# Nonlinear Fourier series and applications to PDE

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# I. Introduction

In this talk, we shall describe a new method to analyze Fourier series. The motivation comes from solving nonlinear PDE's. These PDE's are evolution equations, describing evolution in time of physical systems, e.g. So the series are typically space-time Fourier series .

Generally speaking, conservation laws play an important role in the subject. For example, from Mechanics classes, one learns early on that energy conservation plays an important role. Recall how we learned in high school to calculate how high a stone could reach if we throw it up in the air.

While in physics one can almost always invoke energy conservation, this is no longer so in mathematics, because sometimes one needs to work in a function space where there is no known conserved quantities.

For example, the aforementioned energy is usually defined in a space which requires 1 derivative. It could happen, that sometimes the solutions could only be found in a space which requires more than 1 derivative. (In fact, majority of nonlinear PDE's are in this situation.) So even though the energy is conserved, it is not useful!

These equations are called energy supercritical.

Below we start with the motivating example, the nonlinear Schrödinger equation (NLS). I hope that you will see that the new idea required is sufficiently general that it might be applicable to some other equations.

We start with the basics.

# II. Laplacian and Fourier series in space

We consider the Laplacian $\Delta$ on the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. Functions on the torus can be identified with periodic functions on $\mathbb{R}^d$ with period $(2\pi)^d$.

Solving the Laplace equation:

$$-\Delta u = f$$

with a given function $f$,

leads to the eigenvalue-eigenfunction problem

$$-\Delta u = \lambda u.$$

The eigenvalues $\lambda$ are $j^2 := |j|^2$ with corresponding eigenfunction

$$e^{-ij\cdot x},\ j \in \mathbb{Z}^d,$$

forming the basis of space Fourier series. The analysis of which is an old and classical subject.

# III. Linear Schrödinger equation

The linear Schrödinger equation studies the evolution in time of the Laplacian. It is

$$-i\frac{\partial u}{\partial t} = -\Delta u;$$

or more generally with the addition of a potential $V$:

$$-i\frac{\partial u}{\partial t} = -\Delta u + Vu.$$

We note that by comparison, the heat equation is

$$\frac{\partial u}{\partial t} = -\Delta u.$$

With the addition of $i$, Schrödinger is a different, oscillatory problem.

This is a recurrent point in the study, namely how to take care of the first order operator:

$$i\frac{\partial}{\partial t},$$

which is compatible with translation (in time) invariance and therefore completely loses locality .

To solve the first Schrödinger equation (the free Schrödinger equation), one can use the aforementioned space Fourier series. One obtains solutions of the form

$$e^{ij^2 t} e^{-ij\cdot x},$$

which are components of a space-time Fourier series.

# IV. NLS on $\mathbb{T}^d$

The nonlinear Schrödinger equation (NLS) on the torus takes the following form:

$$-i\dot{u} = -\Delta u + |u|^{2p}u,$$

where $p \in \mathbb{N}$ is arbitrary or more generally

$$-i\dot{u} = -\Delta u + |u|^{2p}u + H(x, u, \bar{u}),$$

with the addition of an analytic $H$, for example.

The main reason we mention $H$ is that it has explicit $x$ dependence, breaking translation invariance and could represent a topological obstruction.

(For example, of a kind that one encounters when trying to embed dimension 3 in dimension 2.)

Remark 1. The Laplacian and the resulting space Fourier series reflect translation invariance. Therefore the loss of this invariance could conceivably be a difficulty.

Remark 2. This obstruction already exists in finite dimensions (i.e., classical Hamiltonian systems), cf. Duistermaat (1984).

The method that I will describe is, however, indifferent to the lack of symmetry and gets around this obstruction. So for the rest of the talk, I will take $H = 0$.

# III. The lift

The free Schrödinger equation has only periodic in time solutions with basic frequency equal to 1. This is because the eigenvalues are integers.

It is therefore natural to see whether some of these periodic solutions could bifurcate to solutions to the nonlinear equation, albeit with several (arbitrary but finite number) frequencies. Let us denote the number of frequencies by $b$.

We wish to continue using Fourier series to find solutions. As a step toward that, we reexamine the free Schrödinger equation and try to find more general solutions of $b$ frequencies.

Remark. Sometimes the reason that one cannot find a solution to a nonlinear equation is because the “solution space” is not large enough and not because the solution does not exist.

One therefore **lifts** the problem and seeks solutions which are appropriate linear combinations of

$$e^{in\cdot\omega t}e^{-ij\cdot x},$$

where $n \in \mathbb{Z}^b$ and $\omega = \{\lambda_k\}_{k=1}^{b}$ with each $\lambda_k = j_k^2$ an eigenvalue of the Laplacian, is a vector in $\mathbb{R}^b$.

In other words, for each frequency in time, an additional dimension is added and one works in $\mathbb{T}^b \times \mathbb{T}^d$ instead.

We note that by restricting to

$$|n| = 1,$$

the base harmonics, this recovers the solutions:

$$e^{ij_k^2 t} e^{-ij_k \cdot x}$$

found earlier.

# IV. The bi-characteristics

Using the above ansatz, the Fourier support of the solutions to the free Schrödinger equation:

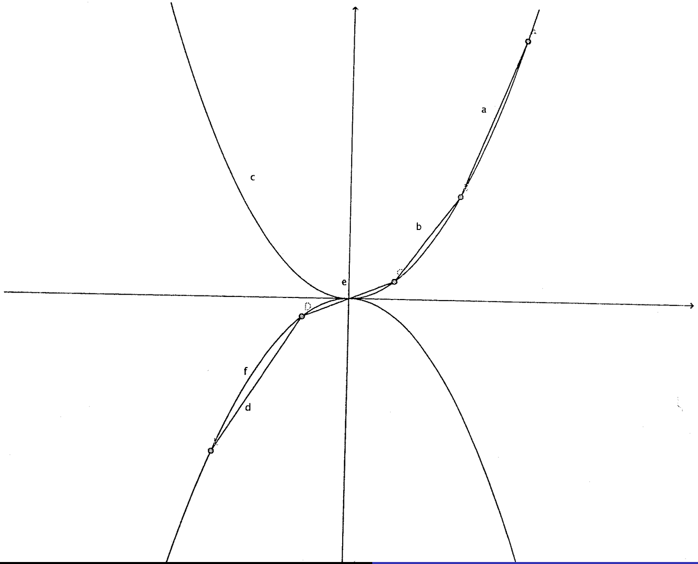
$$-i\frac{\partial u}{\partial t} = -\Delta u,$$

satisfies

$$n \cdot \omega + j^2 = 0.$$

We call this paraboloid the characteristics: $\mathcal{C}^+$.

1d periodic for NLS:



We note that there are many (infinitely) more solutions than before. This should help solving the nonlinear equations as the "solution space" is now much bigger:

$$\mathbb{Z}^d \mapsto \mathbb{Z}^b \times \mathbb{Z}^d,$$

the Fourier dual of $\mathbb{T}^b \times \mathbb{T}^d$.

# V. Nonlinear Fourier series

Returning to the nonlinear equation, as an ansatz, we seek solutions of $b$ frequencies to the nonlinear equation

$$-i\dot{u} = -\Delta u + |u|^{2p}u$$

in the form of a nonlinear Fourier series:

$$u = \sum \hat{u}(n,j)e^{in\cdot\omega t}e^{ij\cdot x}, \quad (n,j) \in \mathbb{Z}^{b+d},$$

where $\omega \in \mathbb{R}^b$ is to be determined.

We note that this is a main difference with the linear equation, where the frequency $\omega$ is fixed.

# IV. The bi-characteristics again

Using the nonlinear Fourier series ansatz, the NLS equation becomes a nonlinear matrix equation:

$$\text{diag}\ (n\cdot\omega+j^2)\hat{u}+(\hat{u}*\hat{v})^{*p}*\hat{u}=0$$

where $(n,j)\in\mathbb{Z}^{b+d}$, $\hat{v}=\bar{\hat{u}}$ and $\omega\in\mathbb{R}^b$ is to be determined. For simplicity we drop the hat and write $u$ for $\hat{u}$ and $v$ for $\hat{v}$ etc.

The solutions to the nonlinear equation will be determined iteratively using a Newton scheme. The initial approximation is the linear solution with the linear frequency, which is composed of $b$ eigenvalues of the Laplacian.

Therefore the initial Fourier support is the same as for the linear solution and we continue to take the paraboloids $\mathcal{C}$ to be the bi-characteristics.

Since $\omega$ is an integer vector at this stage, $\mathcal{C}$ is an infinite set. Considering $\mathcal{C}$ as defining a function on

$$\mathbb{R}^{b+d} \times \mathbb{R}^{b+d},$$

we notice an essential difficulty – the bi-characteristics do not consist of isolated points.

The "isolated point" property is essential to solve PDE. At this stage, this "bad geometry" simply does not permit any meaningful analysis.

Remark (Question). This isolated point property is related to hypo-ellpticity. (?)

# VI. Partitioning the paraboloids

We improve the geometry by making a partition of the integer paraboloids $\mathcal{C}$, so that each set in the partition is "small" (for the most part at most $2d + 2$ lattice points).

The partition here is adapted to the convolution structure generated by the nonlinear terms. This is different from the more standard lattice point partition, which is typically relative to the convolution structure leading to the lattice $\mathbb{Z}^m$.

# *Example.*

The symbol

$$2\cos x = e^{-ix} + e^{ix}$$

leads to the lattice

$$\mathbb{Z}$$

and

$$2\cos 2x + 2\cos 2y = e^{-2ix} + e^{2ix} + e^{-2iy} + e^{2iy}$$

leads to

$$2\mathbb{Z} \times 2\mathbb{Z}.$$

One of the most classical lattice point partition results is on spheres $\mathbb{S}_m$. It is essentially due to Janick (circa 1926). This lemma and its generalizations are in fact convexity bounds. Moreover they are only asymptotic in the large radius limit.

The partition on the paraboloids has to be achieved differently, because the time-$n$ direction is flat. This is the difficulty.

One circumvents the lack of convexity by describing each set in the partition algebraically and proceeds to bound the size of polynomial systems where there is possibly a solution.

For appropriate linear solutions (convolution), one then obtains that the sets in the partition are small. This is the main novelty.

Afterwards, one extracts a parameter from the nonlinearity using amplitude-frequency modulation. Recall that $\omega$ is a variable in the nonlinear Fourier series.

More precisely, $\omega$ is now such that

$$n \cdot \omega + j^2 \neq 0$$

except at $(0, 0)$. (Such $\omega$ is called Diophantine.)

The bi-characteristics $\mathcal{C}$ has therefore been reduced to one point, namely the origin. So one can proceed to the analysis.

But there is a catch – this estimate – this isolated point property is **not** uniform. There is a problem at infinity ($n \to \infty$) leading to small-divisors.

# VII. The Analysis

With the frequency $\omega$ a Diophantine parameter, one can start to adapt the small-divisor analysis of Bourgain, which previously was under the fundamental assumption of a spectral gap, in the form of a spectrally defined Laplacian.

Our work has now removed this assumption and therefore solves the original NLS.

# VIII. Conclusion

Putting everything together using a Newton scheme, the final conclusion is then

**Theorem [W]** There exists a class of **global** solutions to energy supercritical NLS in arbitrary dimension $d$ and for arbitrary nonlinearity $p$.

Remark. These solutions seem to be out of the reach of the current nonlinear Schrödinger theory, which relies on energy conservation laws.

We note that for the cubic NLS ($p = 1$), Procesi and Procesi have related results using a different approach.

# IX. Some comments

The theory sketched above, in some sense, centers around singularity analysis, be it geometrical – characteristics are not isolated points, or analytical – small divisors.

The possible singularities that are dealt with here occur only for long or infinite time and are largely brought on by the first order operator $i\partial/\partial t$. Generally speaking, these singularities do not exist when one can use energy conservation (or when the Kolmogorov non-degeneracy condition is not violated).

This is also reflected in the mathematics used in the theory, which includes notions such as algebraic sets, semi-algebraic sets, variable reductions and subharmonic functions.

When seeking certain types of solutions, this method seems to provide a rather general way to deal with singularities.

# X. A technical note

A large part of the theory is independent of self-adjointness. In fact, in $d \leq 2$, the whole theory goes through without using self-adjointness.

The reason is that most part of the theory uses determinant (polynomial) approximations. For $d \leq 2$, this is ok for controlling the degree of the polynomials since the degeneracy of the Laplacian essentially only contributes a "log", more precisely $e^{\log R/loglogR}$.

For $d > 2$, due to the high degeneracy of the Laplacian, it seems that one way out is to replace determinant variation by eigenvalue variation and hence the need for self-adjointness.

HAPPY BIRTHDAY, JOHANNES !