Eigenvalue clusters for magnetic Laplacians in 2D

San Vũ Ngọc

Univ. Rennes 1

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joint work with Nicolas Raymond (Rennes) and Frédéric Faure (Grenoble)

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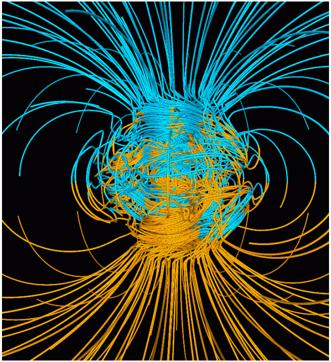

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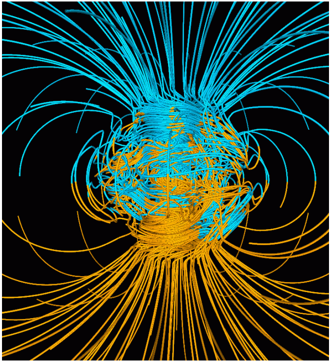

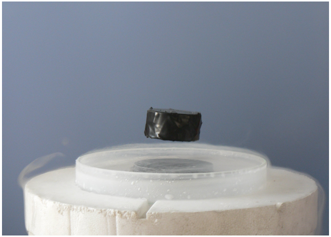

- **Superconductors**
  [Fournais-Helffer, Lu-Pan, etc.]

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  - possibly vanishing magnetic field
  - various geometries
- Dimension 2, non-vanishing B:

Theorem (Helffer-Kordyukov 2009, 2013)

If the magnetic field has a unique and non-degenerate minimum, the j-th eigenvalue admits an expansion in powers of $\hbar^{1/2}$ of the form:

$$\lambda_j(\hbar) \sim \hbar \min_{q\in\mathbb{R}^2} B(q) + \hbar^2(c_1(2j-1) + c_0) + O(\hbar^{5/2}),$$

where c_0 and c_1 are constants depending on the magnetic field.

Averaging method for "Schrödinger" (A. Weinstein, Duke 1977)

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ASYMPTOTICS OF EIGENVALUE CLUSTERS FOR THE LAPLACIAN PLUS A POTENTIAL

ALAN WEINSTEIN

0. Introduction

Let $\triangle$ be the Laplace-Beltrami operator on a compact riemannian manifold X, $V: X \to \mathbb{R}$ a smooth function which we consider as a multiplication operator. The sum $H = \triangle + V$ is called a (reduced) Schrödinger operator, and its eigenvalues represent the energy levels of a quantum mechanical system with kinetic energy given by the riemannian metric, and potential energy V.

If X is the unit n-sphere S^n, the eigenvalues of $\triangle$ are of the form $k(k + n - 1)$, with multiplicities growing as a polynomial of order $n - 1$ in k. These are the energy levels of a "free particle." If a force field with potential V is now applied, each of the multiple eigenvalues splits into a "cluster" of eigenvalues in the interval

$$[k(k + n - 1) + \min V, k(k + n - 1) + \max V].$$

(The Stark effect, in which V is an electrostatic potential, is a physical example of this phenomenon.)

The structure of the clusters has recently been studied by Guillemin in a series of papers [G1][G2][G3] intended to show how the potential function V might be determined by the eigenvalues of $\triangle + V$. In this paper, we extend some of Guillemin's analysis by showing that the distribution of eigenvalues in the k'th cluster approaches a limit as $k \to \infty$, and that the limiting distribution can be expressed in terms of the averages of V along closed geodesics.

Although we begin with differential operators, our constructions immediately require the use of pseudodifferential operators, so we begin in that context as well. In section 1 we show that, modulo a small error, we can replace the potential V by a pseudodifferential operator which commutes with $\triangle$. Section 2, a study of eigenvalues, is a bridge to the principal Section 3, in which we analyze the joint spectrum of two commuting operators. Theorem 3.4 in that section is the main theorem in this paper.* In Section 4, we present some examples, including an application to the spectrum of manifolds all of whose geodesics are closed.

The aforementioned papers of Guillemin, as well as the numerical calculations of Chachere [*C*] (discussed in section 4) were the main impetus behind the

*M. Kac and T. Spencer, as well as H. Widom, have independently proven slightly weaker versions of Theorem 3.4, using completely different methods.

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work described here. I would like to thank V. Guillemin, T. Spencer, M. Taylor, and H. Widom for very helpful conversations.

1. Averaging the potential.

Let A be a first-order, self-adjoint, positive, elliptic pseudo-differential operator (ψDO) on an n-dimensional compact manifold X such that $e^{2\pi iA} = cI$ for some constant c, and let $H = A^2 + B$, where B is a self-adjoint ψDO of order 0. To study the spectrum of H, we introduce the transformed operators

$$B_t = e^{-itA}Be^{itA}$$

and the averaged operator

$$\bar{B} = \frac{1}{2\pi}\int_0^{2\pi} B_t dt.$$

By Theorem 1.1 of [D–G], e^{itA} is a unitary Fourier integral operator, so the B_t's and $\bar{B}$ are self-adjoint ψDO's of order 0.

LEMMA 1.1. $[A,\bar{B}] = 0$.

Proof. We observe first that

$$\frac{d}{dt}B_t = -iAe^{-itA}Be^{itA} + ie^{-itA}Be^{-itA}A,$$

so

$$\frac{d}{dt}B_t = \frac{1}{i}[A,B_t] \tag{1.1}$$

Now $[A,\bar{B}] = \dfrac{i}{2\pi}\displaystyle\int_0^{2\pi}\frac{d}{dt}B_t dt = \dfrac{i}{2\pi}(B_{2\pi} - B_0)$,

which is zero because $e^{2\pi iA}$ commutes with B. ∎

The spectrum of $A^2 + \bar{B}$ is easy to determine in terms of the joint spectrum of A and $\bar{B}$. We will show that $A^2 + \bar{B}$ is a good substitute for $H = A^2 + B$ by proving the following theorem.

THEOREM 1.2. *There is a unitary pseudodifferential operator U such that $U - I$ and $U(A^2 + B)U^{-1} - (A^2 + \bar{B})$ are both of order -1.*

Theorem 1.2 is a realization, in the context of operator theory, of the "averaging method" commonly used in celestial mechanics (see Chapter 10 of [A]). The proof will follow the next few lemmas.

LEMMA 1.3. *Let Q be a skew adjoint ψDO of order -1 such that $[A^2, Q] - (B - \bar{B})$ has order -1. Then, $U = e^Q$ is unitary, and $U - I$ and $U(A^2 + B)U^{-1} - (A^2 + \bar{B})$ have order -1.*

Proof. U is unitary because Q is skew-adjoint. By Seeley's functional calculus [*SE*], U is a ψDO of order 0. Since the 0-order principal symbol of Q is 0,

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Periodic bicharacteristics, more

There have been *many* works on “periodic bicharacteristics”.

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  It is enough that the *principal symbol* is elliptic and has a periodic hamiltonian flow. (+ assmptn on sub-principal)
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Annales Henri Poincaré

Non-Selfadjoint Perturbations of Selfadjoint Operators in 2 Dimensions I

Michael Hitrik and Johannes Sjöstrand

Abstract. This is the first in a series of works devoted to small non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2. In the present work we treat the case when the classical flow of the unperturbed part is periodic and the strength ϵ of the perturbation is $\gg h$ (or sometimes only $\gg h^2$) and bounded from above by h^δ for some $\delta > 0$. We get a complete asymptotic description of all eigenvalues in certain rectangles $[-1/C, 1/C] + i\epsilon[F_0 - 1/C, F_0 + 1/C]$.

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Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ be the Hamilton field of p. In this work, we will always assume that for $E \in \text{neigh}\,(0, \mathbf{R})$:

$$\begin{array}{l}\text{The } H_p\text{-flow is periodic on } p^{-1}(E) \cap T^*M \text{ with} \\ \text{period } T(E) > 0 \text{ depending analytically on } E.\end{array} \tag{1.13}$$

Let $q = \frac{1}{i}\left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} p_\epsilon$, so that

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 m), \tag{1.14}$$

in the case $M = \mathbf{R}^2$ and $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 \langle \xi \rangle^m)$ in the manifold case. Let

$$\langle q \rangle = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} q \circ \exp tH_p dt \text{ on } p^{-1}(E) \cap T^*M. \tag{1.15}$$

Notice that $p, \langle q \rangle$ are in involution; $0 = H_p \langle q \rangle =: \{p, \langle q \rangle\}$. In Section 3, we shall see how to reduce ourselves to the case when

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Periodic bicharacteristics & Harmonic approximation

For semi-excited states, the Harmonic approximation can replace the principal symbol (cf. [Sjöstrand 1992]).

Theorem (Charles, VNS, 2008)

Let $P = -\frac{h^2}{2}\Delta + V(x)$*,* V *has a non-degenerate minimum with eigenvalues* $(\nu_1^2, \ldots, \nu_n^2)$*. Assume that* ν_j *are coprime integers.*

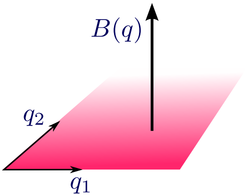
1. *There exists* $\hbar_0 > 0$ *and* $C > 0$ *such that for every* $\hbar \in (0, \hbar_0]$

$$\mathsf{Spec}(P) \cap (-\infty, C\hbar^{\frac{2}{3}}) \subset \bigcup_{E_N \in \mathsf{Spec}(\hat{H}_2)} \left[E_N - \frac{\hbar}{3}, E_N + \frac{\hbar}{3}\right].$$

2. *When* $E_N \leq C\hbar^{\frac{2}{3}}$*, let* $m(E_N, \hbar) = \#\mathsf{Spec}(P) \cap \left[E_N - \frac{\hbar}{3}, E_N + \frac{\hbar}{3}\right]$*. Then* $m(E_N, \hbar)$ *is precisely the dimension of* $\ker(\hat{H}_2 - E_N)$*.*

Classical dynamics for magnetic fields: Lorentz

Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. Our configuration space is $\mathbb{R}^2 = \{q_1 e_1 + q_2 e_2;\ (q_1, q_2) \in \mathbb{R}^2\}$, and the magnetic field is $\vec{B} = B(q_1, q_2) e_3$, $B \neq 0$.



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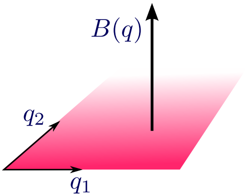

Newton's equation for the particle under the action of the Lorentz force:

$$\ddot{q} = 2\dot{q} \wedge \vec{B}. \tag{1}$$

The kinetic energy $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved.

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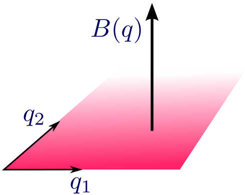

Newton's equation for the particle under the action of the Lorentz force:

$$\ddot{q} = 2\dot{q} \wedge \vec{B}. \tag{1}$$

The kinetic energy $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved.

If the speed $\dot{q}$ is small, we may linearize the system, which amounts to have a constant magnetic field.
$\Rightarrow$ circular motion of angular velocity $\dot{\theta} = -2B$ and radius $\|\dot{q}\|/2B$. Thus, even if the norm of the speed is small, the angular velocity may be very important.

Magnetic drift

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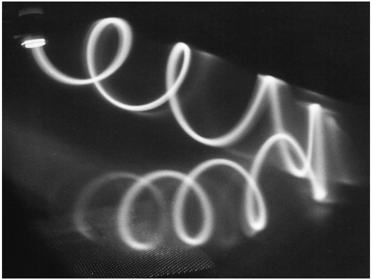
- electron beam in a non-uniform magnetic field



This photograph shows the motion of an electron beam in a non-uniform magnetic field. One can clearly see the fast rotation coupled with a drift. The turning point (here on the right) is called a *mirror point*. Credits: Prof. Reiner Stenzel, http://www.physics.ucla.edu/plasma-exp/beam/BeamLoopyMirror.html

Classical dynamics for magnetic fields: Hamilton

It is known that the system (1) is Hamiltonian.
In terms of canonical variables $(q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ the Hamiltonian (=kinetic energy) is

$$H(q,p) = \|p - A(q)\|^2. \tag{2}$$

We use here the Euclidean norm on $\mathbb{R}^2$, which allows the identification of $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ by

$$\forall (v,p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*, \qquad p(v) = \langle p, v\rangle. \tag{3}$$

Thus, the canonical symplectic structure ω on $T^*\mathbb{R}^2$ is given by

$$\omega((Q_1,P_1),(Q_2,P_2)) = \langle P_1, Q_2\rangle - \langle P_2, Q_1\rangle. \tag{4}$$

It is easy to check that Hamilton's equations for H imply Newton's equation (1). In particular, through the identification (3) we have $\dot{q} = 2(p - A)$.

Fast-slow decomposition: cyclotron & drift

Theorem

There exists a small energy $E_0 > 0$ such that, for all $E < E_0$, for times $t \leq T(E)$, the magnetic flow φ_H^t at kinetic energy $H = E$ is, up to an error of order $\mathcal{O}(E^\infty)$, the Abelian composition of two motions:

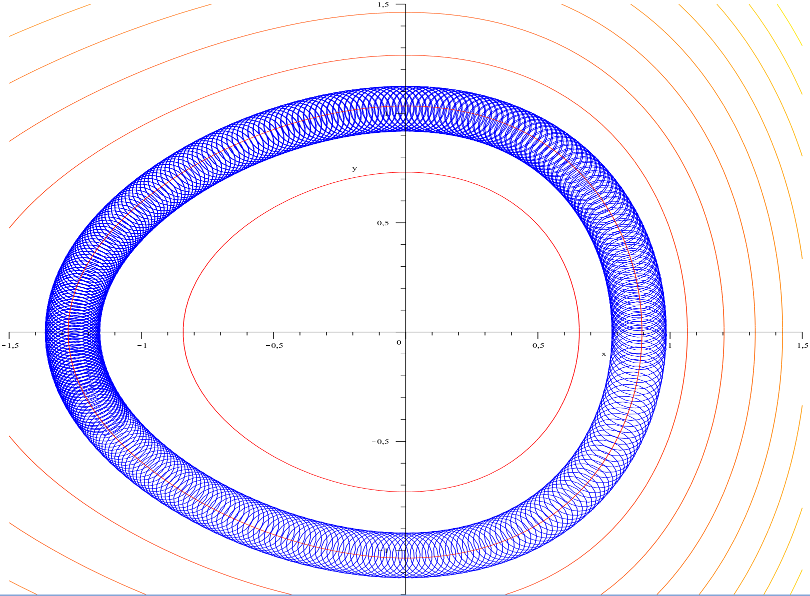
- *[fast rotating motion] a periodic flow with frequency depending smoothly in E;*
- *[slow drift] the Hamiltonian flow of a function of order E on $\Sigma := H^{-1}(0)$.*

- Thus, we can informally describe the motion as a coupling between a fast rotating motion around a center $c(t) \in H^{-1}(0)$ and a slow drift of the point $c(t)$.
- For generic starting points, $T(E) \sim 1/E^N$, arbitrary $N > 0$.

Fast-slow decomposition: numerics

$$B = 2 + q_1^2 + q_2^2 + q_1^3/3$$

Fast-slow decomposition: numerics



A symplectic submanifold

We introduce the submanifold of all particles at rest ($\dot{q} = 0$):

$$\Sigma := H^{-1}(0) = \{(q, p); \qquad p = A(q)\}.$$

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^4$, parameterized by $q \in \mathbb{R}^2$.

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Lemma

Σ is a symplectic submanifold of $\mathbb{R}^4$. In fact,

$$j^*\omega_{\restriction\Sigma} = dA \simeq B,$$

where $j : \mathbb{R}^2 \to \Sigma$ is the embedding $j(q) = (q, A(q))$.

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Proof.

We compute
$j^*\omega = j^*(dp_1 \wedge dq_1 + dp_2 \wedge dq_2) = (-\frac{\partial A_1}{\partial q_2} + \frac{\partial A_2}{\partial q_1})dq_1 \wedge dq_2 \neq 0.$ □

The symplectic orthogonal bundle

We wish to describe a small neighborhood of Σ in $\mathbb{R}^4$, which amounts to understanding the normal symplectic bundle of Σ. (Weinstein, 1971 [9])

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$\Sigma = \{(q, A(q))\} \Rightarrow T_{j(q)}\Sigma = \text{span}\{(Q, T_q\mathbf{A}(Q))\}$.

Lemma

For any $q \in \Omega$, a symplectic basis of $T_{j(q)}\Sigma^\perp$ is:

$$u_1 := \frac{1}{\sqrt{|B|}}(e_1, {}^tT_q\mathbf{A}(e_1)); \quad v_1 := \frac{\sqrt{|B|}}{B}(e_2, {}^tT_q\mathbf{A}(e_2))$$

Proof.

Let $(Q_1, P_1) \in T_{j(q)}\Sigma$ and (Q_2, P_2) with $P_2 = {}^tT_q\mathbf{A}(Q_2)$. Then $\omega((Q_1, P_1), (Q_2, P_2)) = \langle T_q\mathbf{A}(Q_1), Q_2\rangle - \langle {}^tT_q\mathbf{A}(Q_2), Q_1\rangle = 0$. etc. $\square$

The transversal Hessian

Lemma

The transversal Hessian of H, as a quadratic form on $T_{j(q)}\Sigma^{\perp}$, is given by

$$\forall q \in \Omega, \forall (Q,P) \in T_{j(q)}\Sigma^{\perp}, \quad d_q^2 H((Q,P)^2) = 2\|Q \wedge \vec{B}\|^2.$$

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We may express this Hessian in the symplectic basis (u_1, v_1) given by the Lemma:

$$d^2 H_{\restriction T_{j(q)}\Sigma^\perp} = \begin{pmatrix} 2\,|B| & 0 \\ 0 & 2\,|B| \end{pmatrix}. \tag{5}$$

Indeed, $\|e_1 \wedge \vec{B}\|^2 = B^2$, and the off-diagonal term is $\frac{1}{B}\langle e_1 \wedge \vec{B}, e_2 \wedge \vec{B}\rangle = 0$.

Preparation lemma

We endow $\mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$ with canonical variables $z_1 = x_1 + i\xi_1$, $z_2 = (x_2, \xi_2)$, and symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$.

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By Darboux theorem, there exists a diffeomorphism $g : \Omega \to g(\Omega) \subset \mathbb{R}^2_{z_2}$ such that $g(q_0) = 0$ and $g^*(d\xi_2 \wedge dx_2) = j^*\omega$.
In other words, the new embedding $\tilde{j} := j \circ g^{-1} : \mathbb{R}^2 \to \Sigma$ is symplectic.

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$$\mathbb{C} \times \Omega \xrightarrow{\tilde{\Phi}} N\Sigma$$
$$(x_1 + i\xi_1, z_2) \mapsto x_1 u_1(z_2) + \xi_1 v_1(z_2),$$

where $q = g^{-1}(z_2)$. This is an isomorphism between the normal symplectic bundle of $\{0\} \times \Omega$ and $N\Sigma$, the normal symplectic bundle of Σ (for fixed z_2, the map $z_1 \mapsto \tilde{\Phi}(z_1, z_2)$ is a linear symplectic map). Weinstein [9] $\Rightarrow \exists$ symplectomorphism Φ from a neighborhood of $\{0\} \times \Omega$ to a neighborhood of $\tilde{j}(\Omega) \subset \Sigma$ whose differential at $\{0\} \times \Omega$ is equal to $\tilde{\Phi}$.

Preparation lemma: the transformed Hamiltonian

The zero-set $\Sigma = H^{-1}(0)$ is now $\{0\} \times \Omega$, and the symplectic orthogonal $T_{\tilde{\jmath}(0,z_2)}\Sigma^\perp$ is canonically equal to $\mathbb{C} \times \{z_2\}$. By (5), the matrix of the transversal Hessian of $H \circ \Phi$ in the canonical basis of $\mathbb{C}$ is simply $d^2(H \circ \Phi)_{\restriction \mathbb{C} \times \{z_2\}} =$

$$= d^2_{\Phi(0,z_2)} H \circ (d\Phi)^2 = \begin{pmatrix} 2\left|B(g^{-1}(z_2))\right| & 0 \\ 0 & 2\left|B(g^{-1}(z_2))\right| \end{pmatrix}. \quad (6)$$

Therefore, by Taylor's formula in the z_1 variable (locally uniformly with respect to the z_2 variable seen as a parameter), we get
$H \circ \Phi(z_1, z_2) =$
$= H \circ \Phi_{\restriction z_1 = 0} + dH \circ \Phi_{\restriction z_1=0}(z_1) + \frac{1}{2} d^2(H \circ \Phi)_{\restriction z_1 = 0}(z_1^2) + \mathcal{O}(|z_1|^3)$
$= 0 + 0 + \left|B(g^{-1}(z_2))\right| |z_1|^2 + \mathcal{O}(|z_1|^3)$.

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$$= 0 + 0 + \left|B(g^{-1}(z_2))\right| |z_1|^2 + \mathcal{O}(|z_1|^3).$$

Can one do better ?

Magnetic Birkhoff normal form

Theorem

Let $\Omega \subset \mathbb{R}^2$ be an open set where B does not vanish. Then there exists a symplectic diffeomorphism Φ, defined in an open set $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$, with values in $T^*\mathbb{R}^2$, which sends the plane $\{z_1 = 0\}$ to the surface $\{H = 0\}$, and such that

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty), \tag{7}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is smooth. Moreover, the map

$$\varphi : \Omega \ni q \mapsto \Phi^{-1}(q, \mathbf{A}(q)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega} \tag{8}$$

is a local diffeomorphism and $f \circ (\varphi(q), 0) = |B(q)|$.

Long time dynamics

Let $K = |z_1|^2 f(z_2, |z_1|^2)$ (completely integrable).

Theorem

Assume that the magnetic field $B > 0$ is confining: there exists $C > 0$ and $M > 0$ such that $B(q) \geq C$ if $\|q\| \geq M$. Let $C_0 < C$. Then

1. *The flow φ_H^t is uniformly bounded for all starting points (q, p) such that $B(q) \leq C_0$ and $H(q, p) = \mathcal{O}(\epsilon)$ and for times of order $\mathcal{O}(1/\epsilon^N)$, where N is arbitrary.*
2. *Up to a time of order $T_\epsilon = \mathcal{O}(|\ln \epsilon|)$, we have*

$$\|\varphi_H^t(q, p) - \varphi_K^t(q, p)\| = \mathcal{O}(\epsilon^\infty) \tag{9}$$

for all starting points (q, p) such that $B(q) \leq C_0$ and $H(q, p) = \mathcal{O}(\epsilon)$.

Very Long time dynamics

It is interesting to notice that, if one restricts to regular values of B, one obtains the same control for a much longer time, as stated below.

Theorem

Under the same confinement hypothesis, let $J \subset (0, C_0)$ be a closed interval such that dB does not vanish on $B^{-1}(J)$. Then up to a time of order $T = \mathcal{O}(1/\epsilon^N)$, for an arbitrary $N > 0$, we have

$$\|\varphi_H^t(q,p) - \varphi_K^t(q,p)\| = \mathcal{O}(\epsilon^\infty)$$

for all starting points (q,p) such that $B(q) \in J$ and $H(q,p) = \mathcal{O}(\epsilon)$.

Rem: The longer time $T = \mathcal{O}(1/\epsilon^N)$ perhaps also applies for some types of singularities of B; this seems to be an open question.

Quantum spectrum

The spectral theory of $\mathcal{H}_{\hbar,\mathbf{A}}$ is governed at first order by the magnetic field itself, viewed as a symbol on Σ.

Theorem

Assume that the magnetic field B is confining and non vanishing. Let $\mathcal{H}_\hbar^0 = \mathsf{Op}_\hbar^w(H^0)$, where $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$. Then the spectrum of $\mathcal{H}_{\hbar,\mathbf{A}}$ below $C\hbar$ is 'almost the same' as the spectrum of $\mathcal{N}_\hbar := \mathcal{H}_\hbar^0 + Q_\hbar$, i.e.:

$$|\lambda_j(\hbar) - \mu_j(\hbar)| = O(\hbar^\infty).$$

where $Q_\hbar$ is a classical pseudo-differential operator, such that

- *$Q_\hbar$ commutes with $\mathsf{Op}_\hbar^w(|z_1|^2)$;*
- *$Q_\hbar$ is relatively bounded with respect to $\mathcal{H}_\hbar^0$ with an arbitrarily small relative bound;*
- *its Weyl symbol is $O_{z_2}(\hbar^2 + \hbar|z_1|^2 + |z_1|^4)$,*

Microlocal normal form, I

Cf. [Sjöstrand, 1992], [Charles – VNS, 2008], and [Ivrii 1998].

Theorem

For $\hbar$ small enough there exists a Fourier Integral Operator $U_\hbar$ such that

$$U_\hbar^* U_h = I + Z_\hbar, \qquad U_\hbar U_h^* = I + Z'_\hbar,$$

where $Z_\hbar, Z'_\hbar$ are pseudo-differential operators that microlocally vanish in a neighborhood of $\tilde{\Omega} \cap \Sigma$, and

$$U_\hbar^* \mathcal{H}_{\hbar,\mathbf{A}} U_\hbar = \mathcal{I}_\hbar F_\hbar + \hat{\mathcal{O}}(\hbar^\infty), \tag{10}$$

where

1. $\mathcal{I}_\hbar := -\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2$;
2. *$F_\hbar$ is a classical pseudo-differential operator in $S(m)$ that commutes with $\mathcal{I}_\hbar$ (and $\mathcal{I}_\hbar F_\hbar = \mathcal{N}_\hbar = \mathcal{H}_\hbar^0 + Q_\hbar$).*

Microlocal normal form, II

$$[F_\hbar, \mathcal{I}_\hbar] = 0$$

Theorem (Quantization and reduction)

1. *For any Hermite function* $h_n(x_1)$ *such that* $\mathcal{I}_\hbar h_n = \hbar(2n-1)h_n$, *the operator* $F_\hbar^{(n)}$ *acting on* $L^2(\mathbb{R}_{x_2})$ *by*

$$h_n \otimes F_\hbar^{(n)}(u) = F_\hbar(h_n \otimes u)$$

is a classical pseudo-differential operator in $S_{\mathbb{R}^2}(m)$ *with principal symbol* $F^{(n)}(x_2, \xi_2) = B(q)$*;*

Bottom of the magnetic well

We recover the result of Helffer-Kordyukov [4], adding the fact that no odd power of $\hbar^{1/2}$ can show up in the asymptotic expansion.

Corollary (Low lying eigenvalues)

Assume that B has a unique non-degenerate minimum. Then there exists a constant c_0 such that for any j, the eigenvalue $\lambda_j(\hbar)$ has a full asymptotic expansion in integral powers of $\hbar$ whose first terms have the following form:

$$\lambda_j(\hbar) \sim \hbar \min B + \hbar^2(c_1(2j-1) + c_0) + O(\hbar^3),$$

with $c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}$, where the minimum of B is reached at $\varphi^{-1}(0)$.

Magnetic excited states

Corollary (Magnetic excited states)

Let c be a regular value of B, and assume that the level set $B^{-1}(c)$ is connected. Then there exists $\epsilon > 0$ such that the eigenvalues of the magnetic Laplacian in the interval $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ have the form

$$\lambda_j(\hbar) = (2n-1)\hbar f_\hbar(\hbar n(j), \hbar k(j)) + O(\hbar^\infty), \quad (n(j), k(j)) \in \mathbb{Z}^2,$$

where $f_\hbar = f_0 + \hbar f_1 + \cdots$, $f_i \in C^\infty(\mathbb{R}^2; \mathbb{R})$ and $\partial_1 f_0 = 0$, $\partial_2 f_0 \neq 0$. Moreover, the corresponding eigenfunctions are microlocalized in the annulus $B^{-1}([c-\epsilon, c+\epsilon])$.
In particular, if $c \in (\min B, 3\min B)$, the eigenvalues of the magnetic Laplacian in the interval $[\hbar(c-\epsilon), \hbar(c+\epsilon)]$ have gaps of order $O(\hbar^2)$. ($n = 1$)

Proof: semiclassical normal form

Recall $H(z_1, z_2) = H^0 + O(|z_1|^3)$, where $H^0 = B(g^{-1}(z_2))|z_1|^2$.
Consider the space of the formal power series in $\hat{x}_1, \hat{\xi}_1, \hbar$ with coefficients smoothly depending on $(\hat{x}_2, \hat{\xi}_2)$: $\mathcal{E} = \mathcal{C}^\infty_{\hat{x}_2, \hat{\xi}_2}[\hat{x}_1, \hat{\xi}_1, \hbar]$.
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The degree of $\hat{x}_1^\alpha \hat{\xi}_1^\beta \hbar^l$ is $\alpha + \beta + 2l$. $\mathcal{D}_N$ denotes the space of the monomials of degree N. $\mathcal{O}_N$ is the space of formal series with valuation at least N.

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Proposition

Given $\gamma \in \mathcal{O}_3$, there exist formal power series $\tau, \kappa \in \mathcal{O}_3$ such that:

$$e^{i\hbar^{-1}\mathsf{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa,$$

with: $[\kappa, H^0] = 0.$

Open questions

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- $n = 3$!

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  http://www.dynamical-systems.org/stoermer/

Sur les trajectoires des corpuscules électrisés dans l'espace

Applications à l'aurore boréale et aux perturbations magnétiques [1]

Par CARL STÖRMER,

Professeur de Physique à l'Université de Christiania.

L'hypothèse de Birkeland sur l'aurore boréale, hypothèse que M. Störmer s'est proposé d'appuyer par le calcul, est que le phénomène est dû à des rayons cathodiques *émanés du soleil* [1] et attirés dans le champ magnétique terrestre. Birkeland a cherché à vérifier expérimentalement sa théorie, et il a institué quelques expériences remarquables en expo-

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