Isospectral deformations, Mather's β-function and spectral rigidity

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Spectral rigidity

$\widetilde{X}$ smooth manifold of dimension $n \geq 2$
Billiard table in $\widetilde{X}$ - smooth compact Riemannian manifold (X, g), $X \subset \widetilde{X}$, with boundary Γ, dim $X = n$.
C^1 deformation of (X, g) - C^1 family of billiard tables (X_t, g_t) in $\widetilde{X}$, where $X_0 = X$ and $g_0 = g$
Δ_t - the corresponding L-B operator with Dirichlet boundary conditions on Γ_t. The deformation is

- isospectral if $\mathrm{Spec}\,(\Delta_t) = \mathrm{Spec}\,(\Delta_0)$ for each t
- trivial if there is a family of diffeomorphisms $\psi_t : X_0 \to X_t$ such that $\psi_0 = \mathrm{Id}$ and $\psi_t^* g_t = g_0$.

X_0 is spectrally rigid if every isospectral deformation of X_0 is trivial.
Question : Is the ellipse (ellipsoid) spectrally rigid in $\mathbb{R}^2$ $(\mathbb{R}^n)$?

How to relate Spec (Δ) to the the geometry ?

1) Heat invariants :

$$\sigma_h(t) = \sum_j e^{-\lambda_j t} = \operatorname{tr}\left(e^{-t\Delta}\right),\ t > 0,$$

$$\sigma_h(t) \sim c_0 t^{-n/2} + c_1 t^{-(n-1)/2} + \cdots \text{ as } t \searrow 0$$

2) Wave-trace method :

$$\sigma_w(t) = \sum_j \cos(t\sqrt{\lambda_j}) = \operatorname{tr}\left(\cos(t\sqrt{-\Delta})\right),\ t \in \mathbb{R},$$

$$\text{s.s.}\,(\sigma_w) \subset \{\pm\ell :\ \ell \in \mathcal{L}(X)\} \cup \{0\}$$

- Equality for generic domains - Petkov-Stojanov,
- Singular expansions - Chazarain, Andersen-Melrose, Duistermaat-Guillemin, Guillemin-Melrose, Marvizi-Melrose,
- Bikhoff Normal Form - Guillemin, Zelditch, Iantchenko-Sjöstrand-Zworski, Colin de Verdière
- Recovery of the boundary - Zeditch, Hezari-Zelditch

- Closed Riemannian manifolds of negative sectional curvature are spectrally rigid : Guillemin-Kazhdan $n = 2$ (1980), Croke-Sharafutdinov $n \geq 2$ (1998), Anosov flows ($n = 2$) - Paternain-Salo-Uhlmann (2013).
- Infinitesimal rigidity of the ellipse - Hezari-Zelditch (2013).

The wave-trace method requires certain technical assumptions such as simplicity of the length spectrum (a non-coincidence condition) and non-degeneracy of the corresponding closed geodesic and its iterates.
We propose another method which avoids these assumptions.

3) Method based on a quasi-mode construction (Popov-Topalov, CPDE 2012)
Instead of trying to recover the BNF from the coefficients of the complete singular expansion at a given length $t = T$, we are looking for the first Birkhoff invariant for a large family of invariant tori which can be regarded as a Radon transform.

Weak isospectral condition

Fix $d \geq 0$ and $c > 0$. Consider $\mathcal{I} \subset (0, \infty)$ such that

(H$_1$) *$\mathcal{I}$ - union of infinitely many disjoint intervals $[a_k, b_k]$ where*

- $\lim a_k = \lim b_k = +\infty$,
- $b_k - a_k = o\left(\sqrt{a_k}\right)$ *as* $k \to \infty$
- $a_{k+1} - b_k \geq c b_k^{-d}$ *for any* $k \in \mathbb{N}$.

$[a_k, b_k]$ going to infinity, of length $o(\sqrt{a_k})$, and polynomially separated.

(H$_2$) $\exists\, a \geq 1$ *s.t.* $\forall\, t \in [0, 1]$, $\mathrm{Spec}\,(\Delta_t) \cap [a, +\infty) \subset \mathcal{I}$.

Isospectrality implies weak isospectralitry (take $d > n/2$).

K.A.M. tori and Mather's β-function

Billiard ball map - Symplectic map $B : B^*\Gamma \to B^*\Gamma$ associated to the billiard table (X, g). Fix $m \geq 1$ and set $P = B^m$.

Kronecker torus of P with a vector of rotation ω

Embedded Lagrangian submanifold $\Lambda(\omega)$ of $B^*\Gamma$ diffeomorphic to $\mathbb{T}^{n-1}$ such that $\Lambda(\omega)$ is invariant with respect to $P = B^m$ and the restriction of P to $\Lambda(\omega)$ is C^∞ conjugated to the translation

$$R_{2\pi\omega}(\varphi) = \varphi + 2\pi\omega \pmod{2\pi}$$

Embedding $f_\omega : \mathbb{T}^{n-1} \to \Lambda(\omega) \subset B^*\Gamma$ such that

$$\begin{array}{ccc} \mathbb{T}^{n-1} & \xrightarrow{R_{2\pi\omega}} & \mathbb{T}^{n-1} \\ \downarrow f_\omega & & \downarrow f_\omega \\ \Lambda(\omega) & \xrightarrow{P} & \Lambda(\omega) \end{array} \tag{1}$$

Fix $\kappa \in (0, 1]$ and $\tau > n - 1$.

Diophantine vectors of rotation

$\omega \in \mathbb{R}^{n-1}$ is $D(\kappa, \tau)$ if
$\forall\,(k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z},\ k \neq 0 : |\langle \omega, k\rangle + k_n| \ \geq\ \kappa |k|^{-\tau}$;
Ω_κ the set of (κ, τ)-Diophantine vectors in a domain Ω

K.A.M. theorem provides a lot of Kronecker invariant tori $\Lambda(\omega)$ with vectors of rotation $\omega \in \Omega_\kappa$ for small perturbations P of a nondegenerate smooth completely integrable symplectic map. If $t \to P_t$ is C^1 then the corresponding families of invariant tori $t \to \Lambda_t(\omega)$ are C^1 (not at all trivial ! [Popov-Topalov, 2013])

Given $\varrho = (x, \xi) \in B^*\Gamma$ denote the action on the geodesic arc $\gamma(\varrho)$ starting from ϱ and with endpoint $\varrho' = P(\varrho)$ by

$$A(\varrho) := \int_{\gamma(\varrho)} \xi dx.$$

Average action on $\Lambda(\omega)$ when ω is Diophantine

$$\beta(\omega) := -2 \lim_{N\to+\infty} \frac{1}{2N} \sum_{k=-N}^{N-1} A(P^k \varrho) = -2 \int_{\Lambda(\omega)} A d\mu$$

μ is the unique probability measure on $\Lambda(\omega)$ invariant with respect to P. If P is a twist map ($\dim \Gamma = 2$, $m = 1$, Γ-strictly convex), then $\beta(\omega)$ is the value of Mather's β-function at $\omega \in \Omega_\kappa$. The function $t \to \beta_t(\omega)$ is C^1 if $t \to \Lambda_t(\omega)$ is a C^1 family of Kronecker tori of P_t with a Diophantine vector of rotation ω.

Examples :

- Liouville billiard tables : [Popov-Topalov, ETDS 2003, 2008, CMP 2011]

- Elliptic broken geodesics

- Strictly convex billiard tables in $\mathbb{R}^2$ - Lazutkin caustics : $\beta(\omega) = \omega l(\omega) - L(l(\omega))$ where $L(l(\omega))$ is the Lazutkin parameter of the corresponding caustic $C(\omega)$ and $l(\omega)$ is its length.

Isospectral invariants

Theorem (Popov-Topalov, 2013)

Let (X_t, g_t), $t \in [0, 1]$, be a C^1 family of compact billiard tables satisfying the weak isospectral condition $(H_1) - (H_2)$. Let $[0, \delta) \ni t \to \Lambda_t(\omega)$ be a C^1 family of invariant tori of $P_t = B_t^m$ with vectors of rotation $\omega \in \Omega_\kappa$. Then $\beta_t(\omega)$ is independent of $t \in [0, \delta]$ for any $\omega \in \Omega_\kappa$.

Remark. Although the invariant tori $\Lambda_t(\omega)$ may not exist at $t = \delta$ the β-function is well defined and it is continuous in $t = \delta$ for twist maps.
For any $t \in [0, \delta)$ the function β_t is C^∞ on Ω_κ in Whitney sense.

Applications
1. Infinitesimal rigidity of ellipsoidal billiard tables

Theorem (Popov-Topalov, CMP, 2011)

The billiard ball map of an ellipsoidal billiard table in $\mathbb{R}^n$ is a non-degenerate (in Kolmogorov sense) completely integrable symplectic map.

Let X_0 be the ellipsoidal billiard table in $\mathbb{R}^n$. Applying (a variant) of the KAM theorem to B_t near an invariant torus $\Lambda_0(\omega)$, $\omega \in D(\kappa, \tau)$, one obtains a C^1 family of invariant tori $\Lambda_t(\omega)$ of B_t. Suppose that the family is weakly isospectral. Then the main Theorem implies $\beta_t(\omega) = \beta_0(\omega)$ for any $\omega \in \Omega$ and $0 \leq t < \delta$.

Proposition (Popov-Topalov, 2013)

Let $\Gamma_t = \{x + a_t(x)\nu(x) :\ x \in \Gamma_0\}$, *where* $\nu : \Gamma_0 \to S^{n-1}$ *is a normal vector field. Then*

$$\frac{d}{dt}\beta_t(\omega)|_{t=0} = 0 \quad \Leftrightarrow \quad \int_{\Lambda_0(\omega)} \frac{d}{dt} a_t|_{t=0} d\mu = 0$$

Here $d\mu$ *is the unique probability measure on* $\Lambda_0(\omega)$ *which is invariant with respect to* P_0.

Hence, the Radon transform of $\frac{d}{dt} a_t|_{t=0}$ is identically zero

$$\forall\, \omega \in \Omega,\ \int_{\Lambda_0(\omega)} \frac{d}{dt} a_t|_{t=0}\, d\mu = 0\,.$$

Radon transform

Theorem (Popov-Topalov, CMP, 2011)

Let $f \in C(\Gamma_0)$ be invariant under the symmetries of the ellipsoid Γ_0. Then

$$\forall\, \omega \in \Omega,\ \mathcal{R}_f(\Lambda_0(\omega)) := \int_{\Lambda_0(\omega)} f\, d\mu = 0 \quad \Longrightarrow \quad f \equiv 0\,.$$

Ellipse Guillemin-Melrose (1978), Liouville billiard tables $n = 2$
Popov-Topalov (2003), Liouville billiard tables $n = 3$
Popov-Topalov (2011). This implies infinitesimal rigidity : the first variation of a_t at $t = 0$ is 0 if a_t is invariant under the symmetries of the ellipsoid. If $t \to a_t$ is C^∞ then a_t is flat at $t = 0$. Holds for Liouville billiard tables.
For the ellipse ($n = 2$) Hezari-Zelditch (Analysis PDE, 2013).

2. Elliptic geodesics

Let γ be a closed elliptic. Denote by $e^{\pm 2\pi\alpha_k i}$, $1 \le k \le n-1$, the eigenvalues of the linear Poincaré map DP, where $\alpha_k \in (0, 1/2)$ and set $\alpha = (\alpha_1, \dots, \alpha_{n-1})$. Suppose that γ is 4-elementary (no resonances of order ≤ 4) which means that $\langle \alpha, k \rangle \neq 0$ for any $k \in \mathbb{Z}^{n-1}$ with $|k| := |k_1| + \cdots + |k_{n-1}| \le 4$. Then the corresponding Poincaré map P admits BNF

$$P(\theta, r) = (\theta + 2\pi\alpha + Br, r) + O(|r|^{3/2}).$$

Let the BNF be non-degenerate, i.e. $\det B \neq 0$. Then one can apply the K.A.M. theorem for C^1-deformations of P.

Theorem (Popov-Topalov, 2013)

Let (X_t, g_t), $t \in [0,1]$, be a C^1 family of compact Riemannian manifolds (convex billiard tables) satisfying $(H_1)-(H_2)$. Suppose that (X_0, g_0) admits a (broken) closed elliptic 4-elementary geodesic γ_0 with a non-degenerate BNF. Then

1. *there exists a C^1 family of elliptic orbits γ_t, $t \in [0,1]$,*
2. *there is a set $\Omega \subset \mathbb{R}^{n-1}$ of Diophantine vectors such that $\mathrm{meas}\,(\Omega \cap B(\alpha,\epsilon))/\mathrm{meas}\,B(\alpha,\epsilon) = 1 - O(\epsilon)$ and for any $\omega \in \Omega$ a C^1 family of Kronecker invariant tori $\Lambda_t(\omega)$, $t \in [0,1]$, of the corresponding local Poincaré maps P_t*
3. $\forall \omega \in \Omega$ *and* $t \in [0,1]$, $\beta_t(\omega) = \beta_0(\omega)$.

Deformations of Rienannian metrics with the same length spectrum [Popov, Math. Z. 93].
Applications : spectral rigidity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ billiard tables.

Define a class of billiard tables as follows.
Let $(\widetilde{X}, g)$, $\dim \widetilde{X} = 2$ be a real analytic Riemannian manifold of dimension two. Suppose that it admits two commuting involutions $\mathcal{J}_k$, $k = 1, 2$, acting as isometries. Consider the family $\mathcal{B}$ of analytic billiard tables (X, g) in $(\widetilde{X}, g)$ which are invariant with respect to $\mathcal{J}_k$, $k = 1, 2$. Then the set of fixed points of $\mathcal{J}_k$, defines a bouncing ball geodesic γ_k for $k = 1, 2$,

Corollary

Let $(X, g) \in \mathcal{B}$. Suppose that the broken geodesic γ_1 is elliptic, 4-elementary and that the corresponding BNF is non-degenerate. Then (X, g) is spectrally rigid in $\mathcal{B}$.

Exemple : Classical Liouville billiard tables.
It follows from the main theorem and a variant of the above proposition using a simple argument of Popov-Topalov CPDE (2012)

3. Strictly convex billiard tables

Theorem (Popov-Topalov, 2013)

Let $X_t \subset \mathbb{R}^2$, $t \in [0,1]$, be a C^1 family of compact billiard tables in $\mathbb{R}^2$ satisfying the weak isospectral condition $(H_1) - (H_2)$. Suppose that X_0 is strictly convex. Then

1. *X_t is strictly convex for each $t \in [0,1]$*
2. *There is a Cantor set $\Omega \subset (0,1]$ consisting of Diophantine numbers such that meas $(\Omega \cap (0,\varepsilon))/\varepsilon = 1 - O_N(\varepsilon^N)$ as $\varepsilon \to 0^+$ and such that $\forall\, \omega \in \Omega$ there is a C^1 family of Kronecker invariant circles $[0,1] \ni t \to \Lambda_t(\omega)$ of B_t of rotation number ω,*
3. *$\forall \omega \in \Omega$ and $t \in [0,1]$, $l_t(\omega) = l_0(\omega)$, $L_t(l_t(\omega)) = L_0(l_0(\omega))$.*

Recall that $l_t(\omega)$ is the length and $L_t(l_t(\omega))$ is the Lazutkin parameter of the caustic $C_t(\omega)$. Billiard tables with the same length spectrum [Popov, CMP 94]

Idea of the proof

Main ingredients :

- KAM theorem with parameters ($t \to P_t$ a C^1 family)
- Construction of C^1 with respect to t quasi-modes (P-T, CPDE 2012)

Quasi-modes of order $N \geq 0$ associated with $\Lambda_t(\omega)$:
Fix $t \in [0, \delta)$. There is a unbounded index set $\mathcal{M}_t(\omega) \subset \mathbb{Z}^n$ and for any $q \in \mathcal{M}_t(\omega)$ and $s \in [t, t + 2/|q|]$ a quasi-mode $(u_{s,q}, \mu_q(s)^2)$ such that

- $u_{s,q} \in D(\Delta_s) \cap C^\infty(\widetilde{X})$ and $\|u_{s,q}\|_{L^2(X_t)} = 1$,
- $\mu_q \in C^1([t, t + 2/|q|])$ and $\mu_q(s) \geq C|q|$,
- there is $C_N > 0$ such that

$$\left\| \Delta_s u_{s,q} - \mu_q^2(s)\, u_{s,q} \right\|_{L^2(X_s)} \leq C_N\, \mu_q(s)^{-N}$$

$\mu_q(s)$ determines $\beta_s(\omega)$ up to $o(|q|^{-1})$ as $q \to \infty$ for rotation vectors ω in a dense subset $\Omega^t_\kappa \subset \Omega_\kappa$:

$$\mu_q(s) = \mu_q(t) + o_\omega(1) \implies \beta_s(\omega) = \beta_t(\omega) + o_\omega(|q|^{-1})$$

uniformly in $s \in [t, t + 2/|q|]$.

Lemma

Suppose that $(H_1) - (H_2)$ *holds. Then*

$$\mu_q(s) = \mu_q(t) + o_\omega(1) \quad \text{as } q \to \infty$$

uniformly in $s \in [t, t + 2/|q|]$.

We have $\left|\mathrm{Spec}(\Delta_s) - \mu_q(s)^2\right| \le C_N\, \mu_q(s)^{-N}$. Take $N > 2d$. Then $(\mathrm{H}_1) - (\mathrm{H}_2)$ implies $|\mu_q(s)^2 - \mu_q(t)^2| = o(\mu_q(s))$ as $q \to \infty$. Hence, $\frac{d}{dt}\beta_t(\omega) = 0$ on Ω^t_κ and by continuity on Ω_κ.