The Kramers- Fokker-Planck equation with a short-range potential

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Outline

1. Introduction. Motivation
2. The free KFP operator
3. The KFP operator with a potential
4. Low-energy spectral properties in short-range case

The Kramers-Fokker-Planck equation

The Kramers-Fokker-Planck equation is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field $F(x)$:

$$\frac{\partial W}{\partial t} = [-\frac{\partial}{\partial x} v + \frac{\partial}{\partial v}(\gamma v - \frac{F(x)}{m}) - \frac{\gamma k T}{m} \Delta_v] W, \tag{1}$$

where $W = W(x, v; t)$, $x, v \in \mathbb{R}^n$, $t \geq 0$ and $F(x) = -m\nabla V(x)$ is the external force. This equation is a special case of the Fokker-Planck equation.

The Kramers-Fokker-Planck equation

After change of unknowns and suitable normalization of the physical constants, the Kramers-Fokker-Planck (KFP) equation can be written into the form

$$\partial_t u(x,v;t) + Pu(x,v;t) = 0,\ (x,v) \in \mathbb{R}^n \times \mathbb{R}^n, n \geq 1, t > 0 \quad (2)$$

with the initial condition

$$u(x,v;0) = u_0(x,v) \quad (3)$$

where $P = v\cdot\nabla_x - \nabla V(x)\cdot\nabla_v - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$.

Return to the equilibrium

In this talk, we are interested in the time-decay of solutions to the equation (2) in the case $\nabla V(x) \to 0$ as $|x| \to \infty$.

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The case $|\nabla V(x)| \to \infty$ (or at least $|\nabla V(x)| \geq C > 0$ at the infinity) has been studied by several authors: Desvillettes-Villani(CPAM, 2001), Hérau-Nier(ARMA, 2004), Helffer-Nier(LNM, 2005), Hérau-Hitrik-Sjöstrand (AHP, 2008 -), $\cdots$

Return to the equilibrium

In the case $|\nabla V(x)| \to \infty$ and $V(x) > 0$ outside some compact set, the solutions look like

$$u(t) - c(u_0)m_0 = O(e^{-\sigma t}), \sigma > 0,$$

in appropriate spaces where $m_0 = e^{-\frac{1}{2}(\frac{v^2}{2}+V(x))}$ is the Maxwillian.

The existence of a gap between 0 and the remaining part of the spectrum is crucial for such results.

The question

If $V(x) \approx a|x|^{\mu}$ for some $a > 0$ and $0 < \mu < 1$, then $m_0 \in L^2$ and 0 is an eigenvalue of P. If $V(x) \approx a \ln |x|$, m_0 is an eigenfunction if $a > \frac{n}{2}$ and is a resonant state if $\frac{n-2}{2} \leq a \leq \frac{n}{2}$. But now there is no gap between 0 and the remaining part of the spectrum.

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Question. What can one say about the time-decay of solutions if V is slowly increasing or decreasing?

One may say that the case $|\nabla V| \to \infty$ is a non-selfadjoint eigenvalue problem, while the case $|\nabla V| \to 0$ is a non-selfadjoint scattering problem for the pair (P_0, P) where P_0 is the free KFP operator

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$$

Complex harmonic oscillators

$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$ with the maximal demain is an accretive and hypoelliptic operator. It is unitarily equivalent with $\hat{P}_0$ which is a direct integral of $\hat{P}_0(\xi), \xi \in \mathbb{R}^n$,

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4}(v + i2\xi)^2 + \xi^2 - \frac{n}{2}.$$

One can check that $\sigma(\hat{P}_0(\xi)) = \{k + \xi^2, k \in \mathbb{N}\}$. All the eigenvalues are semisimple and the Riesz projection associated with the eigenvalue $k + \xi^2$ is given by

$$\Pi_k^\xi = \sum_{\alpha \in \mathbb{N}, |\alpha| = k} \langle \psi_\alpha^{-\xi}, \cdot \rangle \psi_\alpha^\xi.$$

Complex harmonic oscillators

Here $\psi_\alpha^\xi(v) = \psi_\alpha(v + i2\xi)$ and $\psi_\alpha, \alpha \in \mathbb{N}^n$, are normalized Hermite functions: $(-\Delta_v + \frac{1}{4}v^2 - \frac{n}{2})\psi_\alpha = |\alpha|\psi_\alpha$.

Lemma 1

For any $\xi \in \mathbb{R}^n$ and $t > 0$, one has the following spectral decomposition for the semigroup:

$$e^{-t\hat{P}_0(\xi)} = \sum_{k=0}^{\infty} e^{-t(k+\xi^2)}\Pi_k^\xi, \quad (4)$$

where the series is norm convergent as operators on $L^2(\mathbb{R}_v^n)$.

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To prove this lemma, we show that if $n = 1$,

$$\sum_{k=0}^{\infty} e^{-t(k+\xi^2)}\|\Pi_k^\xi\| = \frac{e^{-\xi^2(t-2)}}{1-e^{-t}} e^{\frac{4\xi^2}{e^t-1}}. \qquad (5)$$

Time-decay for free KFP operator

The free KFP operator is unitarily equivalent with a direct integral of this family of complex harmonic oscillators. One deduces that

$$\sigma(P_0) = \overline{\cup_{\xi\in\mathbb{R}^n}\sigma(\hat{P}_0(\xi))} = [0, +\infty[.$$

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To study the time-decay of e^{-tP_0}, we introduce

$$\mathcal{L}^{2,s}(\mathbb{R}^{2n}) = L^2(\mathbb{R}^{2n}; \langle x\rangle^{2s} dxdv).$$

and

$$\mathcal{L}^p = L^p(\mathbb{R}^n_x; L^2(\mathbb{R}^n_v)), \quad p \geq 1.$$

Time-decay for the free KFP equation

Proposition 1

One has the following dispersive type estimate: $\exists C > 0$ *such that*

$$\|e^{-tP_0}u\|_{\mathcal{L}^\infty} \le \frac{C}{t^{\frac{n}{2}}}\|u\|_{\mathcal{L}^1}, \quad t \ge 3, \tag{6}$$

for $u \in \mathcal{L}^1$. *In particular, for any* $s > \frac{n}{2}$, *one has for some* $C_s > 0$

$$\|e^{-tP_0}u\|_{\mathcal{L}^{2,-s}} \le \frac{C_s}{t^{\frac{n}{2}}}\|u\|_{\mathcal{L}^{2,s}}, \tag{7}$$

for $t \ge 3$ *and* $u \in \mathcal{L}^{2,s}$.

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(6) is deduced from the estimate

$$\|e^{-t\hat{P}_0(\xi)}\|_{\mathcal{B}(L^2(\mathbb{R}^n_v))} \leq \frac{e^{-\xi^2(t-2-\frac{4}{e^t-1})}}{(1-e^{-t})^n}. \tag{8}$$

Limiting absorption principles for P_0

Set $R_0(z) = (P_0 - z)^{-1}$, $z \notin \mathbb{R}_+$. Denote

$$\mathcal{H}^{r,s} = \{u \in \mathcal{S}'(\mathbb{R}^{2n}); (1 - \Delta_v + v^2 + |D_x|^{\frac{2}{3}})^{\frac{r}{2}} u \in \mathcal{L}^{2,s}\}.$$

Let $\mathcal{B}(r, s; r', s')$ be the space of bounded operators from $\mathcal{H}^{r,s}$ to $\mathcal{H}^{r',s'}$. One has the following resolvent estimates

Proposition 2

(a). Let $n \geq 1$. For any $s > \frac{1}{2}$, the boundary values of the resolvent $R_0(\lambda \pm i0) = \lim_{\epsilon \to 0_+} R_0(\lambda \pm i\epsilon)$ exists $\mathcal{B}(-1, s; 1, -s)$ for $\lambda \in \mathbb{R} \setminus \mathbb{N}$ and is continuous in λ.
(b). Let $k \in \mathbb{N}$ and $n \geq 3$. The limits $R_0(k \pm i0) = \lim_{z \to k, \pm \Im z > 0} R_0(z)$ exist in $\mathcal{B}(-1, s; 1, -s)$ for any $s > 1$.

Open Question. Can one establish some high energy estimates for $R_0(\lambda \pm i0)$?

High energy resolvent estimates

Consider now the KFP operator P with a potential $V(x)$ satisfying

$$|\nabla V(x)| \leq C\langle x\rangle^{-\rho-1}, \quad x \in \mathbb{R}^n. \tag{9}$$

One can write $P = P_0 + W$ with $W = -\nabla V(x) \cdot \nabla_v$. If $\rho > -1$, W is relatively compact with respect to P_0. One has $\sigma_{\text{ess}}(P_0) = [0, +\infty[$. If z is an eigenvalue of P, then $\Re z \geq 0$ and 0 is the only possible eigenvalue with $\Re z = 0$.

High energy resolvent estimates

Theorem 2

Let $n \geq 1$ and assume (9) with $\rho \geq -1$. Then there exists $C > 0$ such that $\sigma(P) \cap \{z; |\Im z| > C, \Re z \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}\} = \emptyset$ and

$$\|R(z)\| \leq \frac{C}{|z|^{\frac{1}{3}}}, \tag{10}$$

and

$$\|(1 - \Delta_v + v^2)^{\frac{1}{2}} R(z)\| \leq \frac{C}{|z|^{\frac{1}{6}}}, \tag{11}$$

for $|\Im z| > C$ and $\Re z \leq \frac{1}{C}|\Im z|^{\frac{1}{3}}$.

In the proof, we use a semiclassical resolvent estimate due to Dencker-Sjöstrand-Zworski (CPAM, 2004).

Time-decay of the semigroup

To obtain time-decay estimates, we also need to study the spectrum of P near 0.

Theorem 3

Assume $n = 3$ and $\rho > 1$. Then for any $s > \frac{3}{2}$, one has for some $C > 0$

$$\|e^{-tP}\|_{\mathcal{B}(0,s;0,-s)} \le Ct^{-\frac{3}{2}}, \quad t > 0. \tag{12}$$

If $\rho > 2$, there exists $B_1 \in \mathcal{B}(-1, s; 1, -s)$ and some $\epsilon > 0$ such that

$$e^{-tP} = t^{-\frac{3}{2}} B_1 + O(t^{-\frac{3}{2}-\epsilon}) \tag{13}$$

in $\mathcal{B}(0, s; 0, -s)$ as $t \to +\infty$.

Low-energy resolvent estimates

To prove Theorem 3, we study the spectral properties of P near 0.

Theorem 4

Assume $n = 3$ and $\rho > 1$. Then 0 is not an accumulation point of the eigenvalues of P. one has the following expansions in $\mathcal{B}(-1, s; 1, -s)$

$$R(z) = A_0 + O(|z|^\epsilon), \text{ if } \rho > 1, s > 1, \tag{14}$$

$$R(z) = A_0 + z^{\frac{1}{2}} A_1 + O(|z|^{\frac{1}{2}+\epsilon}), \text{ if } \rho > 2, s > \frac{3}{2}, \tag{15}$$

for $\notin \mathbb{R}_+$ and $|z|$ small.

Ideas of the proof

To prove that P has no eigenvalues near 0, we use techniques of threshold spectral analysis and the supersymetry of P to show that $Pu = 0$ has no nontrivial solution $u \in \mathcal{L}^{2,-s}$ for any $s > 1$.

Ideas of the proof

To prove that P has no eigenvalues near 0, we use techniques of threshold spectral analysis and the supersymetry of P to show that $Pu = 0$ has no nontrivial solution $u \in \mathcal{L}^{2,-s}$ for any $s > 1$.
Theorem 3 on time-decay follows from the resolvent estimates on an appropriate contour in the right half complex plane.

A comment

One often says that the KFP operator P is closely related to the Witten Laplacian

$$-\Delta_V = (-\nabla_x + \nabla V(x)) \cdot (\nabla_x + \nabla V(x))$$

which is selfadjoint and elliptic. This can in particular be illustrated in terms of low-lying eigenvalues in the case when 0 is in the discrete spectrum.

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which is selfadjoint and elliptic. This can in particular be illustrated in terms of low-lying eigenvalues in the case when 0 is in the discrete spectrum.
In our case 0 is the bottom of the essential spectrum of P. In the case $V(x) \simeq c|x|^\mu$ for some $\mu < 1$ and $c > 0$, low-energy behavior of the resolvent $(-\Delta_V - z)^{-1}$ can be well understood. (Joint work in progress with J.-M. Bouclet).

A comment

In particular, if $0 < \mu < 1$, one can show that

$$(-\Delta_V - z)^{-1} = -\frac{\Pi_0}{z} + C_0 + zC_1 + z^2 C_2 + \cdots, \quad z \to 0, \Im z \neq 0, \quad (16)$$

in appropriate spaces, where Π_0 is the spectral projection associated with the eigenvalue 0 of $-\Delta_V$. Consequently,

$$e^{t\Delta_V} = \Pi_0 + O(t^{-\infty}) : L^2_{\mathrm{comp}}(\mathbb{R}^n_x) \to L^2_{\mathrm{loc}}(\mathbb{R}^n_x), t \to +\infty. \quad (17)$$

The model operator used in this case is the Schrödinger operator with a slowly decreasing potential studied by D. Yafaev (1982,1983), S. Nakamura (1994).

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Question. Can one prove similar results for the KFP operator P?

Thanks!