Some numerical and experimental advances in chaotic scattering

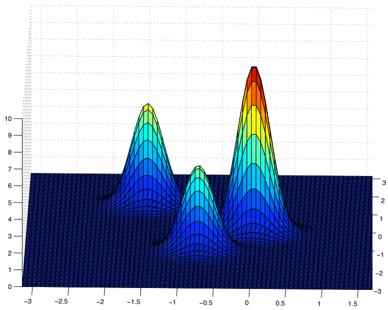
Microlocal Analysis and Spectral Theory 2013

Maciej Zworski

UC Berkeley

September 28, 2013

A scattering problem



$$V(x) = \sum_{j=1}^{3} a_j e^{-|x-x_j|^2/b_j}$$

We consider

$$ih\partial_t u = -h^2 \Delta u + V(x) u$$

$$ih\partial_t u = -h^2 \Delta u + V(x) u$$

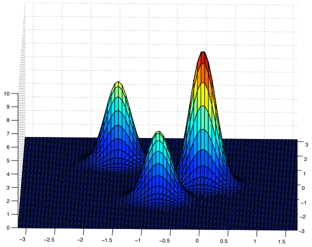
Newtonian dynamics:

$$x'(t) = 2\xi(t), \quad \xi'(t) = -\nabla V(x(t)),$$

$$\varphi_t(x(0), \xi(0)) := (x(t), \xi(t)).$$

Trapped set at energy E:

$$K_E := \{(x, \xi) : \xi^2 + V(x) = E, \ \varphi_t(x, \xi) \not\to \infty, \ t \to \pm\infty\}.$$



In the movies we saw the effects of Newtonian (classical) dynamics but we also saw oscillations, concentration and decay of waves.

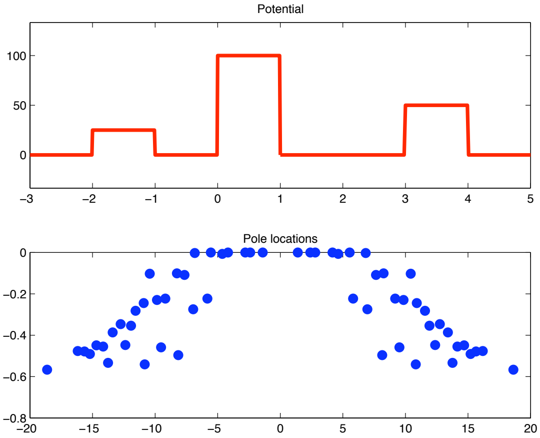
Quantum Resonances describe these waves resonating in interaction regions: there exist complex numbers

$$z_j(h) = E_j(h) - i\Gamma_j(h), \quad \Gamma_j(h) > 0,$$

and $w_j(x) \notin L^2$ (resonant states), such that

$$(P - z_j(h))w_j = 0, \quad w_j \text{ is outgoing } .$$

Quantum Resonances describe the resonating waves:



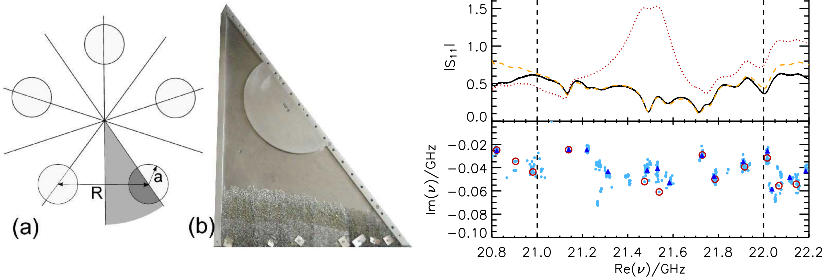
Computed using `squarepot.m`

`http://www.cims.nyu.edu/~dbindel/resonant1d/`

Here is how they sound:

```
time = linspace(0,500,5000);
sound(real(exp(-i*z*time)))
```

A real experimental example

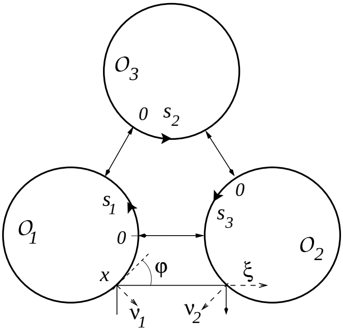


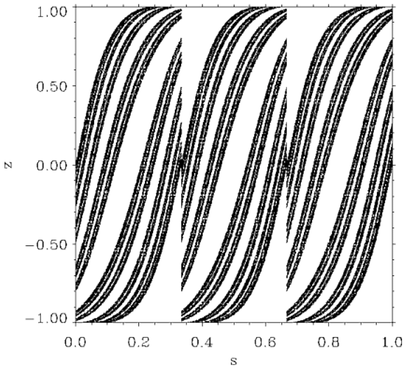
Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Z ’12

Resonances for three discs:

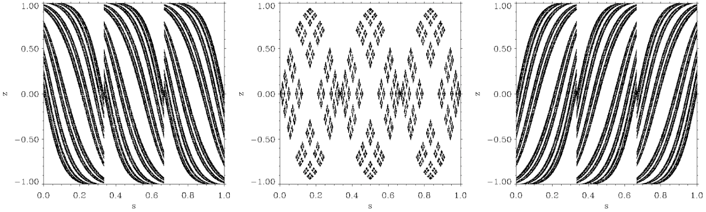
Barkhofen–Kuhl–Weich '13

Resonances for three discs:





Barkhofen–Kuhl–Weich '13



incoming set trapped set outgoing set

Poon–Campos–Ott–Grebogi ’96

Resonances for three discs:

Resonant states are microlocalized on the outgoing set:
Helffer–Sjöstrand '85, Bony–Michel '04, Nonnenmacher–Rubin '07.

Vol. 60, No. 1 DUKE MATHEMATICAL JOURNAL © February 1990

GEOMETRIC BOUNDS ON THE DENSITY OF RESONANCES FOR SEMICLASSICAL PROBLEMS

JOHANNES SJÖSTRAND

0. Introduction. In this paper, we shall give upper bounds on the number of resonances in certain regions in the complex plane close to the real axis, for semiclassical operators like

$$-h^2\Delta + V(x), \tag{0.1}$$

when h is small.

Many of the phenomena are similar (or can be expected to be similar) to the corresponding ones for the exterior Dirichlet (or Neumann) problem for the Helmoltz equation

$$(\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^n \setminus A, \tag{0.2}$$

where A is bounded and has a smooth boundary (and n is odd unless otherwise specified). This case has been a source of inspiration for our study of the semiclassical case, and we also believe that the new results presented in this paper for the semiclassical case, will have analogues for (0.2); so, in order to situate our results, it can be useful to recall some known results also for the exterior problem.

For the problem (0.2), the resonances are usually defined in the framework of the Lax-Phillips scattering theory [LPh] as poles of the scattering matrix, but we can also view them as certain complex values, k in the lower half plane, for which (0.2) has a non-trivial solution in a suitable space. When studying the location of resonances, the geometry usually enters in an essential way. If the obstacle is non trapping in the sense that no maximal optical rays in $\mathbb{R}^n \setminus A$, (reflected according to the rules of optics in ∂A,) can be contained in a bounded set, it follows from the results on propagation of mod(C^∞) singularities of Melrose-Sjöstrand and Ivrii (see the book of Hörmander [Hö] and references given there) combined with the Lax-Philips theory, that there are only finitely many resonances in a logarithmic neighborhood of the real axis. In the case of nontrapping obstacles with analytic boundary, Bardos-Lebeau-Rauch [BLeR] showed that there can only be finitely many resonances inside a parabolic neighborhood of the real axis of the form $\mathrm{Im}(k) \geqslant -C^{-1}|\mathrm{Re}\, k|^{1/3}$. Under additional assumptions, they also determined the optimal value of the constant C. Again this result is based on a result of propagation of singularities now modulo Gevrey 3, due to Lebeau.

Received May 16, 1989. Revision received August 8, 1989.

In the case of trapping obstacles, Ikawa [I1] studied the simplest possible case, when A has two strictly convex components, and he found that in a strip: $0 > \mathrm{Im}(k) > -C_0$, with C_0 conveniently chosen, the resonances form a "string" parallel to the real axis, and he was also able to give asymptotic expansions for the precise location of these resonances. C. Gérard [G] improved this result by determining the asymptotics of the resonances in any fixed strip as above, with C_0 arbitrary. He found several "strings". The geometry in this case is still very simple. (The analogue of this result for the semiclassical case was obtained in [GS1]). There is only one trapped optical ray, which is periodic for the natural optical flow in the cosphere bundle, $S^*(\mathbb{R}^n \setminus A)$, and the corresponding linearized Poincaré map is hyperbolic, that is, has no eigenvalue of modulus 1.

More complicated situations appear if we relax the assumption about strict convexity for the components of A, or if we admit more than 2 components. In these cases, Ikawa [I2, 4] proved the existence of infinitely many resonances in a strip along the real axis. In the case of several strictly convex obstacles, he was also able to determine strips that can host only finitely many resonances ([I3]).

As for estimates on the number of resonances, Melrose [M] showed that the number of resonances in a disc of radius λ is $\mathcal{O}(\lambda^n)$, and he also obtained a similar estimate when we replace the obstacle by a compactly supported potential. The latter result was extended (in a modified form) to the even dimensional case by Intissar [In], and Zworski [Z] was recently able in the odd dimensional potential case to show that we have the same estimate as for the obstacle case.

In the semiclassical case, much of the work so far has been based on the method of complex scaling of Aguilar-Combes [ACo] and Balslev-Combes [BaCo] or on variants of this method. The works of Klein [K], Briet-Combes-Duclos [BrCoDu1], Nakamura [N1, 2] give results on the absence of resonances in certain regions, under certain geometrical "virial" conditions which imply (without being equivalent to) the natural condition, which is absence of trapped trajectories for the corresponding classical Hamiltonian.

A more microlocal approach to semiclassical resonances was developed by Helffer and the author in [HS], and Helffer and Martinez [HMa] showed that this definition is equivalent to the more classical ones in cases of overlap. It might be useful to recall here some ingredients of this approach, in the case of the operator (0.1), and we refer to [HS], for the case of more general elliptic operators. (The main theorems of the present paper apply to these more general operators.)

Let $r, R \in C^\infty(\mathbb{R}^n; \mathbb{R})$ satisfy

$$r \geqslant 1, \quad R > 0, \quad rR \geqslant 1, \quad |\partial^\alpha r(x)| \leqslant C_\alpha r(x) R(x)^{-|\alpha|}, \tag{0.3}$$
$$|\partial^\alpha R(x)| \leqslant C_\alpha R(x)^{1-|\alpha|},$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = (\partial_{x_1})^{\alpha_1} \ldots (\partial_{x_n})^{\alpha_n}$. In this paper we only consider operators with real analytic coefficients (while the general theory of [HS] allows for coefficients which are analytic outside a bounded set), so

Sjöstrand '90:

Suppose $P = -h^2\Delta + V$ where V is analytic (and reasonable). Suppose that the classical flow is hyperbolic on K_E.

Then resonances of of P, $z_j(h)$, satisfy

$$\#\{z_j(h) \in [E-\epsilon, E+\epsilon]-i[0,h]\} \leq Ch^{-m/2}, \quad m > \dim \cup_{|E'-E|<2\epsilon} K_{E'}.$$

Here the dimension is the Minkowski/box dimension: for $M \subset \mathbb{R}^k$,

$$\mathrm{codim}\, M = \sup\{\gamma : \limsup_{\epsilon\to 0} \epsilon^{-\gamma} \mathrm{vol}_{\mathbb{R}^k}(\{\rho : d(\rho, M) < \epsilon\}) < \infty\}.$$

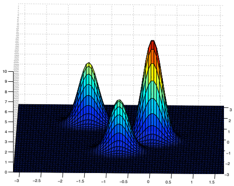
Earlier, non-geometric bounds: Regge '58, Melrose '82, Intissar '86, Z '87,'89.

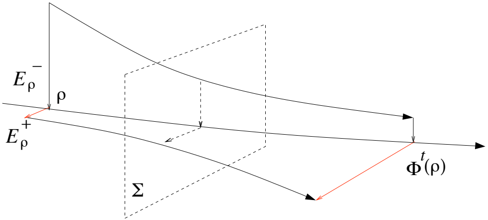
Sjöstrand '90:

$$\#\{z_j(h) \in [E-\epsilon, E+\epsilon] - i[0,h]\} \leq Ch^{-m/2}, \quad m > \dim \cup_{|E'-E|<2\epsilon} K_{E'}.$$

$$K_E := \{(x,\xi) : \xi^2 + V(x) = E,\ \varphi_t(x,\xi) \not\to \infty,\ t \to \pm\infty\}.$$

$$\mathrm{codim}\, M = \sup\{\gamma : \limsup_{\epsilon\to 0} \epsilon^{-\gamma} \mathrm{vol}_{\mathbb{R}^k}(\{\rho : d(\rho, M) < \epsilon\}) < \infty\}.$$





More recently:

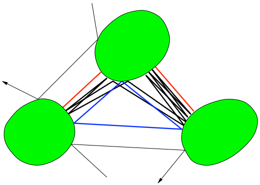
Sjöstrand–Z '07:

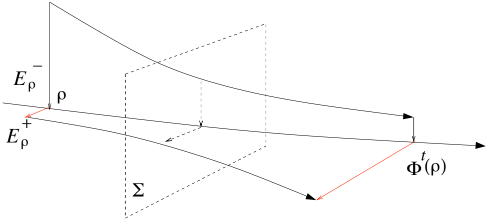
Resonances for $-h^2\Delta + V$ where $V \in \mathcal{C}_c^\infty(\mathbb{R}^n;\mathbb{R})$ (and more general operators)

$$\#\{z_j(h) \in [E-h, E+h] - i[0,h]\} \le Ch^{-\mu}, \quad 2\mu + 1 > \dim K_E.$$

Nonnenmacher–Sjöstrand–Z '13:

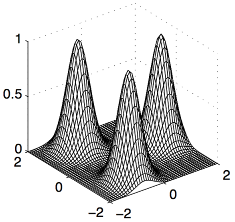
Resonances for $-\Delta$ on $\mathbb{R}^n \setminus \bigcup_{j=1}^J \mathcal{O}_j$ (and more general operators).

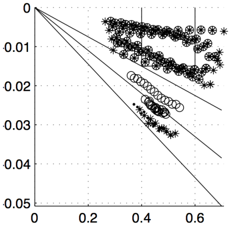




Numerical studies:

Lin '02:





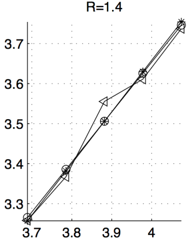


Figure 24: For $R = 1.4$: Triangles represent numerical data, circles least squares regression, and stars the slope predicted by the conjecture. $\hbar$ ranges from 0.025 down to 0.017.

Lu–Sridhar–Z '03:

VOLUME 91, NUMBER 15 PHYSICAL REVIEW LETTERS week ending 10 OCTOBER 2003

Fractal Weyl Laws for Chaotic Open Systems

W. T. Lu,[1] S. Sridhar,[1] and Maciej Zworski[2]

[1]*Department of Physics and Electronic Materials Research Institute, Northeastern University, Boston, Massachusetts 02115, USA*

[2]*Department of Mathematics, University of California, Berkeley, California 94720, USA*

(Received 13 August 2002; revised manuscript received 30 May 2003; published 8 October 2003)

We present a conjecture relating the density of quantum resonances for an open chaotic system to the fractal dimension of the associated classical repeller. Mathematical arguments justifying this conjecture are discussed. Numerical evidence based on computation of resonances of systems of n disks on a plane are presented supporting this conjecture. The result generalizes the Weyl law for the density of states of a closed system to chaotic open systems.

DOI: 10.1103/PhysRevLett.91.154101 PACS numbers: 05.45.Mt, 05.45.Ac, 03.65.Sq, 31.15.Gy

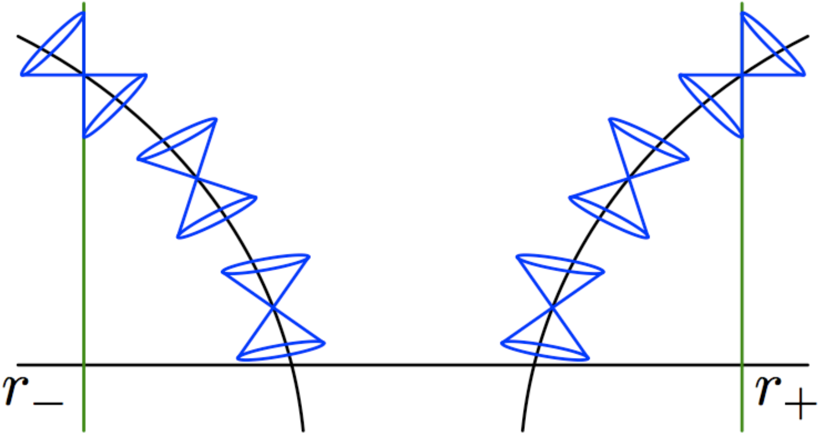
The celebrated Weyl law concerning the density of eigenvalues of bound states is a central result in the spectroscopy of quantum systems [1]. The Weyl formula states that the asymptotic level number $N(k)$, defined as the number of levels with $k_n < k$ (where $k \to \infty$), is given after smoothing by $N(k) \equiv \{k_n : k_n \leq k\} = Vk^D/(D/2)!(4\pi)^{D/2} + \cdots$ for a quantum system bounded in a region R of D-dimensional space whose volume is V. For closed systems with smooth boundaries, the Weyl formula is well established, and although primarily valid in the semiclassical limit, nevertheless can be applied did not restrict ourselves to an energy surface. For closed two-dimensional systems, we have real zeros only and $N(k) = \{k_n : k_n \leq k\} \sim k^2$, which is consistent with (1) as $d_H = 1$. Then everything is trapped.

Our motivation comes from rigorous work on quantum resonances and, in particular, from the work of Sjöstrand [4] on geometric upper bounds on their density. The optimal nature of that bound was recently indicated by a numerical experiment [5] involving a computation of quantum resonances for semiclassical Schrödinger operators with chaotic classical dynamics.

The reason for showing the paper is to indicate that to communicate an idea it helps to publish it in physics.

Dyatlov '13 (math) , Dyatlov–Z '13 (physics)

Weyl law for quasi-normal modes/resonances for perturbations of Kerr-de Sitter metrics (rotating black holes).



Dyatlov '13 (math), Dyatlov–Z '13 (physics)

Weyl law for quasi-normal modes/resonances for perturbations of Kerr-de Sitter metrics (rotating black holes).
The trapped set as a changes from 0 to 1:

Wunsch–Z '11: The key property of this smooth trapped set is the r-normal hyperbolicity for any r.

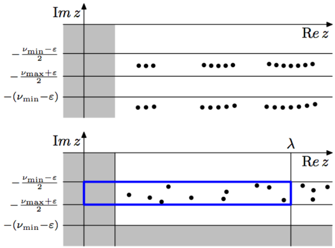
Hirsch–Pugh–Schub '77: stable under small C^r perturbations.

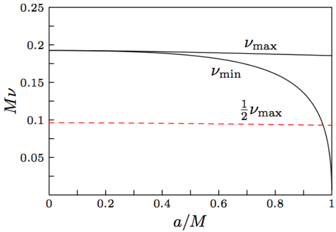
Dyatlov '13 (math), Dyatlov–Z '13 (physics)

Weyl law for quasi-normal modes/resonances for perturbations of Kerr-de Sitter metrics (rotating black holes).

When the transversal expansion rates satisfy $\nu_{\max} < 2\nu_{\min}$ (valid for 98% of rotation speeds of black holes) then

$$\#\{z_j \in \text{the blue box}\} = \frac{\lambda^2}{(2\pi)^2}\mathrm{vol}(\cup_{E<1} K_E)(1 + o(1)),$$





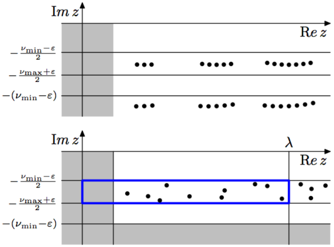
Sjöstrand–Z '99: Asymptotics for resonances for convex obstacles satisfying a pinching condition (cubic bands).

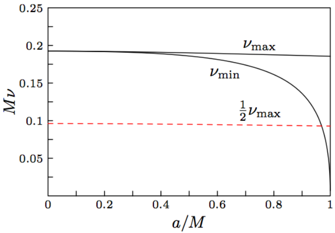
Dyatlov '13 (math), Dyatlov–Z '13 (physics)

Weyl law for quasi-normal modes/resonances for perturbations of Kerr-de Sitter metrics (rotating black holes).

When the transversal expansion rates satisfy $\nu_{\text{max}} < 2\nu_{\text{min}}$ (valid for 98% speeds of rotation of the black hole) then

$$\#\{z_j \in \text{the blue box}\} = \frac{\lambda^2}{(2\pi)^2} \mathrm{vol}(\cup_{E<1} K_E)(1 + o(1)),$$





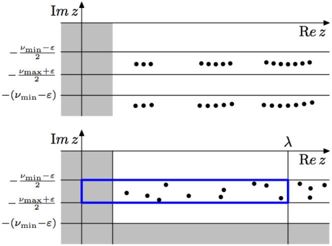
Faure–Tsujii '13: Similar asymptotics for the Policott–Ruelle resonances for contact Anosov flows.

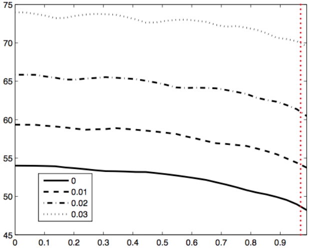
Dyatlov '13 (math), Dyatlov–Z '13 (physics)

Weyl law for quasi-normal modes/resonances for perturbations of Kerr-de Sitter metrics (rotating black holes).

When the transversal expansion rates satisfy $\nu_{\max} < 2\nu_{\min}$ (valid for 98% speeds of rotation of the black hole) then

$$\#\{z_j \in \text{the blue box}\} = \frac{\lambda^2}{(2\pi)^2} \mathrm{vol}(\cup_{E<1} K_E)(1 + o(1)),$$





Faure–Tsujii '13: Similar asymptotics for the Policott–Ruelle resonances for contact Anosov flows.

A simpler model.

Nonnenmacher–Z '05, '07': quantized open Baker maps
(Balazs–Voros '89, Saraceno '90)

Classical relation:

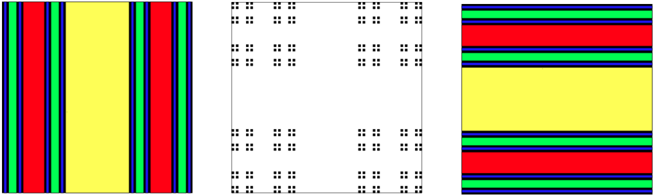
$$(q,p) \sim (q',p') \iff \begin{cases} q' = 3q, & p' = p/3, & 0 \leq q \leq 1/3 \\ q' = 3q-2, & p' = (p+2)/3, & 2/3 \leq q < 1. \end{cases}$$

Quantum operator:

$$M_N = \mathcal{F}_{3N}^* \begin{bmatrix} \mathcal{F}_N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_N \end{bmatrix}.$$

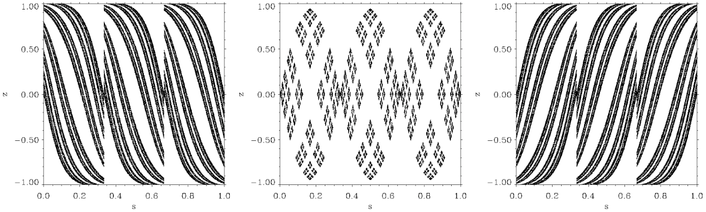
($\mathcal{F}_P$ is the discrete Fourier transform on $\mathbb{C}^P$).

Open Baker map:

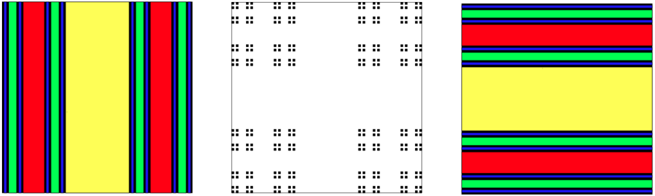


incoming set trapped set outgoing set

Three discs reduced to the boundary:



Open Baker map:

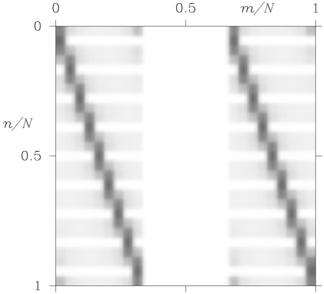


Expected fractal Weyl law: for $0 < r < r_0 < 1$,

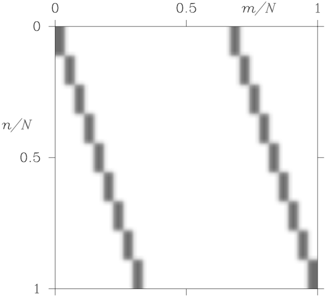
$$\sharp\{\lambda \in \mathrm{Spec}(M_N), |\lambda| > r\} \sim N^{\frac{\log 2}{\log 3}}, \quad M_N = \mathcal{F}_{3N}^* \begin{bmatrix} \mathcal{F}_N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{F}_N \end{bmatrix}.$$

$N = 3^k$	$r = 0.1$	$r = 0.2$	$r = 0.3$	$r = 0.4$	$r = 0.5$	$r = 0.6$	$r = 0.7$	$r = 0.8$
$k = 1$	5	5	5	5	5	4	3	3
$k = 2$	14	14	10	9	8	8	7	6
$k = 3$	32	26	23	19	16	16	14	5
$k = 4$	63	53	45	40	33	33	30	6
$k = 5$	124	103	85	78	71	65	63	11
$k = 6$	237	196	161	150	142	131	128	12

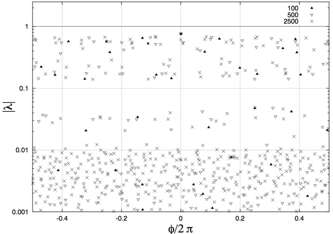
Nonnenmacher–Z '07: for a simplified quantum Baker map corresponding to a complicated classical chaotic relation we have the fractal Weyl law for a sequence $N = 3^k$ (the Walsh model).

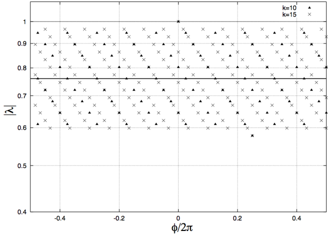


$N = 27$



$N = 27$





Recent works in physics using variants of the quantum open maps (and other methods):

Schomerus–Tworzydło '04, Keating et al '06, Wiersig–Main '08, Ramilowski et al '09, Pedrosa et al '09, Shepelyansky '09, Shomerus–Wiersig–Main '09, Ermann–Shepelyansky '10, Kopp–Schomerus '10, Eberspächer–Main–Wunner '10, Körber et al '13.

An interdisciplinary example:

Fractal Weyl law for Linux Kernel architecture

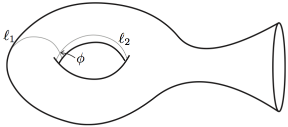
L. Ermann[1], A.D. Chepelianskii[2], and D.L. Shepelyansky[1,a]

[1] Laboratoire de Physique Théorique (IRSAMC), Université de Toulouse, UPS-CNRS, 31062 Toulouse, France
[2] LPS, Université Paris-Sud, CNRS, UMR8502, 91405 Orsay, France

Received 11 October 2010
Published online 21 December 2010 –

Abstract. We study the properties of spectrum and eigenstates of the Google matrix of a directed network formed by the procedure calls in the Linux Kernel. Our results obtained for various versions of the Linux Kernel show that the spectrum is characterized by the fractal Weyl law established recently for systems of quantum chaotic scattering and the Perron-Frobenius operators of dynamical maps. The fractal Weyl exponent is found to be $\nu \approx 0.65$ that corresponds to the fractal dimension of the network $d \approx 1.3$. An independent computation of the fractal dimension by the cluster growing method, generalized for directed networks, gives a close value $d \approx 1.4$. The eigenmodes of the Google matrix of Linux Kernel are localized on certain principal nodes. We argue that the fractal Weyl law should be generic for directed networks with the fractal dimension $d < 2$.

A yet different setting: manifolds with hyperbolic ends



Resonances defined as poles of $(-\Delta_X - (n-1-s)s)^{-1}$, continued from $\operatorname{Im} s > (n-1)/2$; X is a manifold with hyperbolic ends.

Fractal upper bounds:

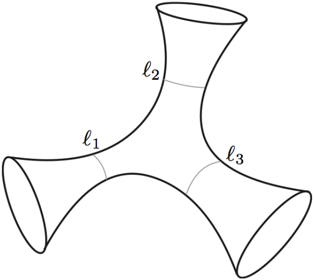
Z '99: $\Gamma\backslash\mathbb{H}^2$, Γ convex co-compact (based on Sjöstrand '90)

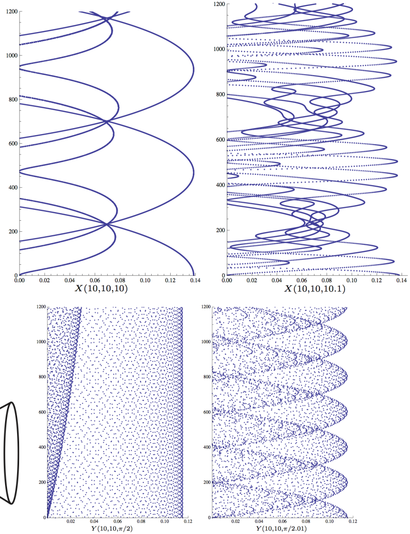
Lin–Guillopé–Z '04: $\Gamma\backslash\mathbb{H}^2$, Γ a Schottky group (based on some new Selberg zeta function techniques)

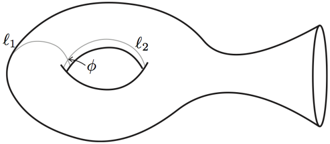
Datchev–Dyatlov '13: any manifold with hyperbolic ends (based on Sjöstrand-Z '07 and a new approach to meromorphic continuation by Vasy '13)

Other models using zeta functions: hyperbolic rational maps. Here the growth of zeros of the zeta function is related to the dimension of the Julia set. Strain-Z '03, Christianson '05.

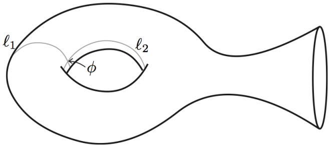
Borthwick '13:

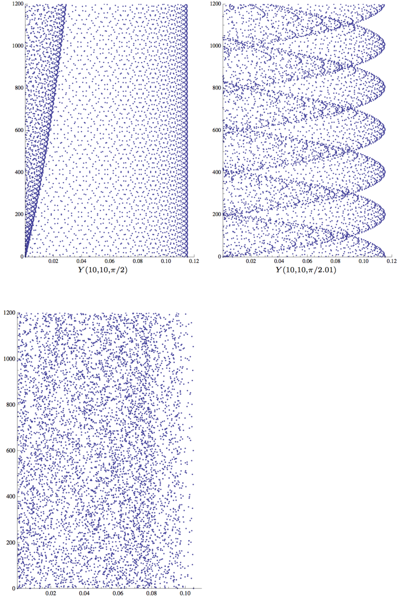






Borthwick '13:

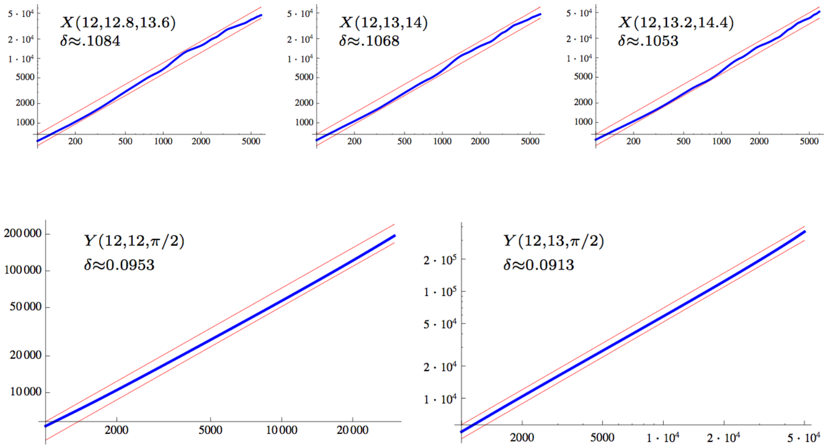




$$\ell_1 = 10, \quad \ell_2 = 12, \quad \varphi = 2\pi/5$$

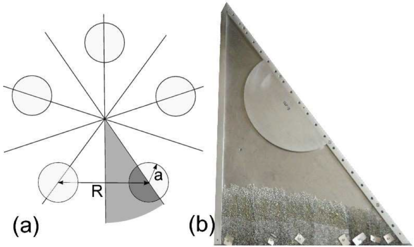
Borthwick '13

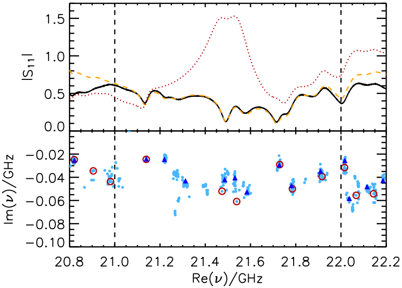
Comparison with the fractal Weyl law:



Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Z ’12

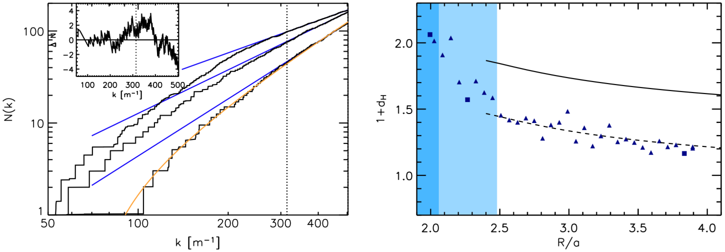
Experimental investigation of fractal Weyl laws.





Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Z '12

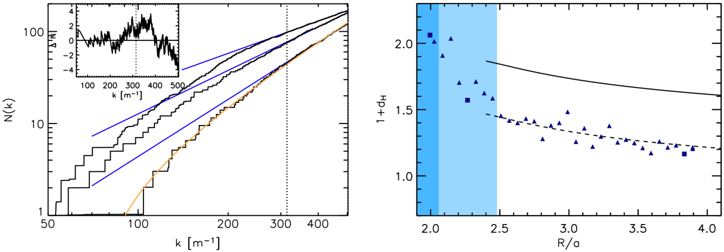
Experimental investigation of fractal Weyl laws.



Left: The counting functions for $R/a = 2, 2.25, 3.9$ Fits of their slope for high frequencies are shown in blue. The orange curve over the lower histogram corresponds to the Weyl formula with 12% loss. Plotted in the inset is the difference between the Weyl formula with 12% loss and the experimental counting function for the closed system ($R/a = 2$).

Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Z '12

Experimental investigation of fractal Weyl laws.



Right: The data points correspond to the fitted exponent of the counting function in dependence of the R/a parameter. The three squares mark the examples which have already been presented in the previous figures. The darker shaded blue region indicates the R/a values without open channels; lighter shaded blue region has only a few open channels.

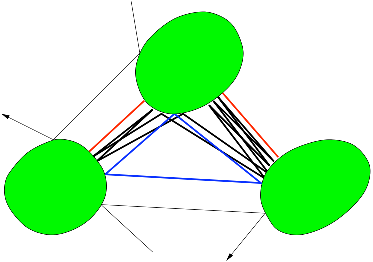
This may not seem to be so succesful but it lead to an interesting experiment about the gap between the real axis and resonances.

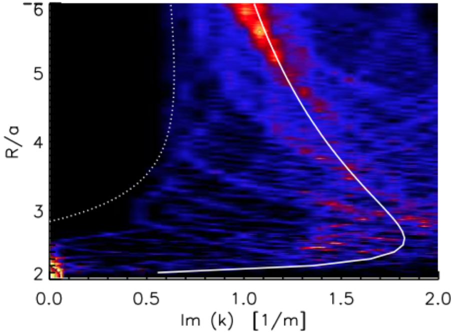
Barkhofen–Weich–Potzuweit–Kuhl–Stöckmann–Z '13

We look for $\gamma > 0$ such that there are no resonances in

$$\operatorname{Im} z > -\gamma, \quad \operatorname{Re} z > C_0$$

How do we determine that gap at the high frequency limit when the dynamics is hyperbolic?

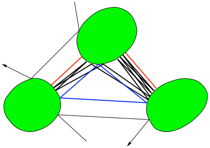


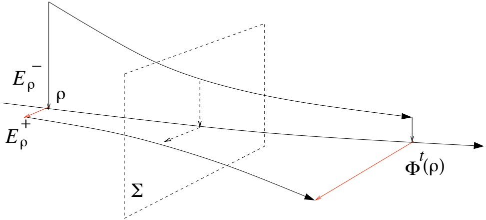


(b)3 Disk Simulation

Gaspard-Rice '89, Lu-Sridhar-Z '03, Barkhofen et al '12

Ikawa '88, Burq '93, Nonnenmacher-Z '09, Naud '04,'12, Petkov-Stoyanov '11





We define the topological pressure associated to the unstable Jacobian:

$$J_t^+(\rho) = \det \left(d\Phi^t_{|E_\rho^+} \right)$$

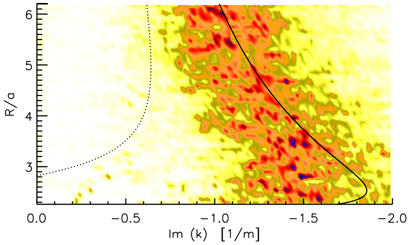
$$\mathcal{P}_E(s) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{T-1<T_\gamma<T} J^+(\gamma)^{-s} \,,$$

where γ are closed orbits with period T_γ.

Ikawa '88, Nonnemacher-Z '09, Petkov-Stoyanov '11:
There are no resonances with $\mathrm{Im}\,\lambda > P_E(1/2)$
(at high energies)

There are no resonances with $\mathrm{Im}\,\lambda > P_E(1/2)$
(at high energies)

The decay of correlations is closely related to *resonance free strips*.

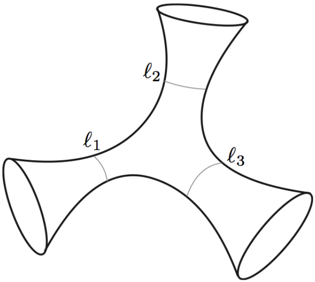


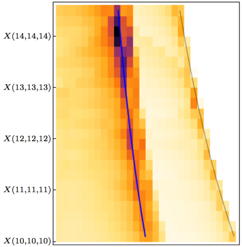
Potzuweit-Weich-Barkhofen-Kuhl-Stöckmann-Z, PRL '13

Lu-Sridhar-Z '03: concentration of decay rates at $P(1)/2$, PRL '03

It is also seen in the case of scattering on hyperbolic sufaces.

Borthwick '13:





Naud '13: If $\dim K_1 = 2\delta + 1$ then

$$\#\{s_j : \sigma < \mathrm{Re} s_j, |\mathrm{Im}\, s_j| < r\} = \mathcal{O}(r^{1+\tau(\sigma)}),$$

where $\tau(\sigma) < \delta$ for $\sigma < \delta/2$.

Fractal Weyl law (Z '99, Lin-Guillopé-Z '04, Datchev-Dyatlov '13) gives the bound $r^{1+\delta}$ for all σ.

Thank you!