Spectral estimates and Random Weighted Sobolev Inequalities

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Content

The simplest model is the 1-D torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with its Sobolev spaces. Let $f(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx}$ then $\|f\|^2_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2s}|c_n|^2$.

By the usual Sobolev embeddings, if $f \in H^{1/2-1/p}(\mathbb{T})$ with $p \geq 2$ then $f \in L^p(\mathbb{T})$.

Paley and Zygmund (1930) have improved this result allowing random coefficients.

Let $f^\omega(x) = \sum_{n\in\mathbb{Z}} X_n(\omega) c_n \mathrm{e}^{inx}$ where $\{X_n\}$ is a sequence of independent Bernoulli random variables.

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Let $f^\omega(x) = \sum_{n\in\mathbb{Z}} X_n(\omega) c_n \mathrm{e}^{inx}$ where $\{X_n\}$ is a sequence of independent Bernoulli random variables. If $f \in L^2(\mathbb{T})$ then for all $p \geq 2$, a.s $f^\omega \in L^p(\mathbb{T})$.

Moreover if for some $\alpha > 1$, $\sum_{n\in\mathbb{Z}} \log^{\alpha}(1+|n|)|c_n|^2 < +\infty$ then a.s $f^{\omega} \in \mathcal{C}(\mathbb{T})$.

Many other results concerning random trigonometric series were obtained by Paley and Zygmund, as it is detailed in the beautiful book of J-P. Kahane (Some random series of functions).

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The setting of random trigonometric series was extended to Riemannian compact manifolds for orthonormal basis of eigenfunctions of the Laplace-Beltrami operator, in particular by Burq, Lebeau, Tvzetkov.

The main motivations and applications are the following :

(1) To get existence and well posedness results for non linear PDE (wave equation or Schrödinger equation) in supercritical cases.

(2) For linear self-adjoint PDE with high multiplicity eigenvalues (Laplace on the 2-sphere; harmonic oscillator with $D \geq 2$) find basis of eigenfunctions satisfying "better" L^∞ estimates or satisfying a quantum ergodic property (Zelditch considered the 2-sphere (1992), recently Burq-Lebeau have improved his result)

In our joint work with A. Poiret and L. Thomann we extend the Burq-Lebeau analysis to Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$. Moreover we consider more general probability measures on the spectral subspaces $\mathcal{E}_h$ satisfying a Gaussian concentration property.

In our joint work with A. Poiret and L. Thomann we extend the Burq-Lebeau analysis to Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$. Moreover we consider more general probability measures on the spectral subspaces $\mathcal{E}_h$ satisfying a Gaussian concentration property. Here we shall only consider the applications (2). For the details, see the link to our preprint: Random weighted Sobolev inequalities and application to Hermite functions and the soon forthcoming paper : Random weighted Sobolev inequalities for Schrödinger operator with superquadratic potentials.
Concerning applications of our results to NLS in supercritical cases with (or without) harmonic potential, see the link to our preprint: Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator.

Let $\{X_n\}$ a sequence of complex i.i.d random variables, of common law ν satisfying the following concentration property:
there exist constants $c, C > 0$ **independent of** $N \in \mathbb{N}$ such that for all Lipschitz and convex function $F : \mathbb{C}^N \longrightarrow \mathbb{R}$

$$\nu^{\otimes N}\Big[X \in \mathbb{C}^N \ : \ \big|F(X) - \mathbb{E}(F(X))\big| \geq r\Big] \leq c\, e^{-\frac{Cr^2}{\|F\|_{Lip}^2}}, \quad \forall r > 0, \tag{1}$$

where $\|F\|_{Lip}$ is the best constant so that $|F(X) - F(Y)| \leq \|F\|_{Lip}\|X - Y\|_{\ell^2}$.
Examples: Gauss law, Bernoulli law and more generally measures with compact support (Talagrand theorem).

Let $\mathcal{K}$ a separable complex Hilbert space and K is a self-adjoint, positive operator on $\mathcal{K}$ with a compact resolvent. We denote by $\{\varphi_j,\ j \geq 1\}$ an orthonormal basis of eigenvectors of K, $K\varphi_j = \lambda_j \varphi_j$, and $\{\lambda_j,\ j \geq 1\}$ is the non decreasing sequence of eigenvalues of K (each is repeated according to its multiplicity). Then we get a natural scale of Sobolev spaces associated with K, defined for $s \geq 0$ by $\mathcal{K}^s = \mathrm{Dom}(K^{s/2})$.

Let $\gamma = \{\gamma_j\}_{j\geq 1}$ a sequence of complex numbers such that $\sum_{j\geq 1} \lambda_j^s |\gamma_j|^2 < +\infty$.

We denote by $v_\gamma = \sum_{j\geq 1} \gamma_j \varphi_j \in \mathcal{K}^s$ and $v_\gamma^\omega = \sum_{j\geq 1} \gamma_j X_j(\omega)\varphi_j$. We have $\mathbb{E}(\|v_\gamma^\omega\|_{\mathcal{K}}^2) < +\infty$, therefore $v_\gamma^\omega \in \mathcal{K}^s$, a.s. We define the measure μ_γ on $\mathcal{K}^s$ as the probability law of the random vector v_γ^ω. These measures were introduced by Burq-Tzvetkov (2008). They are much more flexible than Gibbs measures known before in some particular cases (Lebowitz, Bourgain).

Some properties

(I) If the support of ν is $\mathbb{C}$ and if $\gamma_j \neq 0$ for all $j \geq 1$ then the support of μ_γ is $\mathcal{K}^s$.

(II) If for some $\epsilon > 0$ we have $v_\gamma \notin \mathcal{K}^{s+\epsilon}$ then $\mu_\gamma(\mathcal{K}^{s+\epsilon}) = 0$.

(III) Assume that we are in the particular case where $d\nu(x) = c_\alpha \mathrm{e}^{-|x|^\alpha} dx$ with $\alpha \geq 2$. Let $\gamma = \{\gamma_j\}$ and $\beta = \{\beta_j\}$ be two complex sequences and assume that

$$\sum_{j \geq 1} \left(\left| \frac{\gamma_j}{\beta_j} \right|^{\mathbf{a}/2} - 1 \right)^2 = +\infty.$$

Then the measures μ_γ and μ_β are mutually singular, i.e there exists a measurable set $A \subset \mathcal{K}^s$ such that $\mu_\gamma(A) = 1$ and $\mu_\beta(A) = 0$.

Consider finite dimensional subspaces $\mathcal{E}_h$ of $\mathcal{K}$ defined by spectral localizations depending on a small parameter $0 < h \leq 1$ (h^{-1} is a measure of energy for the quantum Hamiltonian K).
Let $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$, $\Lambda_h = \{j,\ \lambda_j \in I_h\}$, $N_h = \#\Lambda_h$ and $\mathcal{E}_h$ the spectral subspace of K in the interval I_h. Our goal is to find uniform estimates in $h \in]0, h_0[$ for $h_0 > 0$ small enough.
Consider the random vector in $\mathcal{E}_h$:

$$v_\gamma(\omega) := v_{\gamma,h}(\omega) = \sum_{j\in\Lambda_h} \gamma_j X_j(\omega)\varphi_j. \tag{2}$$

Introduce the squeezing condition:

$$|\gamma_n|^2 \leq \frac{K_0}{N_h} \sum_{j\in\Lambda_h} |\gamma_j|^2, \quad \forall n \in \Lambda_h,\ \ \forall h \in]0, h_0]. \tag{3}$$

To get estimates from below we also need:

$$\frac{K_1}{N_h}\sum_{j\in\Lambda_h}|\gamma_j|^2 \le |\gamma_n|^2 \le \frac{K_0}{N_h}\sum_{j\in\Lambda_h}|\gamma_j|^2, \quad \forall n\in\Lambda_h, \ \ \forall h\in]0,1]. \quad (4)$$

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Why this assumption?
For 1-D Hamiltonians the eigenvalues are non degenerate and it is possible to get accurate L^∞ estimates on eigenfunctions. But for $D\geq 2$ eigenvalues may have high multiplicities and it is much more difficult to get accurate L^∞ estimates. With condition (3) or (4) we shall see that it is enough to know good estimates for the spectral functions in small energy windows instead of individual eigenfunctions.

Now consider probabilities on the unit sphere $\mathbf{S}_h$ of the subspaces $\mathcal{E}_h$. The random vector v_γ defines a probability measure $\nu_{\gamma,h}$ on $\mathcal{E}_h$. We define a probability measure $\mathbf{P}_{\gamma,h}$ on $\mathbf{S}_h$ as the image of $\nu_{\gamma,h}$ by $v \mapsto \frac{v}{\|v\|}$.

Examples:

• If $|\gamma_n| = \frac{1}{\sqrt{N}}$ for all $j \in \Lambda$ and if X_n follows the complex normal law $\mathcal{N}_{\mathbb{C}}(0,1)$ then $\mathbf{P}_{\gamma,h}$ is the uniform probability on $\mathbf{S}_h$ considered in Burq-Lebeau.

• Assume that for all $n \in \mathbb{N}$, $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$, then $\mathbf{P}_{\gamma,h}$ is a convex sum of 2^N Dirac measures.

In the first example $\mathbf{P}_{\gamma,h}$ is invariant by e^{-itK}.

To get an optimal lower bound for L^∞ estimates we shall need a stronger normal concentration estimate (Assum2).

(I) The random variables X_j are standard independent Gaussians $\mathcal{N}_{\mathbb{C}}(0, 1)$.

(II) The sequence γ satisfies (4).

Note that conditions (3) and (4) are stable by small perturbations. So assuming that ν is Gaussian we can get an infinite number of pairs of mutually singular probability measures μ_γ.

Let L be a linear form on $\mathcal{E}_h$, and denote by $e_L = \sum_{j \in \Lambda_h} |L(\varphi_j)|^2$.

Then we have the large deviation estimate:

Theorem

Let L be a linear form on $\mathcal{E}_h$. Suppose that (3) holds and that (Assum1) is satisfied. Then there exist $C_2, c_2 > 0$ so that

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : \ |L(u)| \geq t\Big] \leq C_2 \mathrm{e}^{-c_2 \frac{N}{e_L} t^2}, \qquad \forall\, t \geq 0, \ \ \forall\, h \in]0, h_0].$$

If (Assum2) is satisfied, there exist $C_1, C_2, c_1, c_2, \epsilon_0, h_0 > 0$ so that

$$C_1 \, \mathrm{e}^{-c_1 \frac{N}{e_L} t^2} \leq \mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : \ |L(u)| \geq t\Big] \leq C_2 \, \mathrm{e}^{-c_2 \frac{N}{e_L} t^2},$$

$\forall\, t \in [0, \epsilon_0 \sqrt{e_L}\,], \ \ \forall\, h \in]0, h_0].$

In applications considered here there is a Sobolev embedding $\mathcal{K}^s \to C(M)$, for some $s > 0$, where M is a metric space. We have $\mathcal{E} \subseteq \bigcap_{s\in\mathbb{R}} \mathcal{K}^s$, thus we can consider the Dirac evaluation linear form $\delta_x(v) = v(x)$. In this case we have $e_L = \sum_{j\in\Lambda_h} |\varphi_j(x)|^2 = e_x$, which is usually called the spectral function of K in the interval I. Notice that from Cauchy-Schwarz:

$$|L(v)| \leq e_L^{1/2}\|v\|, \ \forall v \in \mathcal{E}_h.$$

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A theorem of P. Levy gives a concentration inequality for the canonical measure on spheres of large dimension. This is generalized as follows. It is a an important basic tool in the Burq-Lebeau approach (see also Zelditch).

The concentration condition on ν gives the following

Proposition

Suppose that (Assum1) is satisfied. Then there exist constants $K > 0$, $\kappa > 0$ (depending only on $C^\star$) such that for every Lipschitz function $F : \mathbf{S}_h \longrightarrow \mathbb{R}$ satisfying

$$|F(u) - F(v)| \leq \|F\|_{Lip} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h,$$

we have

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : |F - \mathcal{M}_F| > r \Big] \leq K e^{-\frac{\kappa N_h r^2}{\|F\|^2_{Lip}}}, \quad \forall r > 0,\ h \in]0,1],$$

where $\mathcal{M}_F$ is a median for F.

Recall that a median $\mathcal{M}_F$ for F is defined by

$$\mathbf{P}_{\gamma,h}\big[u \in \mathbf{S}_h : F \geq \mathcal{M}_F \big] \geq \frac{1}{2}, \quad \mathbf{P}_{\gamma,h}\big[u \in \mathbf{S}_h : F \leq \mathcal{M}_F \big] \geq \frac{1}{2}.$$

Consider the Schrödinger Hamiltonian $\hat{H} = -\triangle + V$ in $\mathbb{R}^d$ for $d \geq 2$.
Assume that V is an elliptic polynomial in $\mathbb{R}^d$, that means:
$V = V_0 + V_1$ where V_0 is an homogeneous elliptic polynomial of degree $2k$ ($V_0(x) > 0$ if $x \neq 0$) and $V_1(x)$ is a polynomial of degree $\leq 2k-1$. We can assume $V(x) > 0$ on $\mathbb{R}^d$.
It is more convenient to work with the normalized Hamiltonian $\hat{H}_{nor} = \hat{H}^{\frac{k+1}{2k}}$. Recall the Weyl law for H_{nor}:

$$N_{H_{nor}}(\lambda) = \mathcal{W}_{nor}(\lambda) + \mathcal{O}(\lambda^{d-1})$$

with $\mathcal{W}_{nor}(\lambda) \approx \lambda^d$ (classical Weyl term).

(Helffer-R, some years ago)

We also need accurate estimates for the spectral function of H (or H_{nor}). Denote $\pi_{H_{nor}}(\lambda; x, y) = \sum_{\omega_j \leq \lambda} \varphi_j(x)\overline{\varphi_j(y)}$.

Proposition

For every $\delta \in [0, 1]$ and $C_0 > 0$, there exists $C > 0$ such that for every $\theta \in [0, \frac{d}{k}]$ and $r \in]1, +\infty]$, there exists $C > 0$ such that

$$\|\pi_{H_{nor}}(\lambda + \mu; x, x) - \pi_{H_{nor}}(\lambda; x, x)\|_{L^{r,k(r-1)\theta}(\mathbb{R}^d)} \leq C\lambda^{\alpha}$$

for $|\mu| \leq C_0\lambda^{1-\delta}$, $\lambda \geq 1$, $\alpha = \frac{d}{k+1}(k + \frac{1}{r}) - \delta + \frac{k\theta}{k+1}(1 - \frac{1}{r})$.

Here $\|u\|^r_{L^{r,s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle x\rangle^s |u(x)|^r dx$.

Comment on spectral function estimates: Proofs are easier if $\delta < 2/3$. For the harmonic oscillator and $\delta = 1$ this is a consequence of results of Thangavelu (1993) Karadzhov (1995) or Koch-Tataru (2005). For $k > 1$ this can be proved using Koch-Tataru-Zworski (2007).

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The Sobolev spaces associated with $\hat{H}$ are here defined as follows. Let $s \geq 0$, $p \in [1, +\infty]$.

$$\mathcal{W}_k^{s,p} := \mathcal{W}_k^{s,p}(\mathbb{R}^d) := \left\{u \in L^p(\mathbb{R}^d),\ \hat{H}_{nor}^s u \in L^p(\mathbb{R}^d)\right\},$$
$$\|u\|_{s,p} = \|\hat{H}_{nor}^s u\|_{L^p(\mathbb{R}^d)}.$$

The Hilbert Sobolev spaces are denoted $\mathcal{H}_k^s = \mathcal{W}_k^{s,2}$.

To prepare a spectral analysis "à la Littlewood-Paley" of this Sobolev spaces, introduce $I_h = [\frac{a_h}{h}, \frac{b_h}{h}[$ such that a_h and b_h satisfy, for some $a, b, D > 0, \delta \in [0, 1]$,

$$\lim_{h \to 0} a_h = a, \quad \lim_{h \to 0} b_h = b, \quad 0 < a \leq b \quad \text{and} \quad b_h - a_h \geq D h^{\delta}.$$

From Weyl asymptotics we have $N_h \sim c h^{-d}(b_h - a_h)$ $(c > 0)$.
In particular ($N_h \overset{h \to 0}{\to} +\infty$ for $d \geq 2$).

Using estimates on the spectral function and interpolation we get Sobolev inequalities with weights:
for every $u \in \mathcal{E}_h$, $\theta \geq 0$, $p \geq 2$.

$$\|u\|_{L^{\infty,k\theta/2}(\mathbb{R}^d)} \leq C\left(N_h h^{\frac{d-k\theta}{k+1}}\right)^{1/2}\|u\|_{L^2(\mathbb{R}^d)}$$

$$\|u\|_{L^{p,k\theta(p/2-1)}(\mathbb{R}^d)} \leq C\left(N_h h^{\frac{d-k\theta}{k+1}}\right)^{\frac{1}{2}-\frac{1}{p}}\|u\|_{L^2(\mathbb{R}^d)}.$$

Notice that N_h is of order $(b_h - a_h)h^{-d} \approx h^{\delta-d}$, $\delta \in [0,1]$.

Theorem

Assume that $0 \leq \delta < 2/3$. For every $\kappa \in]0,1[$, $K > 0$, there exist $C_0 > 0$, $C_1 > 0$, $c_1 > 0$ such that for every $r \in [2, K|\log h|^\kappa]$ we have

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : C_0\sqrt{r}h^{\frac{d}{2(k+1)}(1-\frac{2}{r})}h^{-s} \leq \|u\|_{\mathcal{W}_k^{s,r}(\mathbb{R}^d)} \leq C_1\sqrt{r}h^{\frac{d}{2(k+1)}(1-\frac{2}{r})}h^{-s}\Big] \geq 1 - \mathrm{e}^{-c_1|\log h|^{1-\kappa}},$$

and for $r = +\infty$ we have for all $h \in]0, h_0]$

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : C_0|\log h|^{1/2}h^{\frac{d}{2(k+1)}-s} \leq \|u\|_{\mathcal{W}_k^{s,\infty}(\mathbb{R}^d)} \leq C_1|\log h|^{1/2}h^{\frac{d}{2(k+1)}-s}\Big] \geq 1 - h^{c_1}.$$

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This Theorem shows a gain of $\frac{k+1}{2k}d$ derivatives compared to the usual deterministic Sobolev embeddings.

Recall that the condition $\delta < 2/3$ is used here because to get a lower bound for weighted norms of the spectral function (on compact manifold this lower bound is given, when $\delta = 1$, by the local Weyl asymptotics proved by Hörmander).
Even for the Harmonic oscillator it seems that no good global lower bounds are known for the spectral function when $\delta \in [2/3, 1]$.

Example

• Assume $\gamma_j = \frac{1}{\sqrt{N_h}}$, $1 \leq j \leq N_h$ and ν is a Gaussian law. Then $\mathbf{P}_{\gamma,h}$ is the uniform probability and the Theorem says that for x in a subset Ω_h of $\mathbb{S}_{2N_h-1}$ such that $\lim_{h\to 0} \mathbf{P}_{\gamma,h}\Omega_h = 1$ we have

$$\| \sum_{1\leq j\leq N_h} x_j\varphi_j\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \approx |\log h|^{1/2}.$$

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$$\|\sum_{1\leq j\leq N_h} x_j\varphi_j\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \approx |\log h|^{1/2}.$$

• Assume now that ν is a Bernoulli law. Let $\gamma = \{\gamma_j\}$ as above and $|\gamma| = 1$. $\Omega = \{0,1\}^{N_h}$ is the probability space and we have

$$\frac{1}{2^{N_h}}\#\Big\{\epsilon \in \Omega; \|\sum_{1\leq j\leq N_h} (-1)^{\epsilon_j}\gamma_j\varphi_j\|_{\mathcal{W}^{d/2,\infty}} \approx |\log h|^{1/2}\Big\}$$

converges to 1 as $h \to 0$.

Applying to the Harmonic oscillator a method of Burq-Lebeau, and Zelditch, we get the following consequence for the Hermite functions.
Assume that $\gamma_j = N_h^{-1/2}$ and that $X_j \sim \mathcal{N}_{\mathbb{C}}(0,1)$, so that $\mathbf{P}_h := \mathbf{P}_{\gamma,h}$ is the uniform probability on $\mathbf{S}_h$. We set $h_k = 1/k$ with $k \in \mathbb{N}^*$, and

$$a_{h_k} = 2 + dh_k, \quad b_{h_k} = 2 + (2+d)h_k.$$

So we can take $\delta = 1$ and $D = 2$. In particular, each interval

$$I_{h_k} = \Big[\frac{a_{h_k}}{h_k}, \frac{b_{h_k}}{h_k} \Big[= [2k+d, 2k+d+2[$$

only contains the eigenvalue $\lambda_k = 2k+d$ with multiplicity $N_{h_k} \sim ck^{d-1}$, and $\mathcal{E}_{h_k}$ is the corresponding eigenspace of the harmonic oscillator H.

The space of the orthonormal basis of $\mathcal{E}_{h_k}$ can be identified with the unitary group $U(N_{h_k})$ and we endow $U(N_{h_k})$ with its Haar probability measure ρ_k. Then the space $\mathcal{B}$ of the Hilbertian bases of eigenfunctions of H in $L^2(\mathbb{R}^d)$ can be identified with

$$\mathcal{B} = \times_{k\in\mathbb{N}} U(N_{h_k}),$$

with the probability measure

$$\mathrm{d}\rho = \otimes_{k\in\mathbb{N}} \mathrm{d}\rho_k.$$

Denote by $B = (\varphi_{k,\ell})_{k\in\mathbb{N},\, \ell\in[\![1,N_{h_k}]\!]} \in \mathcal{B}$ a typical orthonormal basis of $L^2(\mathbb{R}^d)$ so that for all $k \in \mathbb{N}$, $(\varphi_{k,\ell})_{\ell\in[\![1,N_{h_k}]\!]} \in U(N_{h_k})$ is an orthonormal basis of $\mathcal{E}_{h_k}$.

Theorem

Let $d \geq 2$. Then, if $M > 0$ is large enough, there exist $c, C > 0$ so that for all $r > 0$

$$\rho\Big[B = (\varphi_{k,\ell})_{k\in\mathbb{N},\, \ell\in[\![1,N_{h_k}]\!]} \in \mathcal{B} : \exists k,\ell;\ \|\varphi_{k,\ell}\|_{\mathcal{W}^{d/2,\infty}(\mathbb{R}^d)} \geq M(\log k)^{1/2} + r \Big] \leq C\mathrm{e}^{-cr^2}.$$

Corollary

For $d \geq 2$ there exists orthonormal basis $\{\varphi_n\}$ of eigenfunctions of the Harmonic oscillator $-\triangle + |x|^2$ such that

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq M\lambda_n^{-d/4}(1+\log\lambda_n)^{1/2},\ \forall n \geq 0.$$

Notice that for general bases we have $\|\varphi_n\|_{L^\infty(\mathbb{R}^d)} \leq M\lambda_n^{d/4-1/2}$ (Koch-Tataru).

Global Estimates

Using a dyadic Littlewood-Paley decomposition, we get probabilistic estimates in Sobolev spaces $\mathcal{W}^{s,r}(\mathbb{R}^d)$. For $s \in \mathbb{R}$, $p, q \in [1, +\infty]$, define the harmonic Besov space by

$$\mathcal{B}^s_{p,q}(\mathbb{R}^d) = \left\{ u = \sum_{n \geq 0} u_n : \ \sum_{n \geq 0} 2^{nqs/2} \|u_n\|^q_{L^p(\mathbb{R}^d)} < +\infty \right\},$$

$u_n = \Pi_{2^{-n}} u$. $\mathcal{B}^s_{p,q}(\mathbb{R}^d)$ is a Banach space with the norm in $\ell^q(\mathbb{N})$ of $\{2^{ns/2}\|u_n\|_{L^p(\mathbb{R}^d)}\}_{n \geq 0}$.
We assume that γ satisfies (4) and

$$\sum_{n \geq 0} |\gamma|_{\Lambda_n} < +\infty, \Lambda_n := \Lambda_{2^{-n}}.$$

Let $v_\gamma(\omega) = \sum_{j=0}^{+\infty} \gamma_j X_j(\omega)\varphi_j$.
Almost surely $v_\gamma \in \mathcal{B}^0_{2,1}(\mathbb{R}^d)$ and its probability law defines a measure μ_γ in $\mathcal{B}^0_{2,1}(\mathbb{R}^d)$. Notice that we have

$$\mathcal{H}^s(\mathbb{R}^d) \subset \mathcal{B}^0_{2,1}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d), \quad \forall s > 0.$$

We have the following result

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We have the following result

Theorem

For every $(s,r) \in \mathbb{R}^2$ such that $r \geq 2$ and $s = d(\frac{1}{2} - \frac{1}{r})$ there exists $c_0 > 0$ such that for all $K > 0$ we have

$$\mu_\gamma\Big[u \in \mathcal{B}^0_{2,1}(\mathbb{R}^d) : \ \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K\|u\|_{\mathcal{B}^0_{2,1}(\mathbb{R}^d)}\Big] \leq \mathrm{e}^{-c_0K^2}.$$

In particular μ_γ-almost all functions in $\mathcal{B}^0_{2,1}(\mathbb{R}^d)$ are in $\mathcal{W}^{s,r}(\mathbb{R}^d)$.

If γ satisfies (4) and the (weaker) condition $\sum_{n\geq 0} |\gamma|^2_{\Lambda_n} < +\infty$, then μ_γ defines a probability measure on $L^2(\mathbb{R}^d)$ and we can prove the estimate

$$\mu_\gamma\Big[u \in L^2(\mathbb{R}^d) : \ \|u\|_{\mathcal{W}^{s,r}(\mathbb{R}^d)} \geq K\|u\|_{L^2(\mathbb{R}^d)} \Big] \leq \mathrm{e}^{-c_0 K^2},$$

whenever $s < d(\frac{1}{2} - \frac{1}{r})$.
From this result it is easy to deduce a probabilistic Strichartz estimate for the linear flow $\mathrm{e}^{-it\hat{H}}$ which is used for surcritical NLSH in [PRT].

About the proof of a.s-L^r-estimate

Let us give a sketch of proof of the following estimate ($k = 2$), following the strategy of Burq-Lebeau.
We have to prove:

$$\mathbf{P}_{\gamma,h}\left[u \in \mathbf{S}_h : C_0\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})} \leq \|u\|_{L^{r,\theta(r/2-1)}} \leq C_1\sqrt{r}h^{\frac{d-\theta}{4}(1-\frac{2}{r})}\right] \geq 1 - h^{c_1},$$

for $r \in [2, K|\log h|]$, and $h \in]0, h_0]$.

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for $r \in [2, K|\log h|]$, and $h \in]0, h_0]$.
Denote by $F_r(u) = \|u\|_{L^{r,\theta(r/2-1)}}$ and by $\mathcal{M}_r$ its median. We have the Lipschitz estimate

$$|F_r(u) - F_r(v)| \le C\left(N_h h^{\frac{d-\theta}{2}}\right)^{\frac{1}{2}-\frac{1}{r}} \|u - v\|_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in \mathbf{S}_h.$$

Denote by

$$\beta_{r,\theta} = \frac{d-\theta}{2}\left(1-\frac{2}{r}\right).$$

By Gaussian concentration we have:

$$\mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : |F_r - \mathcal{M}_r| > \Lambda\Big] \leq 2\exp\big(-c_2 N_h^{2/r} h^{-\beta_{r,\theta}} \Lambda^2\big).$$

The next step is to estimate $\mathcal{M}_r$. Denote by $\mathcal{A}_r^r = \mathbf{E}_h(F_r^r)$ the moment of order r and compute, with $s = \theta(r/2-1)$,

$$\begin{aligned} \mathcal{A}_r^r &= \mathbf{E}_h\left(\int_{\mathbb{R}^d} \langle x\rangle^s |u(x)|^r \,\mathrm{d}x\right) \\ &= r\int_{\mathbb{R}^d} \langle x\rangle^s \Big(\int_0^{+\infty} \tau^{r-1} \mathbf{P}_{\gamma,h}\Big[u \in \mathbf{S}_h : |u(x)| > \tau\Big]\,\mathrm{d}\tau\Big)\,\mathrm{d}x. \end{aligned}$$

We get

$$C_1 r \int_{\mathbb{R}^d} \langle x \rangle^s \Big(\int_0^{\epsilon_0 \sqrt{e_x}} \tau^{r-1} \mathrm{e}^{-c_1 \frac{N}{e_x} \tau^2} \, \mathrm{d}\tau \Big) \mathrm{d}x \leq \mathcal{A}_r^r \leq C_2 r \int_{\mathbb{R}^d} \langle x \rangle^s \Big(\int_0^{+\infty} \tau^{r-1} \mathrm{e}^{-c_2 \frac{N}{e_x} \tau^2} \, \mathrm{d}\tau \Big) \mathrm{d}x.$$

After computations, we get that there exists $\epsilon_1 > 0$ such that for N large and $r \leq \epsilon_1 \frac{N}{\log N}$ we have

$$\mathrm{e}^{-r/2} C^{-1} r \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} \mathrm{d}x \right) N^{-r/2} \Gamma(r/2) \leq \mathcal{A}_r^r \leq C_2 \, r N^{-r/2} \left(\int_{\mathbb{R}^d} \langle x \rangle^s e_x^{r/2} \mathrm{d}x \right) \Gamma(r/2)$$

and $\Gamma(r/2)$ can be estimated thanks to the Stirling formula.

Then, from estimates for the spectral function we get

$$C_0\sqrt{rh^{\beta_{r,\theta}}} \leq \mathcal{A}_r \leq C_1\sqrt{rh^{\beta_{r,\theta}}}, \quad \forall r \geq 2,\ h \in]0, h_0].$$

Now we have to compare $\mathcal{A}_r$ and the median $\mathcal{M}_r$. We have

$$\begin{aligned} |\mathcal{A}_r - \mathcal{M}_r|^r &= \big|\|F_r\|_{L^r(\mathbf{S}_h)} - \|\mathcal{M}_r\|_{L^r(\mathbf{S}_h)}\big|^r \\ &\leq \|F_r - \mathcal{M}_r\|^r_{L^r(\mathbf{S}_h)} \\ &= r\int_0^\infty s^{r-1}\mathbf{P}_{\gamma,h}\big[|F_r - \mathcal{M}_r| > s\big]\mathrm{d}s. \end{aligned}$$

Then using a large deviation estimate we get

$$|\mathcal{A}_r - \mathcal{M}_r| \leq CN^{-1/r}\sqrt{rh^{\beta_{r,\theta}}}, \quad \forall r \geq 2.$$

Choosing $r \leq \delta \log N$, $(\delta < 1)$ and N large, we obtain

$$C_0\sqrt{rh^{\beta_{r,\theta}}} \leq \mathcal{M}_r \leq C_1\sqrt{rh^{\beta_{r,\theta}}}, \quad \forall r \in [2, \delta \log N]$$

and the proof of the (a.s) L^r estimate is done.

Assume that $I_h =]a_h, b_h]$ is such that

$$\lim_{h\to 0} a_h = \lim_{h\to 0} b_h = 1 > 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{b_h - a_h}{h} = +\infty.$$

L_1 is the Liouville measure associated with the classical Hamiltonian $H_0(x,\xi) = \frac{|\xi|^2}{2} + V_0(x)$. Recall that

$$L_1(A) = C_1 \int_{[H_0(z)=1]} \frac{A(z)}{|\nabla H_0(z)|} d\Sigma_1(z)$$

where Σ_1 is the Euclidean measure on the hypersurface $\Sigma_1 := H_0^{-1}(1)$ and $C_1 > 0$ is a normalization constant such that L_1 is a probability measure on Σ_1.
We denote by $S(1,k)$ the class of symbols such that $A \in C^\infty(\mathbb{R}^{2d})$ and A is quasi-homogeneous of degree 0 outside a small neighborhood of $(0,0)$ in $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$.

So that $A(\lambda x, \lambda^k \xi) = A(x, \xi)$ for every $\lambda \geq 1$ and $|(x, \xi)| \geq \epsilon$.
For $A \in S(1, k)$ let us denote by $\hat{A}$ the Weyl quantization of A (here $h = 1$).
Notice that if $\|u\|_{L^2(\mathbb{R}^d)} = 1$ then $A \mapsto \langle u, \hat{A}u \rangle$ defines a semiclassical probability measure on Σ_1. We have

Theorem (quantum large deviation)

Consider a potential V which satisfies conditions (A1). Assume that we are in the isotropic case ($\gamma_j = \frac{1}{\sqrt{N_h}}$ for all $j \in \Lambda_h$), and that ν satisfies the concentration of measure property (1). Then there exist $c, C > 0$ so that for all $r \geq 1$ and $A \in S(1, k)$,

$$\mathbf{P}_h \Big[u \in \mathbf{S}_h : |\langle u, \hat{A}u \rangle - L_1(A)| > r \Big] \leq C e^{-cN_h r^2}.$$

This result can be related with quantum ergodicity which concerns the semi-classical behavior of $\langle \varphi_j, \hat{A}\varphi_j \rangle$ when the classical flow is ergodic on the energy hyper surface Σ_1: for "almost-all" eigenfunctions φ_j, we have $\langle \varphi_j, \hat{A}\varphi_j \rangle \overset{j\to+\infty}{\longrightarrow} L_1(A)$.
The meaning of Theorem 9 is that we have $\langle u, \hat{A}u \rangle \overset{h\to 0}{\longrightarrow} L_1(A)$ for almost all u such that all modes $(\varphi_j)_{j\in\Lambda_h}$ are "almost-equi-present" in u (condition on the γ_j).

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The meaning of Theorem 9 is that we have $\langle u, \hat{A}u \rangle \overset{h \to 0}{\longrightarrow} L_1(A)$ for almost all u such that all modes $(\varphi_j)_{j \in \Lambda_h}$ are "almost-equi-present" in u (condition on the γ_j).
Zelditch have proved that on the standard 2-sphere: a random orthonormal basis of eigenfunctions of the Laplace operator is ergodic. Burq-Lebeau obtained a similar result for the Laplacian on a compact manifold. A modification of their proof allows us to consider more general random variables satisfying the Gaussian concentration assumption instead of the uniform law.

Now we easily get two applications of the quantum deviation inequality, using the Borel-Cantelli Lemma.
Let $\{h_j\}_{j\geq 0}$, $\lim\limits_{j\to+\infty} h_j = 0$. $X = \prod_{j\in\mathbb{N}} \mathbf{S}_{h_j}$ is equipped with the product probability $\mathbf{P} = \otimes_{j\in\mathbb{N}}\mathbf{P}_{h_j}$.
Let $u \in X$, $u = \{u_j\}_{j\in\mathbb{N}}$ where $u_j \in \mathbf{S}_{h_j}$. For any $A \in S(1,k)$, $u \mapsto \langle u_j, \hat{A}u_j\rangle$ defines a sequence of random variables on X.

Corollary

Assume that $d \geq 2$ *and that* $\sum\limits_{j\geq 0} \mathrm{e}^{-\epsilon h_j^{1-d}} < +\infty$ *for every* $\epsilon > 0$.

Then

$$\mathbf{P}\Big[u \in X,\ \lim_{j\to+\infty}\langle u_j, \hat{A}u_j\rangle = L_1(A),\ \forall A \in S(1,k)\Big] = 1.$$

Proof:

Denote by $f_h(u) = \langle \Pi_h u, \hat{A}\Pi_h u\rangle$. Notice that the random variable f_{h_j} depends only on $u_j = \Pi_{h_j} u$ so we get

$$\mathbf{P}\Big[u = \{u_j\} \ : \ |f_{h_j}(u) - L_1(A)| \geq \varepsilon\Big] = \mathbf{P}_{h_j}\Big[|f_{h_j} - L_1(A)| \geq \epsilon\Big].$$

So applying the large deviation estimate and the Borel-Cantelli Lemma to the independent random variables $\{f_{h_j}\}_{j\in\mathbb{N}}$ we get the conclusion. $\square$

As another application of the large deviation estimate is that a random orthonormal basis of eigenfunctions of the Harmonic oscillator $\hat{H}$ is Quantum Uniquely Ergodic (Q.U.E, according the terminology of S. Zelditch).

Corollary

Let $\hat{H}$ be the harmonic oscillator. For $B \in \mathcal{B}$ and $A \in S(1,1)$ denote by

$$D_j(B) = \max_{1 \leq \ell \leq N_{h_j}} \left| \langle \varphi_{j,\ell}, \hat{A}\varphi_{j,\ell} \rangle - L(A) \right|.$$

Then we have

$$\lim_{j \to +\infty} D_j(B) = 0, \;\; \rho - \text{a.s on } \mathcal{B}.$$

Proof:

Every $B \in \mathcal{B}$ can be identified with $\left\{B_j\right\}_{j\geq 1}$ where $B_j \in U(N_{h_j})$. The random variables D_j are independent and D_j depends only on B_j. So for every $r > 0$ we have

$$
\begin{aligned}
\rho\left[D_j(B) > r\right] &= \rho_j\left[\exists j \in [\![1, N_{h_j}]\!],\ \left|\langle \varphi_{j,\ell}, \hat{A}\varphi_{j,\ell}\rangle - L(A)\right| > r\right] \\
&\leq \sum_{1\leq j\leq N_{h_j}} \rho_k\left[\left|\langle \varphi_{j,\ell}, \hat{A}\varphi_{j,\ell}\rangle - L(A)\right| > r\right] \\
&= N_{h_j}\mathbf{P}_{h_j}\left[\left|\langle u\hat{A}(h_j)u\rangle - L(A)\right| > r - Ch_j^M\right].
\end{aligned}
$$

Using the quantum large deviation estimate,

$$\rho\Big[D_j(B) > r\Big] \leq C_1 j^{d-1} \exp\Big[- C_2 j^{d-1}(r - C_3 j^{-M})^2 \Big].$$

In particular for any $d \geq 2$ we get

$$\sum_{j \geq 1} \rho\Big[D_j(B) > r\Big] < +\infty$$

and the result is again a consequence of the Borel-Cantelli Lemma.
$\square$

In conclusion, almost all orthonormal basis of the Harmonic oscillator, for $d \geq 2$, is Quantum Uniquely Ergodic but the natural one (tensor products of the 1-D Hermite functions) is not Q.U.E !