# The thermodynamic limit for interacting quantum fermions in a random environment: the random pieces model


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## The $n$ particle system

- On $\Lambda$ large cube of $\mathbb{R}^d$, consider $H_\omega(\Lambda)$ a random Schrödinger operator (single particle model).
- On $\bigwedge_{j=1}^n L^2(\Lambda) = L^2_-(\Lambda^n)$, consider the free operator

$$H^0_\omega(\Lambda,n) = \sum_{i=1}^n \underbrace{1\otimes\ldots\otimes 1}_{i-1 \text{ times}} \otimes H_\omega(\Lambda) \otimes \underbrace{1\otimes\ldots\otimes 1}_{n-i \text{ times}}.$$

- Pick $U:\mathbb{R}^d \to \mathbb{R}^+$ pair interaction potential
  Define

$$H^U_\omega(\Lambda,n) = H^0_\omega(\Lambda,n) + W_n, \quad \text{where} \quad W_n(x^1,\cdots,x^n) := \sum_{i<j} U(x^i - x^j).$$

## Thermodynamic limit

- Let $E^U_\omega(\Lambda,n)$ be the ground state energy of $H_\omega(\Lambda,n)$.
- Let $\Psi^U_\omega(\Lambda,n)$ be the associated eigenfunction.

### Problem

Describe $E^U_\omega(\Lambda,n)$ and $\Psi^U_\omega(\Lambda,n)$ in the limit $|\Lambda| \to +\infty$ and $\frac{n}{|\Lambda|} \to \rho > 0$.



## Description of the ground state : the (reduced) density matrices :

Let $\Psi \in L^2_-(\Lambda^n)$ be a normalized $n$-fermion wave function.

- One-particle density matrix is an operator on $L^2(\Lambda)$ with kernel

$$\gamma^{(1)}_\Psi(x,y) = n\int_{\Lambda^{n-1}} \Psi(x,\tilde{x})\Psi^*(y,\tilde{x})d\tilde{x}.$$

- Two-particle density matrix is an operator on $L^2_-(\Lambda^2)$ with kernel

$$\gamma^{(2)}_\Psi(x^1,x^2,y^1,y^2) = \frac{n(n-1)}{2}\int_{\Lambda^{n-2}} \Psi(x^1,x^2,\tilde{x})\Psi^*(y^1,y^2,\tilde{x})d\tilde{x}.$$

Both $\gamma^{(1)}_\Psi$ and $\gamma^{(2)}_\Psi$ are positive trace class operators satisfying

$$\mathrm{tr}\gamma^{(1)}_\Psi = n \quad \text{and} \quad \mathrm{tr}\gamma^{(2)}_\Psi = \frac{n(n-1)}{2}.$$

## The non interacting system

Let $(E_p)_{p\geq 1} = (E_p(\omega,\Lambda))_{p\geq 1}$ (resp. $(\psi_p)_{p\geq 1} = (\psi_p(\omega,\Lambda))_{p\geq 1}$) be the eigenvalues (resp. associated eigenfunctions) of $H_\omega(\Lambda)$.

Set $\mathbb{N}_n^+ = \{\alpha = (\alpha_1,\cdots,\alpha_n);\ \forall i,\ \alpha_i < \alpha_{i+1}\}$. Then,

- eigenvalues of $H^0_\omega(\Lambda,n)$ given by $E_\alpha := \sum_{1\leq j\leq n} E_{\alpha_j}$ where $\alpha \in \mathbb{N}_n^+$,
- eigenfunction of $H^0_\omega(\Lambda,n)$ associated to $E_\alpha$ given by Slater determinant

$$\Psi_\alpha(x^1,x^2,\cdots,x^n) = \frac{1}{\sqrt{n!}}\det\left(\psi_{\alpha_k}(x^j)\right)_{1\leq j,k\leq n}.$$

Ground state energy per particle for non interacting particles : Define

$$N_{H_\omega(\Lambda)}(E) = \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ in } (-\infty,E]\}}{|\Lambda|}$$

Thus, as $N_{H_\omega(\Lambda)}(E_n) = n/|\Lambda| \to \rho$, one has

$$\frac{E^0_\omega(\Lambda,n)}{n} = \frac{1}{n}\sum_{j=1}^n E_j = \frac{|\Lambda|}{n}\int_{-\infty}^{E_n} E\,dN_{H_\omega(\Lambda)}(E) \underset{\substack{|\Lambda|\to+\infty\\ n/|\Lambda|\to\rho}}{\longrightarrow} \frac{1}{\rho}\int_{-\infty}^{E_\rho} E\,dN(E)$$

where $N(E_\rho) = \rho$ ; $E_\rho$ is the Fermi energy and $N$ IDS of $H_\omega$.



## The non interacting ground state

We describe the one-particle density matrix of $\Psi^0_\omega(\Lambda,n)$, the non interacting ground state :

$$\gamma^{(1)}_{\Psi^0_\omega(\Lambda,n)} = \sum_{p=1}^n \gamma^{(1)}_{\psi_p(\omega,\Lambda)} = \sum_{p=1}^n \psi_p(\omega,\Lambda)\otimes\overline{\psi_p(\omega,\Lambda)}.$$

One proves that, in the thermodynamic limit, one has

$$\gamma^{(1)}_{\Psi^0_\omega(\Lambda,n)} \underset{\substack{|\Lambda|\to+\infty\\ n/|\Lambda|\to\rho}}{\longrightarrow} \mathbf{1}_{(-\infty,E_\rho]}(H_\omega).$$

Depending on the model under consideration, the limit can be proved in the strong operator topology, or for the trace norm par particle.

## A simple one-dimensional random model

The pieces (or Luttinger-Sy) model

- On $\mathbb{R}$, consider Poisson process $d\mu(\omega)$ of intensity $\mu$ i.e.

$$d\mu(\omega) = \sum_{k\in\mathbb{Z}} \delta_{x_k(\omega)}.$$

- For $\Lambda = [-L/2, L/2]$, on $L^2(\Lambda)$, define

$$H_\omega(L) = \bigoplus_{k\in\mathbb{Z}} -\frac{d^2}{dx^2}\Big|^D_{\Delta_k\cap\Lambda} \quad \text{where} \quad \Delta_k = \Delta_k(\omega) = [x_k, x_{k+1}]$$

- Integrated density of states

$$\begin{aligned} N(E) &:= \lim_{L\to+\infty} \frac{\sharp\{\text{eigenvalues of } H_\omega(L) \text{ in } (-\infty, E]\}}{L} \\ &= \frac{\exp(-\ell_E)}{1-\exp(-\ell_E)} 1_{E\geq 0} \text{ where } \ell_E := \frac{\pi}{\sqrt{E}}. \end{aligned}$$

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## Why did we choose the pieces model ?

It shares the common characteristics of a general random one particle system in the localized phase :

- the model exhibits Lifshitz tail asymptotics,
- the eigenfunctions are localized (on a scale $\log L$),
- the localization centers and the eigenvalues satisfy Poisson statistics,

Description of spectral characteristics are much better than for a general random one particle system in the localized phase

- eigenfunctions and eigenvalues are known explicitly,
- the density of states is known explicitly i.e. given by a closed formula.

Main difference compared to more realistic models :

tunnel effect is missing for a single particle.

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## The $n$ particle system

- On $\bigwedge_{j=1}^{n} L^2([-L/2,L/2]) = L^2_-([-L/2,L/2]^n)$, consider the free operator

$$H^0_\omega(L,n) = \sum_{i=1}^{n} \underbrace{1\otimes\ldots\otimes 1}_{i-1 \text{ times}} \otimes H_\omega(L) \otimes \underbrace{1\otimes\ldots\otimes 1}_{n-i \text{ times}}.$$

- Pick $U:\ \mathbb{R}\to\mathbb{R}^+$ not identically vanishing, even, bounded.

  We assume $U\in L^p(\mathbb{R})$ for some $p\in(1,+\infty]$ and $x^3\cdot\int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} 0$.

  Recall $H^U_\omega(L,n) = H^0_\omega(L,n) + W_n$ where $W_n(x^1,\cdots,x^n) := \sum_{i<j} U(x^i - x^j)$.

Thermodynamic limit at small density : $n/L\to\rho$ as $L\to+\infty$ where $\rho>0$ small.

Note : by scaling $x\to\mu x$, $H^U_\omega(L,n)$ on $[0,L]^n$ becomes $\mu^2 H^{U_\mu}_{\omega_\mu}(\mu L,n)$ on $[0,\mu L]^n$

thus, $(\rho,\mu)\longrightarrow(\rho/\mu,1)$ (up to rescaling energy).

## The non interacting system : the ground state energy per particle

$$\mathscr{E}^0(\rho) = \lim_{\substack{L\to+\infty\\ n/L\to\rho}} \frac{E^0_\omega(L,n)}{n} = E_\rho\left(1+O\left(\sqrt{E_\rho}\right)\right) = \pi^2\left|\log\rho\right|^{-2}\left(1+O\left(\left|\log\rho\right|^{-1}\right)\right).$$



## The non interacting ground state

1. Pick all the pieces $\Delta_k = [x_k(\omega), x_{k+1}(\omega)]$ of length larger than $\ell_\rho = \pi/\sqrt{E_\rho}$.
2. For each piece, take all the states associated to levels below $E_\rho$.
3. Form the Slater determinant to get the non interacting ground state.

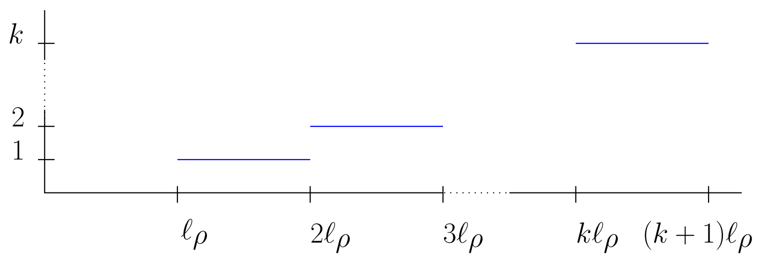
## The reduced one-particle density matrix for the non interacting ground state

$$\gamma^{(1)}_{\Psi^0_\omega(L,n)} = \sum_{j\geq 1}\left[\sum_{j\ell_\rho\leq|\Delta_k|<(j+1)\ell_\rho}\left(\sum_{n=1}^{j}\gamma^{(1)}_{\varphi^n_{\Delta_k}}\right)\right]$$

$$= \sum_{\ell_\rho\leq|\Delta_k|<2\ell_\rho}\gamma^{(1)}_{\varphi^1_{\Delta_k}} + \sum_{2\ell_\rho\leq|\Delta_k|<3\ell_\rho}\left(\gamma^{(1)}_{\varphi^1_{\Delta_k}}+\gamma^{(1)}_{\varphi^2_{\Delta_k}}\right) + R^{(1)}$$

where

- for an interval $I$, we let $\varphi^j_I$ be the $j$-th normalized eigenvector of $-\Delta^D_{|I}$,
- the operator $R^{(1)}$ is trace class and $\|R^{(1)}\|_1 \leq C\rho^2 n$.



## Existence of the ground state energy per particle

**Theorem**

*Under our assumptions on $U$, $\omega-$almost surely, the following limit exists and is independent of $\omega$*

$$\mathscr{E}^U(\rho) := \lim_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{E^U_\omega(L,n)}{n}.$$

## Ground state energy asymptotic expansion

**Theorem**

*Under our assumptions on $U$, one has*

$$\mathscr{E}^U(\rho) = \mathscr{E}^0(\rho) + \frac{\pi^2\gamma_*}{|\log\rho|^3}\rho + o\left(\frac{\rho}{|\log\rho|^3}\right),$$

*where* $\gamma_* = 1 - \exp\left(-\frac{\gamma}{8\pi^2}\right)$.



## Systems of two fermions : within the same piece :

**Lemma**

*Assume that $U \in L^p(\mathbb{R})$ for some $p \in (1,+\infty]$ and that $\int_{\mathbb{R}} x^2 U(x)dx < +\infty$. Consider two fermions in $[0,\ell]$ interacting via the pair potential $U$, i.e., on $L^2([0,\ell]) \wedge L^2([0,\ell])$, consider the Hamiltonian*

$$-\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} + U(x_1 - x_2). \tag{1}$$

*Then, for large $\ell$, $E^{2,U}(\ell)$, its ground state energy admits the following expansion*

$$E^{2,U}(\ell) = \frac{5\pi^2}{\ell^2} + \frac{\gamma}{\ell^3} + o\left(\ell^{-3}\right)$$

*where* $\gamma := \frac{5\pi^2}{2}\left\langle u\sqrt{U(u)}, \left(Id + \frac{1}{2}U^{1/2}(-\Delta_1)^{-1}U^{1/2}\right)^{-1} u\sqrt{U(u)}\right\rangle.$

## Uniqueness of the ground state :

**Theorem**

*Assume $U$ is analytic. Then, for any $L$ and $n$, $H^U_\omega(L,n)$ has a unique ground state $\omega$-almost surely.*

## Interacting ground state : "optimal" approximation

Let $\zeta_I^1$ be the ground state of $-\Delta|_{I^2}^D + U$ acting on $L^2_-(I^2)$. Define

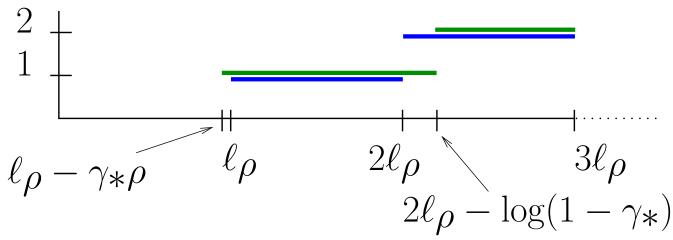
$$\gamma^{(1)}_{\Psi^{\text{opt}}_{L,n}} = \sum_{\ell_\rho - \rho\gamma_* \leq |\Delta_k| \leq 2\ell_\rho - \log(1-\gamma_*)} \gamma^{(1)}_{\varphi^1_{\Delta_k}} + \sum_{2\ell_\rho - \log(1-\gamma_*) \leq |\Delta_k|} \gamma^{(1)}_{\zeta^1_{\Delta_k}},$$

### Theorem

*We assume $U$ cpct support. There exists $\rho_0 > 0$ s.t. for $\rho \in (0, \rho_0)$, $\omega$-a.s., one has*

$$\limsup_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n} \left\| \gamma^{(1)}_{\Psi^U_\omega(L,n)} - \gamma^{(1)}_{\Psi^{opt}_{L,n}} \right\|_1 \lesssim \frac{\rho}{|\log \rho|},$$

$$\limsup_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n^2} \left\| \gamma^{(2)}_{\Psi^U_\omega(L,n)} - \frac{1}{2}(Id - Ex)\left[\gamma^{(1)}_{\Psi^{opt}_{L,n}} \otimes \gamma^{(1)}_{\Psi^{opt}_{L,n}}\right] \right\|_1 \lesssim \frac{\rho}{|\log \rho|}.$$






## Quantification of the influence of interactions

Influence of interactions on the ground state is essentially described by

$$\gamma^{(1)}_{\Psi^0_\omega(L,n)} - \gamma^{(1)}_{\Psi^{\text{opt}}_{L,n}} = \sum_{2\ell_\rho - \log(1-\gamma_*) \leq |\Delta_k|} \left( \gamma^{(1)}_{\varphi^1_{\Delta_k}} + \gamma^{(1)}_{\varphi^2_{\Delta_k}} - \gamma^{(1)}_{\zeta^1_{\Delta_k}} \right)$$

$$- \sum_{\ell_\rho - \rho\gamma_* \leq |\Delta_k| \leq \ell_\rho} \gamma^{(1)}_{\varphi^1_{\Delta_k}} + \sum_{2\ell_\rho \leq |\Delta_k| \leq 2\ell_\rho - \log(1-\gamma_*)} \gamma^{(1)}_{\varphi^2_{\Delta_k}} + \widetilde{R}^{(1)}$$

In particular,

$$\lim_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n} \left\| \gamma^{(1)}_{\Psi^0_\omega(L,n)} - \gamma^{(1)}_{\Psi^U_\omega(L,n)} \right\|_1 = 2\gamma_*\rho + O\left(\frac{\rho}{|\log\rho|}\right),$$

and

$$\lim_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n^2} \left\| \gamma^{(2)}_{\Psi^0_\omega(L,n)} - \gamma^{(2)}_{\Psi^U_\omega(L,n)} \right\|_1 = 2\gamma_*\rho + O\left(\frac{\rho}{|\log\rho|}\right).$$

To be compared with

$$\limsup_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n} \left\| \gamma^{(1)}_{\Psi^U_\omega(L,n)} - \gamma^{(1)}_{\Psi^{\text{opt}}_{L,n}} \right\|_1 \lesssim \frac{\rho}{|\log \rho|}.$$

## Some open questions

1. For $U$ compactly supported, we have a decription of ground state.
   When $x^3 \int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} 0$ not too fast, changes induced by these "long" range interactions difficult to control. Get a good description of the ground state.
2. For $U$ compactly supported, we actually have a better expansion for $\mathscr{E}^U(\rho)$. And we have a more precise description of the ground state.
   Does $\gamma^{(1)}_{\Psi^U_\omega(L,n)}$ converge as $L \to +\infty$ ?
3. What happens if $x^3 \int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} +\infty$ ? One may expect
   - if $\int_{\mathbb{R}} U(t)dt < +\infty$ : interactions at a distance become more important than local interactions in the same piece.
   - if $\int_{\mathbb{R}} U(t)dt = +\infty$, interactions become more important than non interacting energy term.

In our model, no tunneling for a single particle. How to take tunneling into account ?

The work of **Helffer and Sjöstrand** (and successors) on multiple wells !

In dimension 1, preliminary computations suggest same picture.

What happens in higher dimensions ?