Microlocal Analysis and Spectral Theory

in honor of J. Sjöstrand

Luminy, Sept 26, 2013

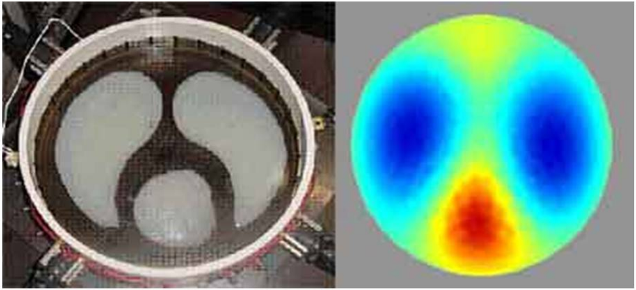
Two Partial Data Problems

Gunther Uhlmann

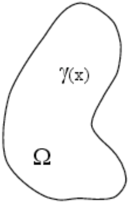
University of Washington and University of Helsinki

Outline:

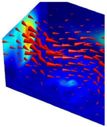
- Calderón's problem with partial data
- Travel time tomography with partial data

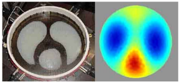


CALDERÓN'S PROBLEM



$\Omega \subset \mathbb{R}^n$
$(n = 2, 3)$

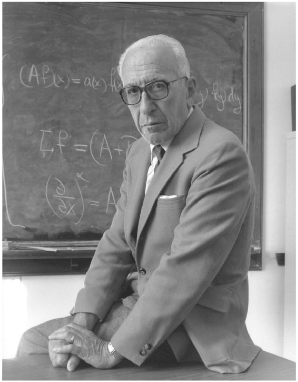




Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho

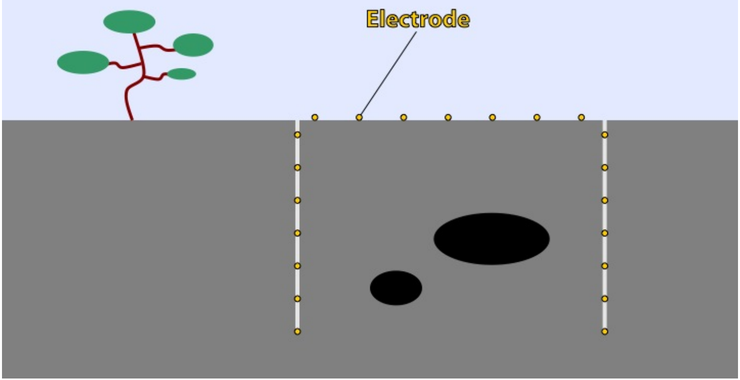


REMINISCENCIA DE MI VIDA MATEMATICA

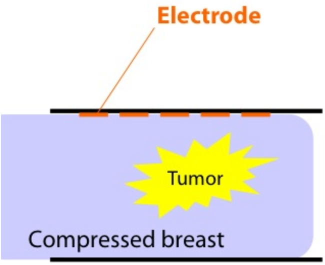
Speech at Universidad Autónoma de Madrid accepting the ‘Doctor Honoris Causa’:

> *My work at “Yacimientos Petroliferos Fiscales” (YPF) was very interesting, but I was not well treated, otherwise I would have stayed there.*

Geological underground probing is the application of EIT considered by Calderón



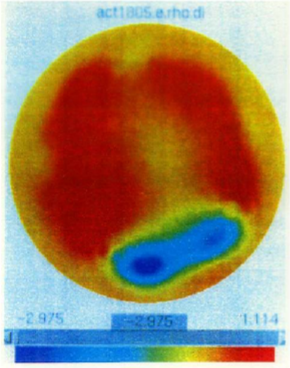
Early detection of breast cancer is effective using combined X-ray mammography and EIT



Cancerous tissue is up to four times more conductive than healthy tissue. [**Jossinet** -98]

X-ray attenuation is almost the same in cancerous and healthy tissue.

David Isaacson and his team have achieved good results in early detection of breast cancer using EIT.



ACT3 imaging blood as it leaves the heart (blue) and fills the lungs (red) during systole.

(Loading DBarPerfMovie1.avi)

Thanks to D. Issacson

CALDERÓN'S PROBLEM (EIT)

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$I(x) = \gamma(x)\nabla u$$

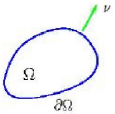
The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

$$\mathrm{div}(\gamma(x)\nabla u) = 0 \text{ in } \Omega$$

$$\begin{aligned} \text{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}$$

$\gamma(x)$ = conductivity,
f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega = (\nu \cdot \gamma\nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\Lambda_\gamma(f) = \nu \cdot \gamma\nabla u\big|_{\partial\Omega}$$

EIT (Calderón's inverse problem)

Does Λ_γ determine γ?

Λ_γ = Dirichlet-to-Neumann map

Theorem $n \geq 3$ (Sylvester-U, 1987)

$$\gamma \in C^2(\overline{\Omega}), \quad 0 < C_1 \leq \gamma(x) \leq C_2 \quad \text{on } \overline{\Omega}$$
$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$$

- Extended to $\gamma \in C^{3/2}(\overline{\Omega})$ (Päivarinta-Panchenko-U, Brown-Torres, 2003)
- $\gamma \in C^{1+\epsilon}(\overline{\Omega}), \gamma$ conormal (Greenleaf-Lassas-U, 2003)
- $\gamma \in C^1(\overline{\Omega})$, (Haberman-Tataru, 2013).

Complex-Geometrical Optics Solutions (CGO)

- Reconstruction (A. Nachman, R. Novikov 1988)
- Stability (G. Alessandrini 1988)
- Numerical Methods (D. Issacson, J. Müller, S. Siltanen)

Reduction to Schrödinger equation

$$\operatorname{div}(\gamma \nabla w) = 0$$

$$u = \sqrt{\gamma} w$$

Then the equation is transformed into:

$$(\Delta - q)u = 0, q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}} \qquad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)$$

$$\begin{aligned} (\Delta - q)u &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}$$

Define $\Lambda_q(f) = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$

ν = unit-outer normal to $\partial\Omega$.

IDENTITY

$$\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} \left((\Lambda_{q_1} - \Lambda_{q_2}) u_1\big|_{\partial\Omega} \right) u_2\big|_{\partial\Omega} dS$$

$$(\Delta - q_i) u_i = 0$$

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{q_1} = \Lambda_{q_2}$ and

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

GOAL: Find **MANY** solutions of $(\Delta - q_i) u_i = 0$.

CGO SOLUTIONS

Calderón: Let $\rho \in \mathbb{C}^n$, $\boxed{\rho \cdot \rho = 0}$

$$\boxed{\rho = \eta + ik} \qquad \eta, k \in \mathbb{R}^n, |\eta| = |k|, \eta \cdot k = 0$$

$$\boxed{u = e^{x\cdot\rho}} = e^{x\cdot\eta} e^{ix\cdot k}$$

$$\Delta u = 0, \qquad u = \begin{cases} \text{exponentially decreasing}, & x \cdot \eta < 0 \\ \text{oscillating}, & x \cdot \eta = 0 \\ \text{exponentially increasing}, & x \cdot \eta > 0 \end{cases}$$

COMPLEX GEOMETRICAL OPTICS

(Sylvester-U) $n \geq 2$, $q \in L^\infty(\Omega)$
Let $\rho \in \mathbb{C}^n$ ($\rho = \eta + ik, \eta, k \in \mathbb{R}^n$) such that $\rho \cdot \rho = 0$
$(|\eta| = |k|, \eta \cdot k = 0)$.

Then for $|\rho|$ sufficiently large we can find solutions of

$$(\Delta - q)w_\rho = 0 \text{ on } \Omega$$

of the form

$$w_\rho = e^{x\cdot\rho}(1 + \Psi_q(x, \rho))$$

with $\Psi_q \to 0$ in Ω as $|\rho| \to \infty$.

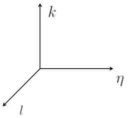
Proof $\boxed{\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2}$

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

$$u_1 = e^{x\cdot\rho_1}(1 + \Psi_{q_1}(x, \rho_1)), \quad u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x, \rho_2))$$

$$\rho_1 \cdot \rho_1 = \rho_2 \cdot \rho_2 = 0, \qquad \begin{aligned} \rho_1 &= \eta + i(k + l) \\ \rho_2 &= -\eta + i(k - l) \end{aligned}$$

$$\eta \cdot k = \eta \cdot l = l \cdot k = 0, \quad |\eta|^2 = |k|^2 + |l|^2$$



$$\int_\Omega (q_1 - q_2) e^{2ix\cdot k}(1 + \Psi_{q_1} + \Psi_{q_2} + \Psi_{q_1}\Psi_{q_2}) = 0$$

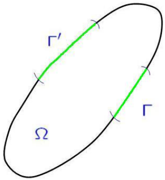
Letting $|l| \to \infty$ $\boxed{\int_\Omega (q_1 - q_2) e^{2ix\cdot k} = 0 \quad \forall k \Longrightarrow q_1 = q_2}$

PARTIAL DATA PROBLEM

Suppose we measure

$$\Lambda_\gamma(f)|_\Gamma, \quad \text{supp} f \subseteq \Gamma'$$

Γ, Γ' open subsets of $\partial\Omega$



Can one recover γ?

Important case $\Gamma = \Gamma'$.

EXTENSION OF CGO SOLUTIONS

$$u = e^{x\cdot\rho}(1 + \Psi_q(x,\rho))$$

$\rho \in \mathbb{C}^n, \rho\cdot\rho = 0$

(Not helpful for localizing)

Kenig-Sjöstrand-U (2007),

$$u = e^{\tau(\varphi(x)+i\psi(x))}(a(x) + R(x,\tau))$$

$\tau \in \mathbb{R}$, φ, ψ real-valued, $R(x,\tau) \to 0$ as $\tau \to \infty$.

φ limiting Carleman weight,

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

Example: $\varphi(x) = \ln|x - x_0|$, $\quad x_0 \notin \overline{ch(\Omega)}$

CGO SOLUTIONS

$$u = e^{\tau(\varphi(x)+i\psi(x))}(a_0(x) + R(x,\tau))$$
$$R(x,\tau) \overset{\tau\to\infty}{\longrightarrow} 0 \text{ in } \Omega$$

$$\varphi(x) = \ln|x - x_0|$$

Complex Spherical Waves

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

$$\Lambda_{q_1}\Big|_\Gamma = \Lambda_{q_2}\Big|_\Gamma, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

Complex Spherical Waves

(Loading reconperfect1.mpg)

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

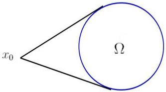
$$\Lambda_{q_1}\Big|_{\Gamma} = \Lambda_{q_2}\Big|_{\Gamma}, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

$u_\tau = e^{\tau(\varphi + i\psi)} a_\tau$ $\quad$ $\varphi(x) = \ln|x - x_0|, x_0 \notin \overline{ch(\Omega)}$

Eikonal: $\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi| = |\nabla\psi|$

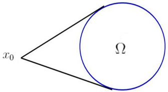
$\psi(x) = d(\frac{x-x_0}{|x-x_0|}, \omega), \omega \in S^{n-1}$: smooth for $x \in \bar{\Omega}$.



Transport: $(\nabla\varphi + i\nabla\psi) \cdot \nabla a_\tau = 0$

(Cauchy-Riemann equation in plane generated by $\nabla\varphi, \nabla\psi$)

$$\varphi(x) = \ln|x - x_0|, \quad x_0 \notin \overline{ch(\Omega)}$$



Carleman Estimates

$$u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\partial\Omega_-} = 0 \qquad \partial\Omega_\pm = \{x \in \partial\Omega; \nabla\varphi\cdot\nu \gtrless 0\}$$

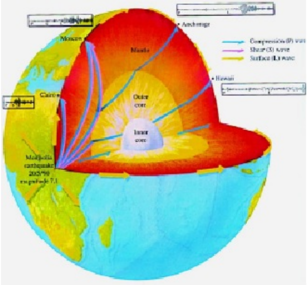
$$\int_{\partial\Omega_+} < \nabla\varphi, \nu > |e^{-\tau\varphi(x)}\frac{\partial u}{\partial\nu}|^2 ds \le \frac{C}{\tau}\int_\Omega |(\Delta - q)ue^{-\tau\varphi(x)}|^2 ds$$

This gives control of $\frac{\partial u}{\partial\nu}|_{\partial\Omega_{+,\delta}}$,

$$\partial\Omega_{+,\delta} = \{x \in \partial\Omega, \nabla\varphi\cdot\nu \ge \delta\}$$

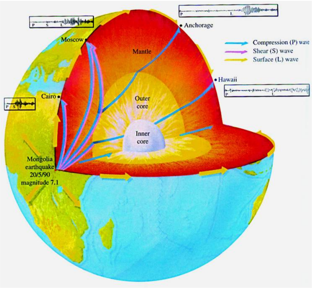
Outline:

- Calderón's problem with partial data
- Travel time tomography with partial data



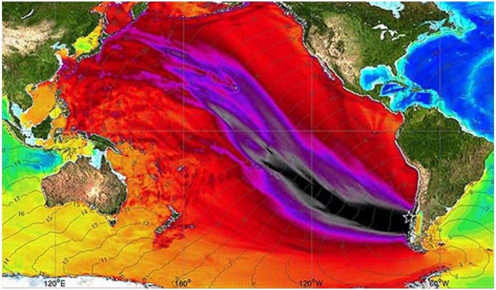
Travel Time Tomography (Transmission)

Global Seismology



Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

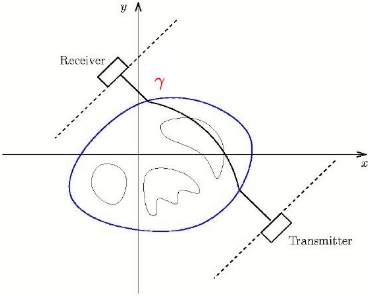
Tsunami of 1960 Chilean Earthquake



Black represents the largest waves, decreasing in height through purple, dark red, orange and on down to yellow. In 1960 a tongue of massive waves spread across the Pacific, with big ones throughout the region.

Human Body Seismology

ULTRASOUND TRANSMISSION TOMOGRAPHY(UTT)



$$T = \int_\gamma \frac{1}{c(x)}\, ds$$ = Travel Time (Time of Flight).

THIRD MOTIVATION

OCEAN ACOUSTIC TOMOGRAPHY

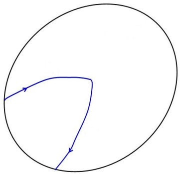
Ocean Acoustic Tomography

Ocean Acoustic Tomography is a tool with which we can study average temperatures over large regions of the ocean. By measuring the time it takes sound to travel between known source and receiver locations, we can determine the soundspeed. Changes in soundspeed can then be related to changes in temperature.

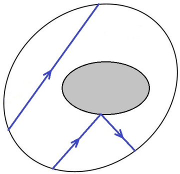
REFLECTION TOMOGRAPHY

Scattering

Points in medium

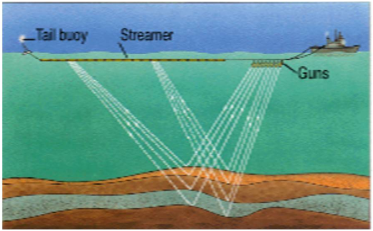


Obstacle



REFLECTION TOMOGRAPHY

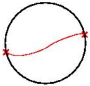
Oil Exploration



Ultrasound

TRAVELTIME TOMOGRAPHY (Transmission)

Motivation:Determine inner structure of Earth by measuring travel times of seismic waves



Herglotz, Wiechert-Zoeppritz (1905)

Sound speed $c(r)$, $r = |x|$

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

Reconstruction method of $c(r)$ from lengths of geodesics

$$ds^2 = \frac{1}{c^2(r)} dx^2$$

More generally $ds^2 = \frac{1}{c^2(x)} dx^2$

Velocity $v(x, \xi) = c(x), \quad |\xi| = 1$ (isotropic)

Anisotropic case

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j$$

$g = (g_{ij})$ is a positive definite symmetric matrix

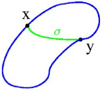
Velocity $v(x, \xi) = \sqrt{\sum_{i,j=1}^{n} g^{ij}(x) \xi_i \xi_j}, \quad |\xi| = 1$

$$g^{ij} = (g_{ij})^{-1}$$

The information is encoded in the
boundary distance function

More general set-up

(M, g) a Riemannian manifold with boundary (compact) $g = (g_{ij})$



$x, y \in \partial M$

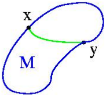
$$d_g(x, y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

$L(\sigma) =$ length of curve σ

$$L(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

Inverse problem

Determine g knowing $d_g(x, y)$ $x, y \in \partial M$



$dg \Rightarrow g$?

(Boundary rigidity problem)

Answer NO $\quad \psi : M \to M$ diffeomorphism

$$\psi\big|_{\partial M} = \text{Identity}$$

$$\boxed{d_{\psi^* g} = d_g}$$

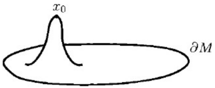
$$\psi^* g = \left(D\psi \circ g \circ (D\psi)^T \right) \circ \psi$$

$$L_g(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

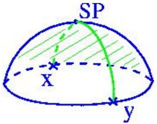
$$\tilde{\sigma} = \psi \circ \sigma \quad \boxed{L_{\psi^* g}(\tilde{\sigma}) = L_g(\sigma)}$$

$$d_{\psi^* g} = d_g$$

Only obstruction to determining g from d_g ? No



$$d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x,y)$$



Can change metric near SP

Def (M, g) is boundary rigid if $(M, \tilde{g})$ satisfies $d_{\tilde{g}} = d_g$. Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} = \text{Identity}$, so that

$$\boxed{\tilde{g} = \psi^* g}$$

Need an a-priori condition for (M, g) to be boundary rigid.

One such condition is that (M, g) is simple

DEF (M,g) is simple if given two points $x, y \in \partial M$, $\exists!$ geodesic joining x and y and ∂M is strictly convex

CONJECTURE

(M,g) is simple then (M,g) is boundary rigid ,that is d_g determines g up to the natural obstruction.
$(d_{\psi^* g} = d_g)$

(Conjecture posed by R. Michel, 1981)

Results (M, g) simple

- R. Michel (1981) Compact subdomains of $\mathbb{R}^2$ or $\mathbb{H}^2$ or the open round hemisphere

- Gromov (1983) Compact subdomains of $\mathbb{R}^n$

- Besson-Courtois-Gallot (1995) Compact subdomains of negatively curved symmetric spaces

(All examples above have constant curvature or special symmetries)

- $\left\{\begin{array}{l}\text{Stefanov-U (1998)}\\ \text{Lassas-Sharafutdinov-U (2003)}\\ \text{Burago-Ivanov (2010)}\end{array}\right\}$ $\boxed{dg = dg_0}$, g_0 close to Euclidean

$n = 2$

- Otal and Croke (1990) $K_g < 0$

THEOREM(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

Theorem ($n \geq 3$) (Stefanov-U, 2005)

(M, g_i) simple $i = 1, 2$, g_i close to $g_0 \in \mathcal{L}$ where $\mathcal{L}$ is a generic set of simple metrics in $C^k(M)$. Then

$$d_{g_1} = d_{g_2} \Rightarrow \exists \psi : \ M \to M \text{ diffeomorphism},$$

$$\psi\big|_{\partial M} = \text{Identity, so that } \boxed{g_1 = \psi^* g_2}$$

Remark

If M is an open set of $\mathbb{R}^n$, $\mathcal{L}$ contains all simple and real-analytic metrics in $C^k(M)$.

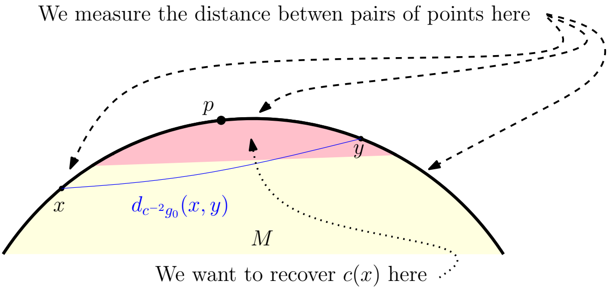
Isotropic Case

Assume that g is isotropic, i.e., $g_{ij}(x) = c^{-2}(x)\delta_{ij}$. Physically, this corresponds to a variable wave speed that does not depend on the direction of propagation. In the class of the isotropic metrics, we do not have the freedom to apply isometries and we would expect g to be uniquely determined.

This is known to be true for simple metrics (Mukhometov, Romanov, et al.) More generally, we can fix g_0 and we have uniqueness of the recovery of the conformal factor $c(x)$ in $c^{-2}g_0$.

Partial Data

Boundary Rigidity with partial data: Does $d_{c^{-2}g_0}$, known on $\partial M \times \partial M$ near some p, determine $c(x)$ near p uniquely?



Theorem (Stefanov-U-Vasy, 2013). Let $\dim M \geq 3$. If ∂M is strictly convex near p for c and $\tilde{c}$, and $d_{c^{-2}g_0} = d_{\tilde{c}^{-2}g_0}$ near (p, p), then $c = \tilde{c}$ near p.

Also stability and reconstruction.

The only results so far of similar nature is for real analytic metrics (Lassas-Sharafutdinov-U, 2003). We can recover the whole jet of the metric at ∂M and then use analytic continuation.

This is the first local result without analyticity assumptions.

Geodesics in Phase Space

$$g = \big(g_{ij}(x)\big) \text{ symmetric, positive definite}$$

Hamiltonian is given by

$$H_g(x,\xi) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j - 1\Big) \qquad g^{-1} = \big(g^{ij}(x)\big)$$

$X_g(s, X^0) = \big(x_g(s, X^0), \xi_g(s, X^0)\big)$ be **bicharacteristics**,

sol. of $$\frac{dx}{ds} = \frac{\partial H_g}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_g}{\partial x}$$

$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_g^{n-1}(x^0)$

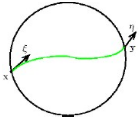
$$\mathcal{S}_g^{n-1}(x) = \big\{\xi \in \mathbb{R}^n;\ H_g(x,\xi) = 0\big\}.$$

Geodesics Projections in x: $x(s)$.

Scattering Relation

d_g only measures first arrival times of waves.

We need to look at behavior of all geodesics



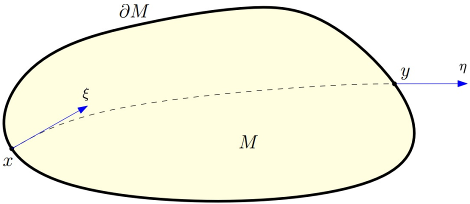
$$\|\xi\|_g = \|\eta\|_g = 1$$

$\alpha_g(x, \xi) = (y, \eta)$, α_g is SCATTERING RELATION

If we know direction and point of entrance of geodesic then we know its direction and point of exit.

Lens Rigidity

Define the scattering relation α_g and the length (travel time) function ℓ:



$$\alpha_g : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change L, ℓ!

Lens rigidity: *Do α_g, ℓ determine g uniquely, up to isometry?*

Lens rigidity: *Do α_g, ℓ determine g uniquely, up to isometry?*

No, in general but the counterexamples are harder to construct.

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! Indeed, then $d_g(x, y)$, known for x, y on ∂M determines α_g, ℓ uniquely, and vice-versa. This is also true locally, near a point p where ∂M is strictly convex.

For non-simple metrics (caustics and/or non-convex boundary), the Lens Rigidity is the right problem to study.

There are fewer results: local generic rigidity near a class of non-simple metrics (Stefanov-U, 2009), for real-analytic metrics satisfying a mild condition (Vargo, 2010), the torus is lens rigid (Croke 2012), stability estimates for a class of non-simple metrics (Bao-Zhang 2012).

Lens Rigidity with partial data

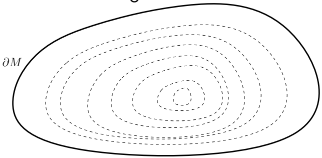
Lens Rigidity with partial data: Does the lens relation known for points near p, and "almost tangent directions" determine $c(x)$ near p uniquely?

As an immediate consequence of our theorem, the answer is affirmative.

Global result under the foliation condition

We could use a layer stripping argument to get deeper and deeper in M and prove that one can determine c in the whole M.

Foliation condition: M is foliated by strictly convex hypersurfaces if, up to a nowhere dense set, $M = \cup_{t\in[0,T)}\Sigma_t$, where Σ_t is a smooth family of strictly convex hypersurfaces and $\Sigma_0 = \partial M$.



A more general condition: several families, starting form outside M.

Global result under the foliation condition

Theorem (Stefanov-U-Vasy, 2013). Let $\dim M \geq 3$, let $c = \tilde{c}$ on ∂M, let ∂M be strictly convex with respect to both $g = c^{-2} g_0$ and $\tilde{g} = \tilde{c}^{-2} g_0$. Assume that M can be foliated by strictly convex hypersurfaces for g. Then if $\alpha_g = \tilde{\alpha}_{\tilde{g}}, l = \tilde{l}$ we have $c = \tilde{c}$ in M.

This is a generalization of Mukhometov's result: one can have conjugate points inside, or even trapped geodesics. Example: a tubular neighborhood of a periodic geodesic on a negatively curved manifold.

Foliation condition is an analog of the Herglotz, Wieckert-Zoeppritz condition for non radial speeds.

Idea of the proof

The proof is based on two main ideas.

First, we use the approach in a recent paper by U-Vasy (2013) on the linear integral geometry problem.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

First step: Linear Problem

U-Vasy result: Consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\, ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". Then they prove that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

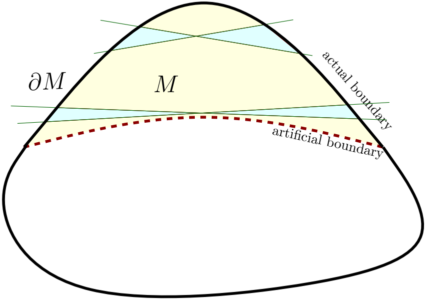
The main trick in U-Vasy is the following idea:

Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melrose's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

Consider

$$Pf(z) := I^*\chi If(z) = \int_{SM} x^{-2}\chi If(\gamma_{z,v})dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary.



Inversion of local geodesic transform

$$Pf(z) := I^*\chi If(z) = \int_{SM} x^{-2}\chi If(\gamma_{z,v})dv,$$

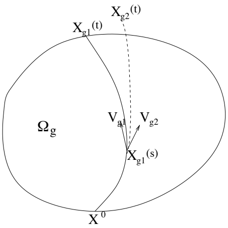
Main result: P is an elliptic pseudodifferential operator in Melrose's scattering calculus.

There exists A such that $AP = I + R$

This is Fredholm and R has a small norm in a neighborhood of p. Therefore invertible near p.

Second Step: Reduction to Pseudolinear Problem

Identity (Stefanov-U, 1998)



$$
\begin{aligned}
&T = d_{g_1},\\
&F(s) = X_{g_2}\big(T-s, X_{g_1}(s,X^0)\big),\\
&F(0) = X_{g_1}(T,X^0), \quad F(T) = X_{g_2}(T,X^0),\\
&\int_0^T F'(s)ds = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)
\end{aligned}
$$

$$
\int_0^T \frac{\partial X_{g_2}}{\partial X^0}\big(T-s, X_{g_1}(s,X^0)\big)\,(V_{g_1}-V_{g_2})\big|_{X_{g_1}(s,X^0)} dS = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)
$$

Identity (Stefanov-U, 1998)

$$\int_0^T \frac{\partial X_{g_2}}{\partial X^0}\left(T - s, X_{g_1}(s, X^0)\right)(V_{g_1} - V_{g_2})\Big|_{X_{g_1}(s,X^0)} dS$$
$$= X_{g_1}(T, X^0) - X_{g_2}(T, X^0)$$

$V_{g_j} := \left(\frac{\partial H_{g_j}}{\partial \xi}, -\frac{\partial H_{g_j}}{\partial x}\right)$ the Hamiltonian vector field.

Particular case:

$$(g_k) = \frac{1}{c_k^2}\left(\delta_{ij}\right), \quad k = 1, 2$$

$$V_{g_k} = \left(c_k^2 \xi, \ -\frac{1}{2}\nabla(c_k^2)|\xi|^2\right)$$

Linear in c_k^2!

Reconstruction

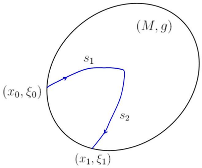
$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\left(T-s, X_{g_2}(s,X^0)\right) \times \left((c_1^2-c_2^2)\xi,\ -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\right)\Big|_{X_{g_2}(s,X^0)} dS = \underbrace{X_{g_1}(T,X^0)}_{\text{data}} - X_{g_2}(T,X^0)$$

Inversion of weighted geodesic ray transform and use similar methods to U-Vasy.

REFLECTION TRAVELTIME TOMOGRAPHY
Broken Scattering Relation

(M, g): manifold with boundary with Riemannian metric g

$$((x_0, \xi_0), (x_1, \xi_1), t) \in \mathcal{B}$$
$$t = s_1 + s_2$$



Theorem (Kurylev-Lassas-U)

$n \geq 3$. Then ∂M and the broken scattering relation $\mathcal{B}$ determines (M, g) uniquely.

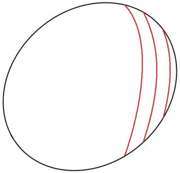
Numerical Method

(Chung-Qian-Zhao-U, IP 2011)

$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\big(T-s, X_{g_2}(s,X^0)\big)\times \Big((c_1^2-c_2^2)\xi,\, -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{g_2}(s,X^0)} dS = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)$$

Adaptive method

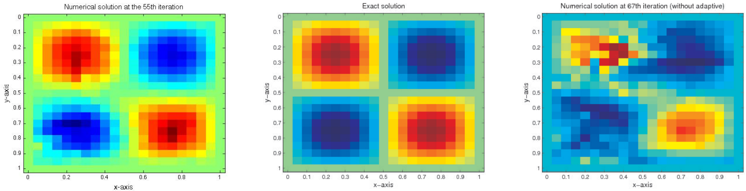
Start near $\partial\Omega$ with $c_2 = 1$ and iterate.



Numerical examples

Example 1: An example with no broken geodesics,

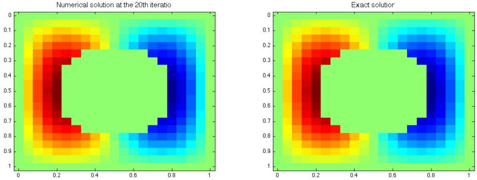
$$c(x,y) = 1 + 0.3\sin(2\pi x)\sin(2\pi y),\ c_0 = 0.8.$$



Left: Numerical solution (using adaptive) at the 55-th iteration. Middle: Exact solution. Right: Numerical solution (without adaptive) at the 67-th iteration.

Example 2: A known circular obstacle enclosed by a square domain. Geodesic either does not hit the inclusion or hits the inclusion (broken) once.

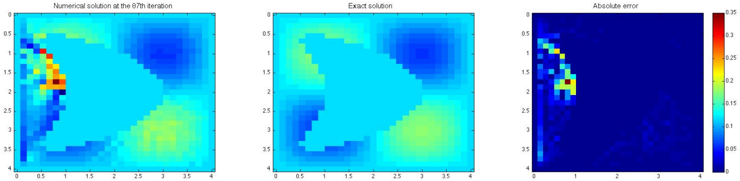
$c(x,y) = 1 + 0.2\sin(2\pi x)\sin(\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 20-th iteration. The relative error is 0.094%. Right: Exact solution.

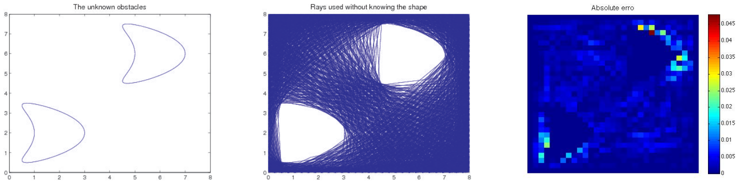
Example 3: A concave obstacle (known).

$c(x, y) = 1 + 0.1 \sin(0.5\pi x) \sin(0.5\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 117-th iteration. The relative error is 2.8%. Middle: Exact solution. Right: Absolute error.

Example 4: Unknown obstacles and medium.

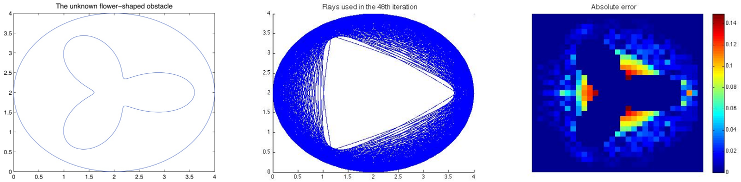


Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

Example 4: Unknown obstacles and medium (continues).

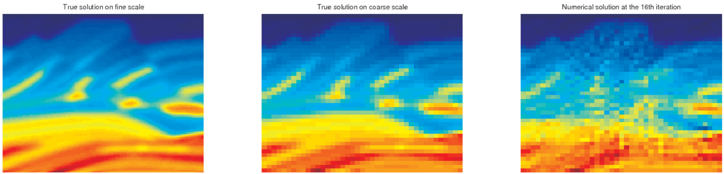
$r = 1 + 0.6\cos(3\theta)$ with $r = \sqrt{(x-2)^2 + (y-2)^2}$.

$$c(r) = 1 + 0.2\sin r$$



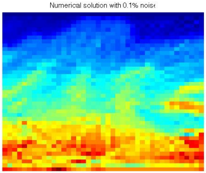
Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

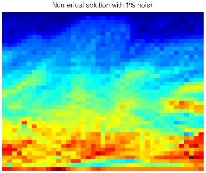
Example 5: The Marmousi model.



Left: The exact solution on fine grid. Middle: The exact solution projected on a coarse grid. Right: The numerical solution at the 16-th iteration. The relative error is 2.24%.

Example 5: The Marmousi model (with noise).





Left: The numerical solution with 0.1% noise. The relative error is 4.16%. Right: The numerical solution with 1% noise. The relative error is 5.53%.

Open problem: Partial Data in $n = 2$ for d_g.

Pestov-U (2005): from d_g one can recover Λ_g.

Question: from $d_g\big|_{\Gamma\times\Gamma}$ can one recover $\Lambda_g\big|_{\Gamma}$?
Carleman estimate?

THANKS JOHANNES FOR THE WONDERFUL MATHEMATICS!