Strichartz inequalities for waves in a strictly convex domain

Oana Ivanovici (†), Richard Lascar (‡), Gilles Lebeau (†) and Fabrice Planchon (†)

(†) Université Nice Sophia Antipolis
(‡) Université Paris 7
lebeau@unice.fr

In honor of

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Outline

Outline

Dispersion

$$\|\chi(hD_t)e^{\pm it\sqrt{|\triangle|}}(\delta_a)\|_{L^\infty_x} \leq Ch^{-d} min(1, (\frac{h}{t})^{\alpha_d}) \tag{1.1}$$

Strichartz

$$(\partial_t^2 - \triangle)u = 0$$

$$h^\beta \|\chi(hD_t)u\|_{L^q_{t\in[0,T]}(L^r_x)} \leq C(\|u(0,x)\|_{L^2} + \|hD_t u(0,x)\|_{L^2}) \tag{1.2}$$

$$q \in]2,\infty[, \quad r \in [2,\infty]$$

$$\frac{1}{q} = \alpha_d(\frac{1}{2} - \frac{1}{r}), \quad \beta = (d - \alpha_d)(\frac{1}{2} - \frac{1}{r})$$

with $\alpha_d = \frac{d-1}{2}$ **in the free space** $\mathbb{R}^d$

Main result

Let (M, g) be a Riemannian manifold. Let Ω be an open relatively compact subset of M with smooth boundary $\partial\Omega$. We assume that Ω is **Strictly Convex** in (M, g), i.e any (small) piece of geodesic tangent to $\partial\Omega$ is exactly tangent at order 2 and lies outside Ω.
We denote by $\triangle$ the Laplacian associated to the metric g on M.

Theorem

For solutions of the mixed problem $(\partial_t^2 - \triangle)u = 0$ on $\mathbb{R}_t \times \Omega$ and $u = 0$ on $\mathbb{R}_t \times \partial\Omega$, the Strichartz inequalities hold true with

$$\alpha_d = \frac{d-1}{2} - \frac{1}{6}, \quad d = dim(M)$$

Remark

This was proved by M. Blair, H.Smith and C.Sogge in the case $d = 2$ for arbitrary boundary (i.e without convexity assumption). The above theorem improves all the known results for $d \geq 3$.

Outline

The problem is local near any point p_0 of the boundary. In geodesic coordinates normal to $\partial\Omega$ and after conjugation by a non vanishing smooth function $e(x, y)$, one has for $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$ near $(0, 0)$

$$\tilde{\triangle} = e^{-1} \cdot \triangle \cdot e = \partial_x^2 + R(x, y, \partial_y)$$

$$\Omega = \{x > 0\}, \quad p_0 = (x = 0, y = 0)$$

0n the boundary, in geodesic coordinates centered at $y = 0$, one has

$$R_0(y, \partial_y) = R(0, y, \partial_y) = \sum \partial_{y_j}^2 + O(y^2)$$

Let $R_1(y, \partial_y) = \partial_x R(0, y, \partial_y) = \sum R_1^{j,k}(y)\partial_{y_j}\partial_{y_k}$. The quadratic form $\sum R_1^{j,k}(y)\eta_j\eta_k$ is positively define. We introduce the

Model Laplacian

$$\triangle_M = \partial_x^2 + \sum \partial_{y_j}^2 + x\Big(\sum R_1^{j,k}(0)\partial_{y_j}\partial_{y_k}\Big)$$

Set

$$\rho(\omega,\eta) = (\eta^2 + \omega q(\eta)^{2/3})^{1/2}, \quad q(\eta) = \sum R_1^{j,k}(0)\eta_j\eta_k$$

The following theorem is due to Melrose-Taylor, Eskin, Zworski, ...

Theorem

There exists two phases $\psi(x,y,\eta,\omega)$ homogeneous of degree 1, $\zeta(x,y,\eta,\omega)$ homogeneous of degree $2/3$, and symbols $p_{0,1}(x,y,\eta,\omega)$ of degree 0 (ω is $2/3$ homogeneous, and $|\omega|\eta|^{-2/3}|$ is small) such that

$$G(x,y;\eta,\omega) = e^{i\psi}\Big(p_0 Ai(\zeta) + x p_1 |\eta|^{-1/3} Ai'(\zeta)\Big)$$

satisfy

$$-\tilde{\triangle} G = \rho^2 G + O_{C^\infty}(|\eta|^{-\infty}) \text{ near } (x,y) = (0,0)$$

$$\zeta = -\omega + x|\eta|^{2/3} e_0(x,y,\eta,\omega)$$

with p_0 and e_0 elliptic near any point $(0,0,\eta,0)$ with $\eta \in \mathbb{R}^{d-1} \setminus 0$.

Let $(X-x)u+(Y-y)v+\Gamma(X,Y,u,v)$ be a generating function for a (Melrose) canonical transformation χ_M such that

$$\chi_M(x=0,\xi^2+\eta^2+xq(\eta)=1)=(X=0,\Xi^2+R(X,Y,\Theta)=1)$$

near $\Sigma_0=\{(x,y,\xi,\eta),x=0,y=0,\xi=0,|\eta|=1\}$. One has

$$\Gamma(0,Y,u,v) \text{ is independent of } u$$

There exists a symbol $q(x,y,\eta,\omega,\sigma)$ of degree 0 (σ is $1/3$ homogeneous) compactly supported near

$$N_0=\{x=0,y=0,\omega=0,\sigma=0,\eta\in\mathbb{R}^{d-1}\setminus 0\}$$

and elliptic on N_0 such that

$$G(x,y;\eta,\omega)=\frac{1}{2\pi}\int e^{i(y\eta+\sigma^3/3+\sigma(xq(\eta)^{1/3}-\omega)+\rho\Gamma(x,y,\frac{\sigma q(\eta)^{1/3}}{\rho},\frac{\eta}{\rho}))}q\,d\sigma$$

Let $\mathcal{G}(t, x, y; a)$ be the Green function solution of the mixed problem, with $a \in]0, a_0]$, $a_0 > 0$ small

$$(\partial_t^2 - \tilde{\Delta})\mathcal{G} = 0 \text{ in } x > 0, \quad \mathcal{G}|_{x=0} = 0$$

$$\mathcal{G}|_{t=0} = \delta_{x=a,y=0}, \quad \partial_t \mathcal{G}|_{t=0} = 0$$

Definition

Let $\chi(x, t, y, hD_t, hD_y)$ be a h-pseudo differential (tangential) operator of degree 0, compactly supported near $\tilde{\Sigma}_0 = \{x = 0, t = 0, y = 0, \tau = 1, |\eta| = 1\}$ and equal to identity near $\tilde{\Sigma}_0$.

A "parametrix" is an approximation (near $\{x = 0, y = 0, t = 0\}$) mod $0_{C^\infty}(h^\infty)$, and uniformly in $a \in]0, a_0]$ of $\chi(x, t, y, hD_t, hD_y)(\mathcal{G}(.; a))$.

Set $\omega = h^{-2/3}\alpha$. Recall $\rho(\alpha,\theta) = (\theta^2 + \alpha q(\theta)^{2/3})^{1/2}$. Let $\Phi(x,y,\theta,\alpha,s)$ be the phase function

$$\Phi = y\theta + s^3/3 + s(xq(\theta)^{1/3} - \alpha) + \rho(\alpha,\theta)\Gamma(x,y,\frac{sq(\theta)^{1/3}}{\rho(\alpha,\theta)},\frac{\theta}{\rho(\alpha,\theta)})$$

and let $q_h(x,y,\theta,\alpha,s) = h^{-1/3}q(x,y,h^{-1}\theta,h^{-2/3}\alpha,h^{-1/3}s)$ Then

$$J(f)(x,y) = \frac{1}{2\pi}\int e^{\frac{i}{h}(\Phi - y'\theta - t'\alpha)} q_h f(y',t')\; dy'dt'd\theta d\alpha ds$$

is a semiclassical OIF associated to a canonical transformation χ such that

$$\chi(\{y'=0, t'=0, |\eta'|=1, \tau'=0\}) = \{y=0, x=0, |\eta|=1, \xi=0\}$$

Moreover, J is elliptic on the above set and

$$-h^2\tilde{\triangle}J(f) = J(\rho^2(hD_{t'},hD_{y'})f) \mod O_{C^\infty}(h^\infty)$$

Airy-Poisson summation formula

Let $A_\pm(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3} z)$. One has $Ai(-z) = A_+(z) + A_-(z)$. For $\omega \in \mathbb{R}$, set

$$L(\omega) = \pi + i \log\left(\frac{A_-(\omega)}{A_+(\omega)}\right)$$

The function L is analytic, strictly increasing, $L(0) = \pi/3$, $\lim_{\omega \to -\infty} L(\omega) = 0$, $L(\omega) \simeq \frac{4\omega^{3/2}}{3} (\omega \to +\infty)$, and one has $\forall k \in \mathbb{N}^*$

$$L(\omega_k) = 2\pi k \Leftrightarrow Ai(-\omega_k) = 0, \quad L'(\omega_k) = \int_0^\infty Ai^2(x - \omega_k)\, dx$$

Lemma

The following equality holds true in $\mathcal{D}'(\mathbb{R}_\omega)$.

$$\sum_{N \in \mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \delta_{\omega = \omega_k}$$

Let $g_{h,a}(y', t')$ such that $J(g_{h,a}) - \frac{1}{2}\delta_{x=a,y=0} = R$ with $WF_h(R) \cap W = \emptyset$, where W is a fixed neighborhood of $\{(x = 0, y = 0, \xi = 0, \eta), \ |\eta| = 1\}$. For $\omega \in \mathbb{R}$, set (recall $\alpha = h^{2/3}\omega$)

$$K_\omega(f)(t, x, y) = \frac{h^{2/3}}{2\pi} \int e^{\frac{i}{h}(t\rho(h^{2/3}\omega,\theta)+\Phi-y'\theta-t'h^{2/3}\omega)} q_h f(y', t') \, dy' dt' d\theta ds$$

One has $J(f) = \int_{\mathbb{R}} K_\omega(f)|_{t=0} \, d\omega$. Finally, set

$$< \sum_{N\in\mathbb{Z}} e^{-iNL(\omega)}, K_\omega(g_{h,a}) >_{\mathcal{D}'(\mathbb{R})} = \mathcal{P}_{h,a}(t, x, y) = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)} K_{\omega_k}(g_{h,a})$$

Proposition

$\mathcal{P}_{h,a}(t, x, y)$ *is a parametrix.*

The proof uses the left formula for $a \geq h^{2/3-\epsilon}$, and the right formula for $a \leq h^{4/7+\epsilon}$.

Outline

The special case $\tilde{\triangle} = \triangle_M$, with $q(\eta) = |\eta|^2$ (Friedlander model), where one has of course $\Gamma = 0$, has been studied by Ivanovici-Lebeau-Planchon in Dispersion for waves inside strictly convex domains I: the Friedlander model case. (http://arxiv.org/abs/1208.0925 and to appear in Annals of Maths). The analysis of phase integrals are (essentially) the same in the general case, and leads to the following result.

Theorem

$$|\mathcal{P}_{h,a}(t,x,y)| \leq Ch^{-d} min\Big(1, (\frac{h}{t})^{\frac{d-2}{2}} \mathbf{C}\Big) \tag{3.1}$$

$$\mathbf{C} = (\frac{h}{t})^{1/2} + a^{1/8} h^{1/4} \ \ for\ a \geq h^{2/3-\epsilon}$$

$$\mathbf{C} = (\frac{h}{t})^{1/3} \ \ for\ a \leq h^{1/3+\epsilon}$$

Corollary

Strichartz holds true in any dimension $d \geq 2$ *with* $\alpha_d = \frac{d-1}{2} - \frac{1}{4}$

Swallowtails

The bad factor $h^{1/4}$ occurs only near the projection $SW_n, n \geq 1$ of the swallowtails. This are smooth submanifold of codimension 3 in $\mathbb{R}^{1+d}$, parametrized by a tangential initial direction $\nu \in S^{d-2}$. In the $\triangle_M$ model, SW_n is given by

$$t_n(a,\nu) = 4na^{1/2}(1+aq(\nu))^{1/2}q(\nu)^{-1/2}$$

$$x_n(a,\nu) = a, \quad y_n(a,\nu) = 4na^{1/2}(\nu + aq'(\nu)/3)q(\nu)^{-1/2}$$

with an estimation of $\mathbf{C}$ for t near $t_n(a,\nu)$, for a given ν, by

$$\mathbf{C} \leq (\frac{h}{t})^{1/2} + h^{1/3} + \frac{a^{1/8}h^{1/4}}{|n|^{1/4} + h^{-1/12}a^{-1/24}(t^2 - t_n^2(a,\nu))^{1/6}}$$

For $|t^2 - t_n^2(a,\nu)| \geq \epsilon a$, i.e $t \notin]t_n(a,\nu) - \frac{\epsilon' a^{1/2}}{|n|}, t_n(a,\nu) - \frac{\epsilon' a^{1/2}}{|n|}[$ the last factor is $\leq h^{1/3}$.

Outline

Let us now give a sketch of the proof of the Strichartz estimate with $\alpha_d = (d-1)/2 - 1/6$. Let us denote by $\mathcal{P}_{h,a,b}(t,x,y)$ the above parametrix where the source point is located at $x = a, y = b$ (the above estimates for $b = 0$ apply uniformly for any value of b). For f compactly supported in $(s, a \geq 0, b)$, define

$$A(f)(t,x,y) = \int \mathcal{P}_{h,a,b}(t-s,x,y)f(s,a,b)dsdadb$$

Let us first consider the case $d = 3$. Our dispersive exponent is $\alpha_d = \alpha_3 = 1 - 1/6 = 5/6$. We have to prove the estimate (end point estimate for $r = \infty$ and $q = 12/5$), for some $T_0 > 0$,

$$h^{2\beta}\|A(f); L^{12/5}_{t\in[0,T_0]}(L^\infty_{x,y})\| \leq C\|f; L^{12/7}_s(L^1_{a,b})\| \tag{4.1}$$

$$2\beta = (d - \alpha_d) = 2 + 1/6 = 13/6$$

We write $\mathcal{P}_{h,a,b}(t,x,y) = \mathcal{P}^0_{h,a,b}(t,x,y) + \mathcal{P}^S_{h,a,b}(t,x,y)$ where $\mathcal{P}^S$ is the singular part, associated to a cutoff of $\mathcal{P}_{h,a,b}$ centered at the swallowtail singularities ($|n| \geq 1$)

$$|x - x_n(a,b,\nu)| \leq \frac{a}{|n|^2}, \quad |t - t_n(a,b,\nu)| \leq \frac{\sqrt{a}}{|n|}, \quad |y - y_n(a,b,\nu)| \leq \frac{\sqrt{a}}{|n|}$$

Lemma

$$h^{2\beta} \sup_{x,y,a,b} |\mathcal{P}^0_{h,a,b}(t,x,y)| \leq C|t|^{-5/6} \tag{4.2}$$

Let $A = A^0 + A^S$.
The estimate for A^0 follows easily, since the convolution by $|t|^{-5/6}$ maps $L^{12/7}$ in $L^{12/5}$.

The estimate for A^S will thus follows from the following lemma.

Lemma

$h^{2\beta} A^S$ is bounded from $L_t^{12/7}(L_{x,y}^1)$ into $L_t^{12/5}(L_{x,y}^\infty)$.

In the Friedlander model, an explicit computation shows that $h^{2\beta} A^S$ is bounded from $L_t^1(L_{x,y}^1)$ into $L_t^{2-\epsilon}(L_{x,y}^\infty)$. Since the cutoff in balls near the swallowtails singularities is symmetric in (x, a), by duality, $h^{2\beta} A^S$ is bounded from $L_t^{2+\epsilon}(L_{x,y}^1)$ into $L_t^\infty(L_{x,y}^\infty)$, and by interpolation, we get

$$h^{2\beta} A^S \text{ is bounded from } L_t^{12/7}(L_{x,y}^1) \text{ into } L_t^{12-\epsilon}(L_{x,y}^\infty)$$

which is far sufficient.

In the case $d \geq 4$, the end point Strichartz estimate we have to prove is

$$h^{(d-\alpha_d)/\alpha_d} \|A(f); L^2_{t\in[0,1]}(L^r_{x,y})\| \leq C\|f; L^2_s(L^{r'}_{a,b})\|, \quad r = \frac{6d-8}{3d-10} \quad (4.3)$$

The decomposition $\mathcal{P} = \mathcal{P}^0 + \mathcal{P}^S$ satisfy

Proposition

One can construct $\mathcal{P}^S$ such that $f(a,b) \mapsto \int \mathcal{P}^0_{h,a,b}(t,x,y)(a,b)dadb$ is bounded on L^2 uniformly in $t \in [-T_0, T_0]$. Moreover, $\mathcal{P}^0$ satisfies

$$\sup_{x,y,a,b} |\mathcal{P}^0_{h,a,b}(t,x,y)| \leq Ch^{-(d-\alpha_d)}(\frac{1}{|t|})^{\alpha_d} \quad (4.4)$$

With $A = A^0 + A^S$, the estimate for the part A^0 follows now by the classical proof of Strichartz estimates in the free space: interpolation between the energy estimate and the $L^1 \to L^\infty$ estimate. The estimate for A^S follows as above by the precise estimation near the swallowtails.

Outline

Ivanovici counter examples

For simplicity, we consider here only the 3-d case. In the free space $\mathbb{R}^3$, Strichartz reads, with $\alpha_3 = 1$ and $1/2 - 1/r = 1/q > 2$

$$h^{2(\frac{1}{2}-\frac{1}{r})}\|\chi(hD_t)u\|_{L^q_{t\in[0,T]}(L^r_x)} \leq C(\|u(0,x)\|_{L^2} + \|hD_t u(0,x)\|_{L^2}) \quad (5.1)$$

The following theorem is due to O. Ivanovici

Theorem

In any domain of $\mathbb{R}^3$ with at least one strictly convex geodesic on the boundary, for waves with Dirichlet boundary conditions, **and any** $r > 4$ *and* $\epsilon > 0$, *the following inequality fails to be true*

$$h^{2(\frac{1}{2}-\frac{1}{r})+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-\epsilon}\|\chi(hD_t)u\|_{L^q_{t\in[0,T]}(L^r_x)} \leq C(\|u(0,x)\|_{L^2} + \|hD_t u(0,x)\|_{L^2}) \quad (5.2)$$

Optimality

The above results motivate the following question in 3-d:

For waves inside a domain with strictly convex boundary, does (free space) Strichartz holds true for the pair

$$(q, r) = (4, 4)$$

???

Outline

Comments

How to deal with general boundaries??

- 1. Propagation of singularities: The **Melrose-Sjöstrand theorem**
- 2. Strichartz estimates.
- 3. Dispersive estimates.
- 4. Parametrix construction.

The Melrose-Sjöstrand theorem involves a micro-hyperbolic argument.

How to get Strichartz (or bilinear) estimates without any weak form for a parametrix ?