss!

(1.15) $$s^{-1/2} F\left(a - \tfrac{1}{2}, a' - \tfrac{1}{2}, \tfrac{1}{2}; s\right)$$

$$= \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1-a)\Gamma(1-a')} F\left(a, a', \tfrac{3}{2}; 1-s\right)$$

$$+ \frac{\Gamma(\frac{1}{2})^2}{\Gamma(a-\frac{1}{2})\Gamma(a'-\frac{1}{2})} = \frac{\pi\cos(\pi\sqrt{1+4\beta})}{\pi}$$

$$\times (1-s)^{-1/2} F\left(\tfrac{3}{2} - a, \tfrac{3}{2} - a', \tfrac{1}{2}; 1-s\right)$$

(1.16) $$\psi_{-,B}(x, y; \beta)$$

$$= \frac{1}{\sqrt{-4\pi}} (1-s)^{-1/2}$$

$$\times F\left(\tfrac{3}{2} - a, \tfrac{3}{2} - a', \tfrac{1}{2}; 1-s\right)$$

(1.17) $$\Delta_{y=2\sqrt{x}} \psi_{+,B} = \underset{y=2\sqrt{x}}{\operatorname{disc}} \psi_{+,B}(x, y; \beta)$$

$$= 2i\cos\left(\pi\sqrt{1+4\beta}\right) \psi_{-,B}(x, y; \beta)$$

[A] Discontinuity formula (Borel plane) $\Rightarrow$ Stokes phenomena

[B] $\psi_{\pm, B}(x, y; \beta)$ are infra exponential entire in β.

$\Rightarrow$ $\psi_{\pm}$: symbol of l.d.o

($\beta = b\sigma_1(\partial/\partial y)$)

(1.18) anti-Wick $\mathop{\circ}\limits_{\circ} \psi_{\pm, \beta} \mathop{\circ}\limits_{\circ}$

(1.19) $\mathop{\circ}\limits_{\circ} \cos(\pi\sqrt{1+4\beta})\, \psi_{-, B}(x, y; \beta) \mathop{\circ}\limits_{\circ}$

$= \mathop{\circ}\limits_{\circ} \cos(\pi\sqrt{1+4\beta}) \mathop{\circ}\limits_{\circ} \mathop{\circ}\limits_{\circ} \psi_{-, B}(x, y; \beta) \mathop{\circ}\limits_{\circ}$

(1.20) sing. part of $\mathop{\circ}\limits_{\circ} \psi_{+, B} \mathop{\circ}\limits_{\circ}$ $\underbrace{\text{near}}_{y = 2\sqrt{x}}$

$= i \underbrace{\mathop{\circ}\limits_{\circ} \cos(\pi\sqrt{1+4\beta}) \mathop{\circ}\limits_{\circ}}_{\text{diff. op of inf. order; sheaf hom on } \mathcal{O}} \mathop{\circ}\limits_{\circ} \psi_{-, B}(x, y; \beta) \mathop{\circ}\limits_{\circ}$

(1.21) $\mathfrak{Z}\, w\, \mathop{\circ}\limits_{\circ} \psi_{\pm, B} \mathop{\circ}\limits_{\circ}$

(1.22) w : Wick product

(1.23)

$$\mathfrak{Z} : \mathcal{D}^\infty \longrightarrow \mathcal{D}^\infty / (\underbrace{\mathcal{D}^\infty(\partial/\partial x) + \mathcal{D}^\infty(\partial/\partial y)}_{})$$

$$(1.24)\quad \mathcal{F}\left(W \,{}^\circ_\circ\, \beta^k \mathcal{B}(\eta^{-n-1/2}) \,{}^\circ_\circ\right)$$

$$= \mathcal{F}\left(W \,{}^\circ_\circ\, \beta^k \, y^{n-1/2} / \Gamma(n+\tfrac{1}{2})\right)$$

$$= \mathcal{F}\left(W\left(b^k \frac{\partial^k}{\partial y^k}\left(y^{n-1/2} / \Gamma(n+\tfrac{1}{2})\right)\right)\right)$$

$$= b^k \, \mathcal{F}\Big(\sum_{0 \le l \le k} \binom{k}{l} \underbrace{y^{n-l-1/2} / \Gamma(n-l+1/2)}_{\frac{\partial^{k-l}}{\partial y^{k-l}}}$$

$$= b^k \mathcal{F}\Big(y^{n-k-1/2} / \Gamma(n-k+1/2)$$

$$+ \sum_{0 \le l \le k-1} \binom{k}{l} (y^{n-l-1/2} / \Gamma(n-l+1/2) \, \partial_y^{k-l} \Big)$$

$$= b^k y^{n-k-1/2} / \Gamma(n-k+1/2)$$

$$= \mathcal{B}(b^k \eta^k \eta^{-n-1/2})$$

$$= \mathcal{B}(\beta^k \eta^{-n-1/2})$$

<0>

$$(1.25)\quad \left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{x}\frac{\partial^2}{\partial y^2} + \frac{\beta}{x^2}\right)\right] \psi_{\pm,\mathcal{B}} = 0$$

$$(1.26)\quad \left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{x}\frac{\partial^2}{\partial y^2} + \frac{b}{x^2}\frac{\partial}{\partial y}\right] {}^\circ_\circ\, \psi_{\pm,\mathcal{B}} \underbrace{(b\sigma_1) \,{}^\circ_\circ}_{=0}$$

(1.27) $\varphi_\pm(x, y; b) \underset{\text{def}}{=} \mathcal{J}\omega_0^0 \psi_{\pm,B}(x, y; b\sigma_1)_0^0$

(1.28) $\left[\dfrac{\partial^2}{\partial x^2} - \left(\dfrac{1}{x}\dfrac{\partial^2}{\partial y^2} + \dfrac{b}{x^2}\dfrac{\partial}{\partial y}\right)\right.$

Borel tr of the boosted simple.p.

(1.29) $\dfrac{d^2}{dx^2} - \eta^2\left(\dfrac{1}{x} + \dfrac{b}{x^2}\eta^{-1}\right)$

(1.30) $$\tilde{\chi}_\pm = \left(\sum_{n\geq 0} \psi_{\pm,n}(\beta) \times \sqrt{x}^{\,-n+1/2}\, \eta^{-n-1/2}\right)\Big|_{\beta = b\eta}$$

(1.31) $\hat{\psi}_\pm = \exp\left(\pm\eta\displaystyle\int_0^x \frac{dx}{\sqrt{x}}\right)\tilde{\chi}_\pm$:

WKB sol. of the boosted s.p. eq.

(1.32) $\mathcal{B}(\hat{\psi}_\pm) = \varphi_\pm(x, y, b)$

(1.20) entails:

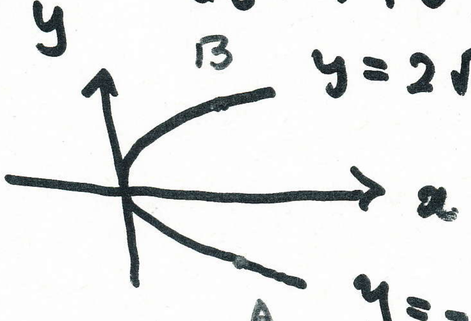
(1.33) $$\Delta_{y=2\sqrt{x}}\varphi_{+,B} = 2i\cos\left(\pi\sqrt{1+4b\,\partial/\partial y}\right)\varphi_{-,B}.$$

@ $\cos(\pi\sqrt{1+4b\,\partial/\partial y})$ acts on $\mathcal{O}$ as a sheaf homomorphism. Thus Figure can be used in the same manner.

$$(1.33)\quad \left(\frac{\partial^2}{\partial x^2} - \frac{1}{x}\frac{\partial^2}{\partial y^2} - \frac{b}{x^2}\frac{\partial}{\partial y}\right)\mathfrak{X}$$

$$= \mathcal{Y}\left(\frac{\partial^2}{\partial x^2} - \frac{Q_0(x)}{x}\frac{\partial^2}{\partial y^2} - \frac{Q_1(x)}{x^2}\frac{\partial}{\partial y}\right)$$

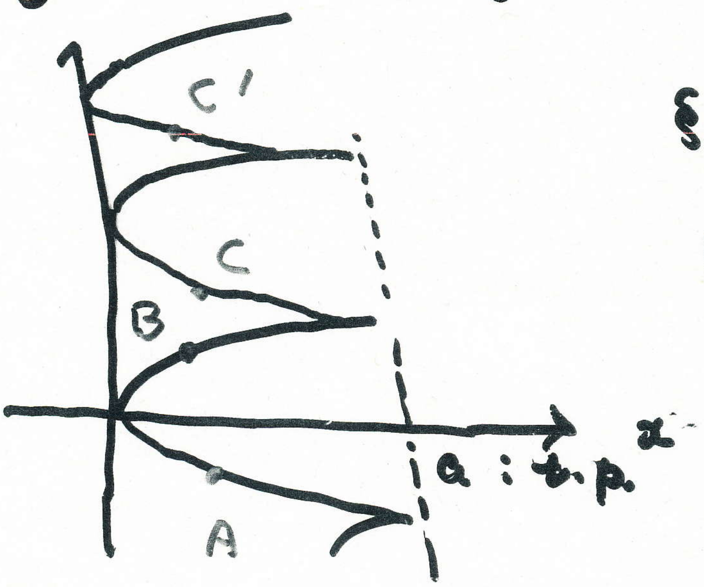
$Q_0(0)\neq 0$, $Q_1(0)=b$



$y=2\sqrt{x}$

$y=-2\sqrt{x}$

($x\neq 0$): bich. curve for $x\xi^2-\eta^2$



§ 2:

Fixed singularities versus movable singularities

AB: $2\int_0^x \frac{dx}{\sqrt{x}}$

When C, C', ... hits the path....

AC: $2\int_0^a \sqrt{Q_0}\,dx$

Whittaker eq.

$$(2.1)\quad \left[\frac{d^2}{dx^2} - \eta^2 Q(x,\eta;\alpha,\beta)\right]\psi = 0$$

$$(2.2)\quad Q = \frac{1}{4} - \frac{\alpha}{x} + \eta^{-2}\frac{\beta}{x^2}$$

Boosted Whittaker equation

(2.3) $$\left[\frac{d^2}{dx^2} - \eta^2 \hat{Q}(x, \eta; \alpha, \tilde{\beta})\right]\tilde{\psi} = 0$$

(2.4) $$\tilde{Q} = \frac{1}{4} - \frac{\alpha}{x} + \eta^{-1}\frac{\tilde{\beta}}{x^2}$$

(2.5) $$y_\pm(x) = \pm\int_{4\alpha}^{x} S_{-1}\,dx$$

$\langle 0 \rangle$

(2.6) $$\psi_+ = \exp(V)\,\psi_+^{(\infty)}$$

(2.7) $$V = \int_{4\alpha}^{\infty}(S_{odd} - \eta S_{-1})\,dx$$

(2.8) $$\psi_+^{(\infty)} = \frac{1}{\sqrt{S_{odd}}}\exp(y_+(x)\eta) \times \exp\left(\int_{\infty}^{x}\underbrace{(S_{odd} - \eta S_{-1})}_{S_j\ (j \geq 1)\text{ is integral at }\infty}\,dx\right)$$

(2.9) $$\psi_{+,B} = \mathcal{B}(\exp(V)) * \psi_{+,B}^{(\infty)}$$

(2.10) $$\Delta_{y=-y_+(x)+2m\pi i\alpha}(\psi_{+,B})$$

$$= \Delta_{y=-y_+(x)+2m\pi i\alpha}(\mathcal{B}(e^V) * \psi_{+,B}^{(\infty)})$$

$$= \Delta_{y=2m\pi i\alpha} (\mathcal{B}(e^{v_1}) * \psi_{+,B}^{(\infty)}$$

$$= (\Delta_{y=2m\pi i\alpha}(V_B)\,\mathcal{B}(e^{v})) * \psi_{+,B}^{(\infty)}$$

$$= \Delta_{y=2m\pi i\alpha}(V_B)\,\psi_{+,B}\,.$$

The same for $\tilde{\psi}_{+,B}$.

(2.11) $$\tilde{V} = \int_{4\alpha}^{\infty} (\tilde{S}_{odd} - \eta S_{-1})\,dx$$

(2.12) $$\tilde{S}_{odd} = (\tilde{S}^{(+)} - \tilde{S}^{(-)})/2$$

(2.13) $\tilde{S}_{odd,j}$ $(j \geq 0)$: integrable near ∞

e.g.

(2.14) $$\tilde{S}_{odd,0} = \frac{2\tilde{\beta}}{x^{3/2}(x-4\alpha)^{1/2}}$$

—— (1) ——

V_B is known : Koike - Takei

(Publ. RIMS. 47 (2011), 375 ~ 39

(2.15) V_B

$$= \frac{1}{2y}\,\frac{e^{-\gamma y/\alpha} + e^{(\gamma+1)y/\alpha}}{e^{y/\alpha}} - \frac{\alpha}{y^2}$$

[$\gamma^2+\gamma=\beta$; V_B is non-sing at $y=0$]

(2.16) $\operatorname{Res}_{y=\underline{2m\pi i\alpha}} V_B \quad \underset{\text{def}}{=} y_m$

$$= \frac{(-1)^m \cos\left(m\pi\sqrt{1+4\beta}\right)}{2m}$$

Rem : $\cos(2m\pi\gamma)$

$= \cos\left(m\pi\left(-1 \pm \sqrt{1+4\beta}\right)\right) = (-1)^m \cos(\quad)$

(2.17) $(\Delta_{y_m} V_B)(y)$

$$= \frac{(-1)^m}{m} \cos\left(m\pi\sqrt{1+4\beta}\right) \frac{1}{2\pi i(y-y_m)}$$

$$+ \Phi_m(y, \alpha, \beta) \quad (\text{hol. } y=y_m)$$

(2.18) $\tilde{V}(\eta ; \alpha, \hat{\beta}) = V(\eta, \alpha, \beta)\Big|_{\beta=\tilde{\beta}}$

(2.19) $W \underset{\mathrm{def}}{=} V - \frac{\beta}{2\alpha} \eta^{-1}$

$\tilde{W} \underset{\mathrm{def}}{=} \tilde{V} - \frac{\beta}{2\alpha}$

(2.20) $\Delta_{y_m} \tilde{V}_B = \Delta_{y_m} \tilde{W}_B$

(2.21) $\tilde{W}(\eta, \alpha, \beta)$

$= W(\eta, \alpha, \beta)|_{\beta = \tilde{\beta}\eta} \in \eta^{-1} \mathbb{C}[[\eta^{-1}]]$

(2.22) $\mathcal{F} \, \omega \colon W_B(y; \alpha, \beta\sigma_1(\partial/\partial y)) \colon$

[The same as in (1.24)] $= \tilde{W}_B(y; \alpha, \tilde{\beta})$

(2.23) $\Delta_{y_m} (\mathcal{F}(\omega \colon W_B(\tilde{\beta}\sigma_1) \colon)$

$= \mathcal{F}(\omega \colon \Delta_{y_m} W_B(\tilde{\beta}\sigma_1) \colon)$

$= \mathcal{F}(\omega \colon \frac{(-1)^m}{2m\pi i} \cos(m\pi\sqrt{})$

$\times \frac{1}{y - y_m} + \Phi_m(y; \alpha, \tilde{\beta}\sigma_1(\partial/\partial y)) \colon$

$$= \frac{(-1)^m}{2m\pi i} \cos\left(m\pi\sqrt{1+4\tilde{\beta}\,\partial/\partial y}\right)$$

$$\times \frac{1}{y-y_m} + \varphi_m(y; \alpha, \tilde{\beta})$$

$\Delta\, \hat{V}_B$: simple pole acted upon by an diff. op. of ∞ order

Thus we have encountered an essential singularity in a concrete problem in exact WKB analysis