Introduction to

(some aspects in)

quantum chaos

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Chaos in quantum systems?

Classical chaos:

sensitive dependence on initial conditions for trajectories

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origin: nonlinearity

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Study of quantum systems, whose classical dynamics is “chaotic”

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no trajectories

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Linear time evolution: No chaos!?

Quantum chaos: Pragmatic “definition”:

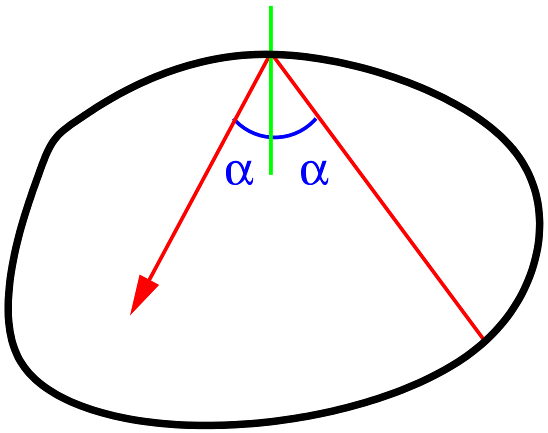
Study of quantum systems, whose classical dynamics is “chaotic”

What are the properties of eigenvalues and eigenfunctions

if the corresponding classical system is chaotic?

Classical billiards

Free motion of a point particle in some Euclidean domain $\Omega \subset \mathbb{R}^2$ with elastic reflections at the boundary $\partial\Omega$.



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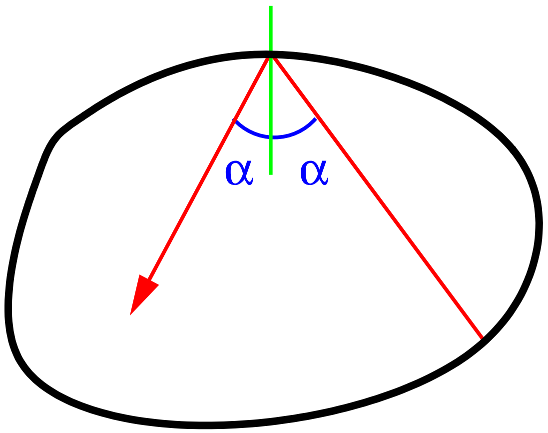
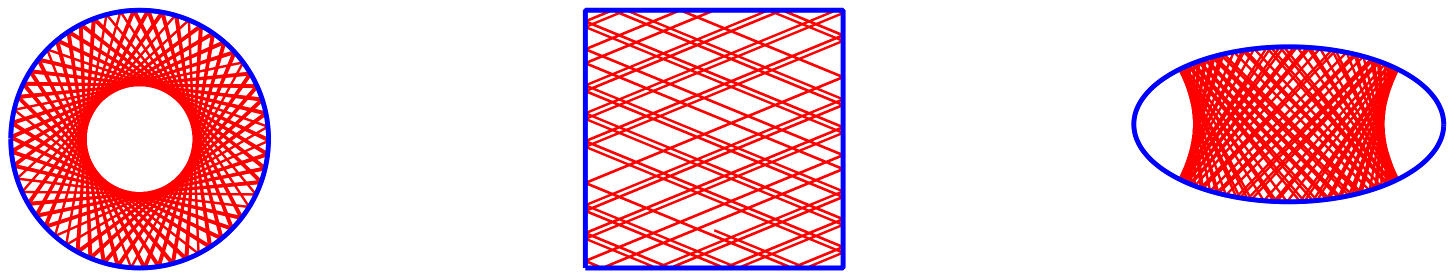
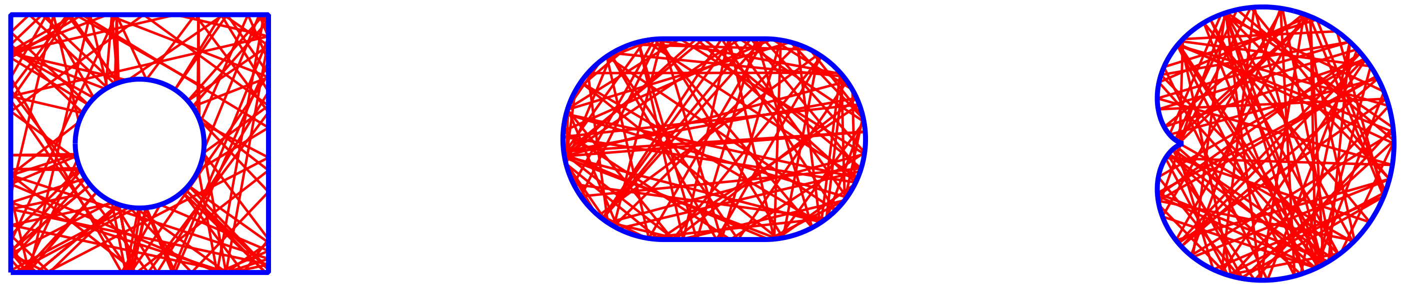


Depending on the boundary one obtains completely *different dynamical behaviour*:

Integrable systems (regular motion)



Chaotic systems



Properties of “chaotic“ billiards

Degrees of chaoticity

- ergodicity
- mixing
- K-systems
- Bernoulli

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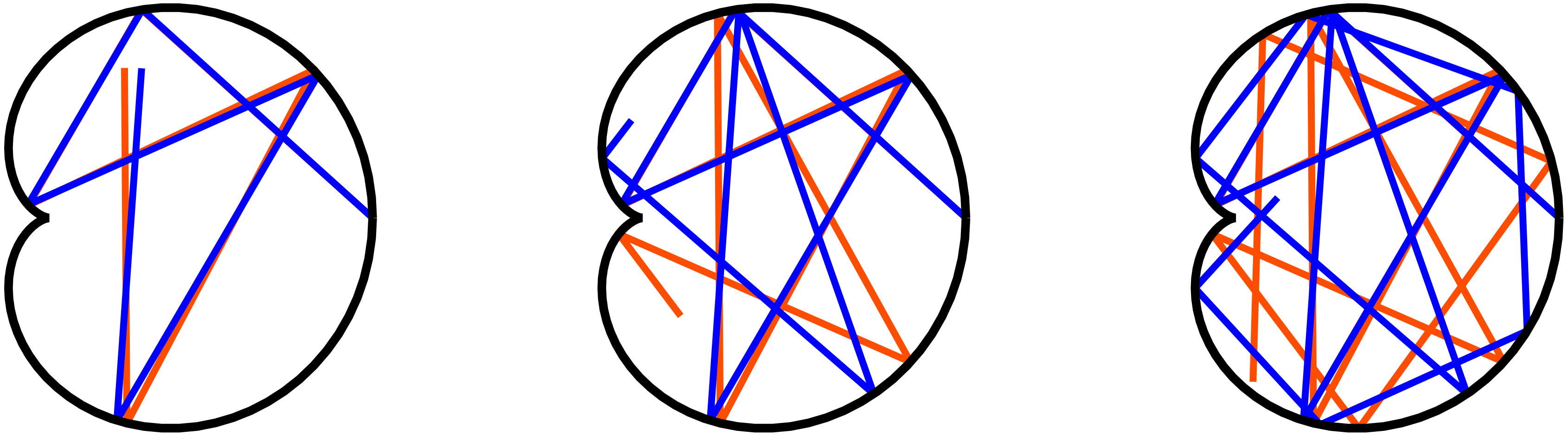
“Sensitive dependence on the initial conditions”



$t = 10s$ $t = 15s$ $t = 20s$

Classical billiards – Mathematical description

Phase space: $T^\star\Omega = \{(\boldsymbol{p}, \boldsymbol{q}) \mid \boldsymbol{p} \in \mathbb{R}^2, \ \boldsymbol{q} \in \Omega\}$

Billiard flow: $\Phi^t(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}(t), \boldsymbol{q}(t))$

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Surfaces of constant energy E

$$\Sigma_E := \left\{(\boldsymbol{p},\boldsymbol{q}) \in \mathbb{R}^2 \times \Omega \mid \boldsymbol{p}^2 = E\right\} \quad .$$

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Invariant measure: is the **Liouville measure**

$$\mathrm{d}\nu = \frac{1}{\mathrm{vol}(\Sigma_E)} \delta(E - H(\boldsymbol{p}, \boldsymbol{q}))\, \mathrm{d}^2 p\, \mathrm{d}^2 q \ .$$

Remark: A measure ν is called invariant if $\nu(A) = \nu(\phi^t A)$ for all measurable $A \subset T^\star\Omega$.

Classical billiards – time and spatial averages

Consider a one-parameter group of automorphisms $\{\Phi^t\}$ of a measure space M with invariant probability measure ν.

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Definition *The time average* $\overset{*}{f}$ *of a function of a function* $f : M \to \mathbb{R}$ *(if it exists) is given by*

$$\overset{*}{f}(X) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\Phi^t X)\, dt \ , \qquad X \in M \ .$$

The spatial average (if it exists) is defined by

$$\bar{f} = \int_M f(X)\, d\nu \ .$$

Remark: The Birkhoff ergodic theorem shows that for $f \in L^1(M, \nu)$ the time average exists (ν a.e.).

Ergodicity

Definition *A system is called ergodic if for any function* $f \in L_1(M, \nu)$ *the time average equals the spatial average*

$$\overset{*}{f}(X) = \bar{f} \qquad \text{for almost every point } X \in M \quad .$$

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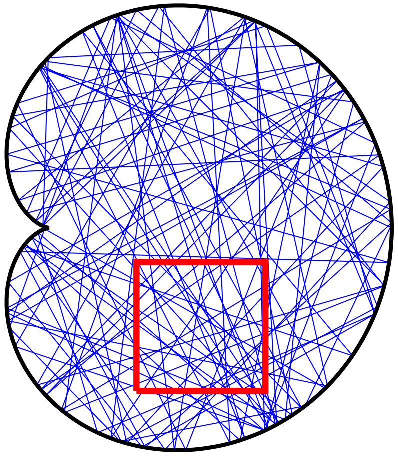
Therefore, ergodicity means that a typical trajectory fills M (eg. the equi–energy surface Σ_E) densely in a uniform way.

However, it does not mean that a typical trajectory hits every point in M.

Classical ergodicity of a flow $\{\phi^t\}$, position space

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_D(\phi^t(\boldsymbol{p},\boldsymbol{q}))\,\mathrm{d}t = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$



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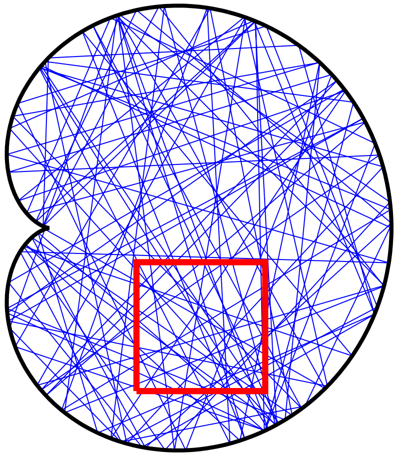

Remarks:

- $\chi_D(X)$: characteristic function of D: $\chi_D(X) = \begin{cases} 1 & X \in D \\ 0 & X \notin D \end{cases}$
- "Almost everywhere": Let $\mathcal{S}$ be the set of initial conditions for which the above does not hold.
  Then $\nu(\mathcal{S}) = 0$.

Definition *The time–correlation function of two functions* $f_1, f_2 \in L^2(M, \nu)$ *is defined by*

$$C(t) = \int_M f_1(\Phi^t X) f_2(X) \, d\nu \quad .$$

Definition *A flow is called mixing if*

$$\lim_{t \to \infty} C(t) = \int_M f_1(X) \, d\nu \, \int_M f_2(X) \, d\nu = \overline{f}_1 \overline{f}_2 \quad .$$

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Remarks:
- A mixing system is ergodic
- An ergodic system is not necessarily mixing (Eg.: irrational translations on S^1)

Definition (Periodic orbits)

A periodic orbit γ *is a trajectory which returns to its initial point in phase space after some time* $t > 0$.

I.e.: for a point $(\boldsymbol{p}, \boldsymbol{q})$ *on a periodic orbit* $\exists t \in \mathbb{R}^+$ *s.t.*

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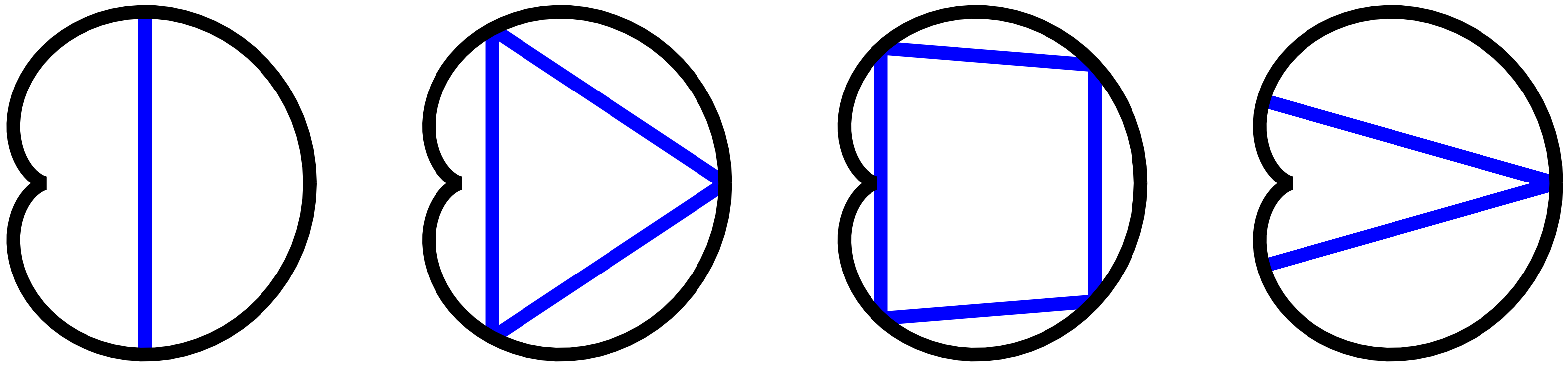
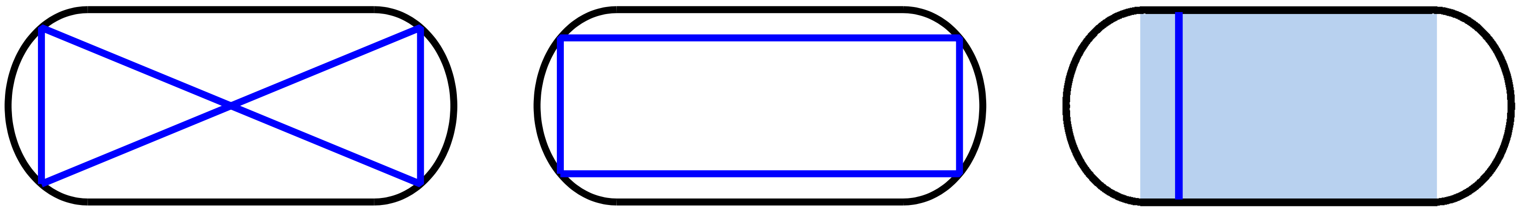
Examples:

Cardioid:

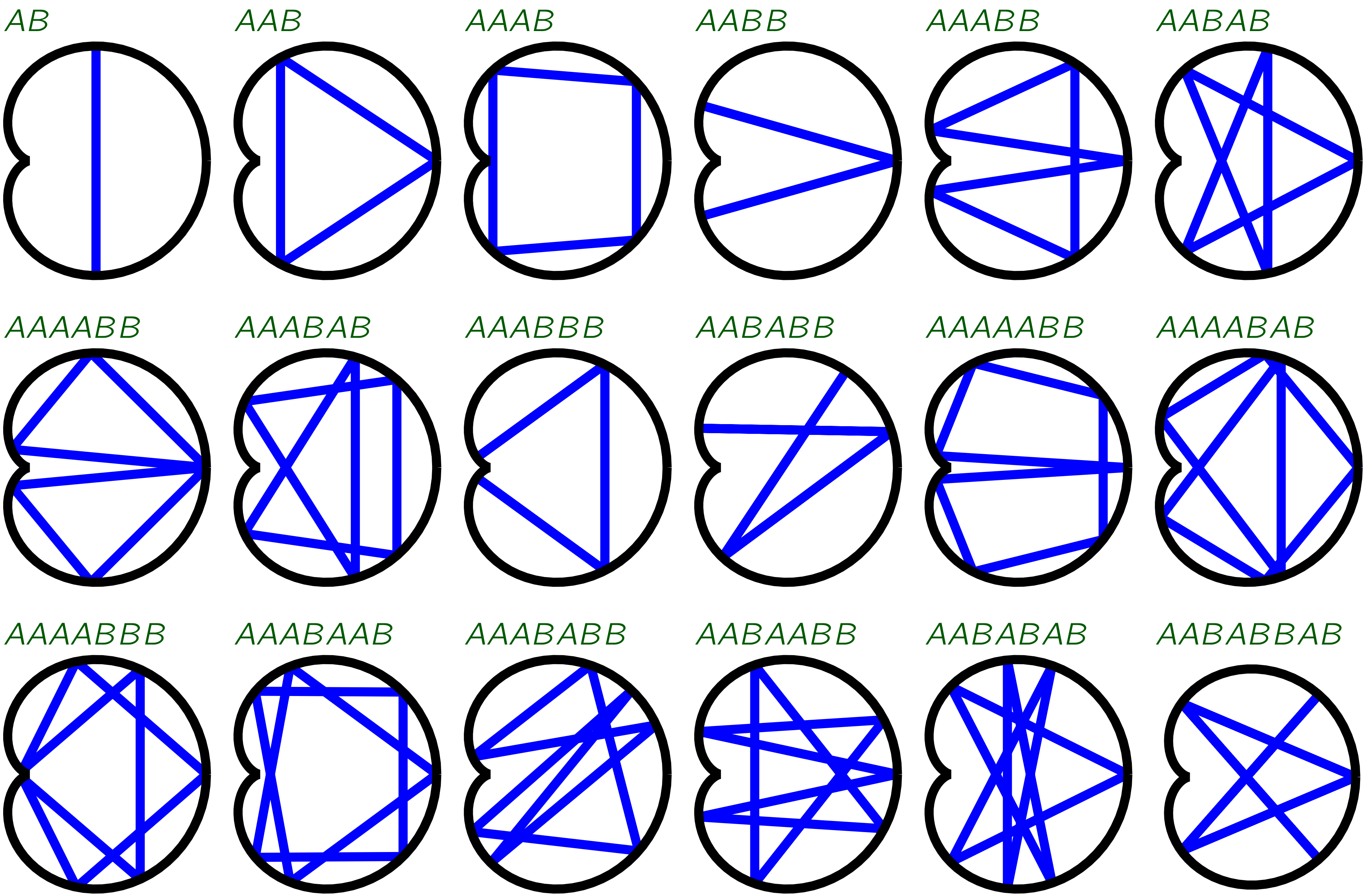


Stadium:



Classical billiards — periodic orbits

To compute periodic orbits systematically: **symbolic dynamics**



Classically: State: point in phase space $(\boldsymbol{p}, \boldsymbol{q}) \in T^{\star}\Omega$

Time-evolution: Hamilton-equations

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Quantum mechanically: State: wave function $\Psi(\boldsymbol{q}, t)$

Interpretation: $|\Psi(\boldsymbol{q}, t)|^2$ is the probability density of finding the particle in the point $\boldsymbol{q}$ at time t.

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$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\boldsymbol{q}, t) = \widehat{H} \psi(\boldsymbol{q}, t)$$

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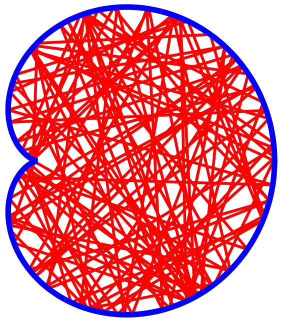
Illustration: wave-packet, initially centered in point $\boldsymbol{q}$

QM: Time evolution

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(\boldsymbol{q}, t) = \widehat{H}\psi(\boldsymbol{q}, t)$$



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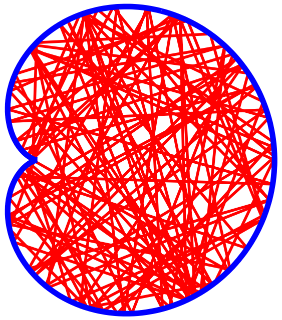
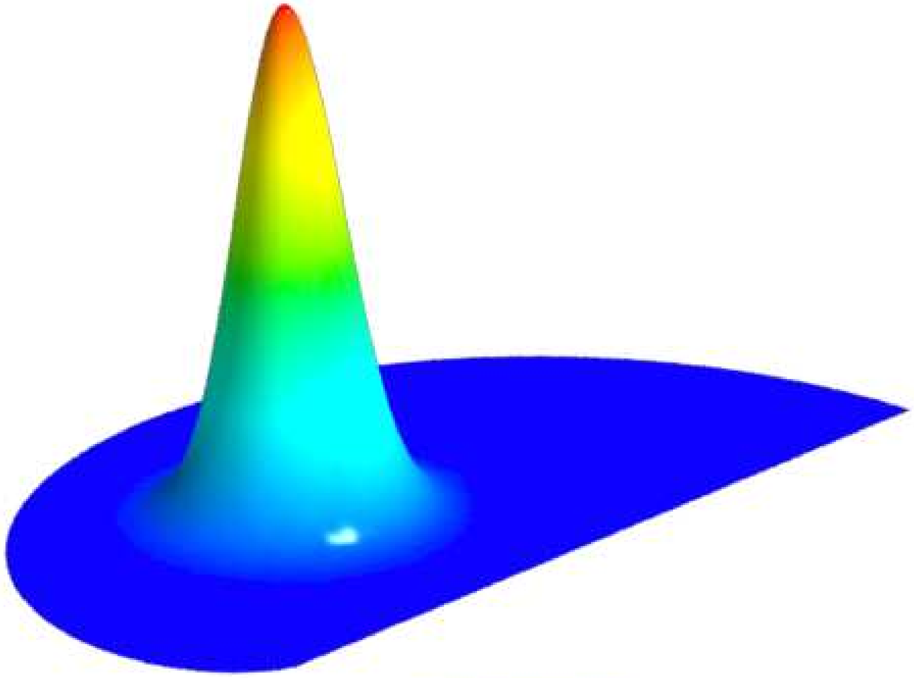
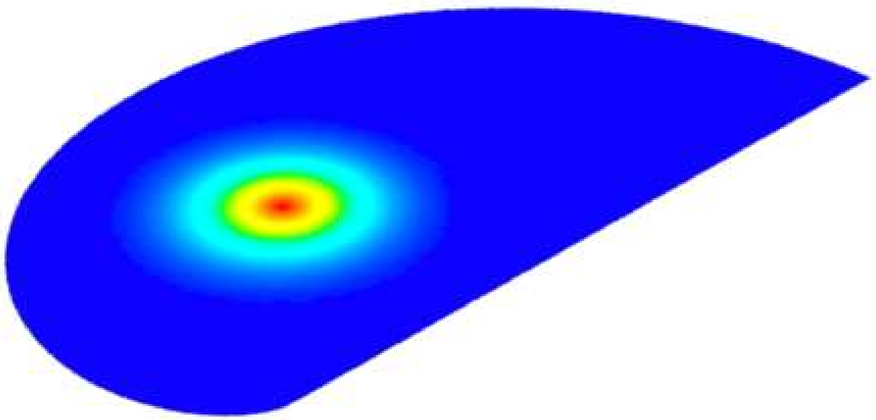




Start with coherent state

$$\mathrm{Coh}_{(\boldsymbol{p},\boldsymbol{q})}(\boldsymbol{x}) := \left(\frac{1}{\hbar\pi}\right)^{1/4} \mathrm{e}^{\frac{i}{\hbar}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{q})-\frac{1}{2\hbar}(\boldsymbol{x}-\boldsymbol{q})^2} ,$$



localized around the point

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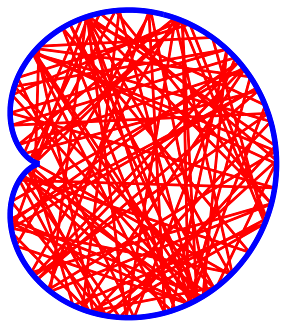

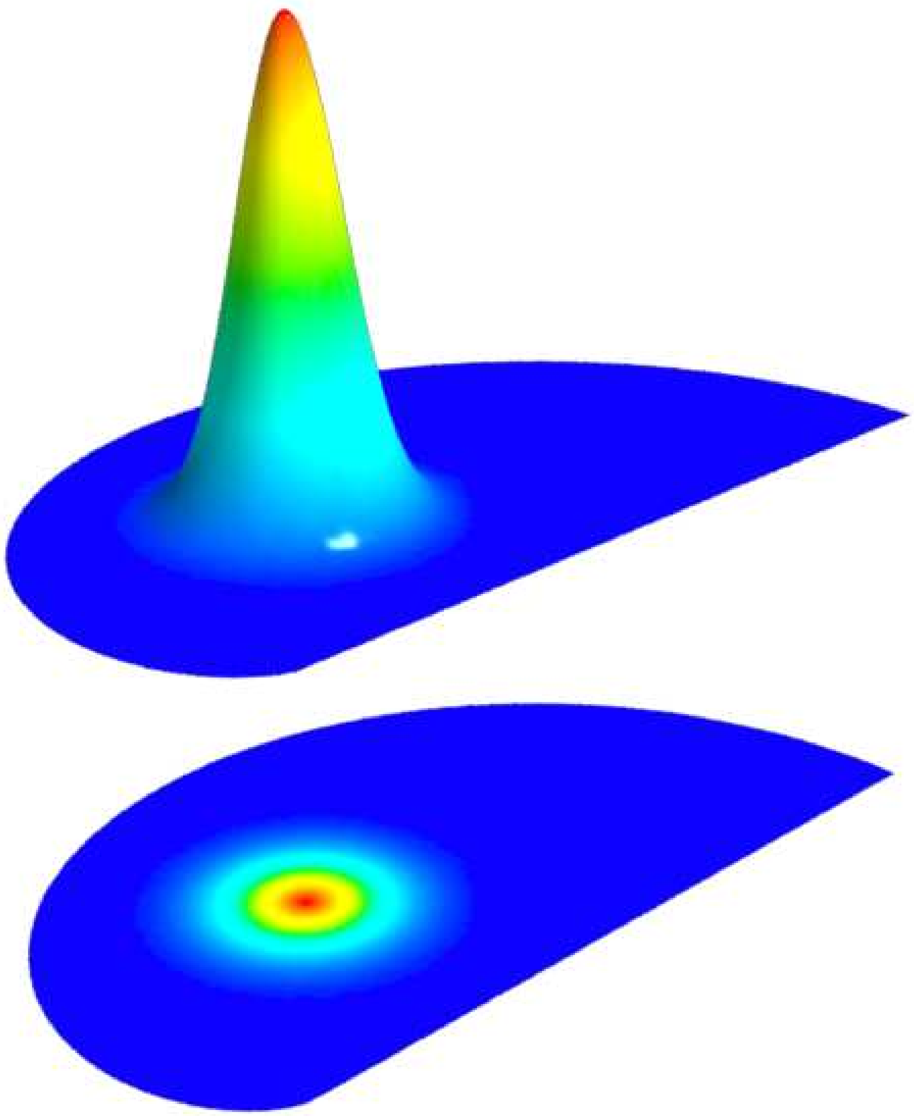

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Observation: follows classical trajectory for some time

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use the ansatz: $\psi(\boldsymbol{q},t)=\psi(\boldsymbol{q})\exp\left(-\frac{\mathrm{i}}{\hbar}Et\right)$

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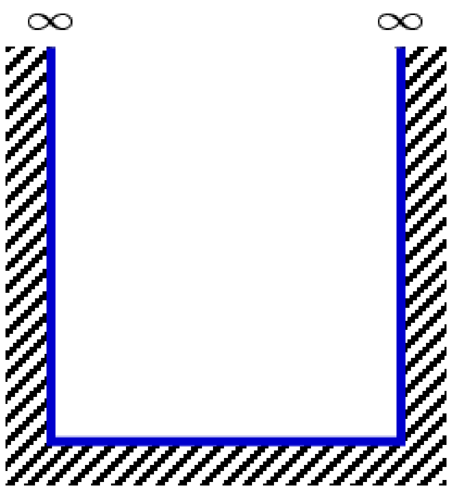
First: Eigenvalues and Eigenfunctions of simple systems

QM: One–dimensional box

Quantum mechanics in a 1D box

$$-\frac{\partial^2}{\partial x^2}\psi_n(x) = E_n\psi_n(x)$$

with $\psi(0) = \psi(\pi) = 0$.



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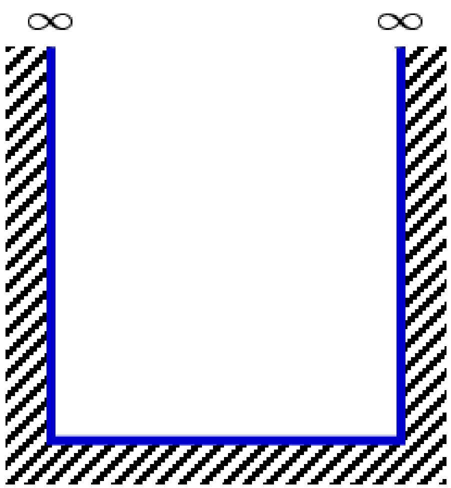

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Solution (length of the box: π)

$$\psi_n(x) = \sin(nx) \qquad\qquad E_n = n^2$$

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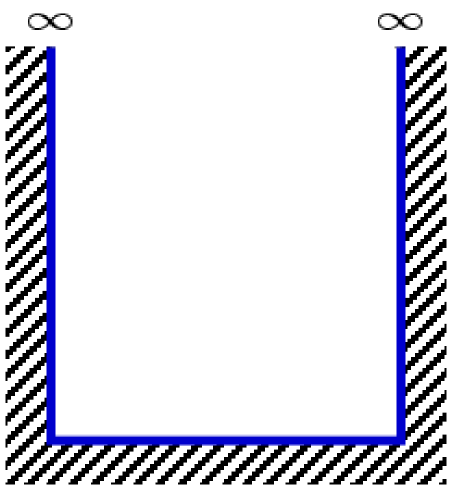

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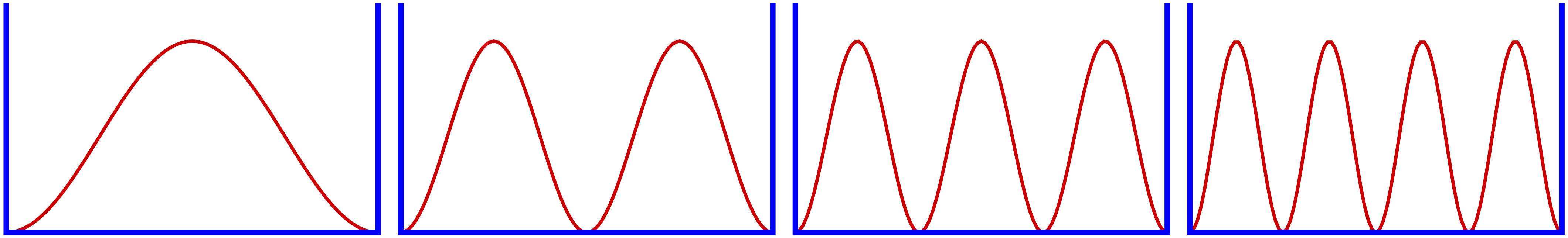
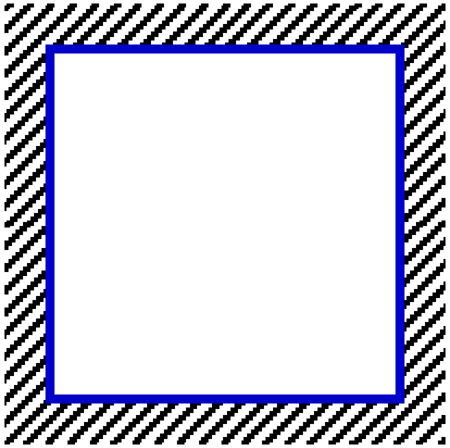


$n = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$

QM: Rectangular box (billiard)

Quantum mechanics in a 2D box



$$-\Delta\psi_n(\boldsymbol{q}) = E_n\psi_n(\boldsymbol{q}) \quad , \qquad \boldsymbol{q} \in \Omega$$

with Dirichlet boundary conditions,

i.e. $\psi_n(\boldsymbol{q}) = 0$ for $\boldsymbol{q} \in \partial\Omega$.

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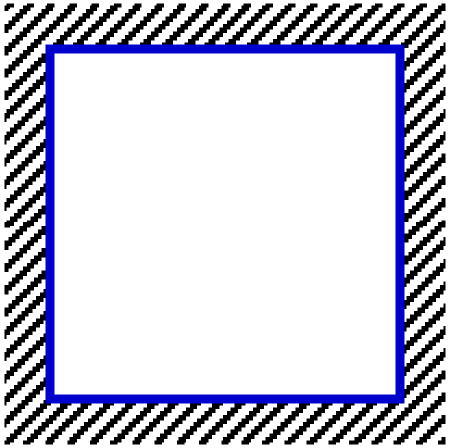

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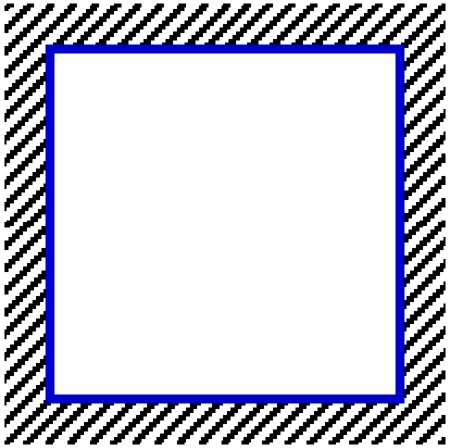

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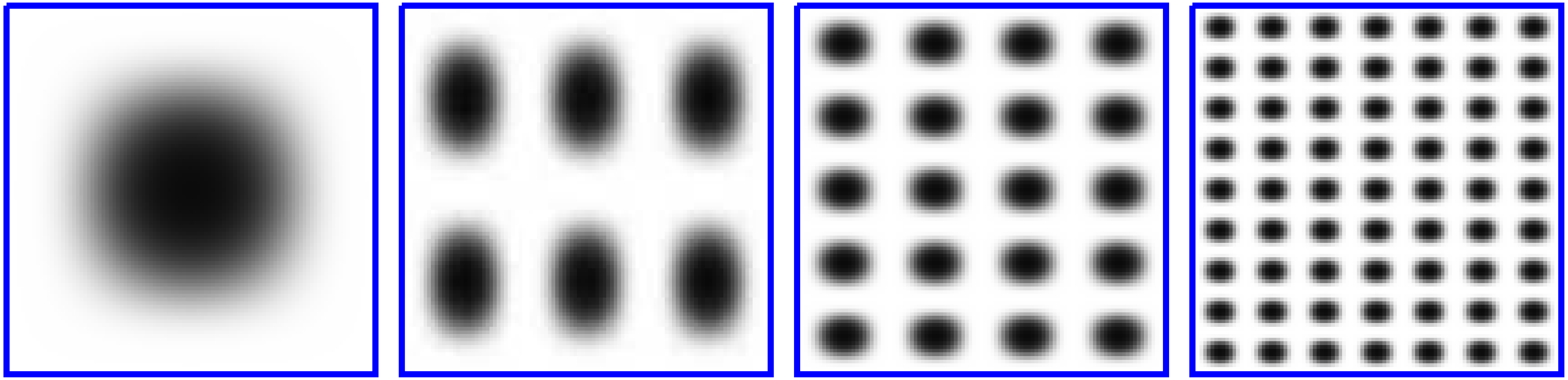

$(k,l) = (1,1)$ $\quad (3,2)$ $\quad (4,5)$ $\quad (7,9)$

Stationary Schrödinger equation (in units $\hbar = 2m = 1$)

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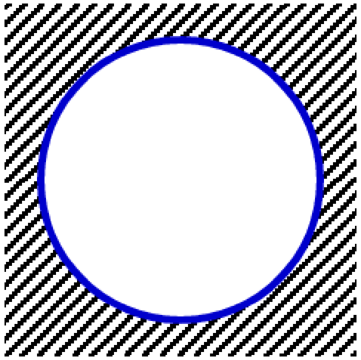
Question: What is the behaviour of the eigenvalues and eigenfunctions in the *semiclassical limit* $E \to \infty$?

QM: another solvable system: circular billiard



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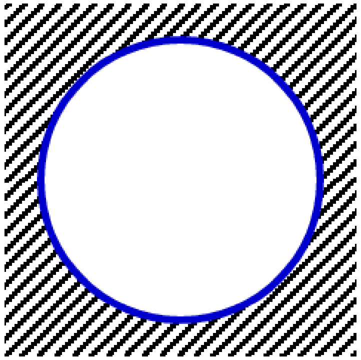
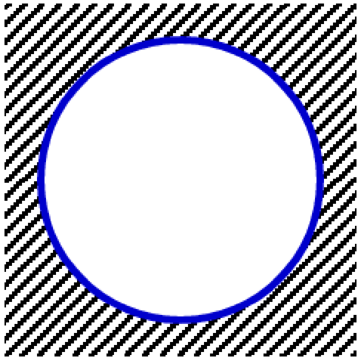


Eigenfunctions in polar coordinates

$$\psi_{kl}(r,\phi) = J_k(j_{kl} r) \begin{cases} \cos(k\phi), & k = 0, 1, 2, \ldots \\ \sin(k\phi), & k = 1, 2, \ldots \end{cases}$$

Here j_{kl} is the l-th zero of the Bessel function $J_k(x)$.



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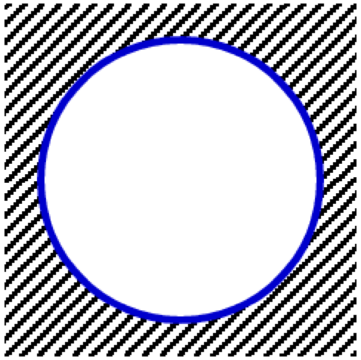

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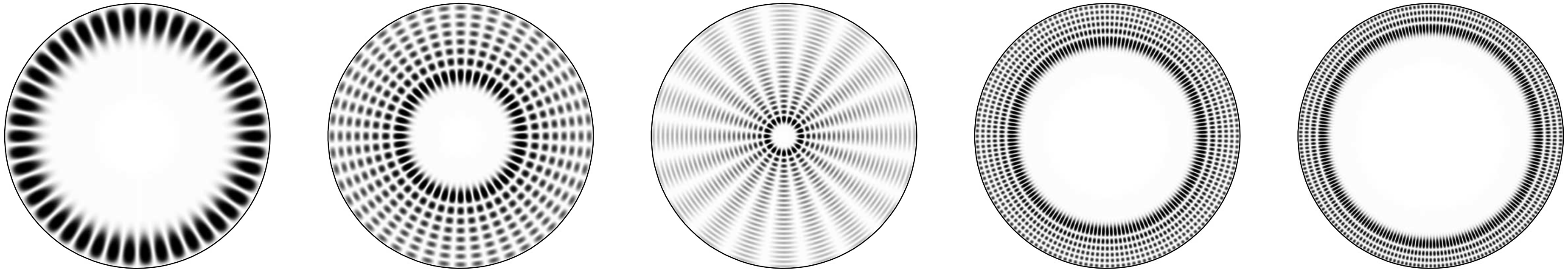

$n = 100$ $\quad n = 400$ $\quad n = 1000$ $\quad n = 1500$ $\quad n = 2000$

Semiclassical determination of eigenvalues

For d–dimensional integrable systems:

EBK Quantisation rule with quantum numbers $n_i = 0, 1, 2, \ldots$

$$I_i := \oint_{\gamma_i} \mathbf{p}\, d\mathbf{q} = 2\pi\hbar \left(n_i + \frac{\mu_i}{4} \right) \qquad i = 1, \ldots, d$$

and $E_{n_1, n_2, \ldots, n_d} = H(I_1, \ldots, I_d)$.

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Comparison:

k	0	0	0	1	1	1
l	1	2	3	1	2	3
E_{kl} exact	5.78	30.47	74.88	14.68	49.21	103.49
E_{kl} semiclassics	5.55	30.22	74.63	14.39	48.95	103.24

QM: stadium billiard

Instead of the circle consider a stadium

How will the eigenfunctions look like?

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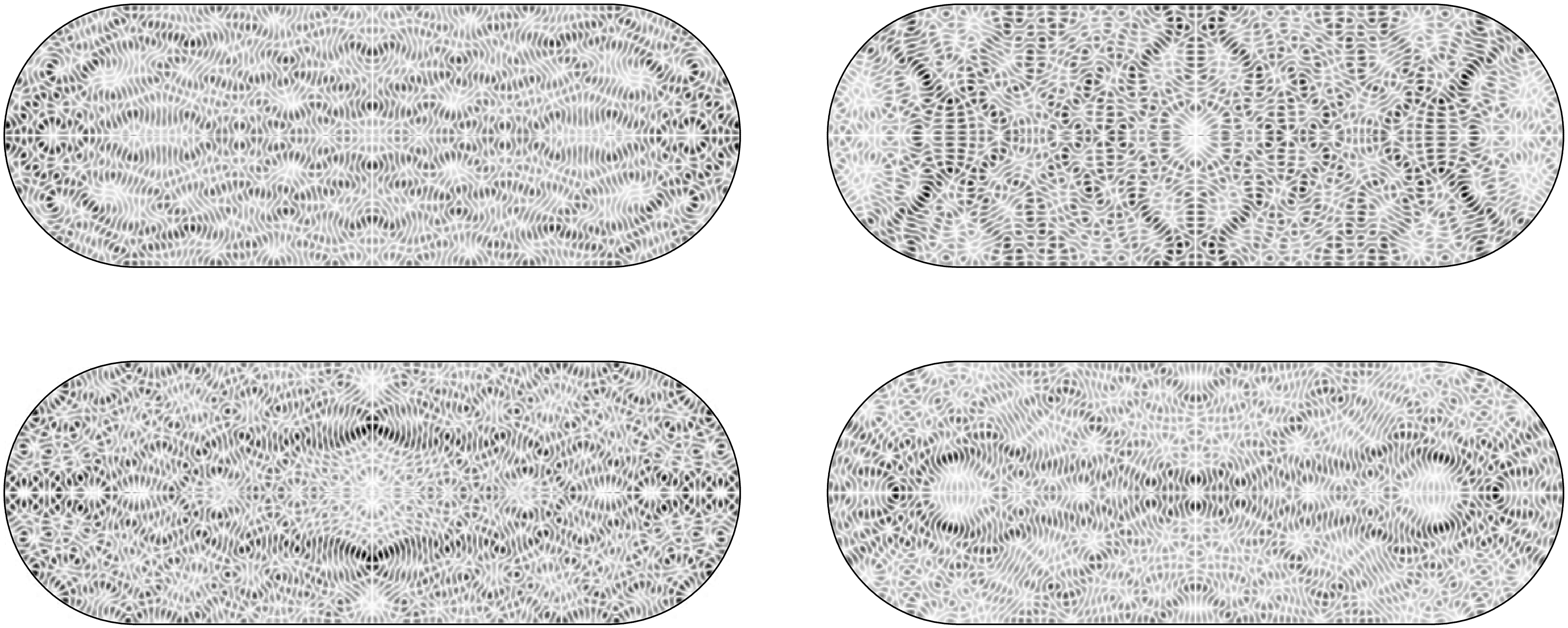

Helmholtz equation

$$-\Delta\psi_n(\boldsymbol{q}) = E_n\psi_n(\boldsymbol{q}) \quad , \qquad \boldsymbol{q} \in \Omega$$

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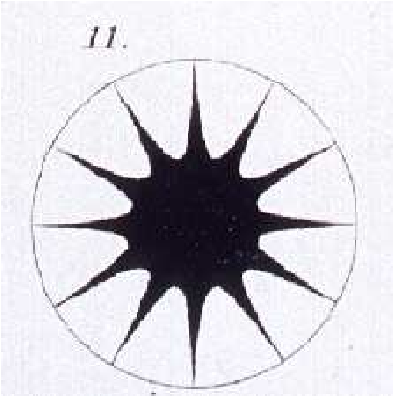
Chladni figures



[Chladni]

[ ]

QM: some examples of quantum billiards in the real world

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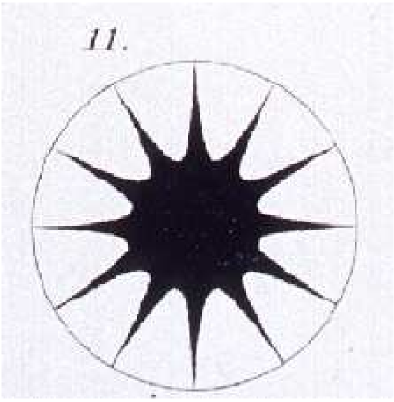
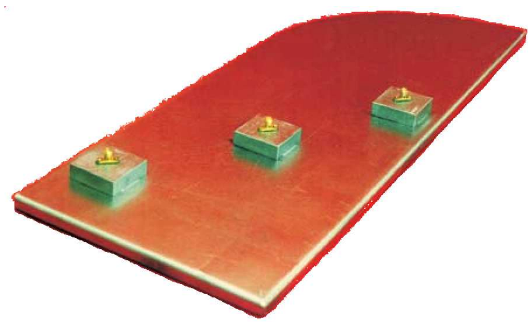


[Chladni]

[ ]

Microwave cavities



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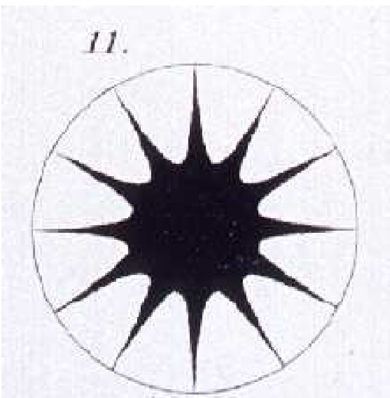

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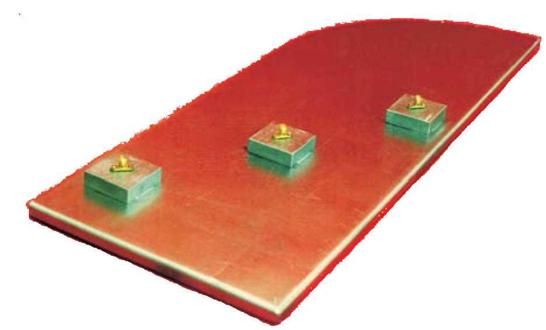

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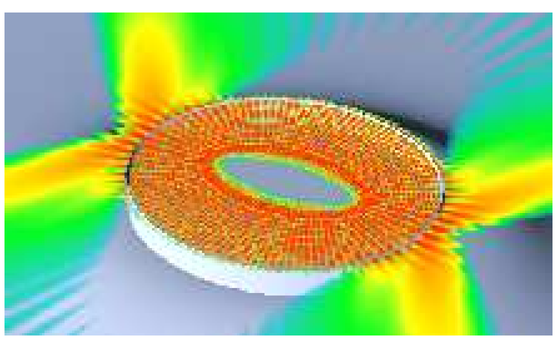

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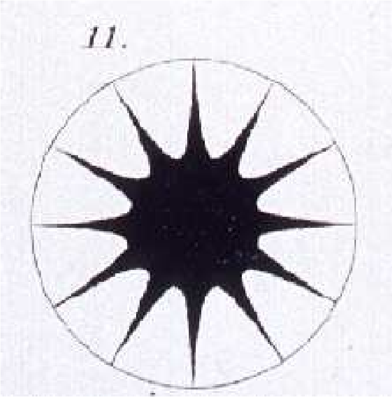

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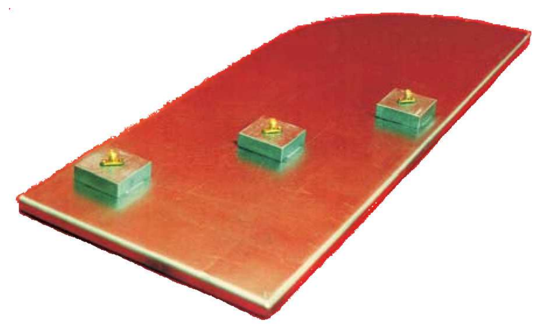

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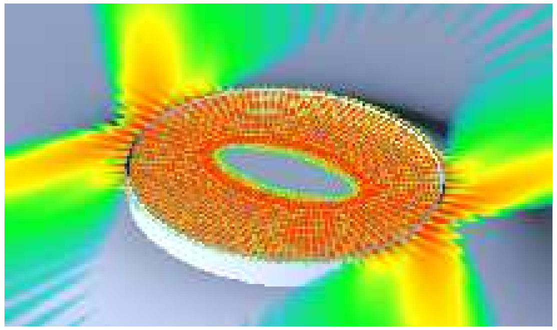

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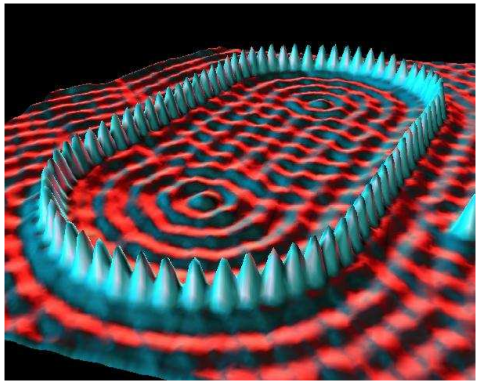

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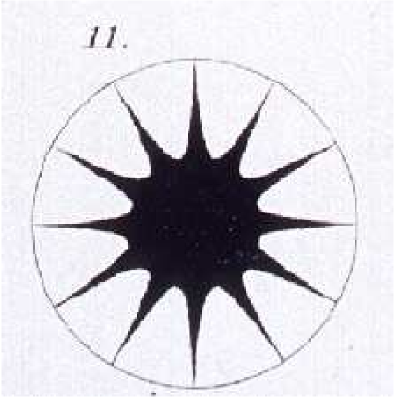

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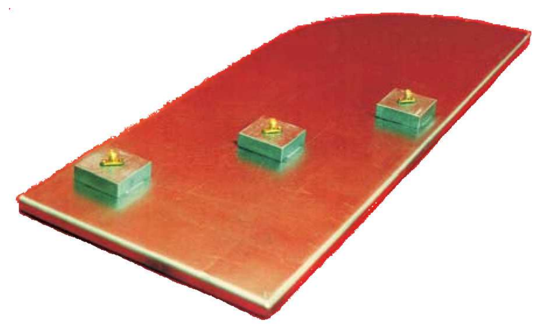

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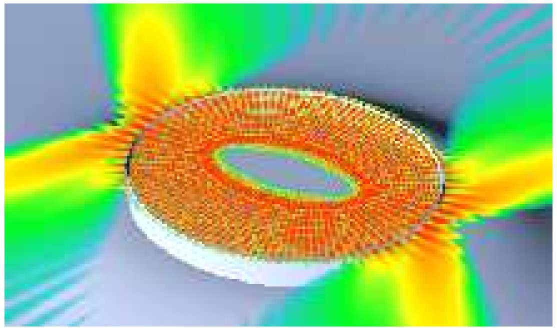

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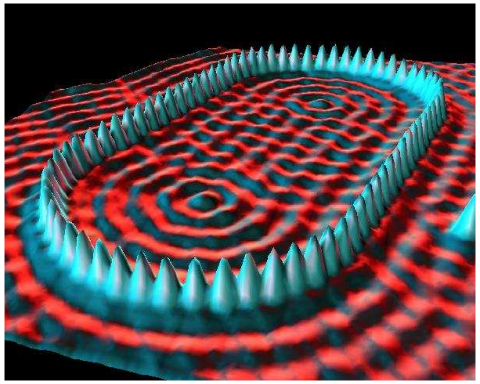

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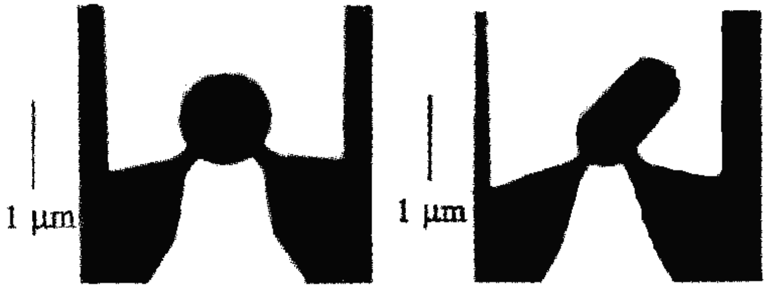
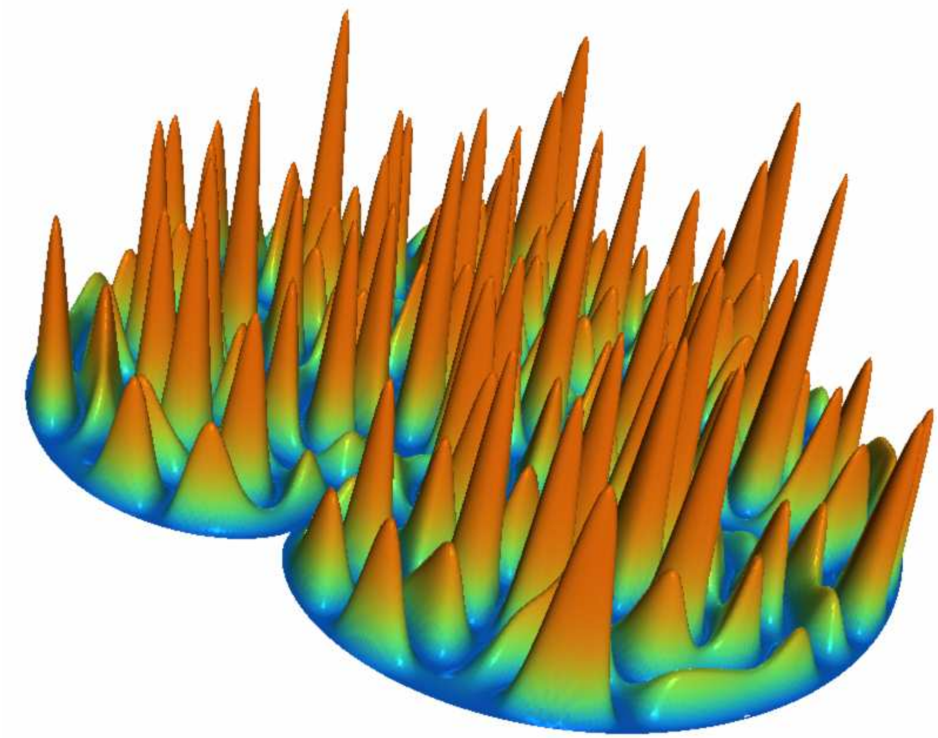
Mesoscopic systems



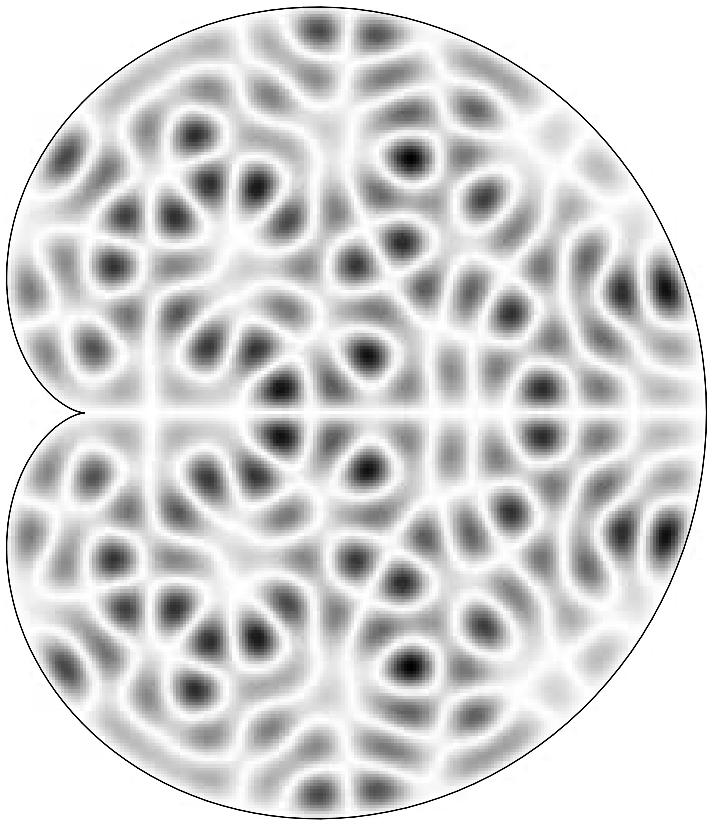
[Marcus *et.al.*]

Visualization of eigenstates

3D Plot of $|\psi_n(\boldsymbol{q})|^2$



Density plot of $|\psi_n(\boldsymbol{q})|^2$



Remark: Numerical computations via boundary integral method.

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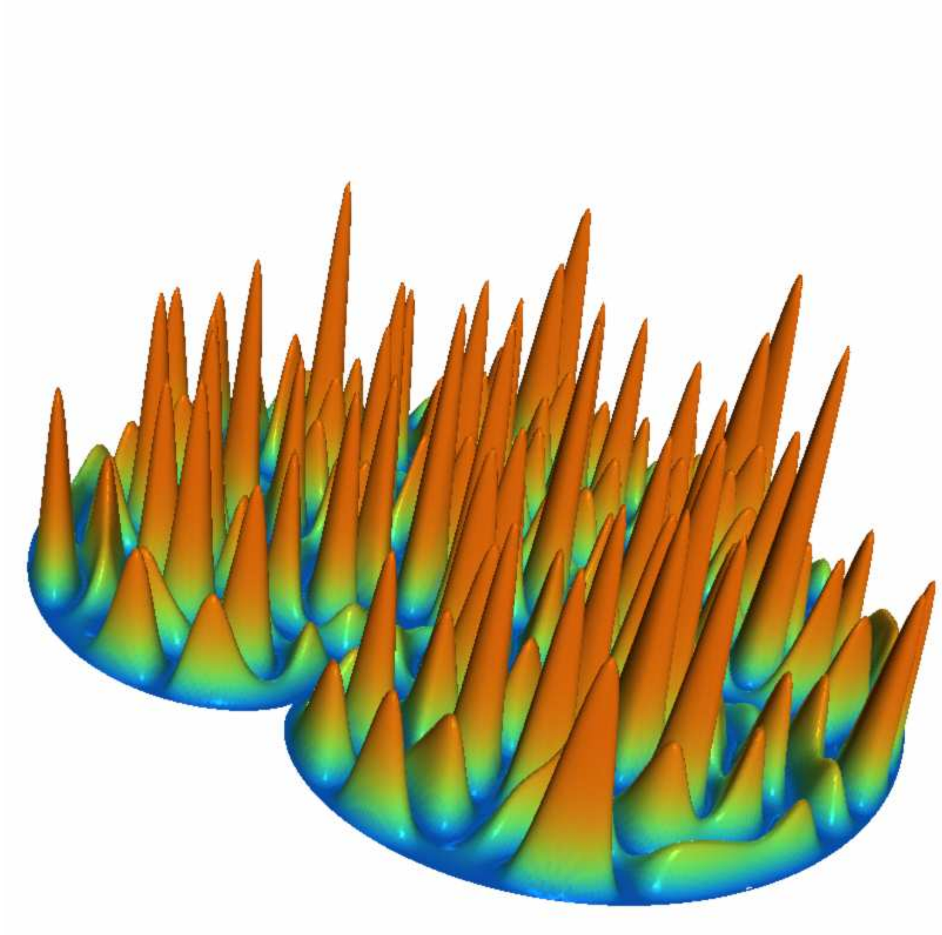

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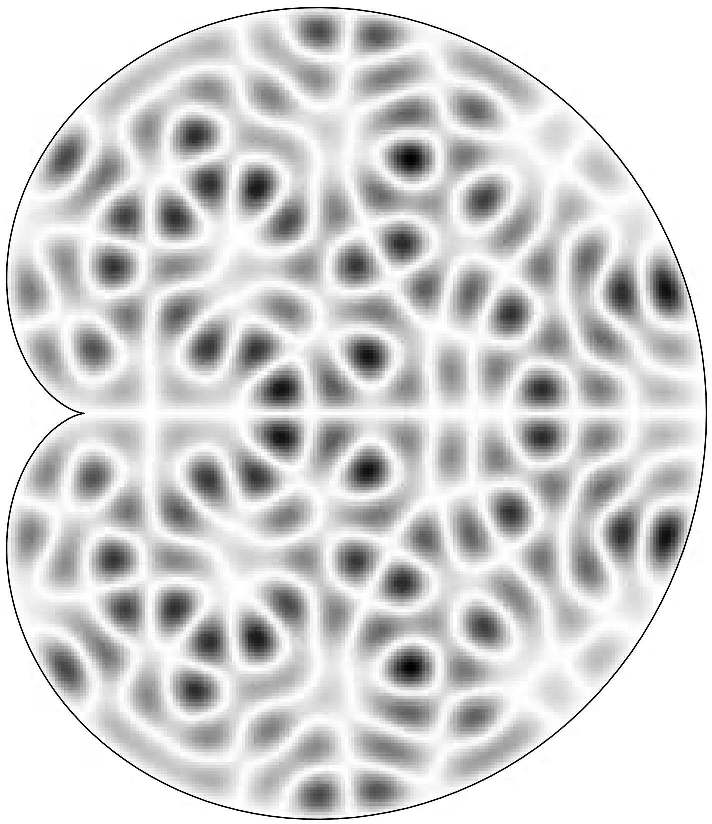

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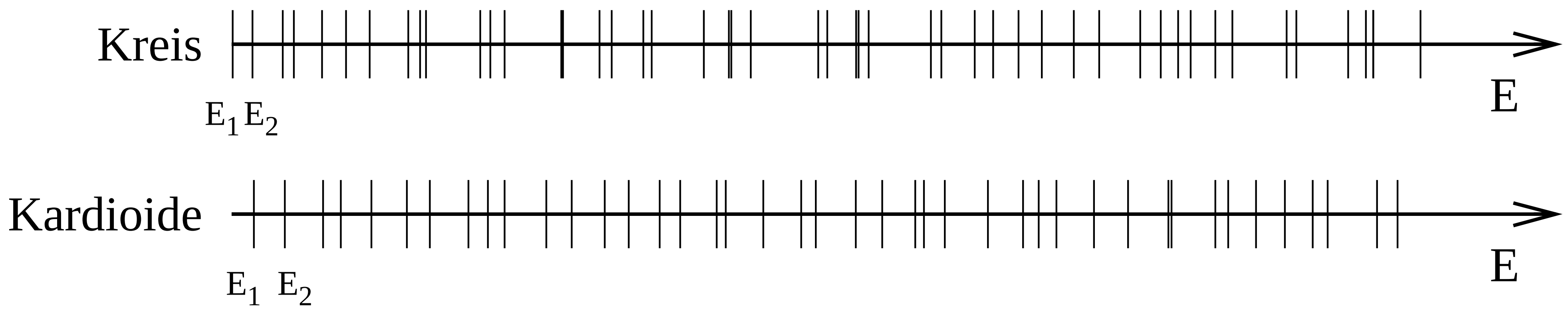
More on eigenfunctions: later ....

Eigenvalues for circle and cardioid billiard



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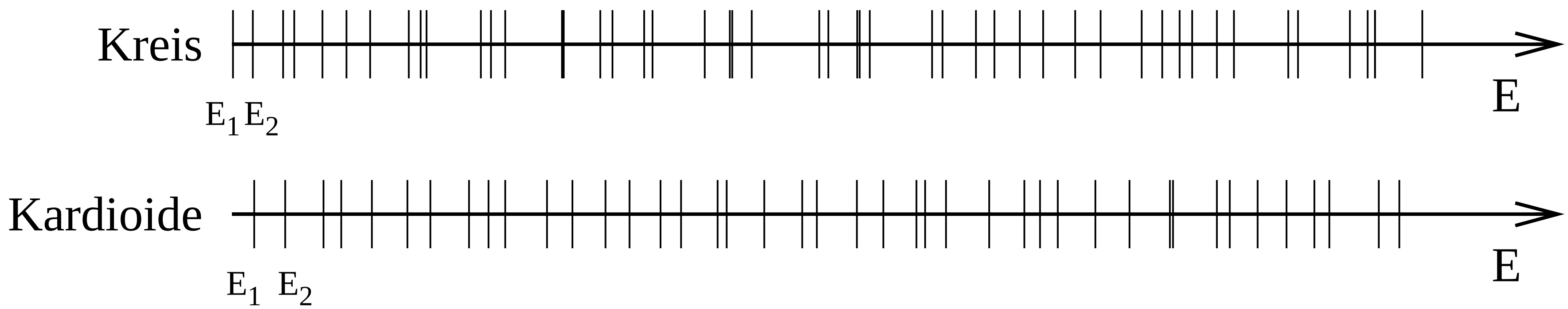

Before doing statistics:

make spectra independent of the size of the billiard:

– spectral staircase function,

– unfolding the spectrum

Spectral statistics – counting eigenvalues

The spectral staircase function $N(E)$ (integrated level density)

$$N(E) := \#\{n \mid E_n \leq E\} \quad ,$$

counts the number of energy levels E_n below a given energy E.

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Mean behaviour $\overline{N}(E)$ of $N(E)$ for 2D billiards:

Weyl formula

$$\overline{N}(E) = \frac{\mathcal{A}}{4\pi}E - \frac{\mathcal{L}}{4\pi}\sqrt{E} + \mathcal{C} \quad ,$$

where

- $\mathcal{A}$: area of the billiard
- $\mathcal{L}$: length of the boundary

Spectral staircase function (integrated density of states)

$$N(E) = \#\{n \mid E_n \leq E\} \tag{1}$$

Mean behaviour (Weyl formula)

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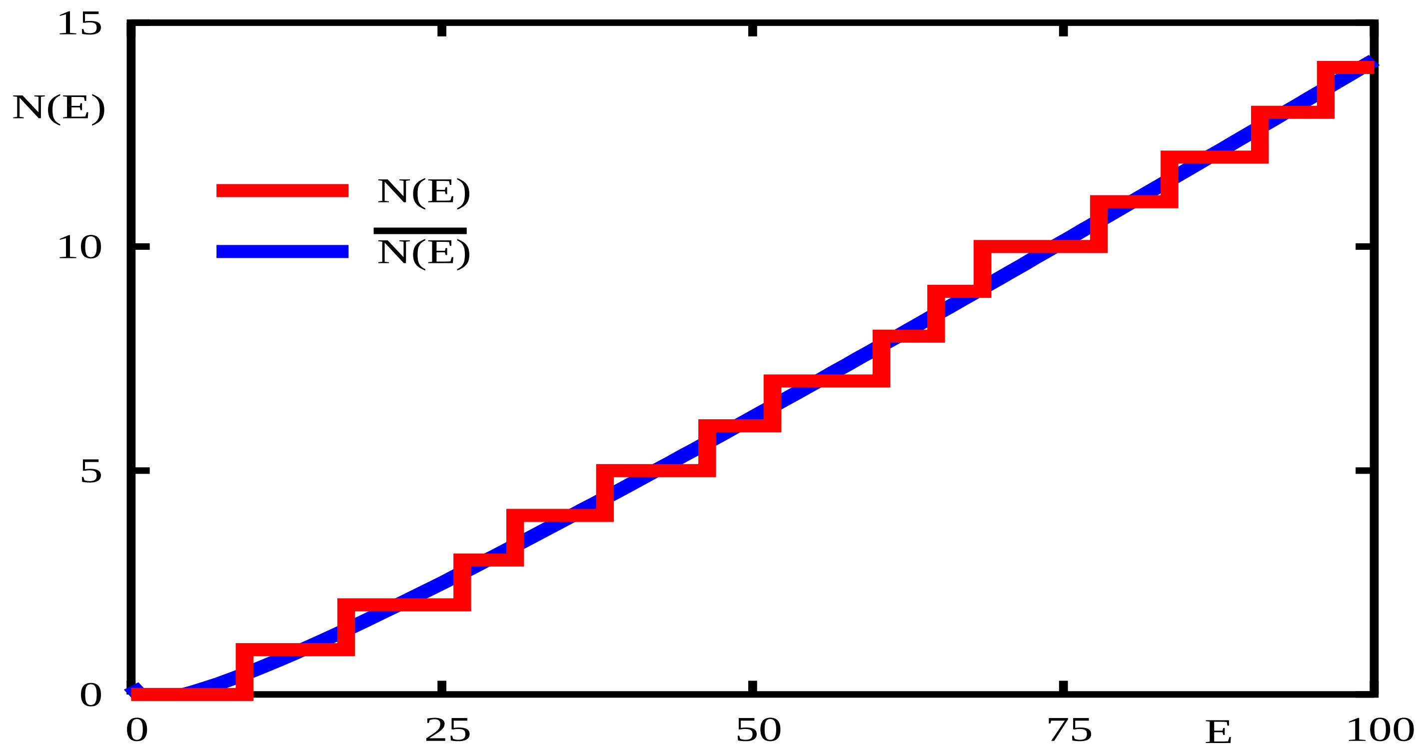
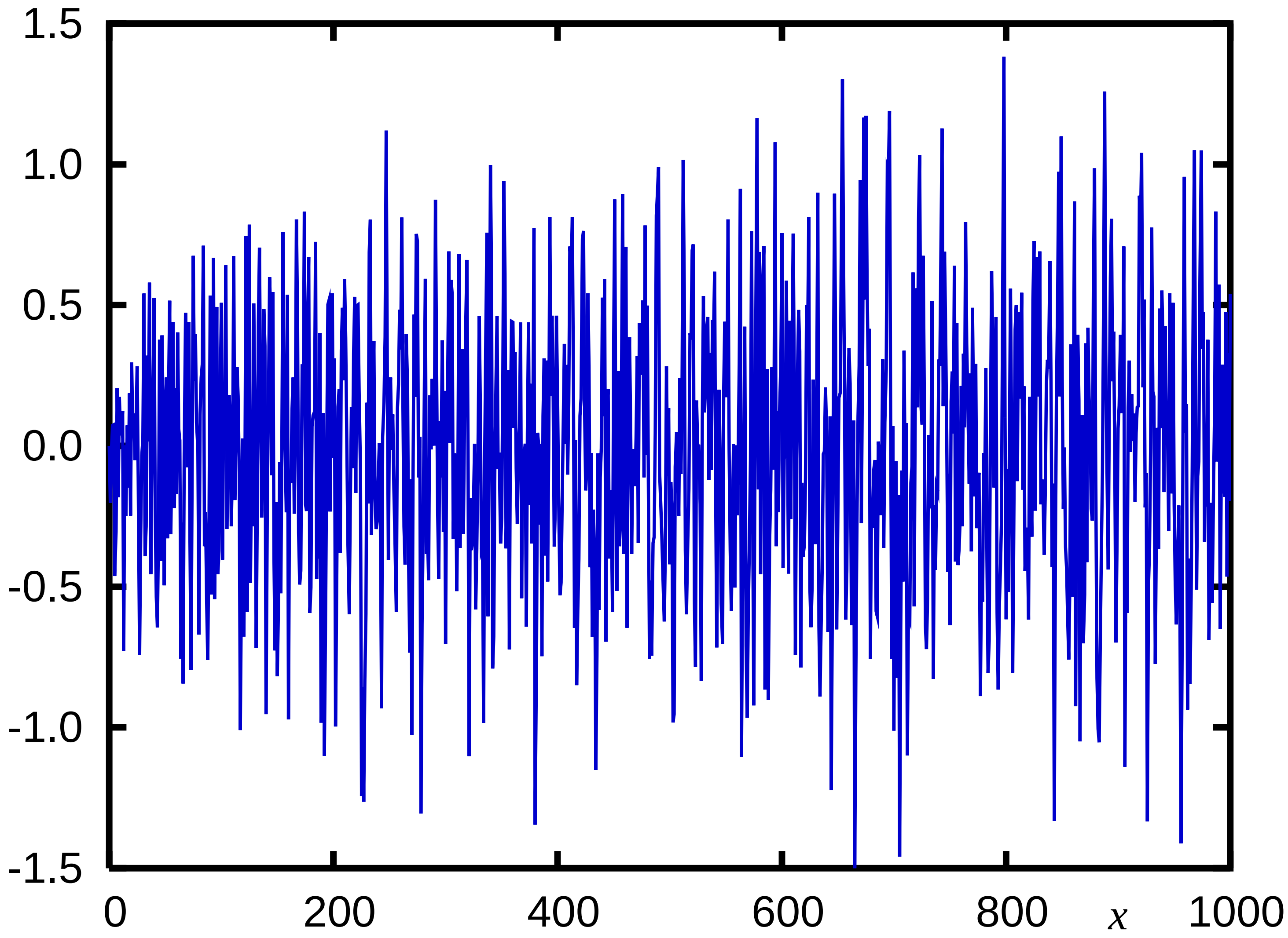


Define unfolded spectrum $x_n := \overline{N}(E_n)$. Measure of completeness

$$\delta_n := N(x_n) - \overline{N}(x_n) = n - \frac{1}{2} - x_n \quad , \tag{3}$$



Unfolding the spectrum (Are computed spectra complete?)

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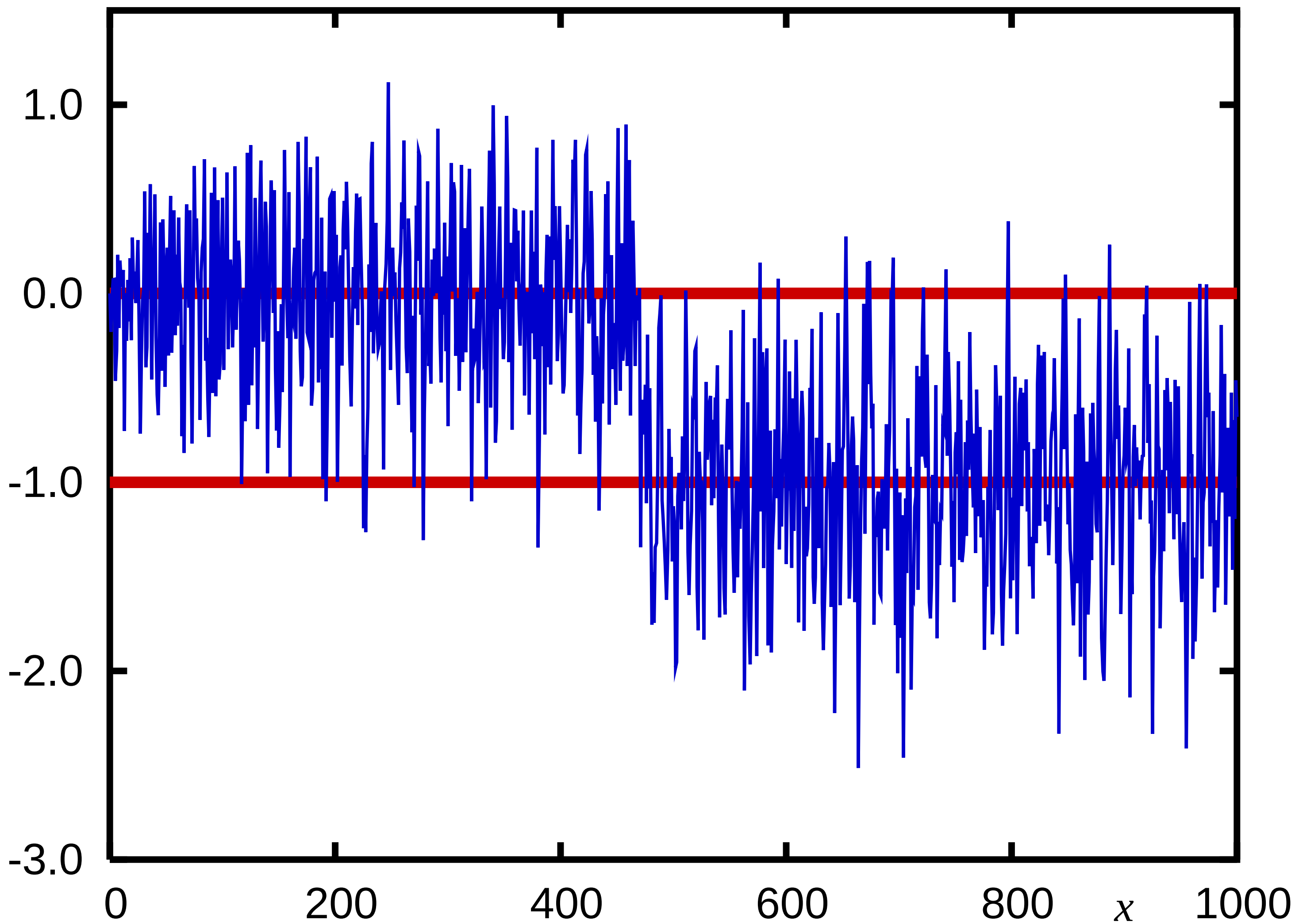

Counting for the square: circle problem

Eigenvalues (for square $[0, \pi] \times [0, \pi]$):

$$E_{k,l} = k^2 + l^2 \qquad k, l \in \mathbb{N}$$

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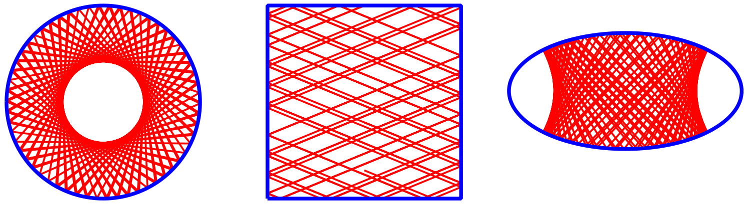
$$\overline{N}(E) = \frac{\mathcal{A}}{4\pi}E - \frac{\mathcal{L}}{4\pi}\sqrt{E} + \mathcal{C}$$

with $\mathcal{A} = \pi^2$ and $\mathcal{L} = 4\pi$

Spectral statistics: Conjectures

Integrable systems



Conjecture [Berry, Tabor '77]:

The energy level statistics of generic integrable systems can be described by those of a Poissonian random process.

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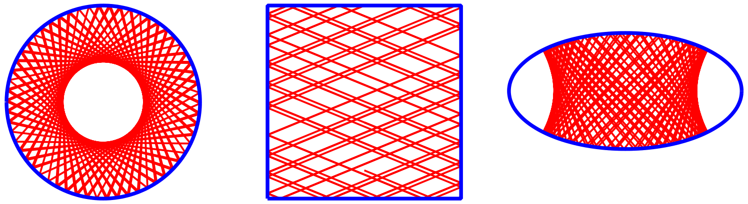

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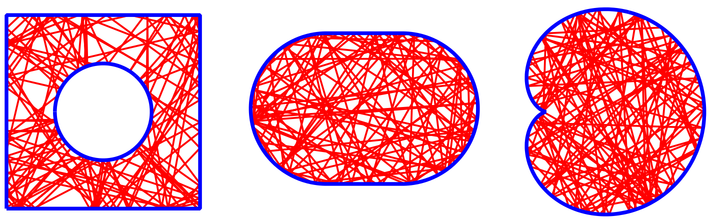
Chaotic systems



Conjecture [Bohigas, Giannoni, Schmit '84]:

The energy level statistics of classically strongly chaotic systems should be reproduced by the corresponding random matrix distributions (GOE, GUE, . . .).

Spectral statistics: Level spacing distribution

Distribution of the distances between neighboured eigenvalues

$$s_n := x_{n+1} - x_n$$

- Regular systems

$$P(s) \to \exp(-s)$$ Poisson distribution

"level attracttion"

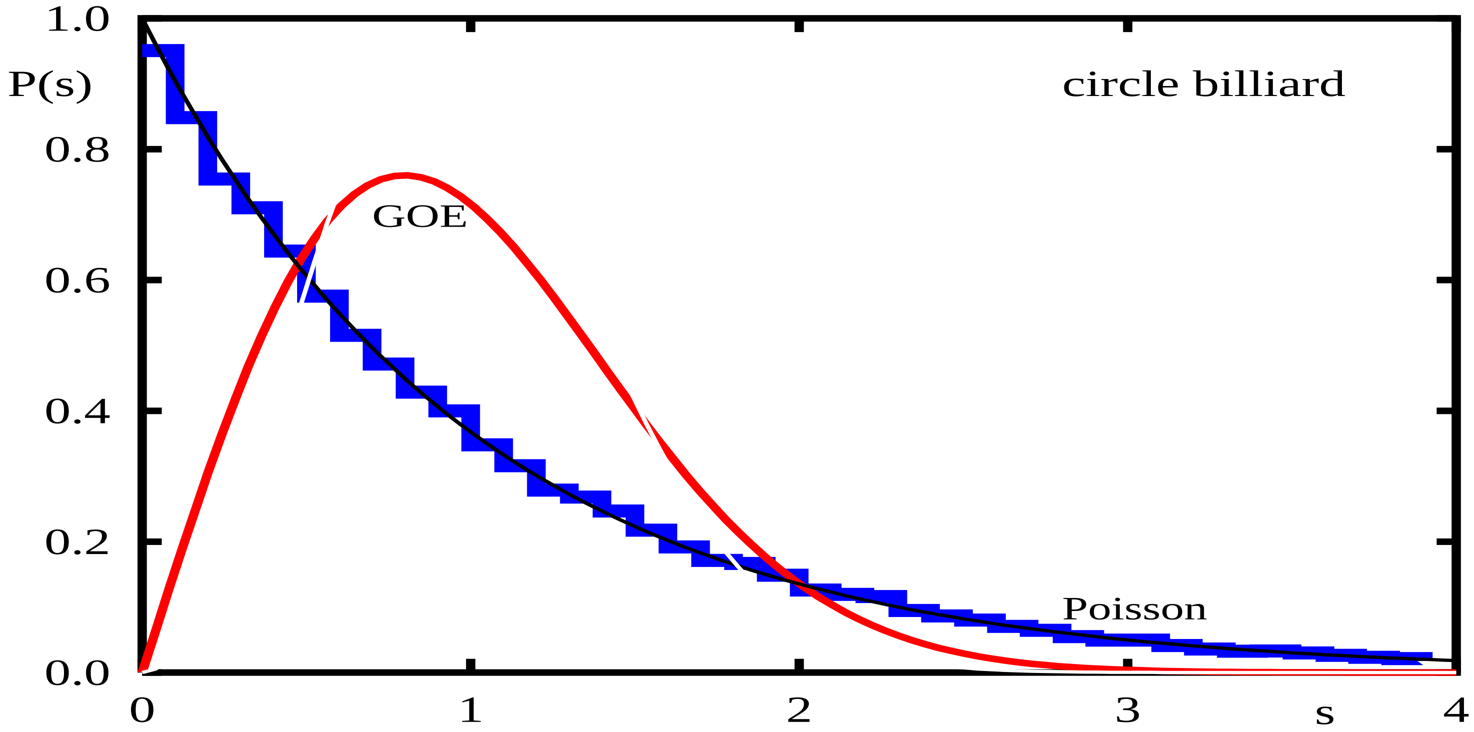
- Chaotic systems

$$P(s) \to \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)$$ Wigner distribution

(random matrix distribution)

"level repulsion"

Spectral statistics – level spacing distribution



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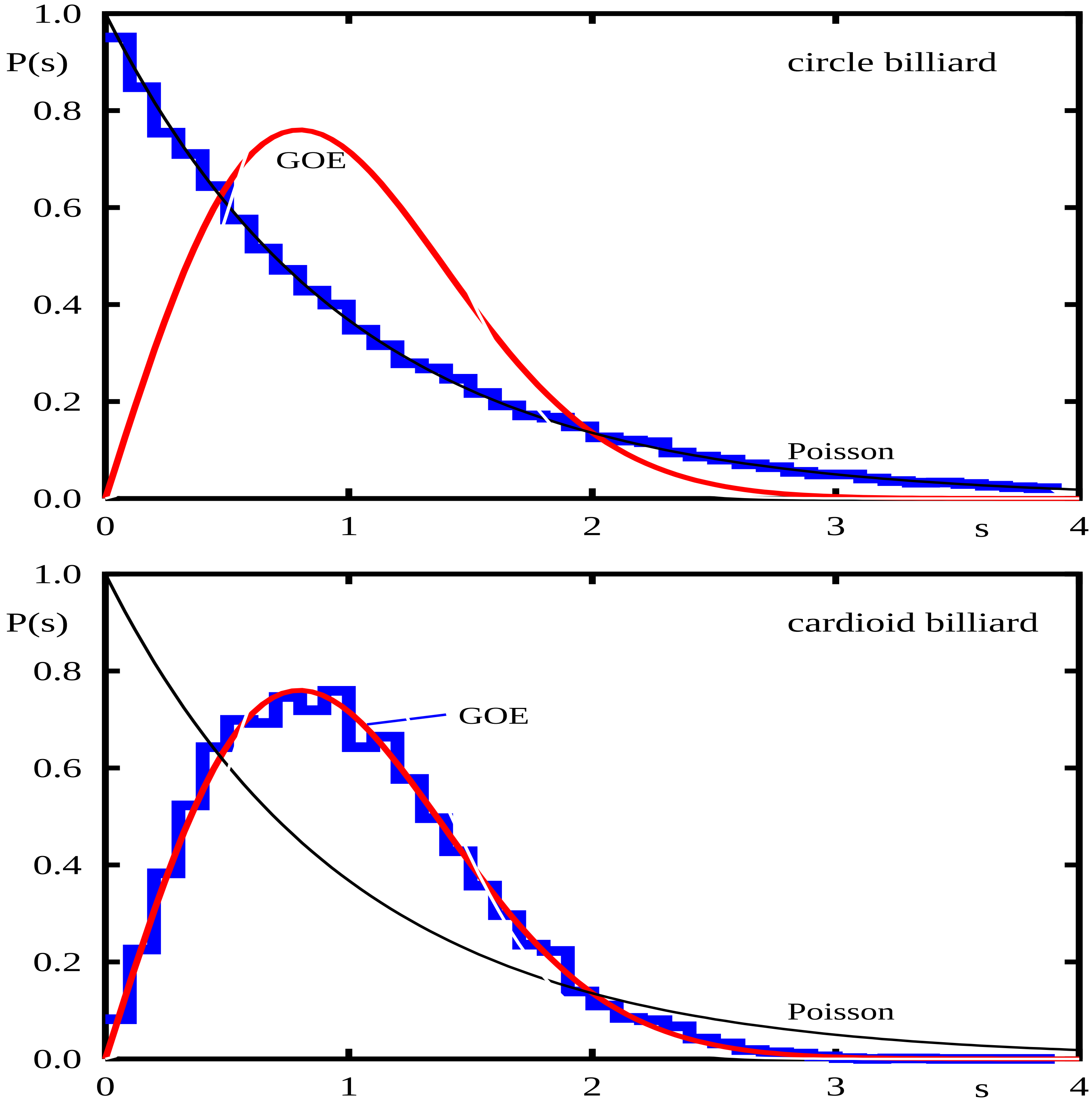

How to explain/understand? Trace formulae

For integrable systems: WKB/EBK quantization ...

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Trace formula provide a

fundamental connection between classical and quantum system

Application in two directions:

- periodic orbits $\longrightarrow$ eigenvalues : Quantisation
- Eigenvalues $\longrightarrow$ periodic orbits : Fourieranalysis

Of central importance to describe/understand spectral statistics!

Asymptotically, in the semiclassical limit $p^2 := E \to \infty$,

the density of states is ("Gutzwiller trace formula")

$$d(p) := \sum_{n=1}^{\infty} \delta(p - p_n)$$

$$\sim \bar{d}(p) + \sum_{\gamma} \sum_{k=1}^{\infty} \frac{l_\gamma}{\sqrt{|2 - \mathrm{Tr}\, M_\gamma^k|}} \cos\left(k l_\gamma p - \frac{\pi}{2} k \nu_\gamma\right)$$

Here

- $p_n := \sqrt{E_n}$
- $\bar{d}(p)$: average density of states
- γ: periodic orbits
- l_γ: geometric length of γ
- ν_γ: Maslov–index of γ
- $\mathrm{Tr}\, M_\gamma$: trace of the linearized map around γ

Fourier transformation of the trace formula gives

$$F(L) := \sum_{n=1}^{N} \cos(\sqrt{E_n} L) e^{-E_n t} \quad ,$$

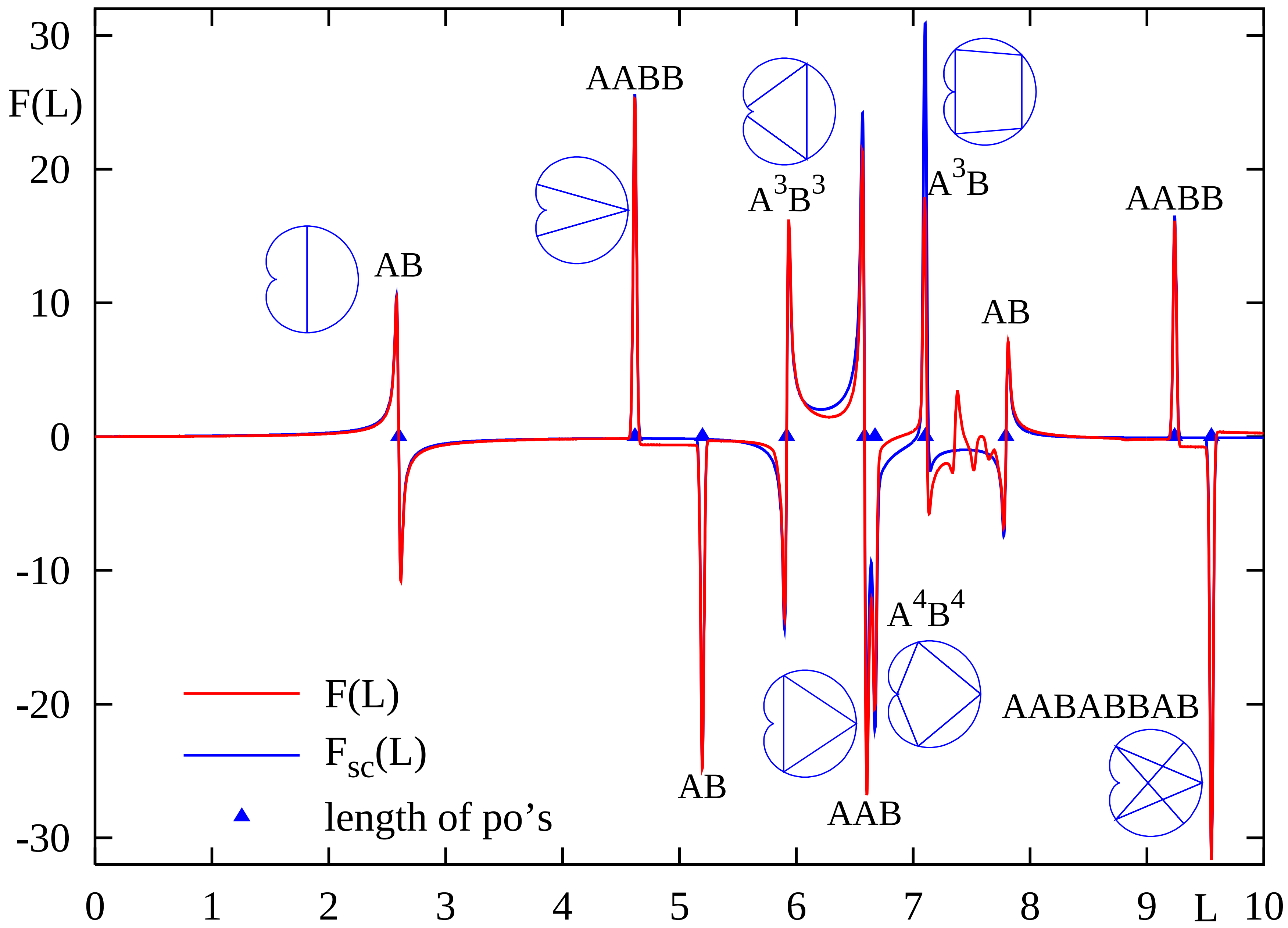
and

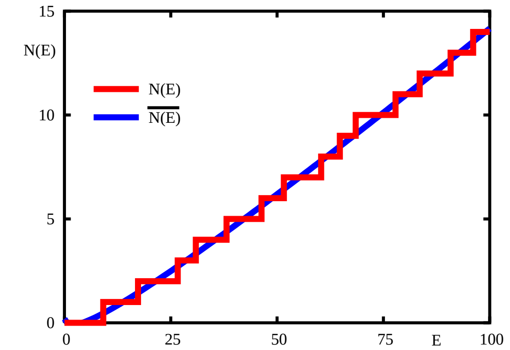
$$F_{\mathrm{sc}}(L) := \sum_{\gamma} \sum_{k=1}^{\infty} \frac{l_\gamma}{\sqrt{|2 - \mathrm{Tr} M_\gamma^k|}} \mathcal{F}_\gamma\{h(k l_\gamma)\} \quad ,$$

where

$$\mathcal{F}_\gamma(x) = \begin{cases} \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{(x-L)^2}{4t}\right) & \text{for } k\nu_\gamma = 2m \\ \frac{(-1)^m}{4} \frac{1}{\pi t} (x - L) \, {}_1F_1\left(1, \frac{3}{2}, -\frac{(x-L)^2}{4t}\right) & \text{for } k\nu_\gamma = 2m+1 \end{cases}$$

Fourieranalysis: cardioid billiard





Starting from the spectral staircase function

$$N(E) := \#\{n \mid E_n \leq E\}$$

define the quantisation rule [Aurich et al. '92]

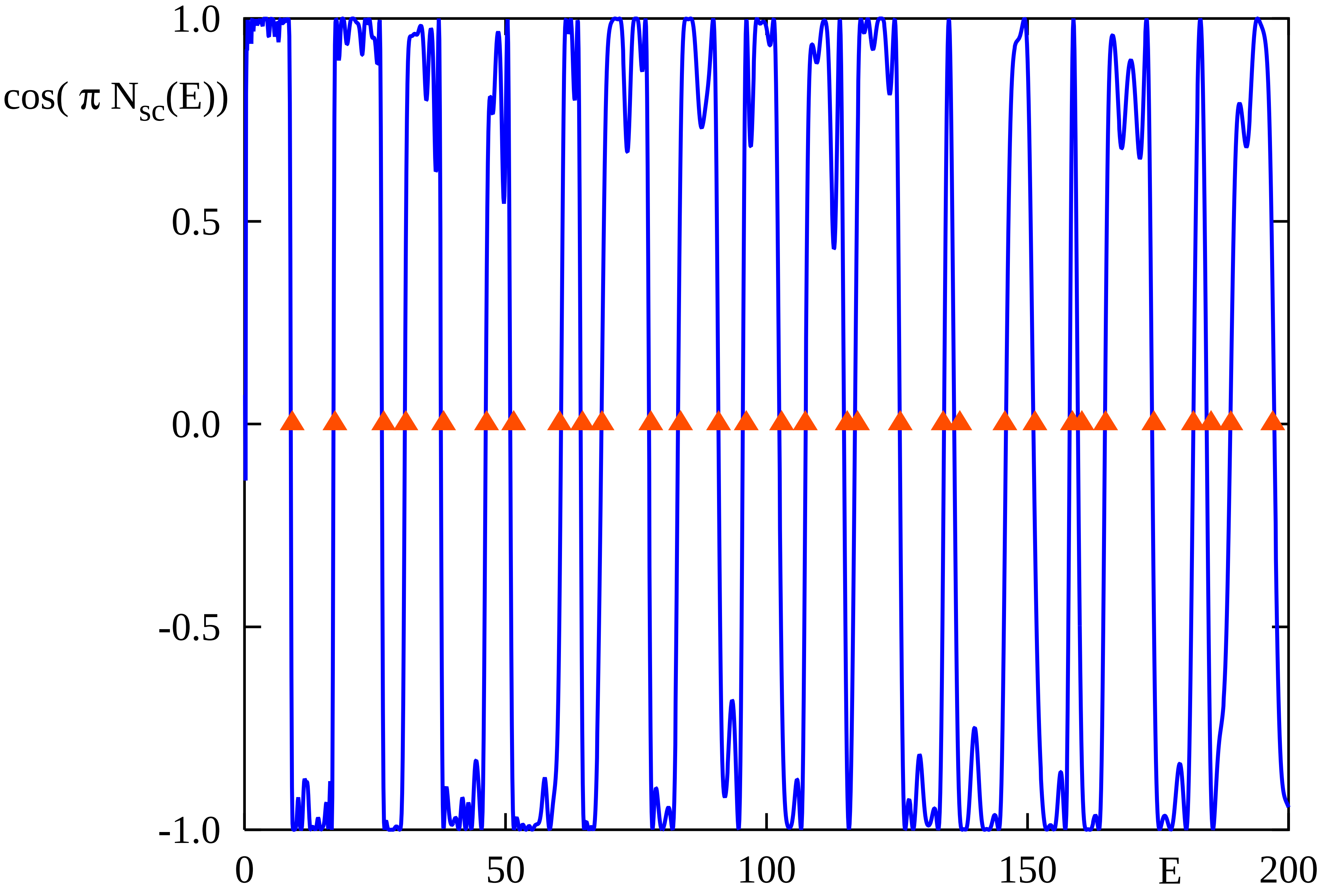
$$N(E) \stackrel{!}{=} n + \frac{1}{2} \qquad \Longleftrightarrow \qquad \cos\big(\pi N(E)\big) = 0$$

The trace formula gives a semiclassical approximation for $N(E)$,

$$N_{\mathrm{sc}}(E = p^2) := \frac{\mathcal{A}}{4\pi}p^2 - \frac{\mathcal{L}}{4\pi}p + \mathcal{C} + \frac{1}{\pi}\sum_{\gamma}\sum_{k=1}^{\infty}\frac{1}{\sqrt{|2 - \operatorname{Tr} M_\gamma^k|}}\frac{\sin\left(k l_\gamma p - \frac{\pi}{2}k\nu_\gamma\right)}{k} .$$

Semiclassical quantization: cardioid billiard

Using all periodic orbits up to 20 reflections



Summary I: Spectral statistics

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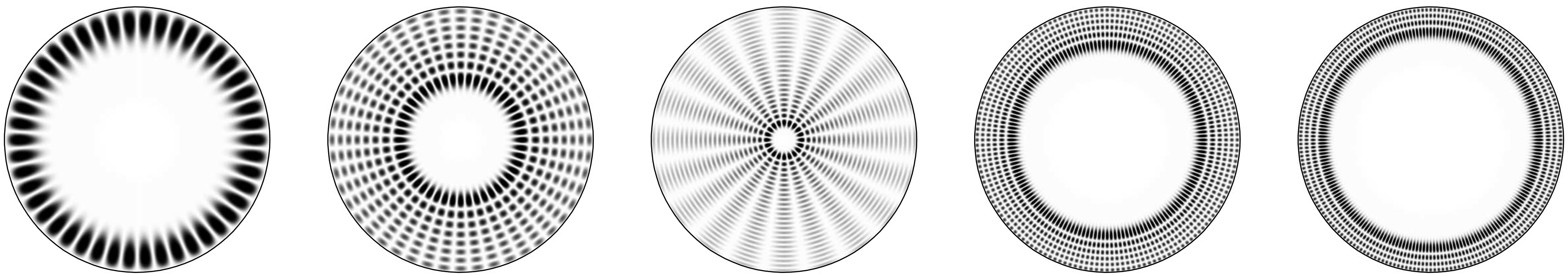
- Quantum billiards
  prototypical class of systems to study quantum chaos
- Compare different spectra by unfolding
  (using the Weyl formula)
- Central conjectures
  - Regular systems: described by Poissonian statistics
  - Strongly chaotic systems: follow random matrix statistics
- Trace formulae
  Fundamental connection: classical $\longleftrightarrow$ quantum system
  - Fourier analysis
  - quantization
  - explanation of spectral statistics

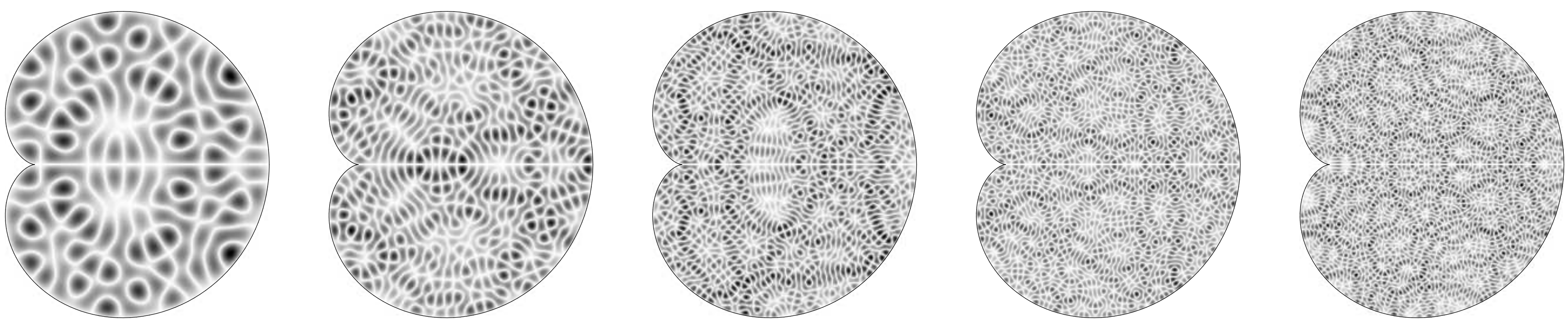
Back to Eigenfunctions: Integrable vs. chaotic

Eigenstates circular billiard

$n = 100$ $\quad n = 400$ $\quad n = 1000$ $\quad n = 1500$ $\quad n = 2000$



Eigenstates cardioid billiard



Random wave model [Berry 1977]

For **chaotic systems** eigenfunctions behave like a superposition of plane waves with random amplitude, phase and direction.

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On a domain $\Omega \subset \mathbb{R}^2$ a random wave may be written as

$$f(\boldsymbol{q}) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} a_n \cos(\boldsymbol{k}_n \boldsymbol{q} + \varepsilon_n)$$

where:

- $a_n \in \mathbb{R}$ are independent Gaussian random variables,
- momenta $\boldsymbol{k}_n \in \mathbb{R}^2$ are randomly equidistributed on the circle of radius $\sqrt{E}$, i.e. $|\boldsymbol{k}_n| = \sqrt{E}$,
- ε_n are equidistributed random variables on $[0, 2\pi[$,
- f is a normalized random function on D when $\mathrm{vol}(D) = 1$.

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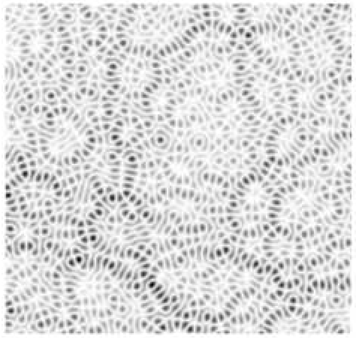
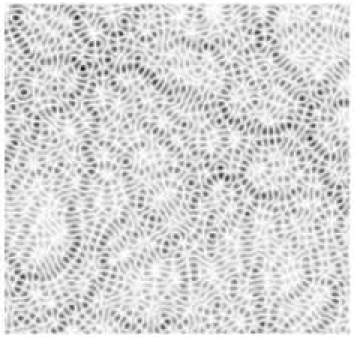
Random wave model – illustration





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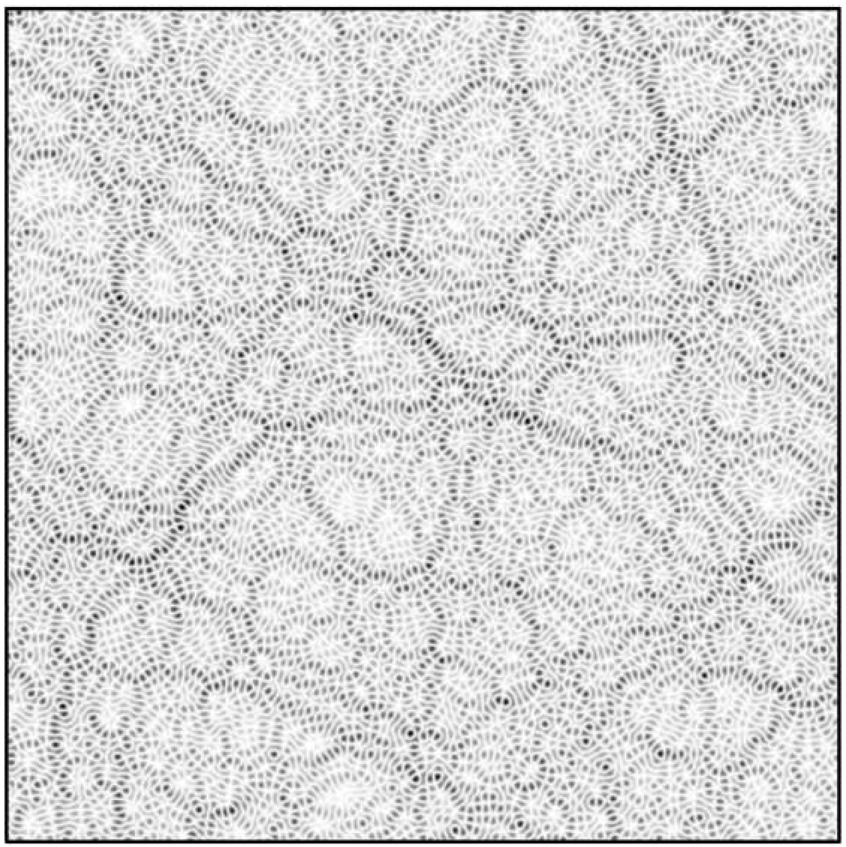
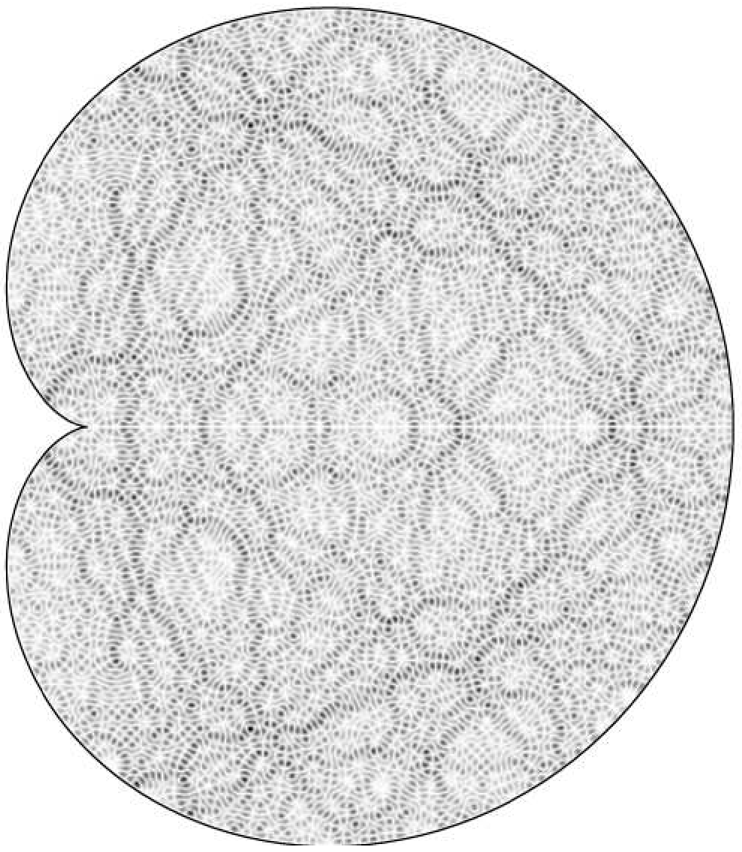


Random wave

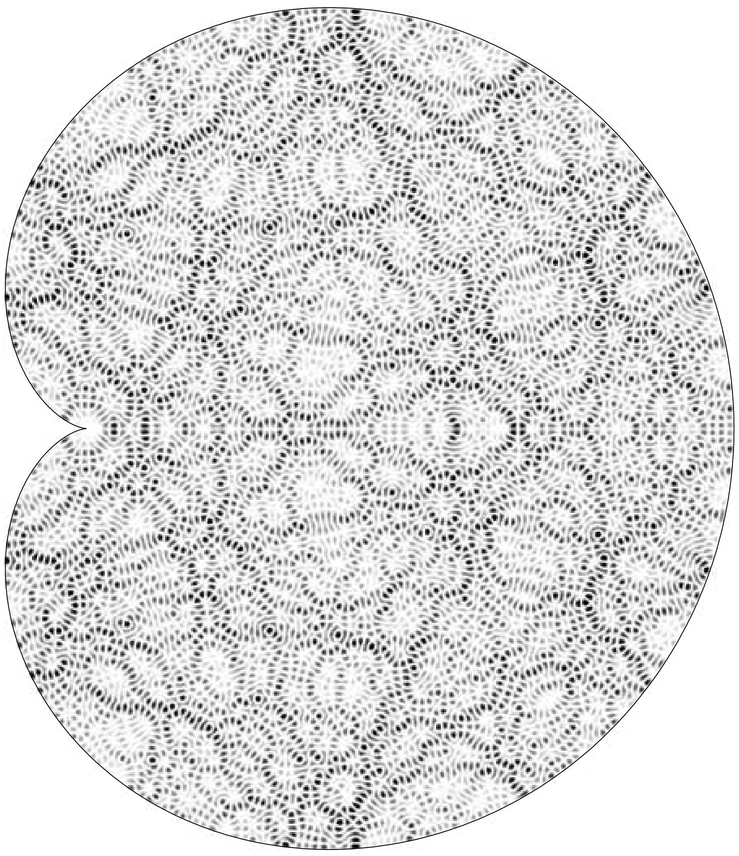


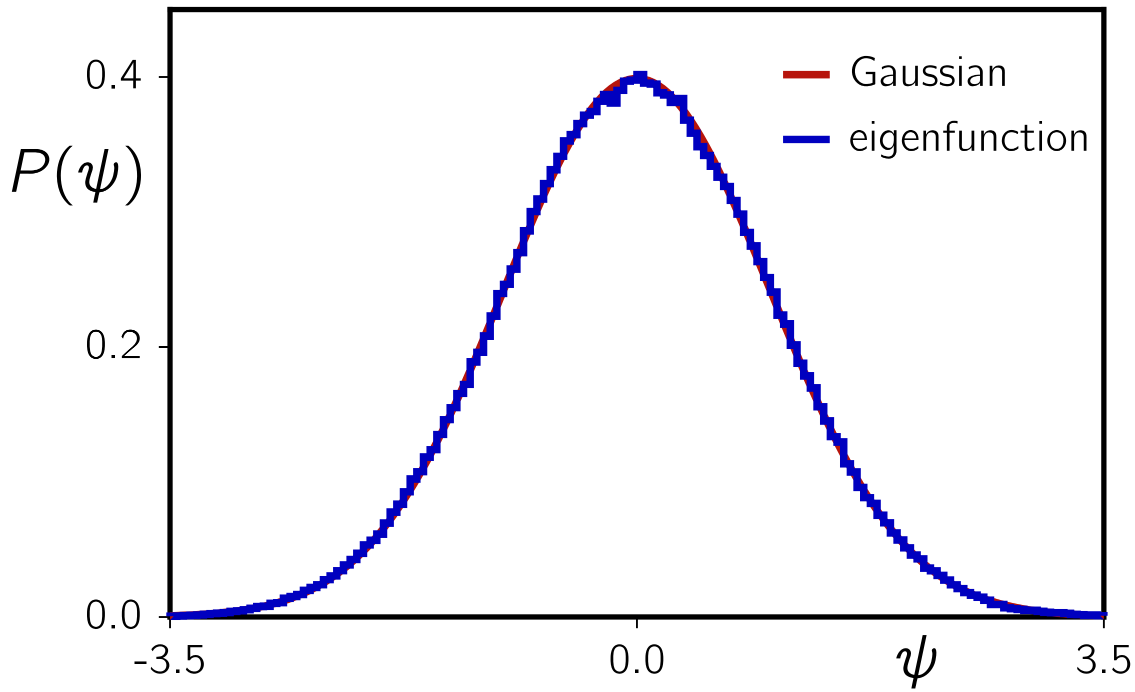
6000$^{\text{th}}$ eigenfunction

Random wave model – consequences

RWM $\Longrightarrow$ Amplitude distribution is Gaussian

$$P(\psi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\psi^2}{2\sigma^2}\right)$$





Semiclassical eigenfunction hypothesis [Berry '77, '83, Voros '79]

The Wigner function

$$W_n(\boldsymbol{p}, \boldsymbol{q}) := \frac{1}{(2\pi)^2} \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{q}'} \psi_n^*(\boldsymbol{q} - \boldsymbol{q}'/2)\psi_n(\boldsymbol{q} + \boldsymbol{q}'/2)\, \mathrm{d}q' \ ,$$

semiclassically concentrates on those regions in phase space, which a typical orbit explores in the long time limit $t \to \infty$.

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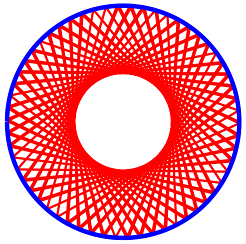
Implications for

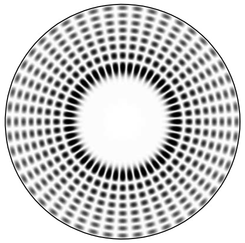
- integrable systems
- chaotic systems

Consequences of the semiclassical eigenfunction hypothesis

- For **integrable systems**: Localization on invariant tori

$$W(\boldsymbol{p}, \boldsymbol{q}) \sim \frac{\delta(I(\boldsymbol{p}, \boldsymbol{q}) - I)}{(2\pi)^2}$$





(here: $I(\boldsymbol{p}, \boldsymbol{q})$: action variable)

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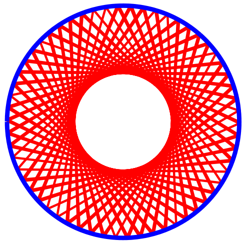
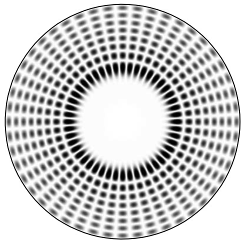
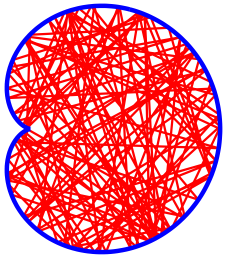
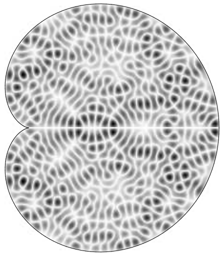



(here: $I(\boldsymbol{p}, \boldsymbol{q})$: action variable)

- For **chaotic systems**

$$W_n(\boldsymbol{p}, \boldsymbol{q}) \to \frac{1}{\text{vol}(\Sigma_E)} \delta(H(\boldsymbol{p}, \boldsymbol{q}) - E) \quad ,$$

i.e. semiclassical condensation on the energy surface Σ_E.

Quantum ergodicity theorem

QET [Shnirelman '74, Colin de Verdière '85, Zelditch '87, Zelditch/Zworski '96, ...]

For ergodic systems there exists a subsequence $\{n_j\}$ of density one such that

$$\lim_{j\to\infty} \langle \psi_{n_j}, A\psi_{n_j} \rangle = \overline{\sigma(A)} \ ,$$

for every classical pseudodifferential operator A of order zero.

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Here $\sigma(A)$ is the principal symbol of A.

And $\overline{\sigma(A)}$ is its classical expectation value,

$$\overline{a} = \frac{1}{\mathrm{vol}(\Sigma_1)} \iint\limits_{\mathbb{R}^2\times\Omega} a(\boldsymbol{p}, \boldsymbol{q})\, \delta(\boldsymbol{p}^2 - 1)\, \mathrm{d}p\, \mathrm{d}q \ .$$

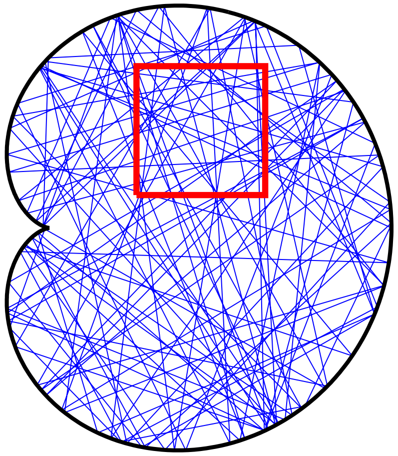
A subsequence $\{n_j\} \subset \mathbb{N}$ has density one if $\displaystyle\lim_{E\to\infty} \frac{\#\{n_j \mid E_{n_j} < E\}}{N(E)} = 1$,
where $N(E) := \#\{n \mid E_n < E\}$ is the spectral staircase function.

Quantum ergodicity theorem – Special Case

Classical ergodicity of a flow $\{\phi^t\}$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \chi_D(\phi^t(\boldsymbol{p}, \boldsymbol{q}))\, \mathrm{d}t = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

for almost all initial conditions in phase space, $(\boldsymbol{p}, \boldsymbol{q}) \in T^*\Omega$.



Quantum ergodicity theorem – Special Case

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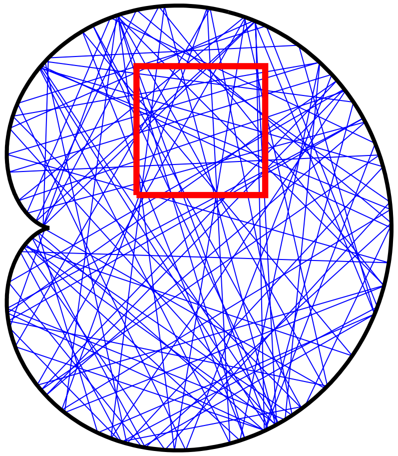

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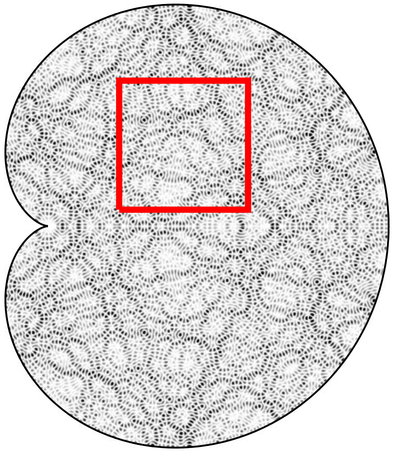
Quantum ergodicity in position space



$$\lim_{j\to\infty} \int_\Omega \chi_D(\boldsymbol{q})\, |\psi_{n_j}(\boldsymbol{q})|^2\, \mathrm{d}^2 q = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

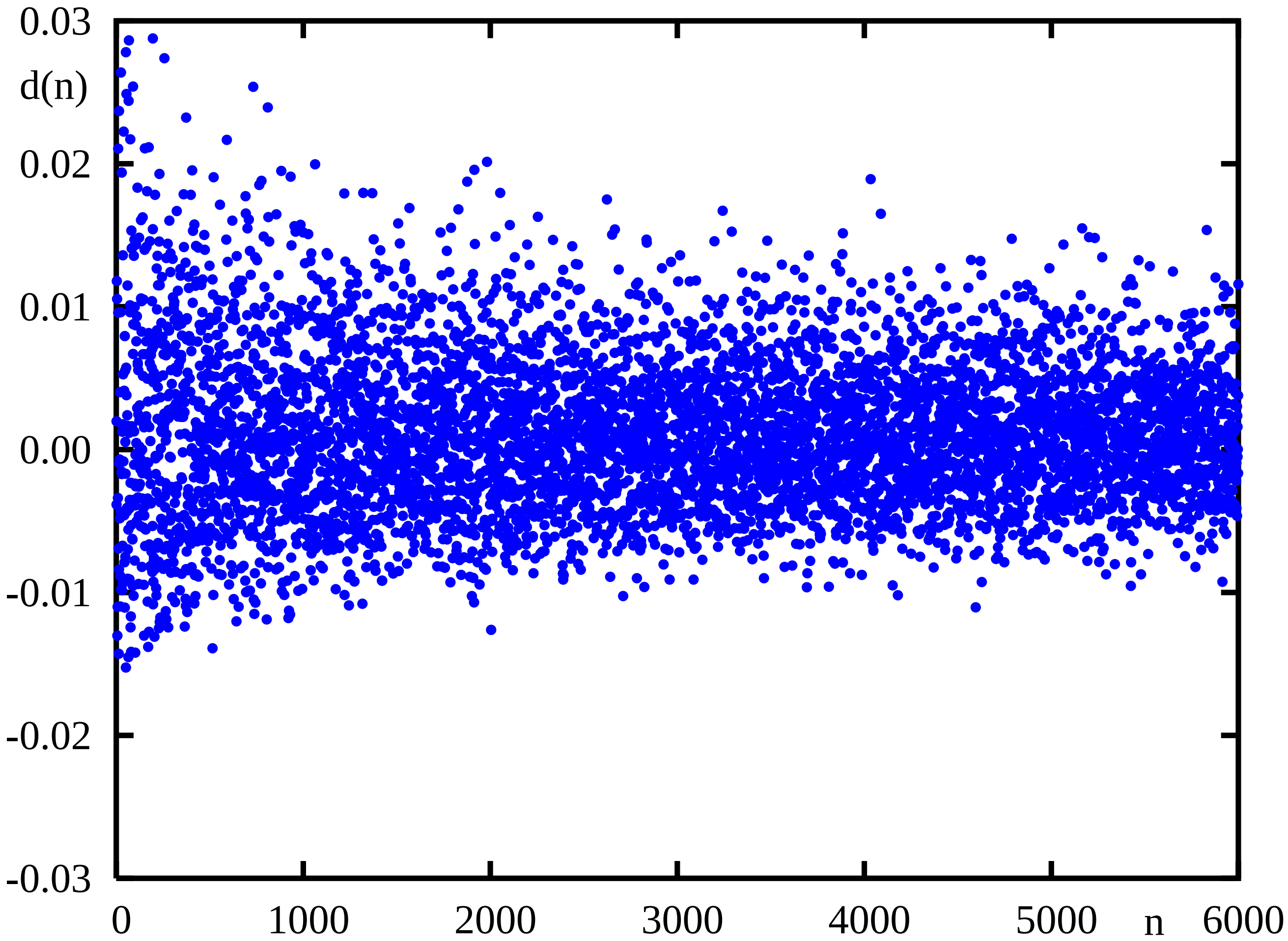
for a subsequence of density one.

Quantum ergodicity theorem makes statement about sequences of eigenfunctions (weak limit!).

QET – Example – observable in position space

Consider as observable $A = \chi_D(\boldsymbol{q})$ and plot

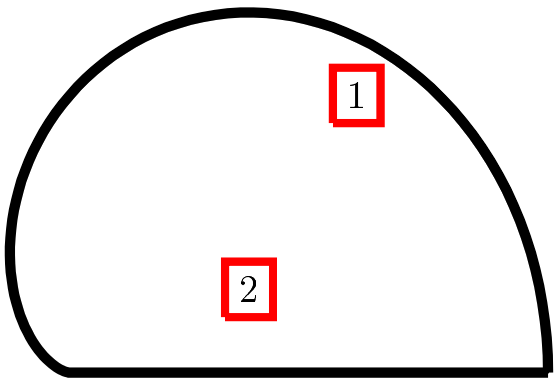
$$\int_\Omega \chi_D(\boldsymbol{q})\, |\psi_{n_j}(\boldsymbol{q})|^2 \, \mathrm{d}^2 q - \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

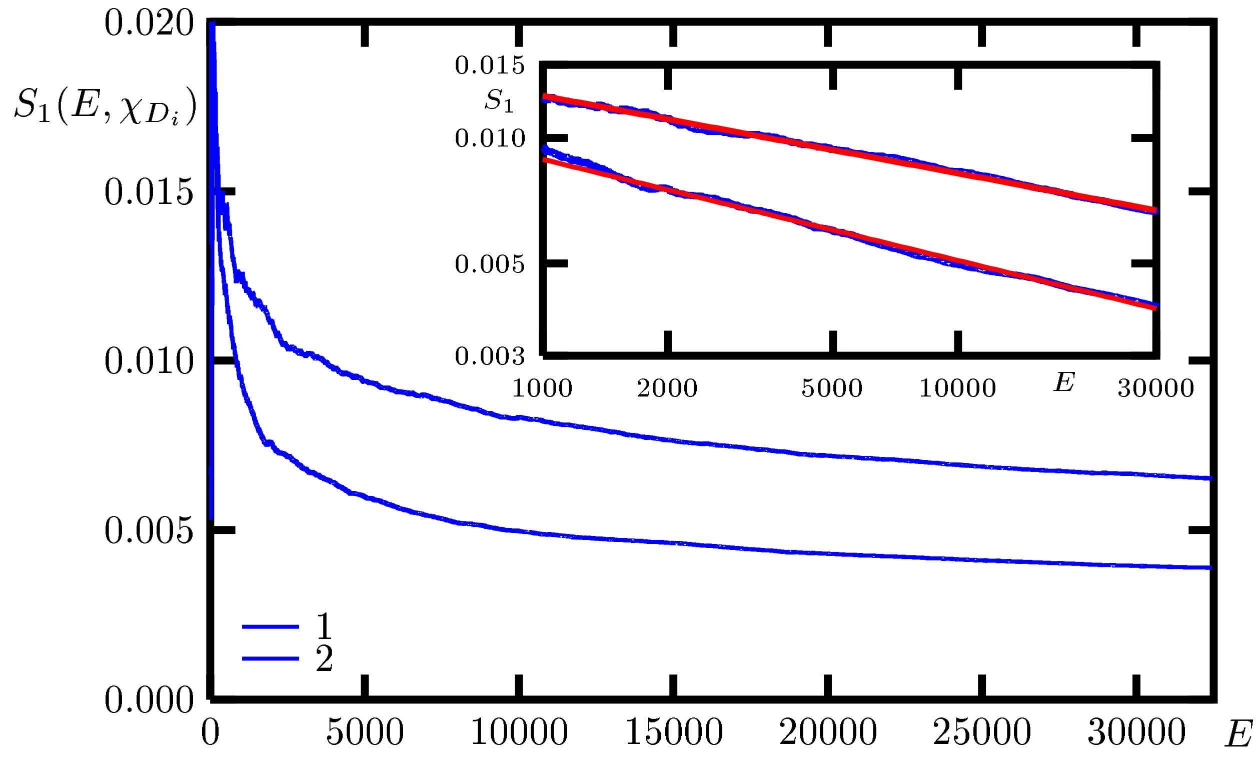


QET – Example – observable in position space

Thus consider cumulative differences

$$S_1(E, A) := \frac{1}{N(E)} \sum_{n: E_n \le E} \left| \int_\Omega \chi_D(\boldsymbol{q}) \, |\psi_{n_j}(\boldsymbol{q})|^2 \, \mathrm{d}^2 q - \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)} \right| .$$

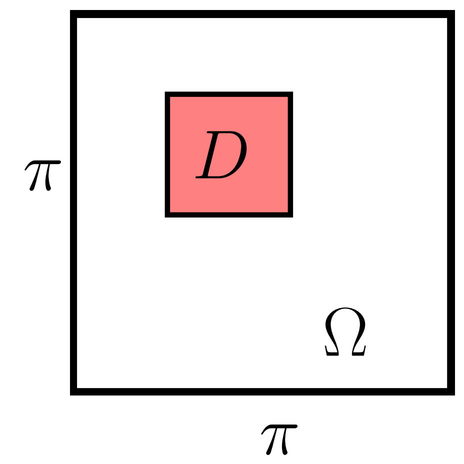




Remark: QET is equivalent to $S_1(E, A) \to 0$ as $E \to \infty$.

Example: Square billiard (to confuse you ... ;-):

$$\psi_{kl}(x,y) = \frac{2}{\pi}\sin(kx)\sin(ly) \quad k,l \in \mathbb{N}$$



Then one gets

$$\iint_D |\psi_{kl}(x,y)|^2 \, \mathrm{d}x\mathrm{d}y$$

$$= \frac{4}{\pi^2}\int_{x_0}^{x_1}\mathrm{d}x\int_{y_0}^{y_1}\mathrm{d}y\,\sin^2(kx)\sin^2(ly)$$

$$\to \frac{(x_1-x_0)(y_1-y_0)}{\pi^2} \equiv \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

for a subsequence of density one.

Quantum ergodicity theorem — “example” 2

Consider the observable $a(\boldsymbol{p}, \boldsymbol{q}) = a(\boldsymbol{p})$. Then

$$\langle \psi_n, A\psi_n \rangle = \int\limits_{\mathbb{R}^2} |\widehat{\psi}_n(\boldsymbol{p})|^2 a(\boldsymbol{p}) \, \mathrm{d}^2 p \ ,$$

with

$$\widehat{\psi}_n(\boldsymbol{p}) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{q}} \psi(\boldsymbol{q}) \, \mathrm{d}^2 q$$

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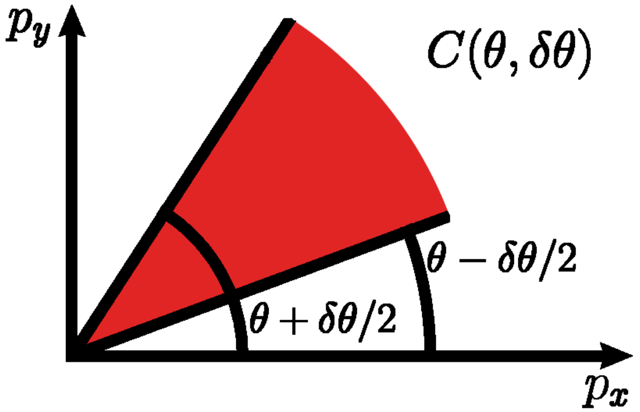
Characteristic function in momentum space



$$a(\boldsymbol{p}) = \chi_{C(\theta,\delta\theta)}(\boldsymbol{p})$$

where

$$C(\theta, \delta\theta) := \left\{(r, \varphi) \;\middle|\; r \in \mathbb{R}^+, \varphi \in [\theta - \delta\theta/2, \theta + \delta\theta/2]\right\}$$

Quantum ergodicity theorem — “example” 2 (cont’d)

QET implies for a subsequence of density one:

$$\lim_{n_j\to\infty} \iint_{C(\theta,\delta\theta)} |\widehat{\psi}_{n_j}(\boldsymbol{p})|^2 \, \mathrm{d}^2 p = \frac{\delta\theta}{2\pi}$$

Quantum ergodicity theorem — "example" 2 (cont'd)

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Example: Circle billiard (to confuse you even more ... ;-):

$$\psi_{kl}(r,\phi) = J_k(j_{kl} r) \cos(k\phi)$$

One can show that a subsequence of density one of eigenfunctions is quantum ergodic in momentum space.

Several interesting questions

- Do exceptional eigenfunctions exist ?
  E.g.: scars, bouncing ball modes, ...
  (quantum limit has to be invariant under the flow!)

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- If yes, how many are there ?
  The quantum ergodicity theorem implies

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  Can one say more about $N_{\text{exceptional}}(E)$?

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  Can one say more about $N_{\text{exceptional}}(E)$?
- How fast do quantum expectation values tend to the corresponding classical limit ?
  I.e., what is the rate of quantum ergodicity ?

Quantum limits (in position space)

Consider the sequence of probability measures on Ω

$$\mathrm{d}\mu_n := |\psi_n(\boldsymbol{q})|^2 \, \mathrm{d}^2 q$$

Definition *A measure μ^{ql} is called quantum limit if a subsequence of the μ_n converges to μ^{ql}.*

QET: for a subsequence of density one the quantum limit is

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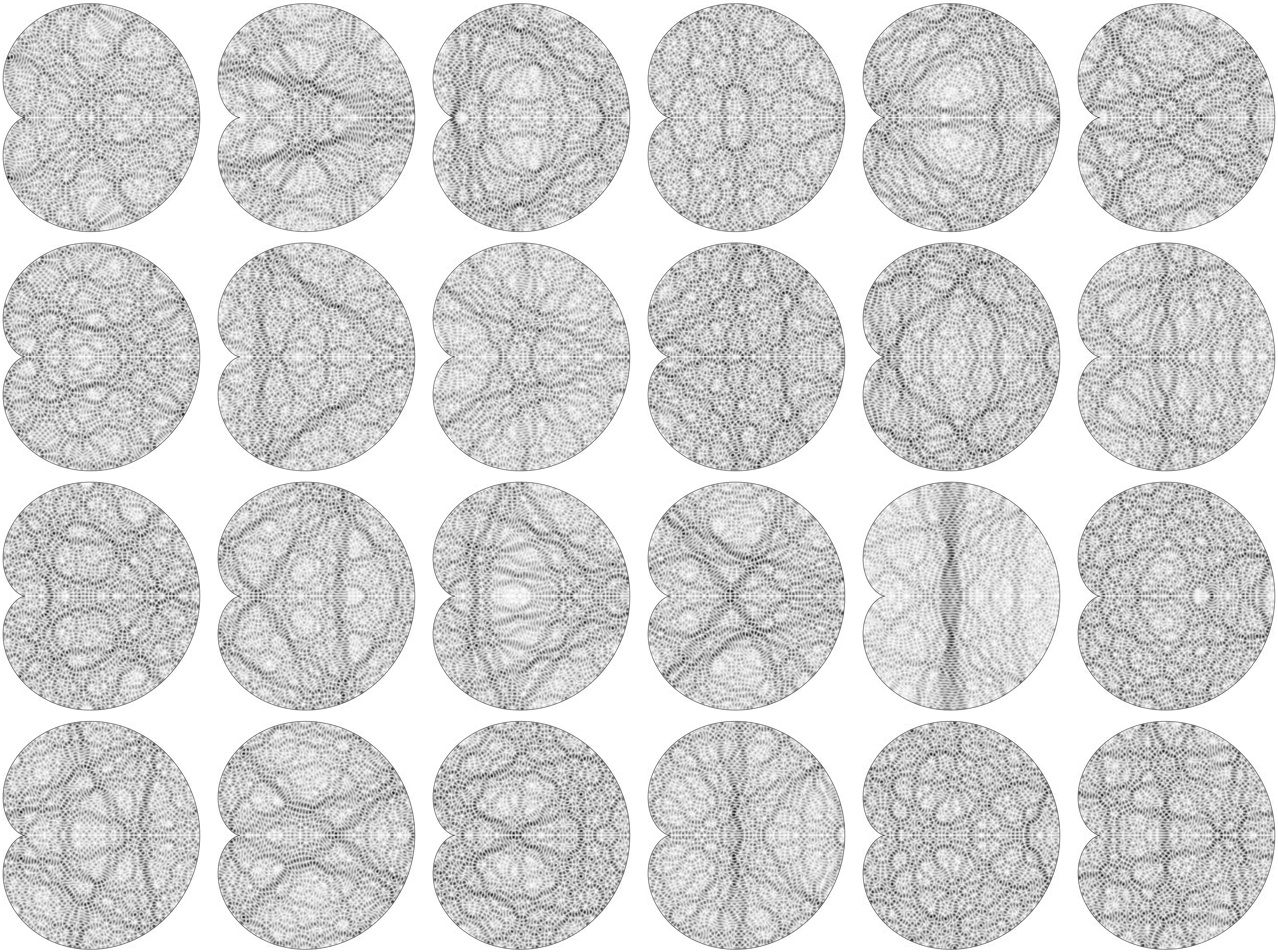
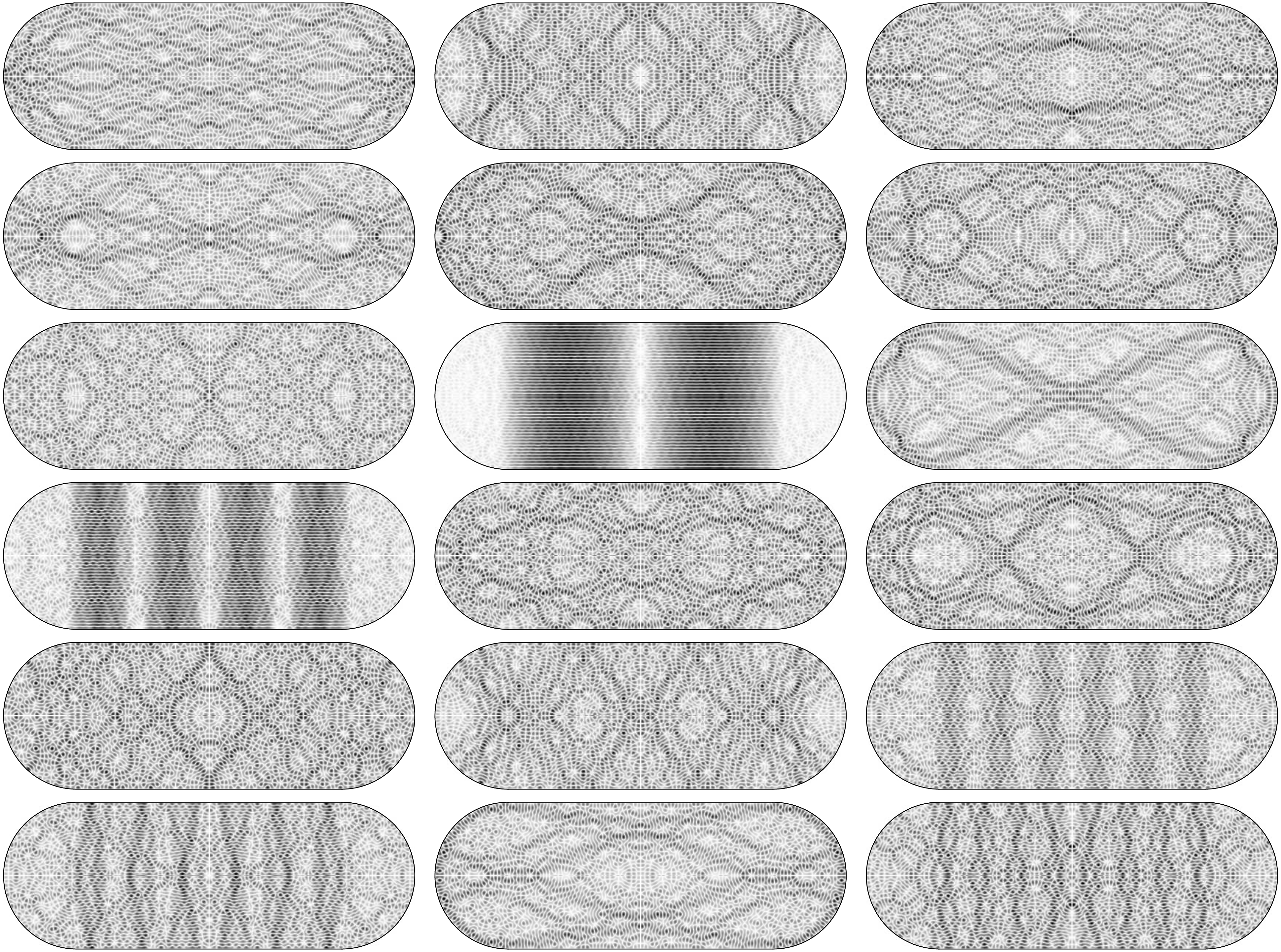
What quantum limits can occur?

They have to be invariant under the flow!

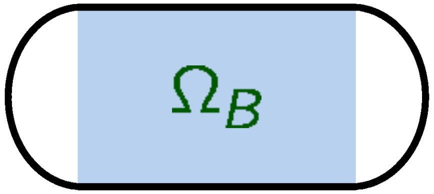
QET – eigenfunctions cardioid



QET – eigenfunctions stadium



Quantum limit for bouncing ball modes:



In position space

$$\lim_{n_j\to\infty} \mathrm{supp}(\psi_{n_j}) \subset \Omega_B$$

and in momentum space

$$\lim_{n_j\to\infty} |\widehat{\psi}_{n_j}|^2 = \delta(p_x)\frac{\delta(p_y-1)+\delta(p_y+1)}{2}$$

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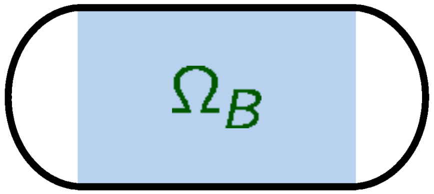

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$$\lim_{n_j \to \infty} |\widehat{\psi}_{n_j}|^2 = \delta(p_x) \frac{\delta(p_y - 1) + \delta(p_y + 1)}{2}$$

Consider counting function

$$N_{\mathrm{bb}}(E) := \left\{ n \;\middle|\; \psi_n \text{ is a bouncing ball mode} \right\}$$

The QET implies for $E \to \infty$

$$\frac{N_{\mathrm{bb}}(E)}{N(E)} \to 0$$

QET – BBMs – Counting function

One can show ([G. Tanner '97], [AB, R. Schubert, P. Stifter '97])

- Stadium billiard

$$N_{\mathrm{bb}}(E) \sim cE^{3/4}$$

- Cosine billiard

$$N_{\mathrm{bb}}(E) \sim cE^{9/10}$$

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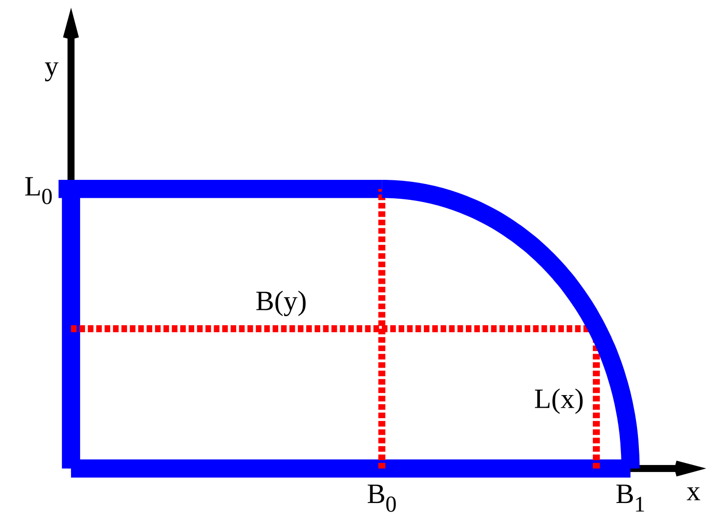


$$L(x) \sim L_0 - C(B_0 + x)^{\gamma}$$

$$\delta = \frac{1}{2} + \frac{1}{2+\gamma} \ .$$

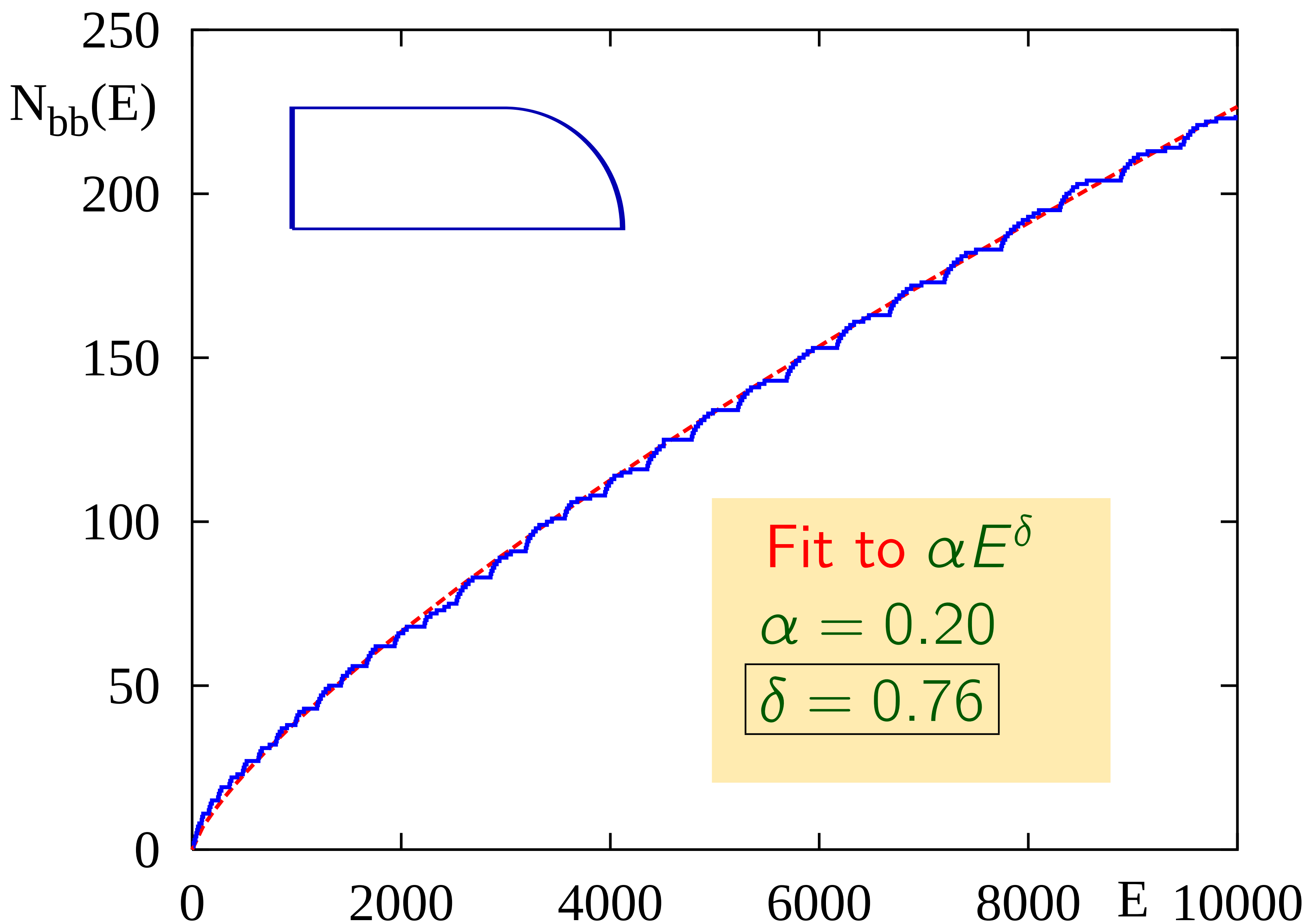
Remark:

- For every $\frac{1}{2} < \delta < 1$ one can find an ergodic Sinai billiard, s.t. $N_{\mathrm{bb}}(E) \sim cE^{\delta}$.

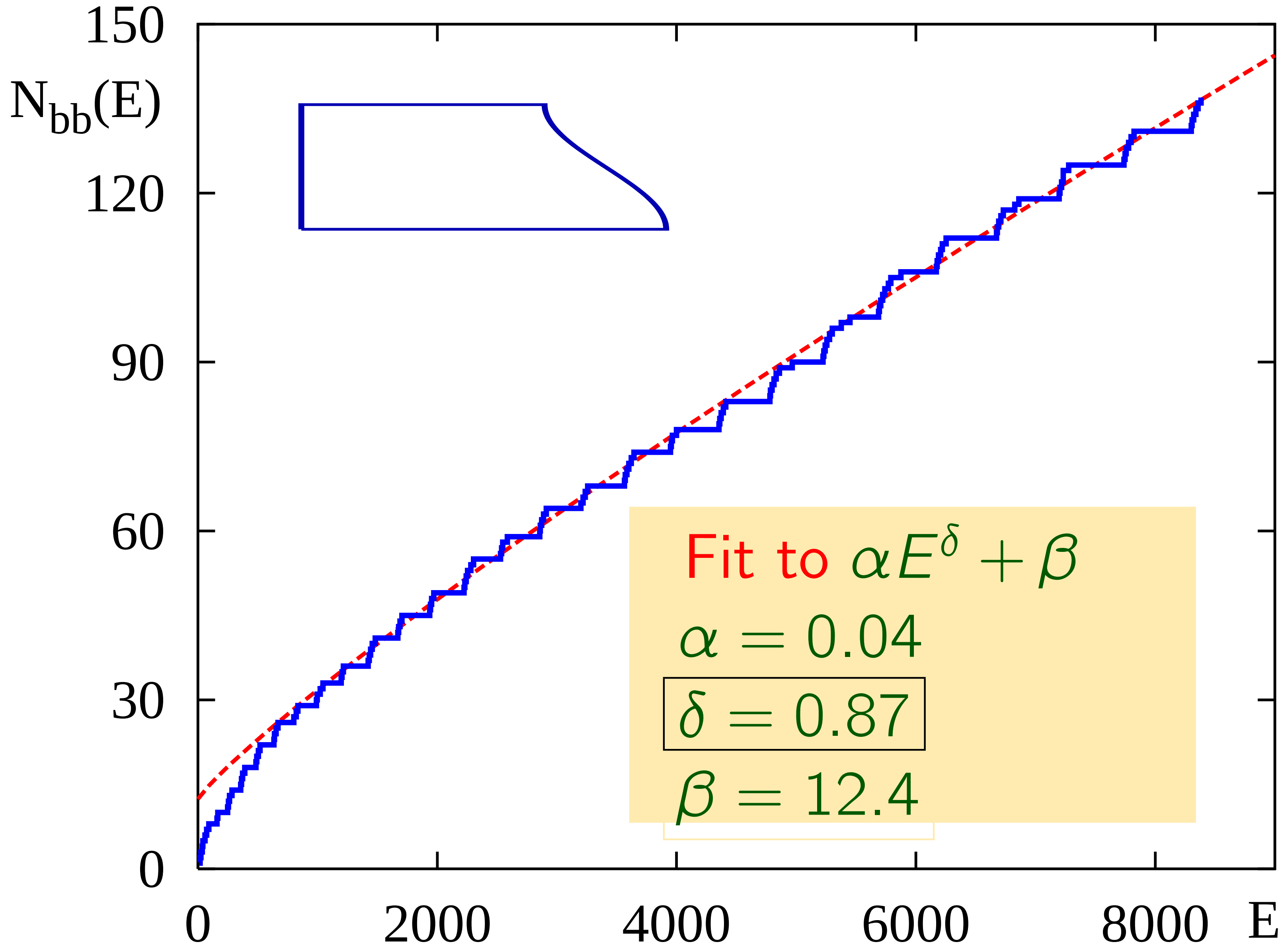
  This suggests: **the QET is sharp**

- Recent results: [Burq,Zworski 2003], [Zelditch 2003] [Hassel 2008]

Counting function for bouncing ball modes, stadium billiard



Counting function for bouncing ball modes, cosine billiard



Basic idea:

Scars are eigenfunctions showing an enhanced density around an unstable periodic orbit

Theoretical studies:

- Heller '84, Kaplan/Heller '98
- Bogomolny '88, Berry '89
- Ozorio de Almeida '98
- ... many others ...

Problems

- not really a definition
- not constant in time ;-)

QET — Scars: Some more details

Expectation: scars should occur at around energies $E_n^{\mathrm{scar}} = (k_n^{\mathrm{scar}})^2$ where

$$k_n^{\mathrm{scar}} = \frac{2\pi}{l_\gamma}\left(n + \frac{\nu_\gamma}{4}\right)$$

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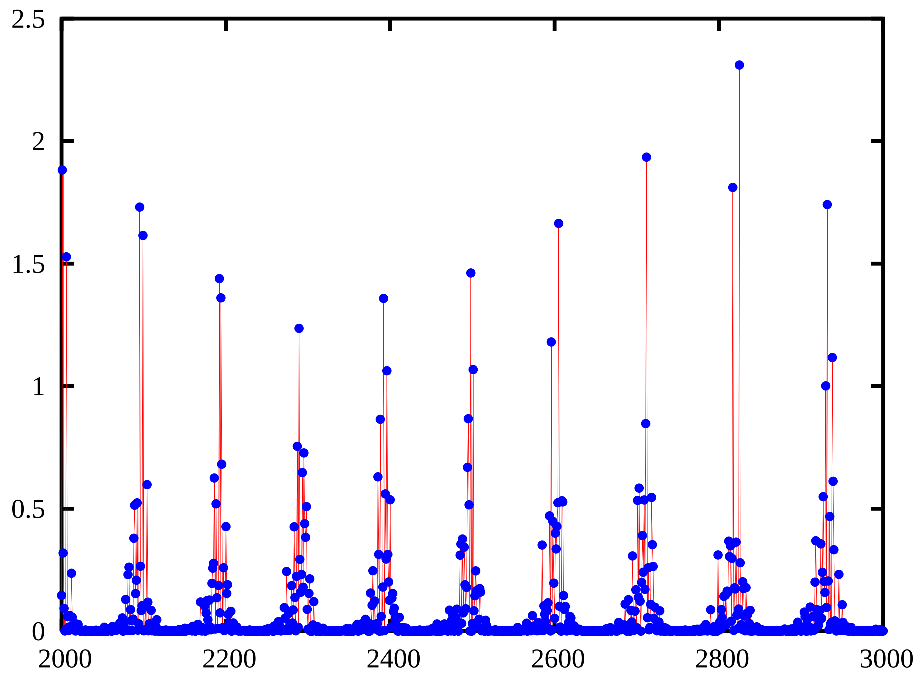
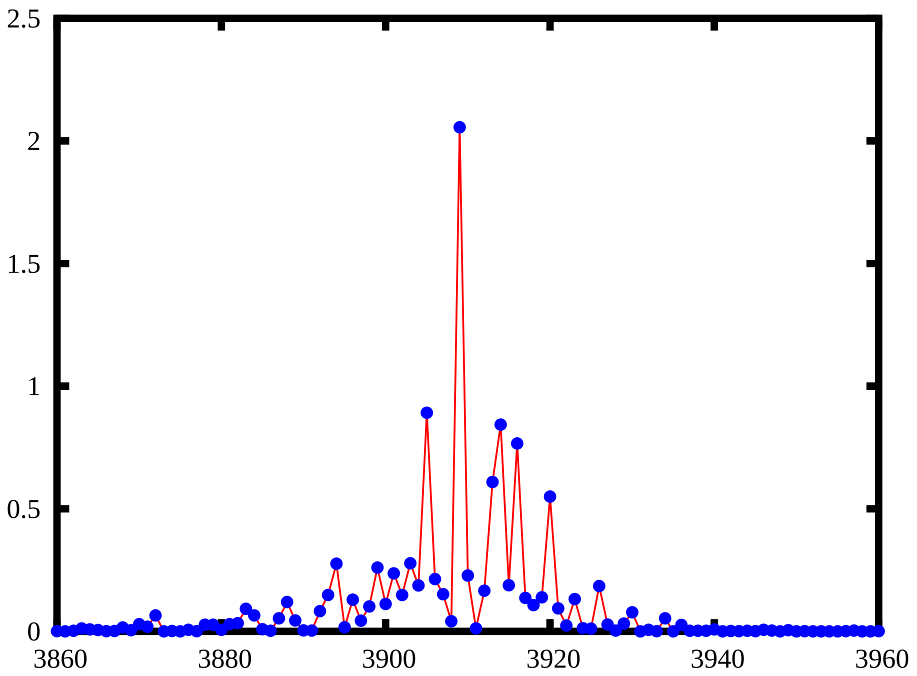
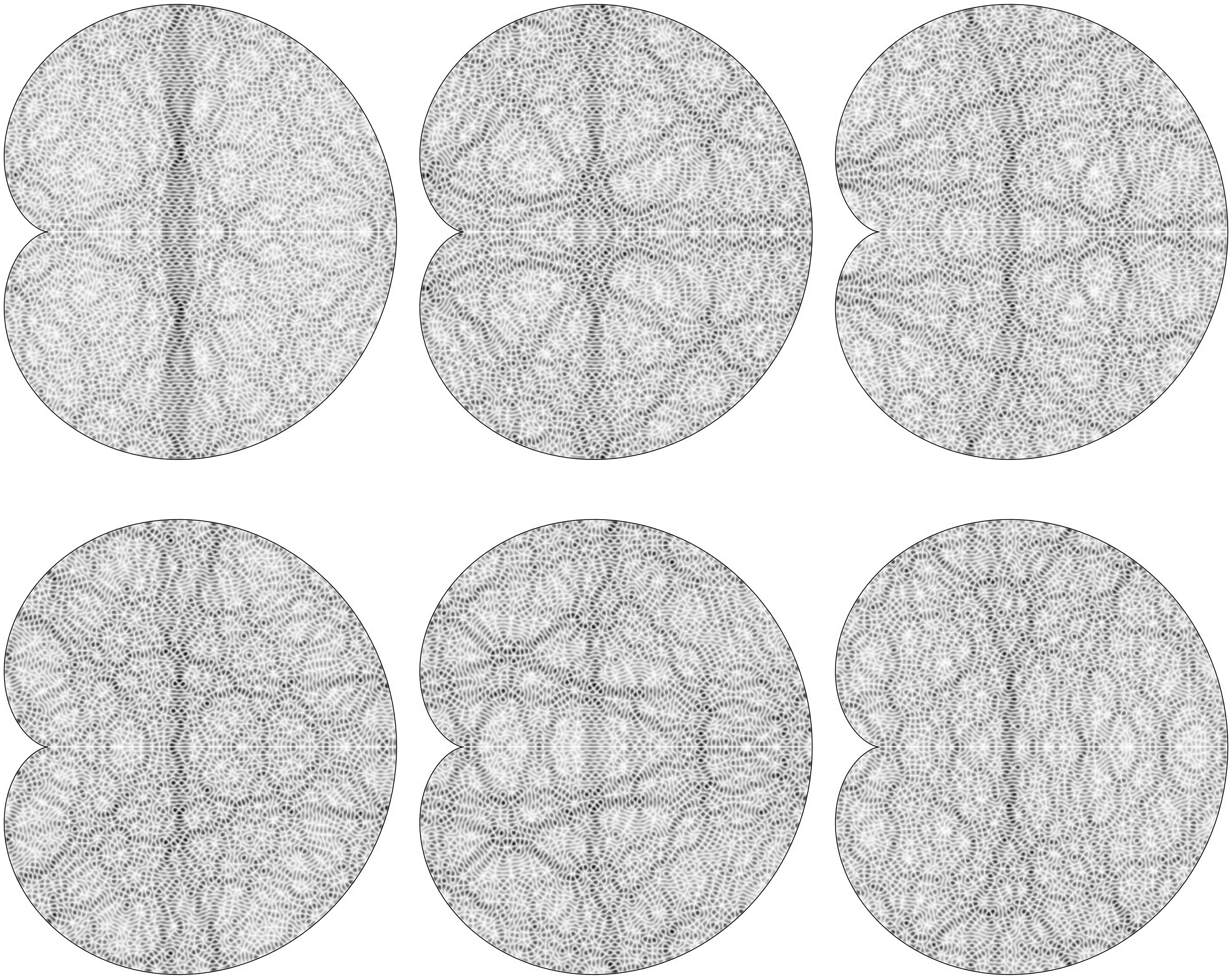
Plot of a scar measure: (via Poincaré Husimi function)

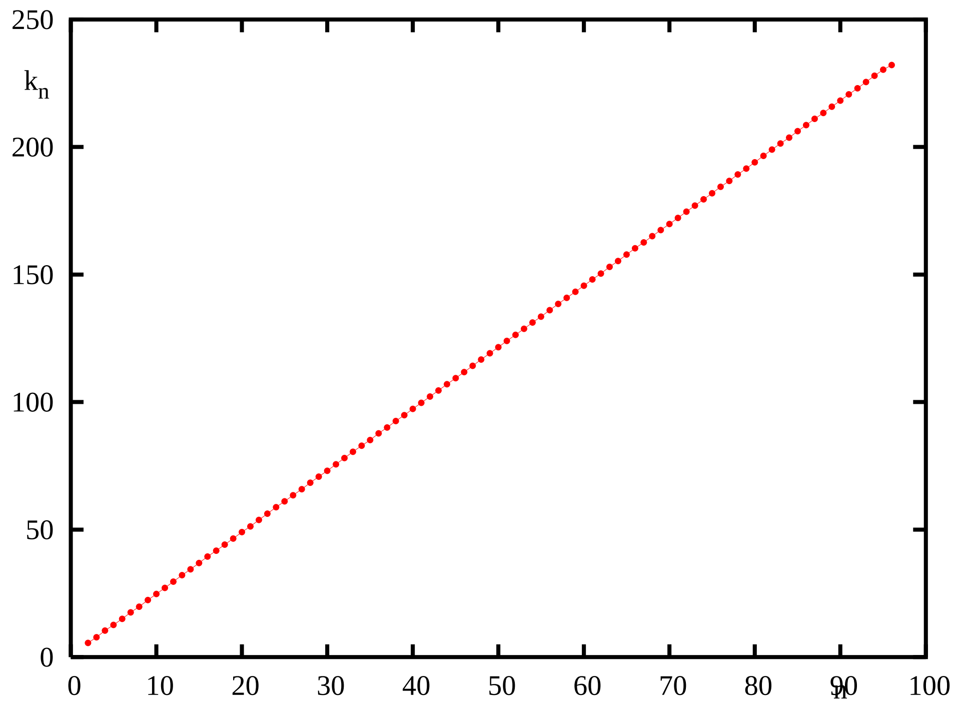


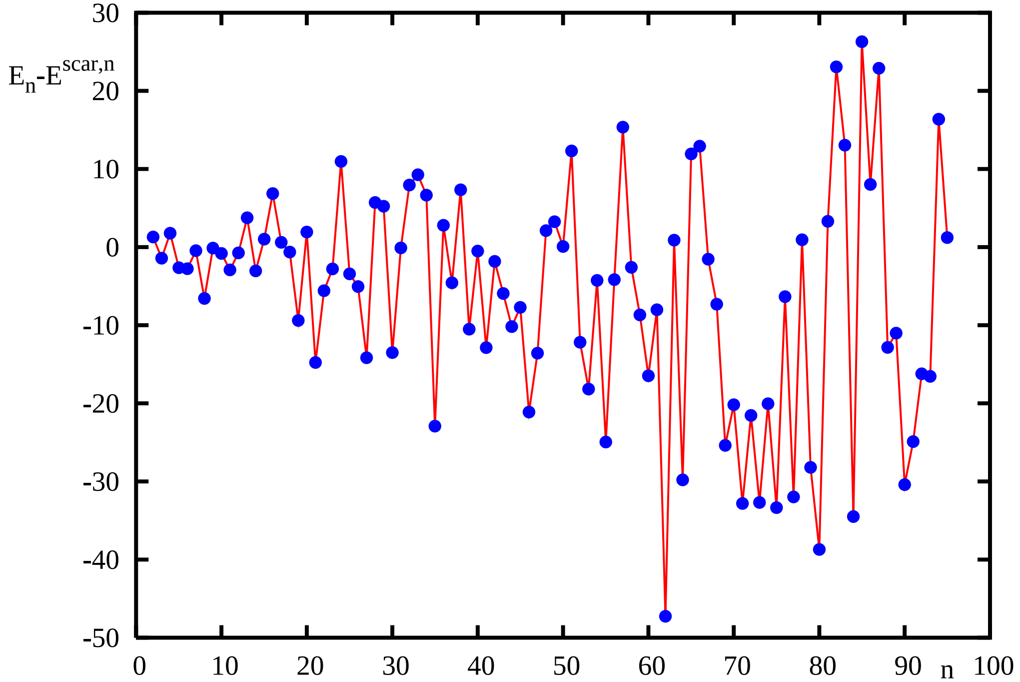


Eigenfunctions in the cluster:



Energies of scars: difference to $k_n^{\mathbf{scar}}$





Quite strong fluctuations !!

(mean spacing: $\frac{4\pi}{\mathcal{A}} = \frac{4\pi}{3\pi/4} = \frac{16}{3} = 5.333\ldots$)

Remark: For surfaces of constant negative curvature:

no scars observed, see

- Aurich, Steiner '95
- Auslaender, Fishman '98

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Possible types of scars:

- Strong scarring:
  quantum limit is a δ function on a periodic orbit
- Not that strong scarring:
  quantum limit is a δ function on a periodic orbit
  $+$ Liouville measure
- "soft scarring":
  quantum limit is the Liouville measure

Some results on this:

- For any Ansov map on the torus: weight for scars (on a finite union of periodic orbit): $< (\sqrt{5} - 1)/2$

  ([Bonechi, De Biévre 2003])

- Explicit construction of a sequence of states for the cat map for which the quantum limit is the sum of

  $1/2$ Lebesgue $+ 1/2\ \delta$ on any periodic orbit.

  ([F. Faure, S. Nonnenmacher, S. De Bièvre 2003])

- weight for scars (on a finite or countable union of p.o.): $< 1/2$,

  ([F. Faure, S. Nonnenmacher 2003])

QET — Quantum unique ergodicity

- proven for ergodic linear parabolic maps on $\mathbb{T}^2$

  ([Marklof, Rudnick 2000])

- for certain cat maps: QUE for joint eigenstates with Hecke operators, ([Rudnick, Kurlberg 2000])

  (not all eigenstates are of this type)

- for sequences of joint eigenstates of the Laplacian and Hecke operators on arithmetic surfaces ([Lindenstrauss 2003])

  (all eigenstates are conjectured to be of this type)

- Soundararajan/Holowinsky 2008: proof for Hecke eigenforms

Other extreme:

- class of ergodic piecewise affine transformation on $\mathbb{T}^2$:
  all classical invariant measures appear as quantum limits.
  ([Chang, Krüger, Schubert])

Summary II: Eigenfunctions in strongly chaotic systems

- Random wave model

  statistical description of eigenfunction

  Prediction: Gaussian distribution of eigenstates

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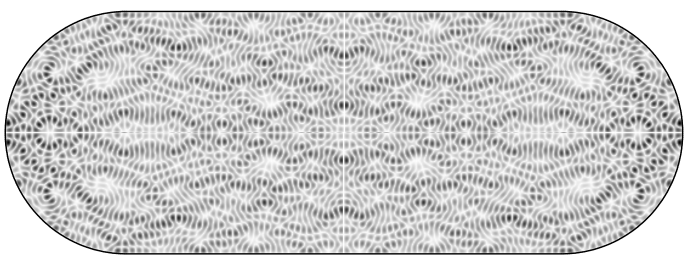
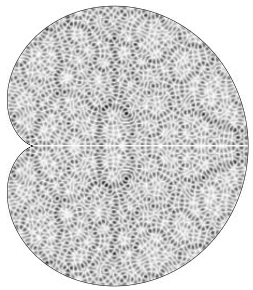
  almost all eigenfunctions become equidistributed





Summary II: Eigenfunctions in strongly chaotic systems

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  statistical description of eigenfunction

  Prediction: Gaussian distribution of eigenstates

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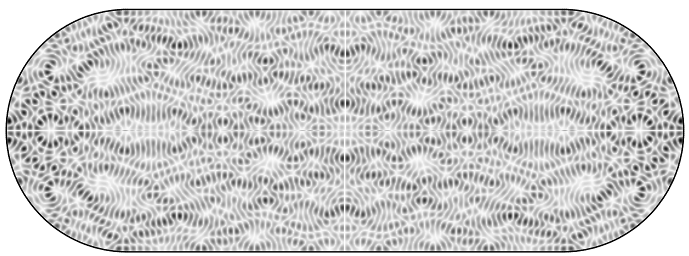
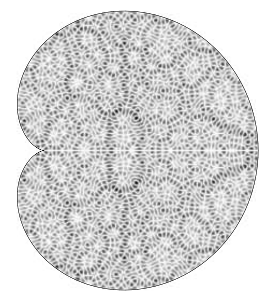
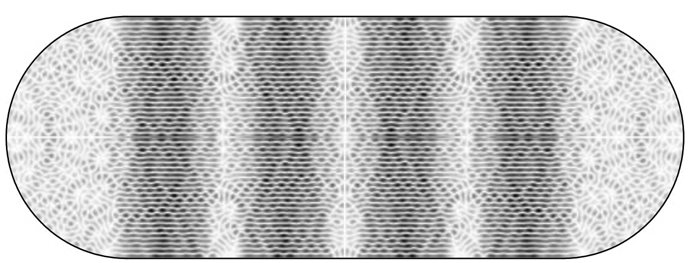
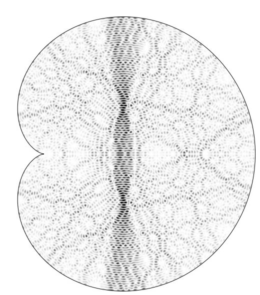



Possible exceptional eigenfunctions:

bouncing ball modes, scars, . . .

Example: limaçon billiards: [Robnik '83]

START

Systems with mixed phase space

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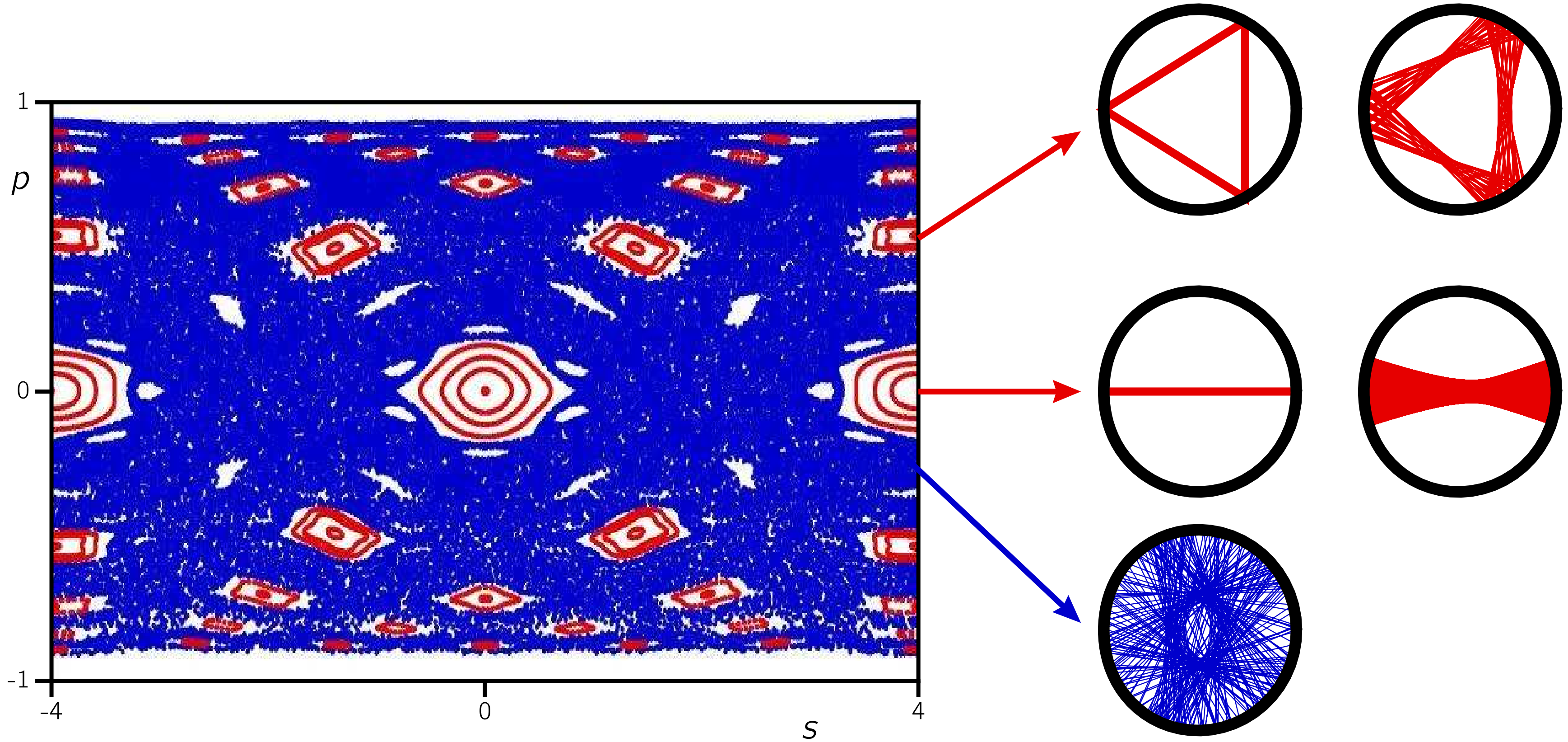
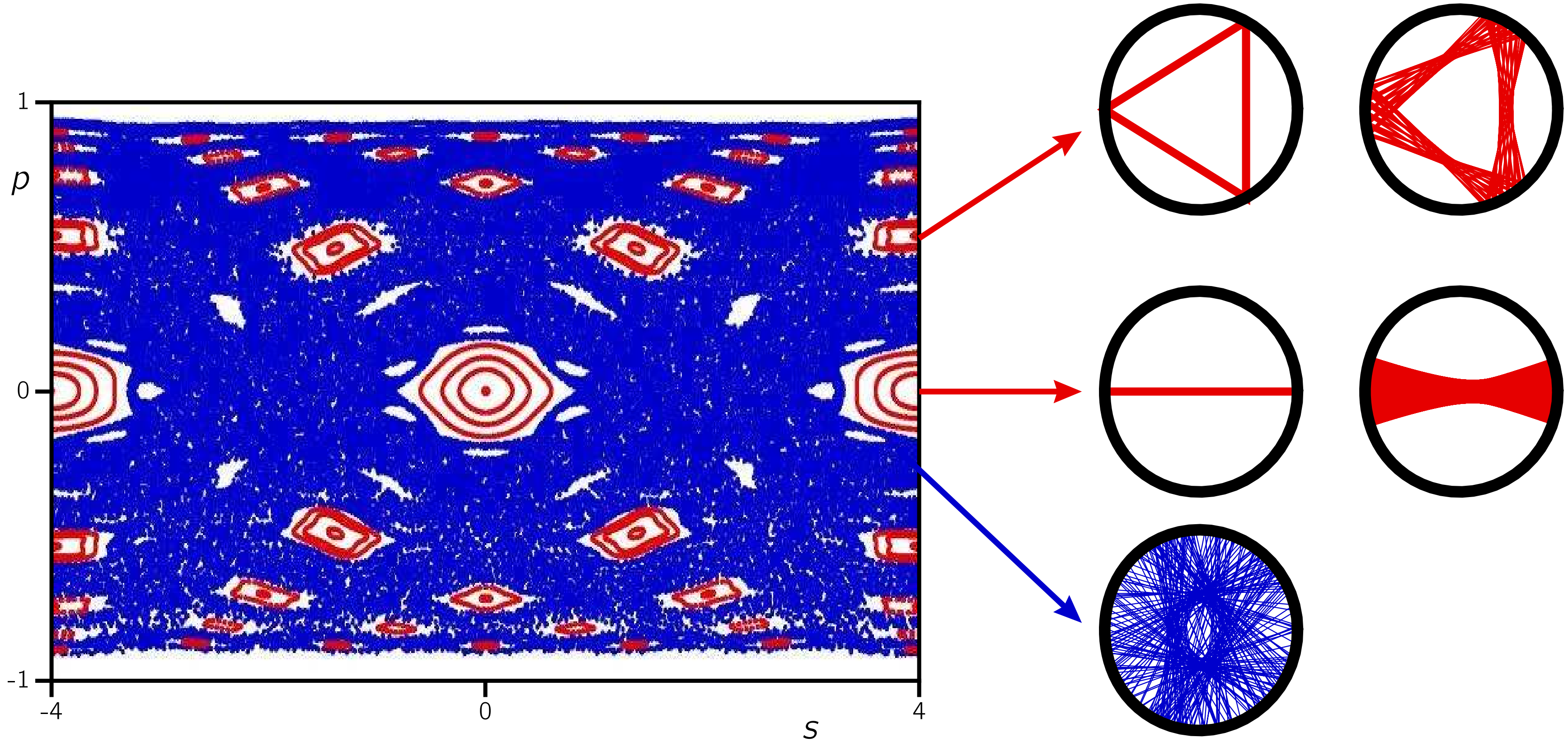


Coexistence of

- regular motion (islands)
- irregular motion (chaotic sea)

Systems with mixed phase space

Example: limaçon billiards: [Robnik '83]



Coexistence of

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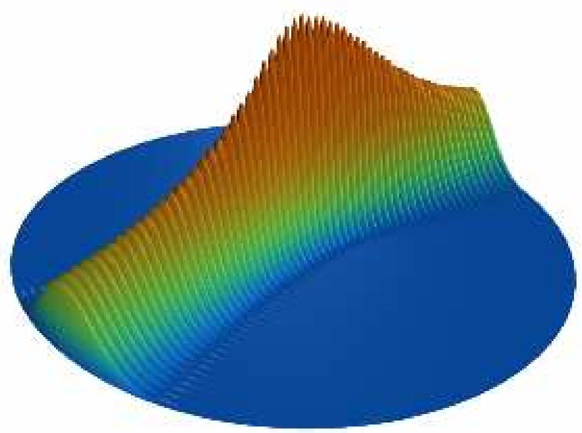
How do eigenfunctions look like in mixed systems?

Eigenfunctions in mixed systems



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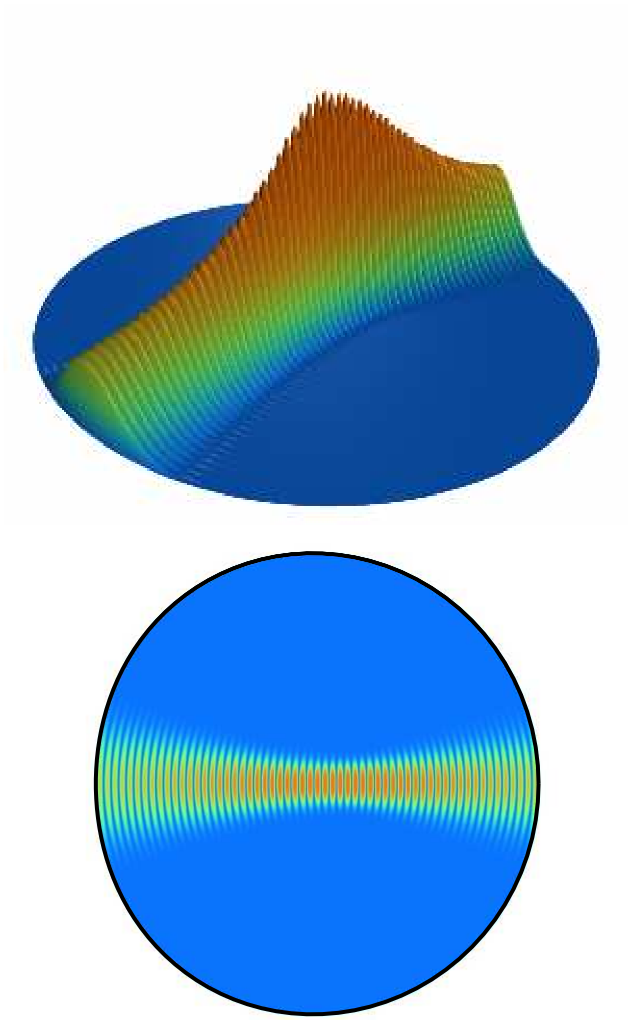

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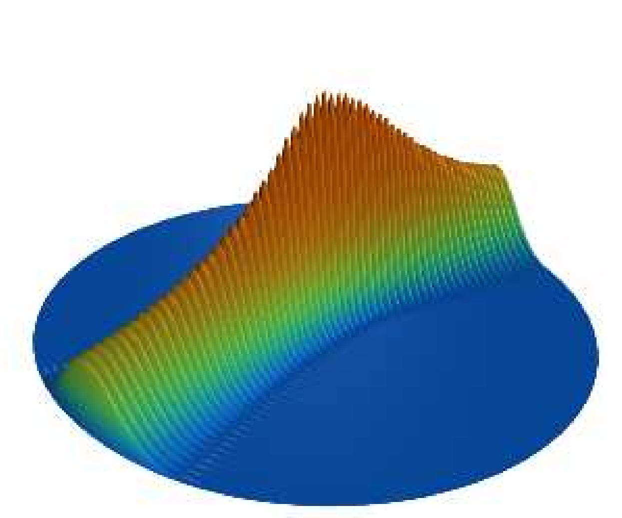
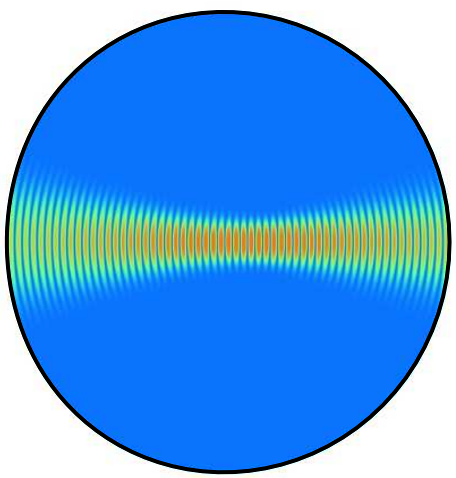
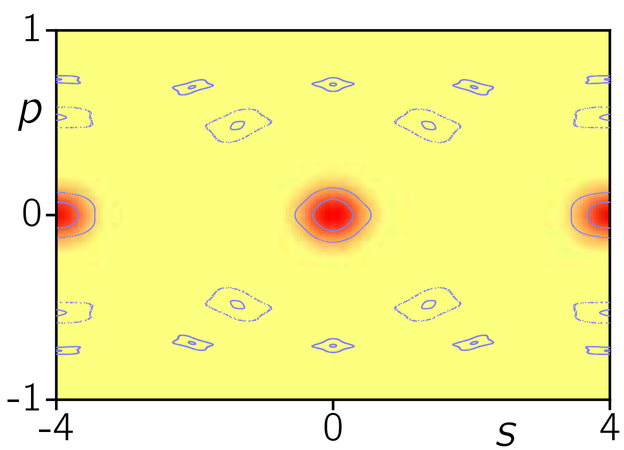
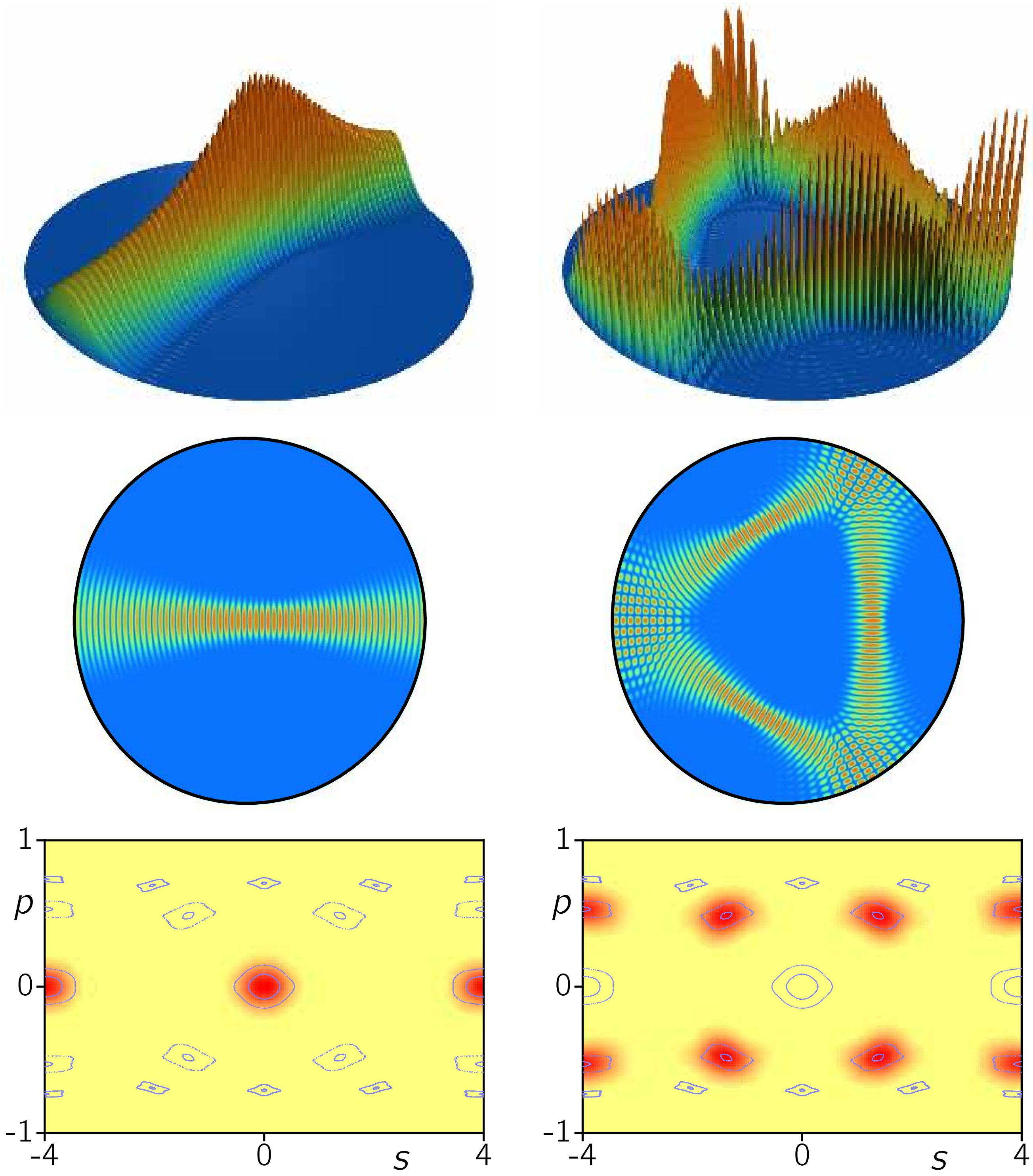






Poincaré Husimi representation

Eigenfunctions in mixed systems



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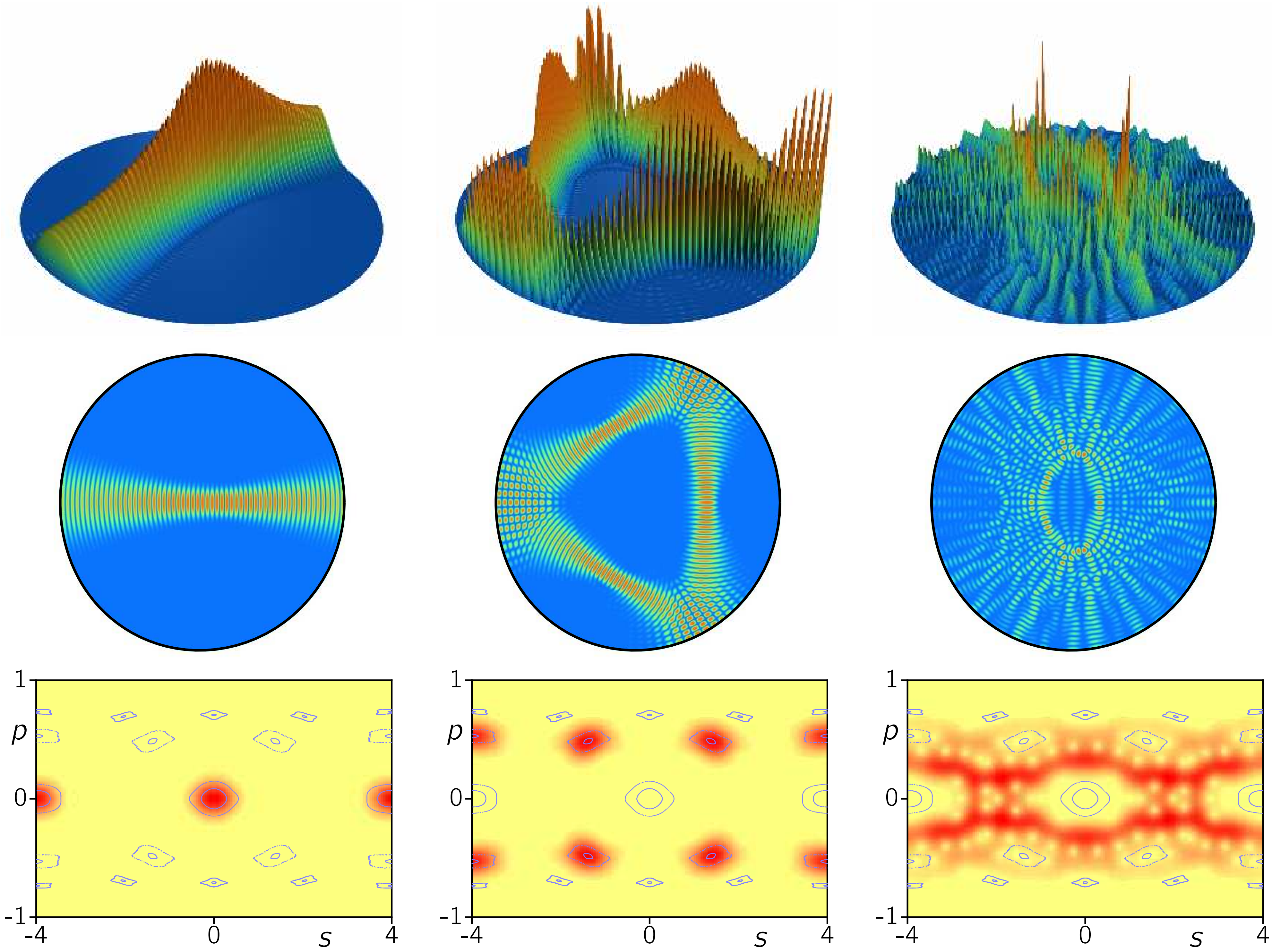

Eigenfunction statistics in mixed systems

To describe the statistical properties of irregular states in mixed systems introduce the *restricted random wave model*: [AB, Schubert, JPA 2002]

$$\psi_{\mathrm{RRWM},D}(\boldsymbol{q}) = \sqrt{\frac{4\pi}{\mathrm{vol}(D)N}} \sum_{n=1}^{N} \chi_D(\boldsymbol{k}_n, \boldsymbol{q}) \cos(\boldsymbol{k}_n \boldsymbol{q} + \varepsilon_n) \quad .$$

where:

- $\chi_D(\cdot)$ is the characteristic function of D
- D: domain in phase space
- the phases ε_n are independent random variables equidistributed on $[0, 2\pi]$
- momenta $\boldsymbol{k}_n \in \mathbb{R}^2$ are independent random variables equidistributed on the circle of radius $\sqrt{E}$.

RRWM: Local value distribution

The variance is given by

$$\begin{aligned}\sigma^2(\boldsymbol{q}) &= \mathbb{E}\left(\frac{4\pi}{\mathrm{vol}(D)}\left(\chi_D(\boldsymbol{k}_n, \boldsymbol{q})\cos(\boldsymbol{k}_n\boldsymbol{q} + \varepsilon_n)\right)^2\right) \\ &= \frac{1}{\mathrm{vol}(D)}\int_0^{2\pi} \chi_D(\mathbf{e}(\phi), \boldsymbol{q})\, \mathrm{d}\phi \quad ,\end{aligned} \tag{4}$$

where $\mathbf{e}(\phi) := (\cos(\phi), \sin(\phi))$

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For the distribution of $P_{\boldsymbol{q}}(\psi)$ of $\psi_{\mathrm{RRWM},D}(\boldsymbol{q})$ in a given point $\boldsymbol{q}\in\Omega$ one gets as $E\to\infty$ by the central limit theorem

$$P_{\boldsymbol{q}}(\psi) \longrightarrow \sqrt{\frac{1}{2\pi\sigma^2(\boldsymbol{q})}}\exp\left(-\frac{\psi^2}{2\sigma^2(\boldsymbol{q})}\right) \quad , \tag{5}$$

Mixed systems – Global amplitude distribution

Integrating the local result over Ω gives

$$P_{\mathrm{RRWM},D}(\psi) = \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} P_{\boldsymbol{q}}(\psi)\, \mathrm{d}^2 q$$

$$= \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} \sqrt{\frac{1}{2\pi\sigma^2(\boldsymbol{q})}} \exp\left(-\frac{1}{2\sigma^2(\boldsymbol{q})}\psi^2\right)\, \mathrm{d}^2 q\ .$$

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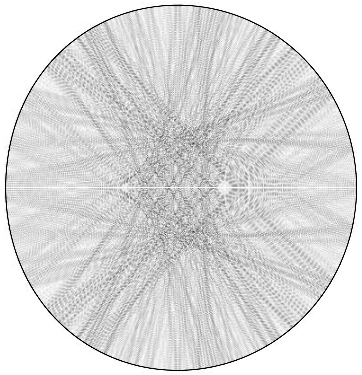
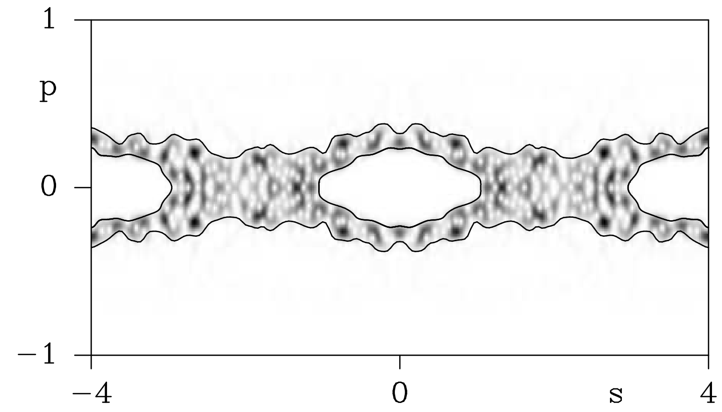
Remarks:

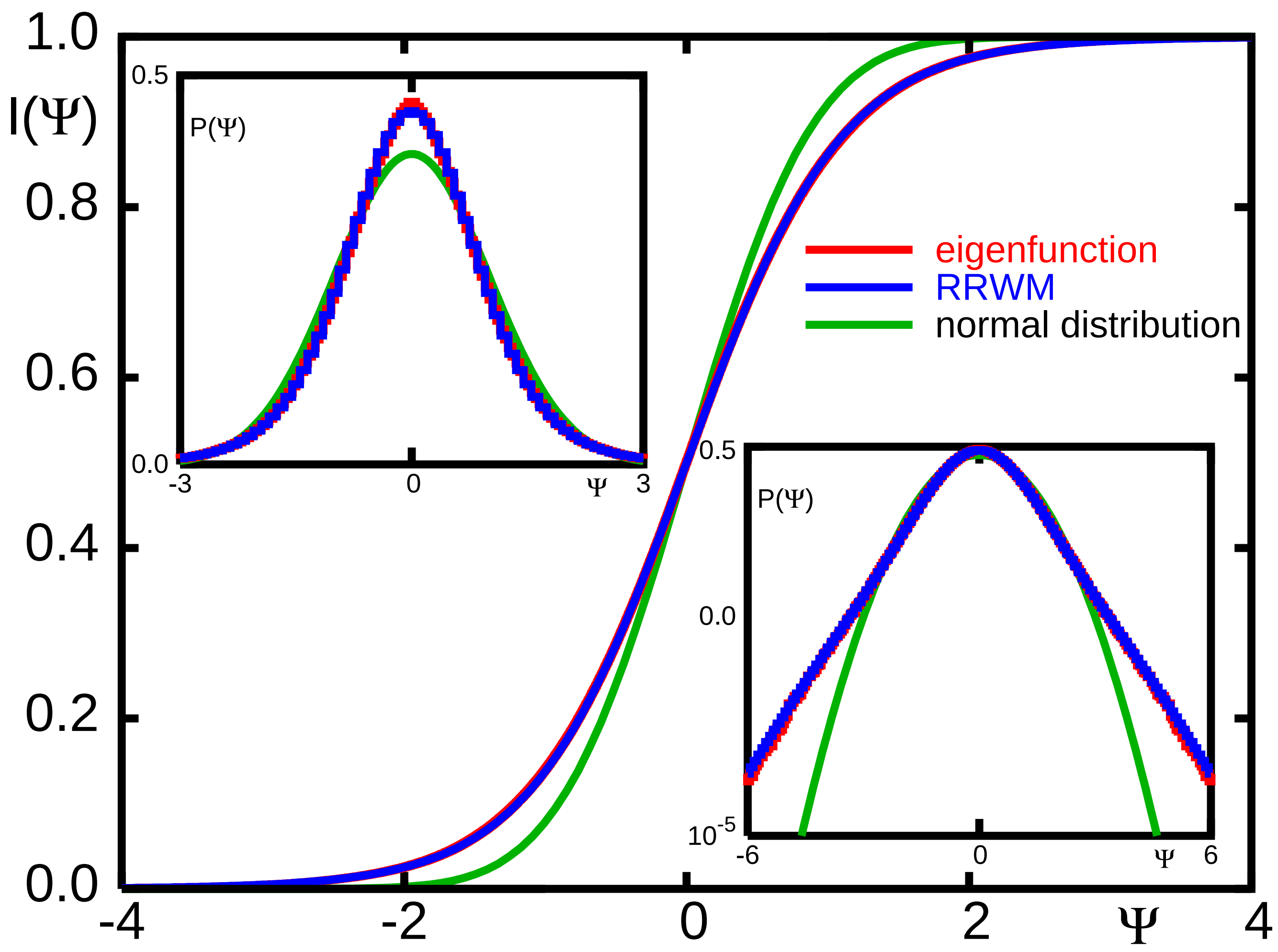
- If $\sigma^2(\boldsymbol{q})$ is not constant, $P_{\mathrm{RRWM},D}(\psi)$ differs from a Gaussian.

- Ergodic systems: $\sigma^2(\boldsymbol{q}) = \frac{1}{\mathrm{vol}(\Omega)}$.
  Thus $P_{\mathrm{RRWM},D}(\psi)$ is Gaussian with variance $\sigma^2 = \frac{1}{\mathrm{vol}(\Omega)}$.
  are given by

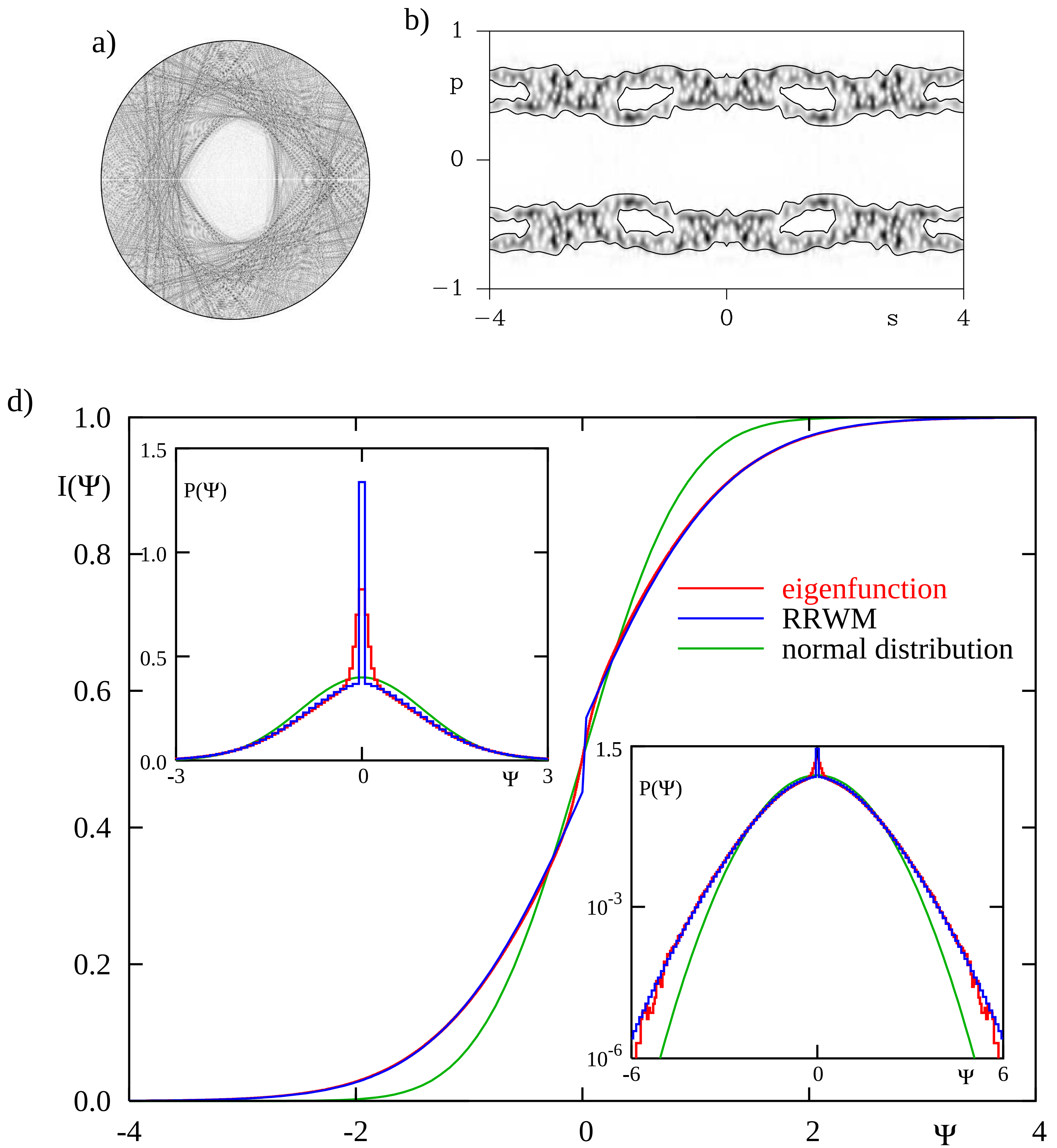
High lying eigenfunction (approx. 130568th state)



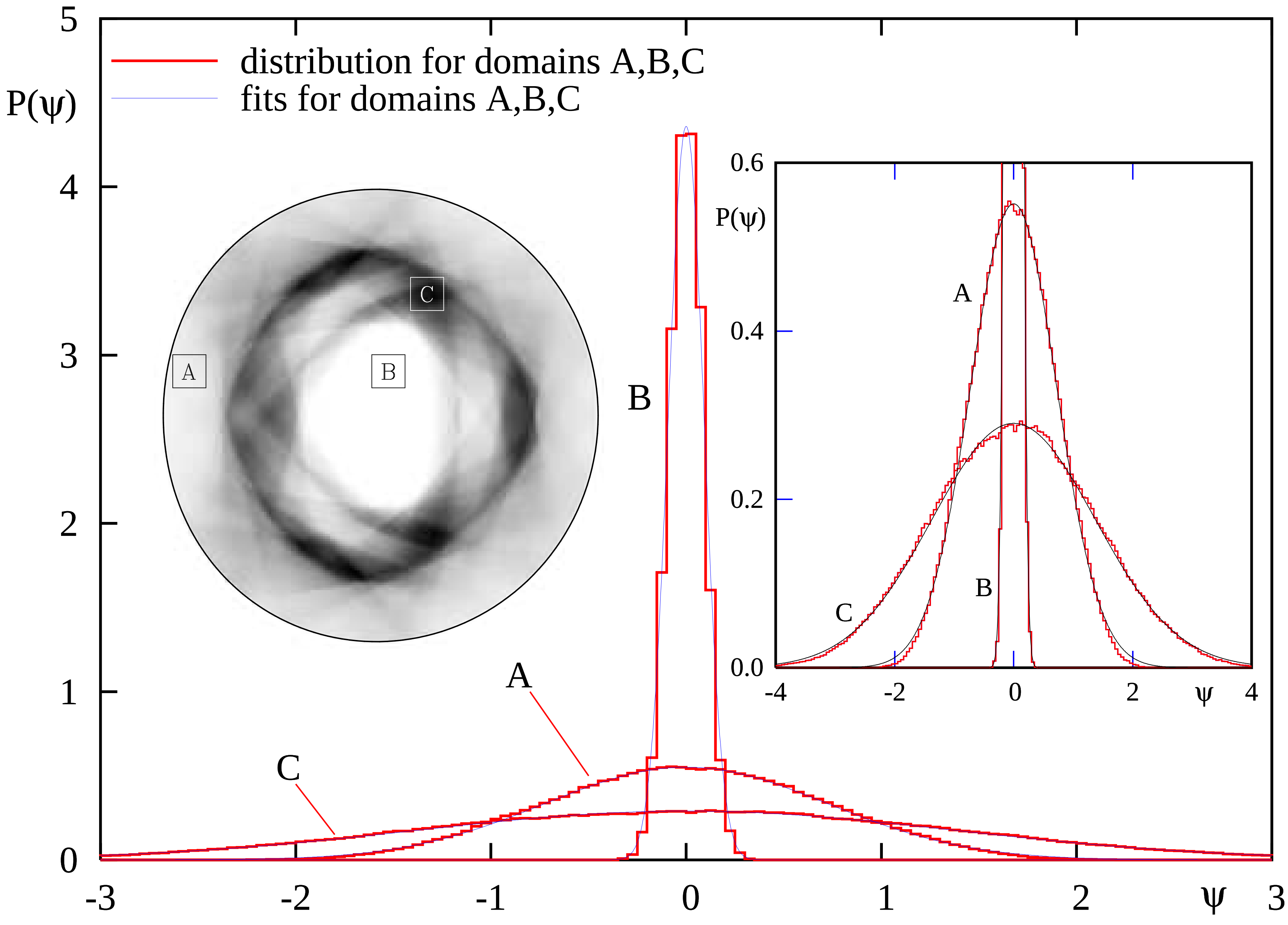




High lying eigenfunction (approx. 130607th state)



Local amplitude distribution



Simpler class of systems: **Area preserving maps**

Classical system: one-dimensional kicked Hamiltonians,

$$H(p, x, t) = T(p) + V(x) \sum_n \delta(t - n)$$

Corresponding map: $P(x_t, p_t) = (x_{t+1}, p_{t+1})$,

$$x_{t+1} = x_t + T'(p_t), \qquad \pmod 1$$
$$p_{t+1} = p_t - V'(x_{t+1}) \qquad \pmod 1$$

Example1, Example2

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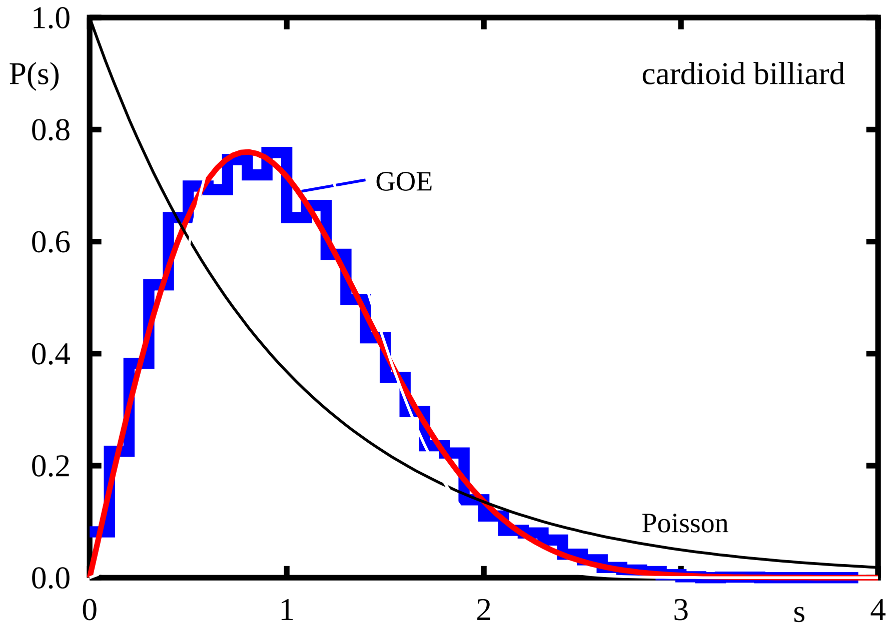
Example1, Example2

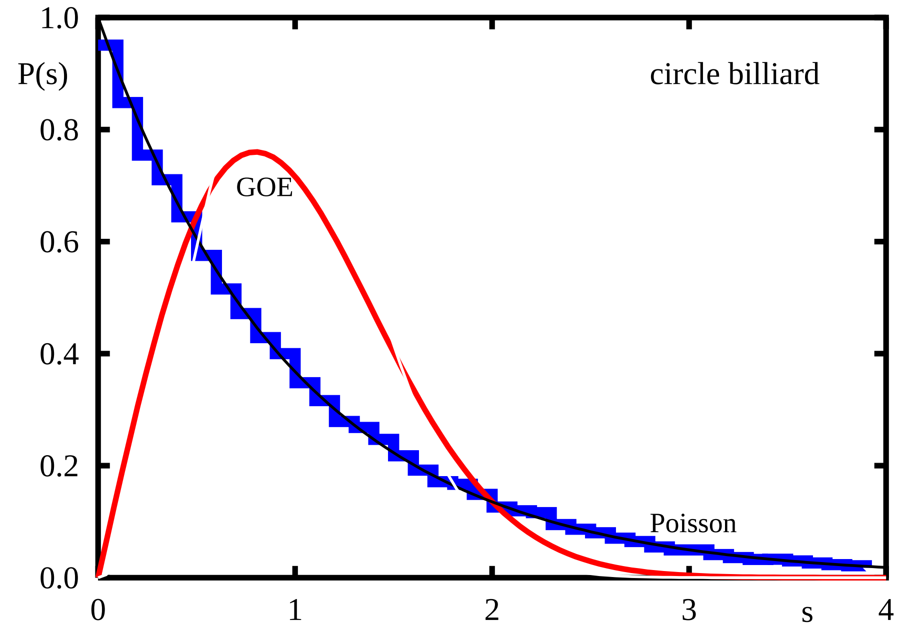
Quantization: Unitary operator on finite dimensional Hilbertspace

$$\hat{U} = \exp[-2\pi i T(\hat{p})/h_{\text{eff}}] \exp[-2\pi i V(\hat{q})/h_{\text{eff}}]$$

Spectral statistics – level spacing distribution

Recall: For integrable/fully chaotic systems:





- Regular systems (integrable)

$$P(s) \to \exp(-s) \qquad \text{Poisson distribution}$$

- Chaotic systems

$$P(s) \to \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) \qquad \text{Wigner distribution}$$

(random matrix distribution)

Spectral statistics – mixed systems

Regular and chaotic regions $\longrightarrow$ two sub-spectra:

- regular spectrum: Poissonian $P(s)$
  (fraction of levels: ρ_1)
- chaotic spectrum: GOE $P(s)$
  (fraction of levels: $\rho_2 = 1 - \rho_1$ levels)

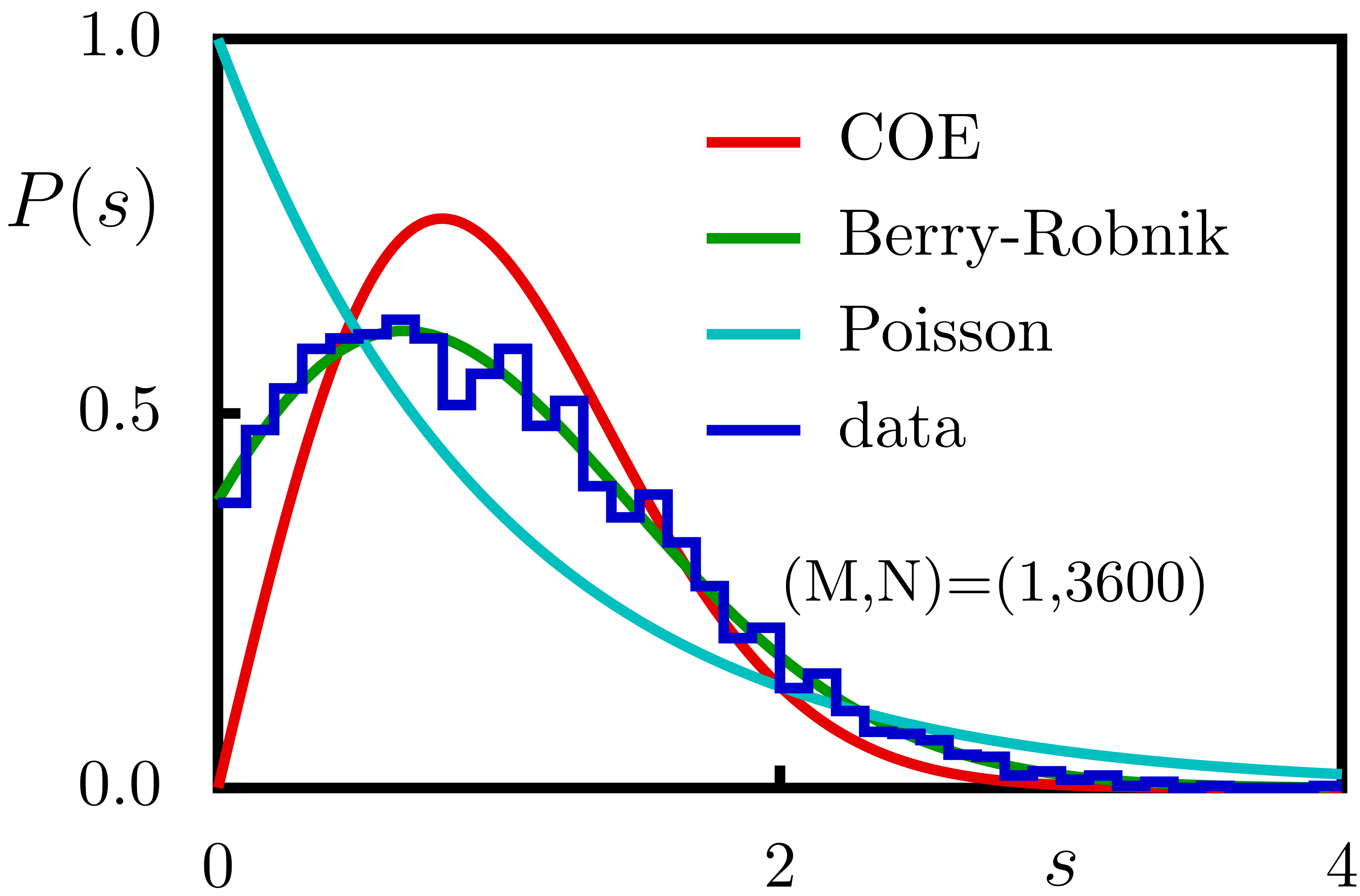
Independent spectra: gap probability $E(s)$ factorizes.

$$P(s) = \frac{\mathrm{d}^2 E(s)}{\mathrm{d}s^2} \quad .$$

Result: Berry-Robnik formula [Berry, Robnik 1984]

$$P_{\rho_1}^{\text{Berry-Robnik}}(s) = \left[\rho_1^2 \mathrm{erfc}\left(\frac{1}{2}\sqrt{\pi}\rho_2 s\right) + \left(2\rho_1\rho_2 + \frac{1}{2}\pi\rho_2^3 s\right) \mathrm{e}^{-\pi\rho_2^2 s^2/4}\right] \mathrm{e}^{-\rho_1 s} \quad .$$

Spectral statistics – mixed systems



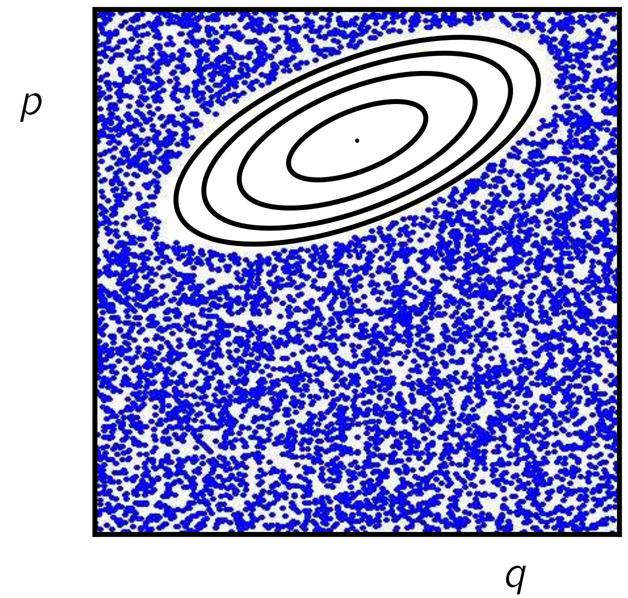
When do regular states exist?

Simplest situation:

1 island

and

chaotic sea



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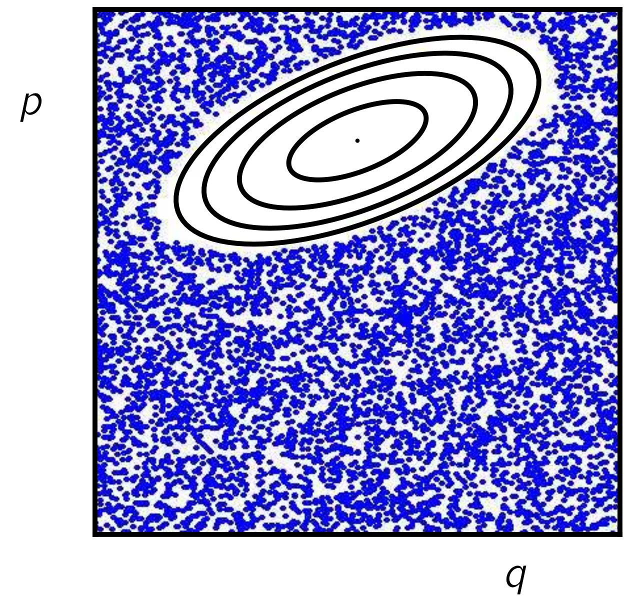
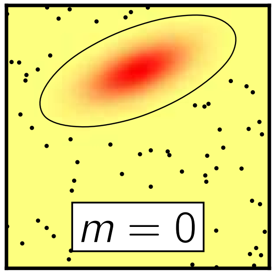
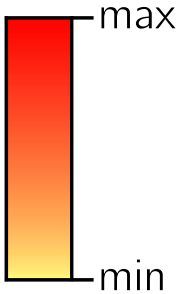


Husimi functions of the quantum system





- regular states

quantization cond. $\oint p \,\mathrm{d}q = \left(m + \frac{1}{2}\right) h$

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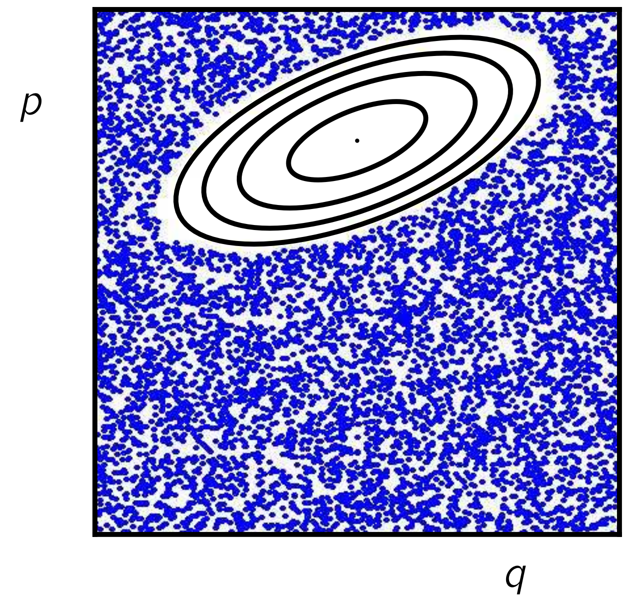

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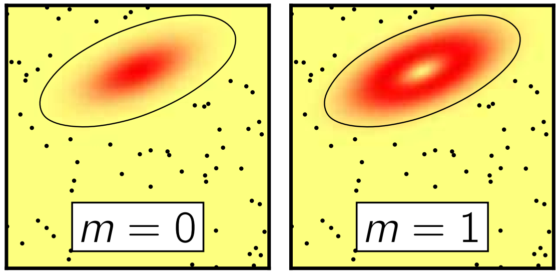

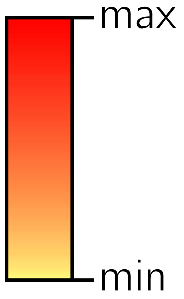

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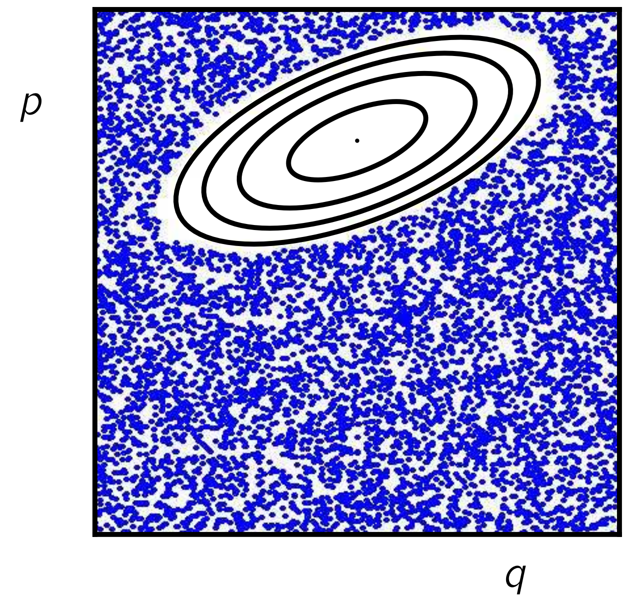

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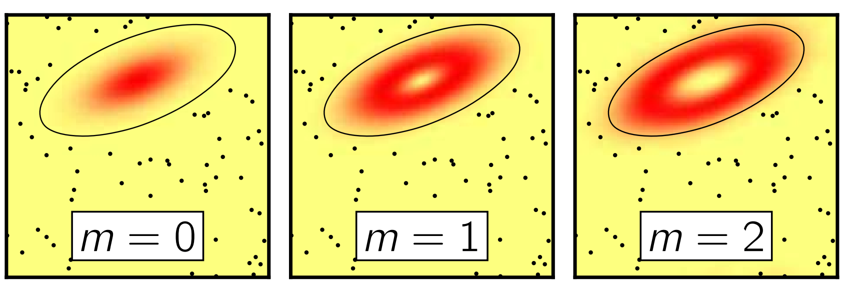

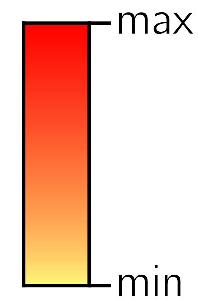

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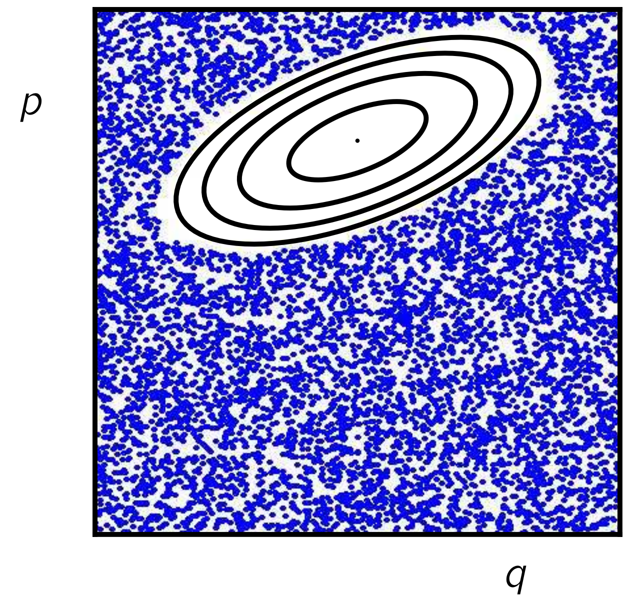

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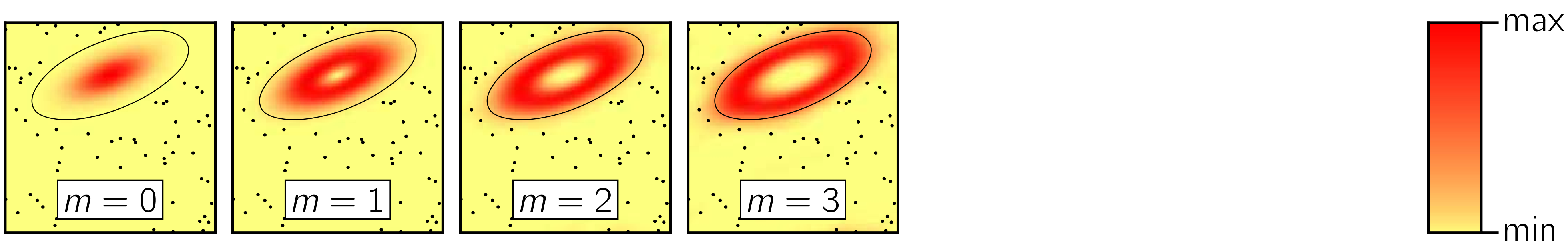

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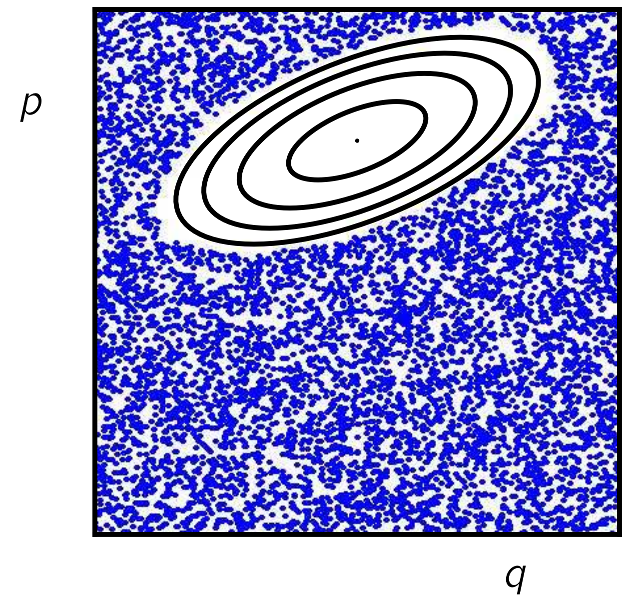

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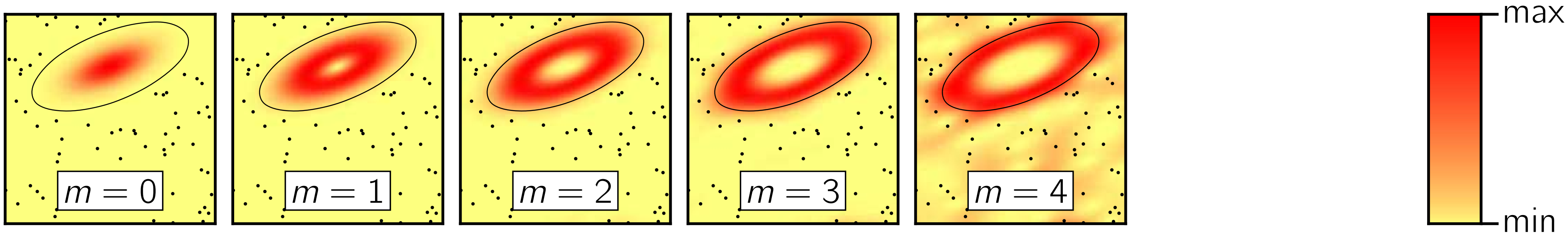

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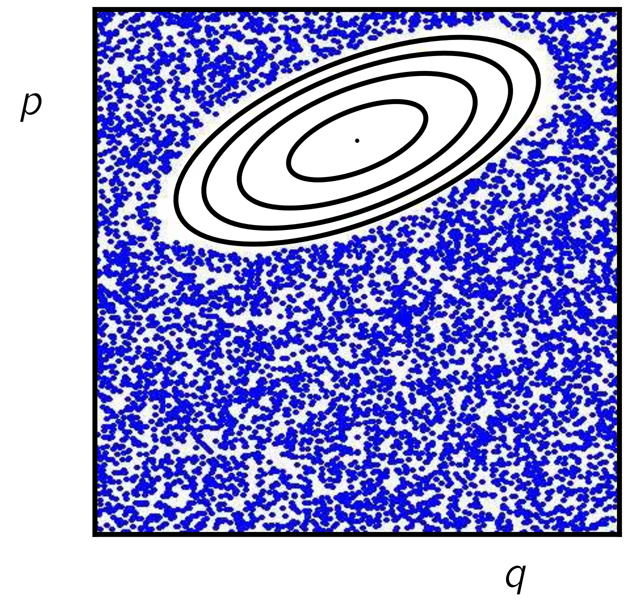

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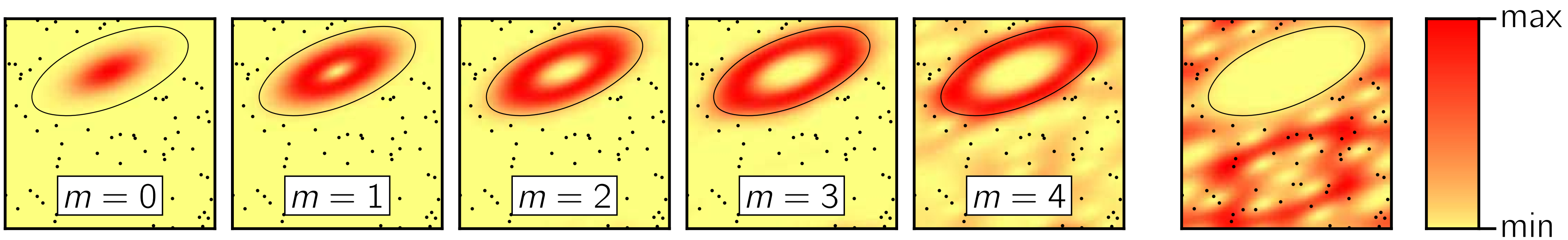

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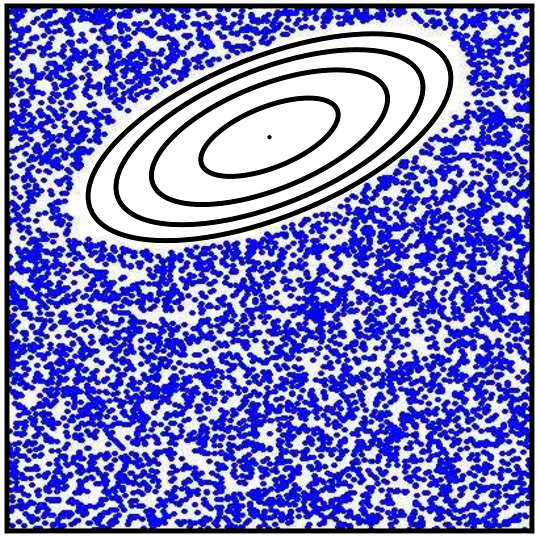

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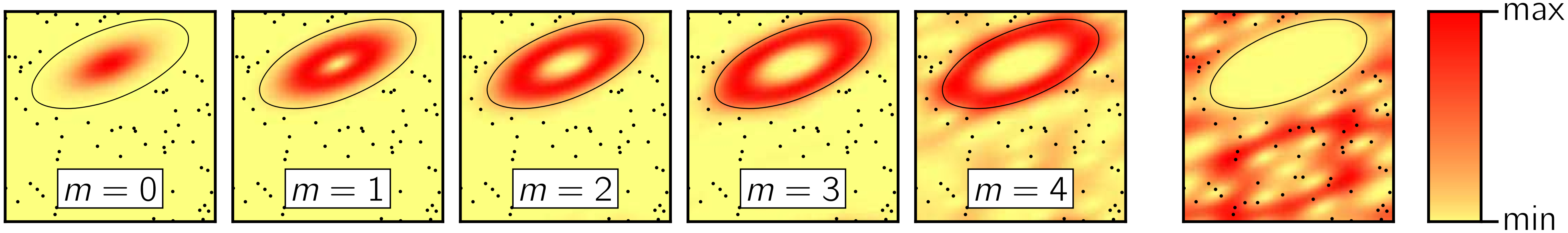

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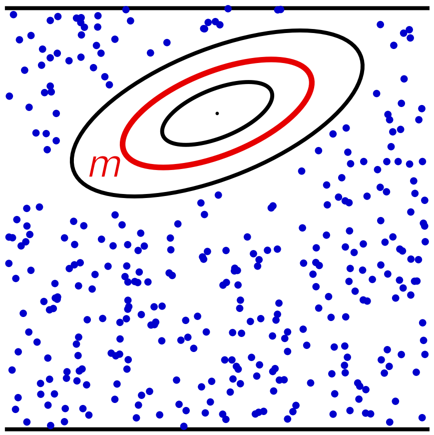
- chaotic states

Outmost torus separates regular and chaotic states

When do regular states exist?

Quantization condition:

$$\oint p \, \mathrm{d}q = \left(m + \tfrac{1}{2}\right) h$$



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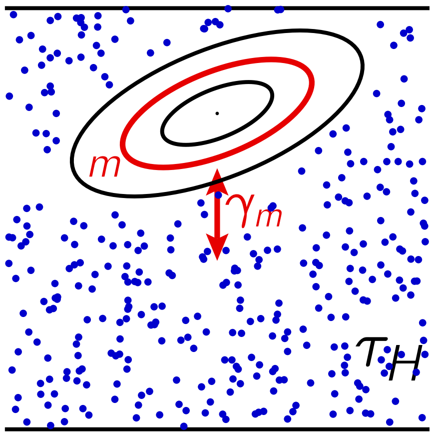


New additional condition: [AB, Ketzmerick, Monastra, PRL 2005]

A regular state on the m-th quantized torus can only exist if

$$\gamma_m < \frac{1}{\tau_H}$$

- γ_m: decay rate of the regular state

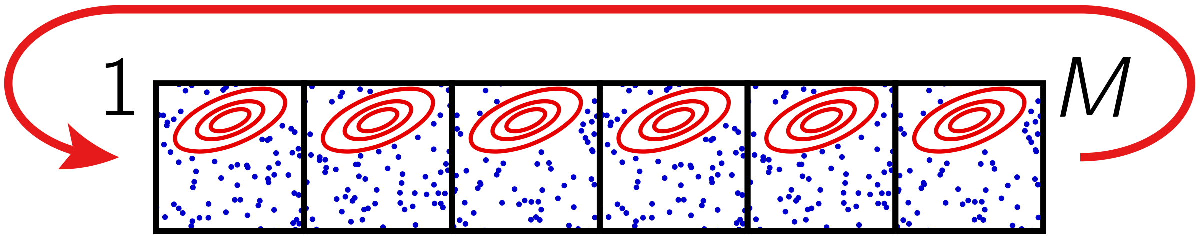
  (when the chaotic sea is infinite)

- $\tau_H = \frac{h}{\Delta_{\mathrm{ch}}}$: Heisenberg time

  (Δ_{ch}: mean level spacing of chaotic states)

Specific model system, vary τ_H

- Periodic boundary conditions after M cells: →



Kicked Hamiltonian →

$$H(p,q,t) = T(p) + V(q)\sum_n \delta(t-n)$$

$$q_{t+1} = q_t + T'(p_t),$$

$$p_{t+1} = p_t - V'(q_{t+1})$$

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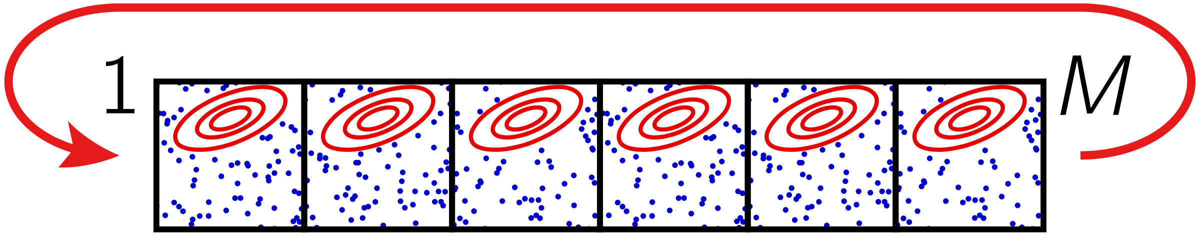

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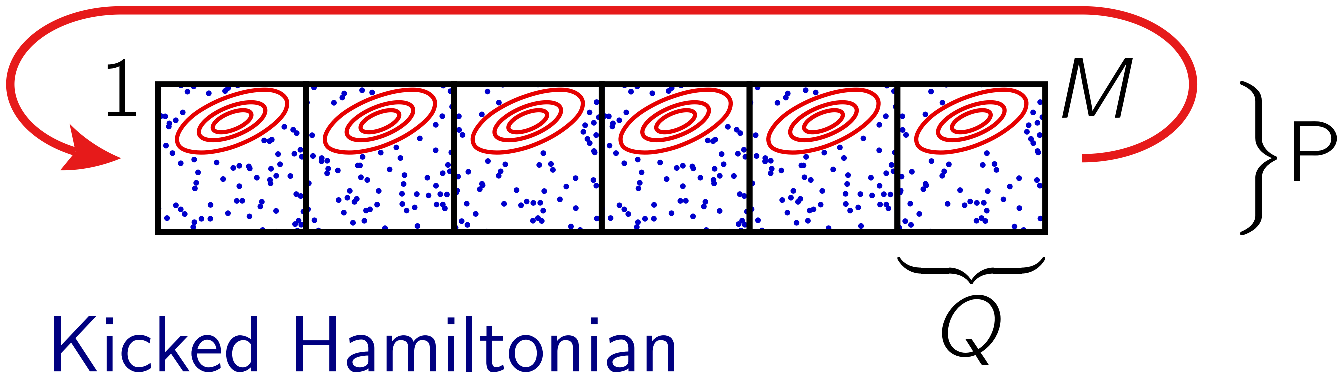

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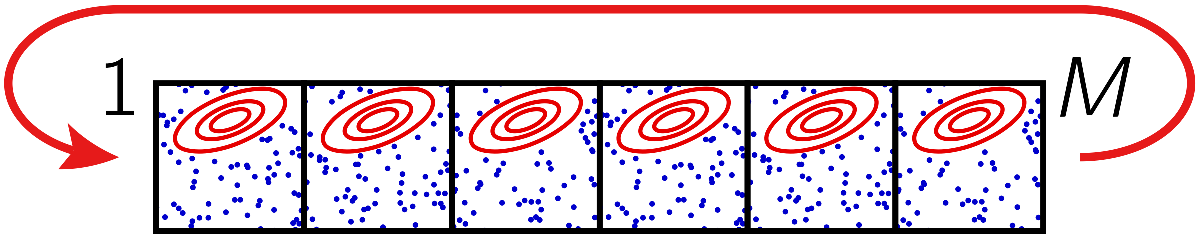
Semiclassical parameter: $h_{\text{eff}} = \dfrac{h}{PQ}$

Specific model system, vary τ_H

- Periodic boundary conditions after M cells: $\rightarrow$



- Quantum map $\hat{U} = \exp[-2\pi i T(\hat{p})/h_{\text{eff}}] \exp[-2\pi i V(\hat{q})/h_{\text{eff}}]$
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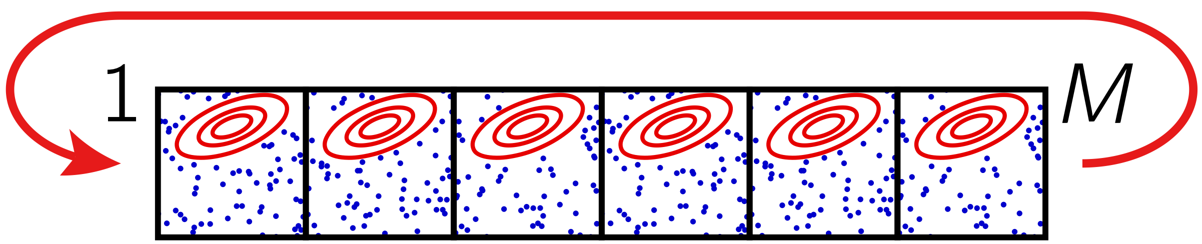
- Periodic boundary conditions after M cells: →



- Quantum map $\hat{U} = \exp[-2\pi i T(\hat{p})/h_{\text{eff}}]\, \exp[-2\pi i V(\hat{q})/h_{\text{eff}}]$
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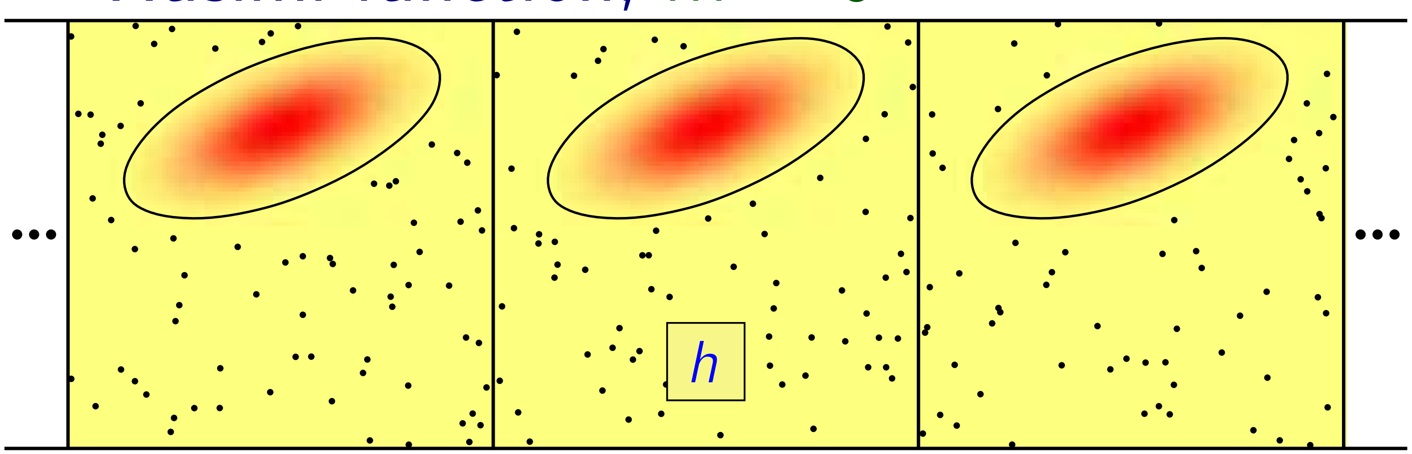
- Consider $M = 1597$ and $h_{\text{eff}} = 1/30$

Husimi function, $m = 0$

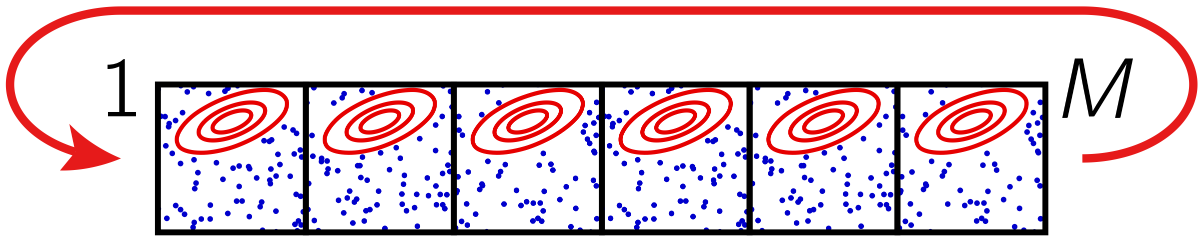


EBK quantization

$$\oint p\, \mathrm{d}q = \left(m + \tfrac{1}{2}\right) h$$

Specific model system, vary τ_H

- Periodic boundary conditions after M cells: →



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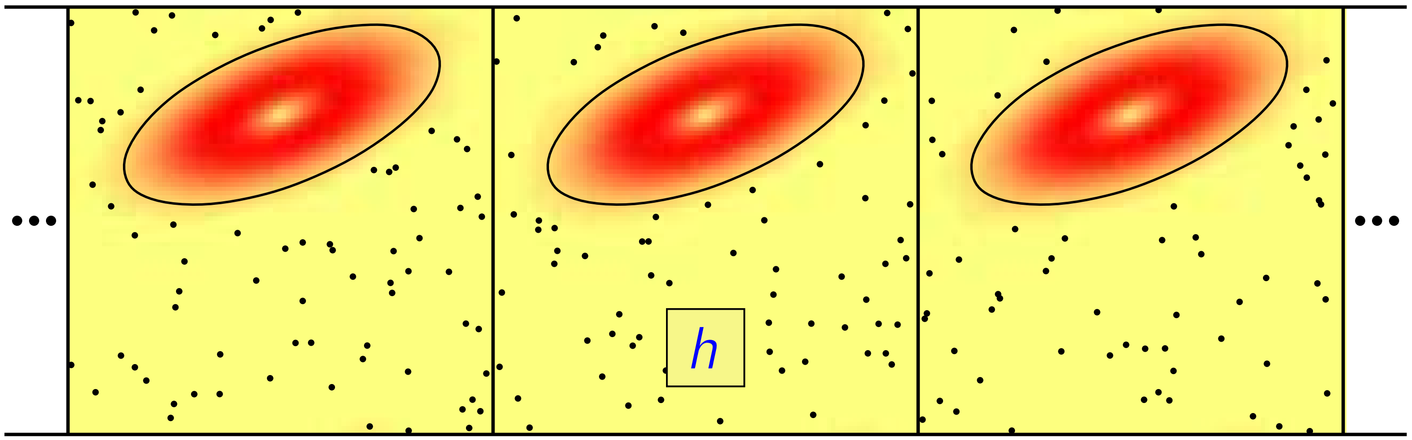
Husimi function, $m = 1$



EBK quantization

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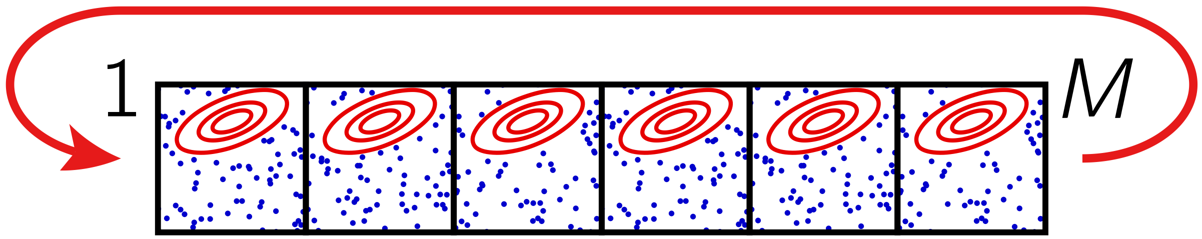
Specific model system, vary τ_H

- Periodic boundary conditions after M cells: $\rightarrow$



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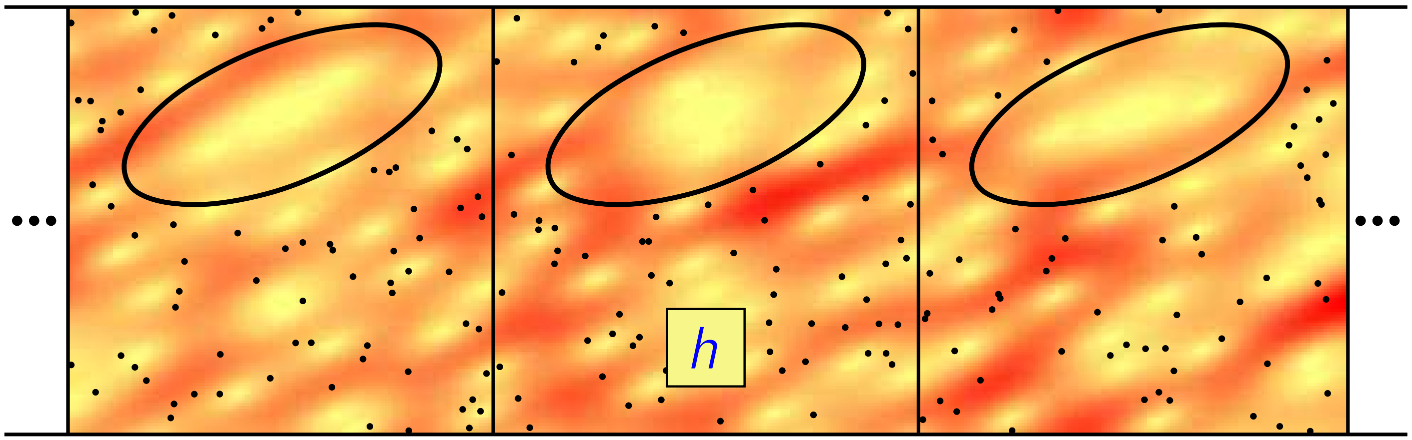

Specific model system, vary τ_H

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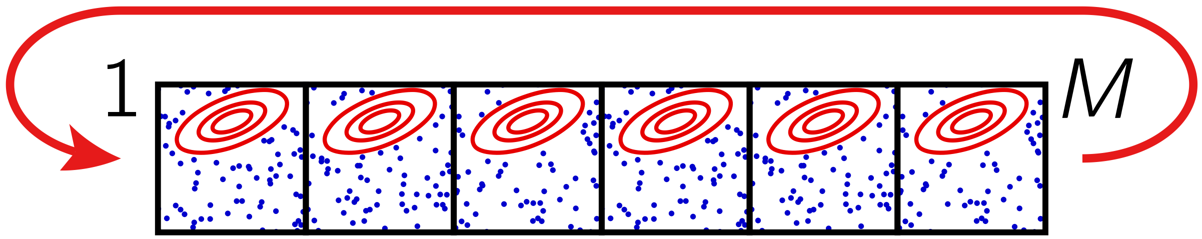

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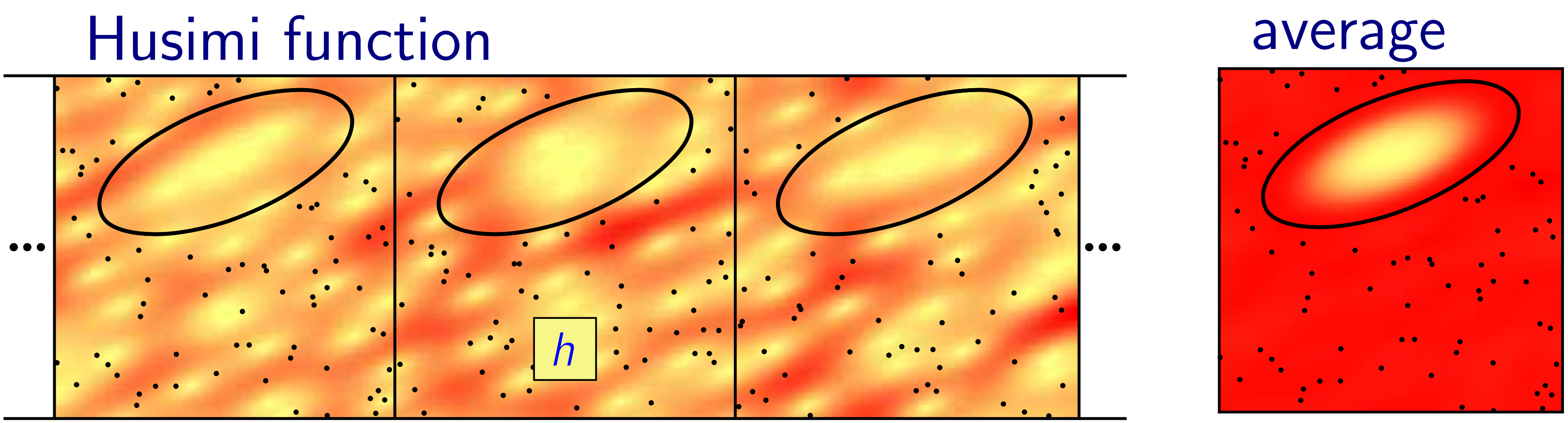
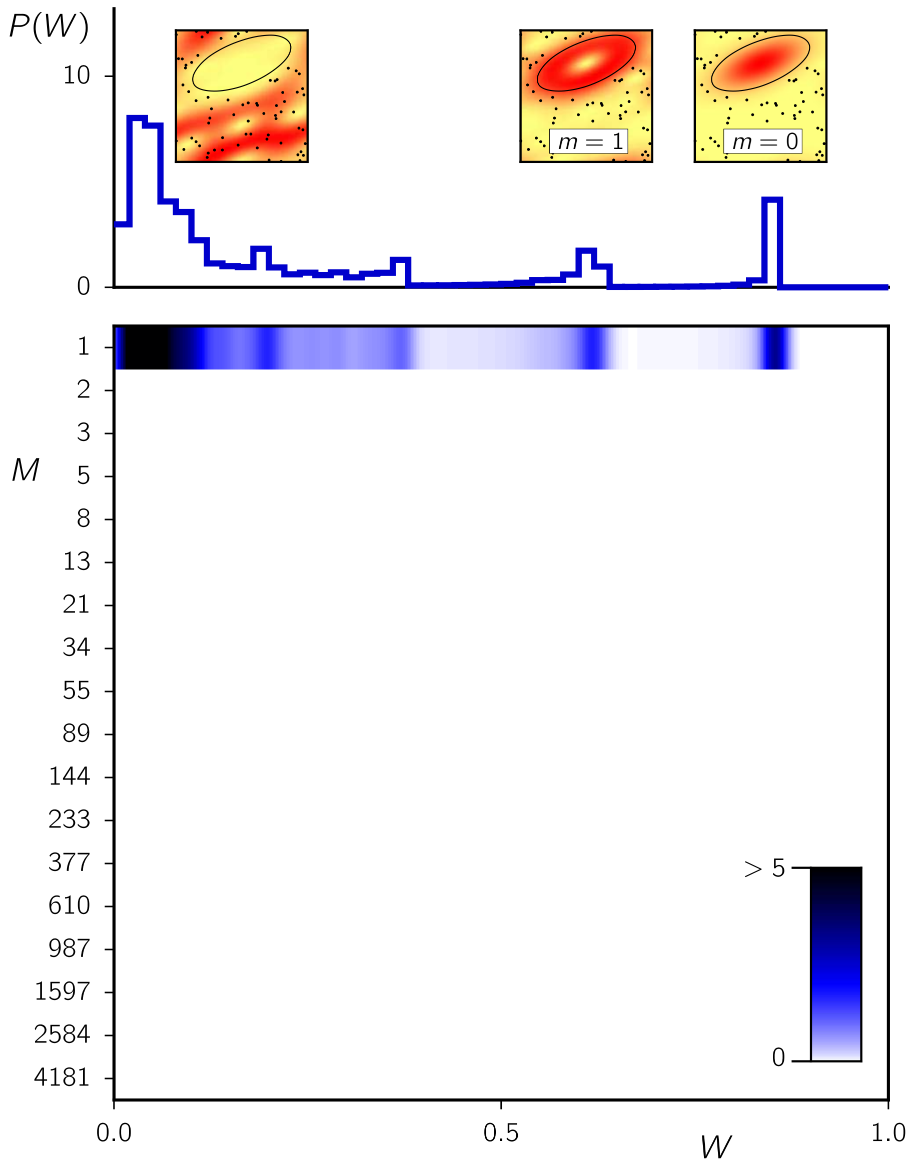


Chaotic states flood the island!

Weight W of states inside the islands

1 cell:
chaotic and
regular states



– all states
– phase variation
– $h_{\text{eff}} = \frac{1}{10}$

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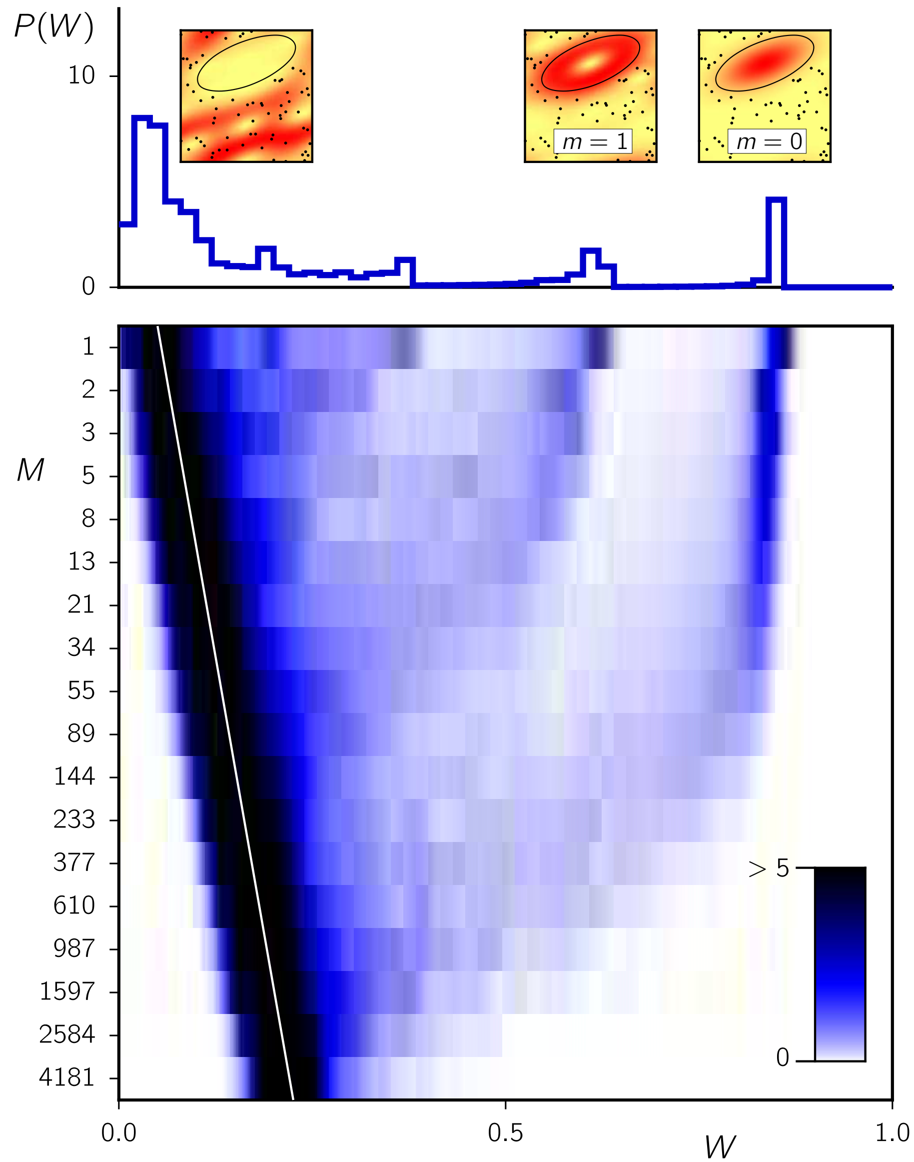

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disappearance of regular states

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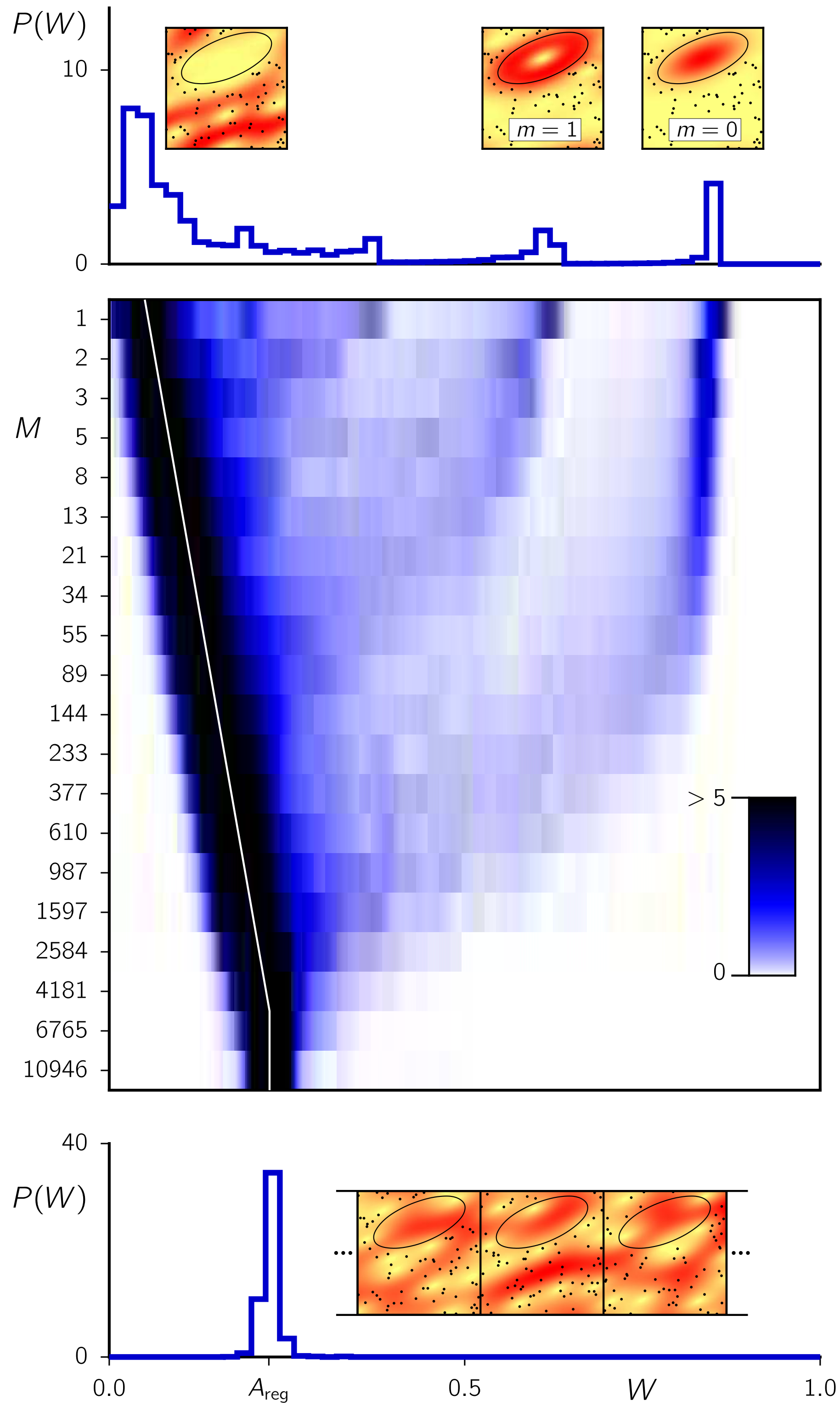

– all states
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chaotic states
flood the islands

disappearance
of regular states

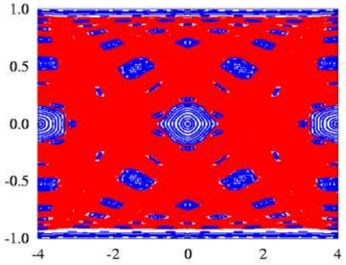
complete flooding!

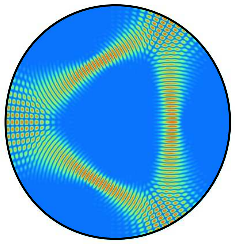
amphibious eigenstates

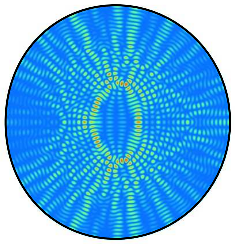
[Hufnagel *et.al* PRL 2002]

Summary III: Systems with a mixed phase space

Regular and chaotic states







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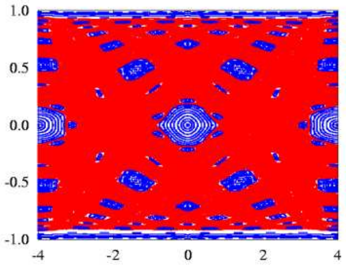
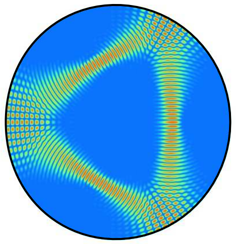
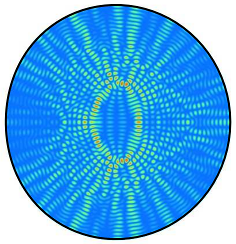
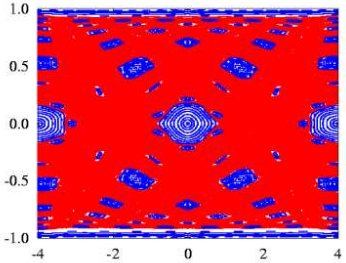


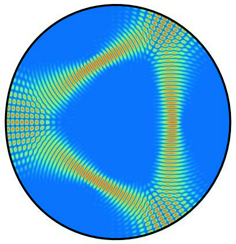


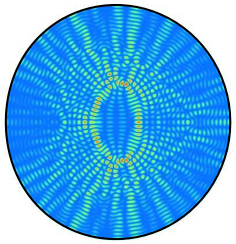


- Eigenfunction statistics: RRWM

Regular and chaotic states







- Eigenfunction statistics: RRWM
- Spectral statistics: Berry-Robnik distribution
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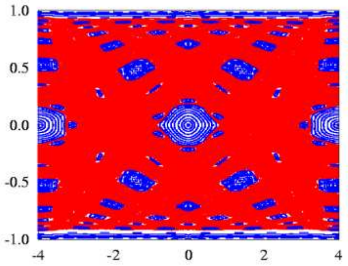
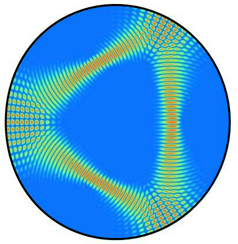
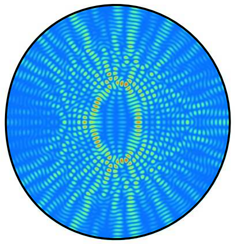
Summary III: Systems with a mixed phase space

Regular and chaotic states







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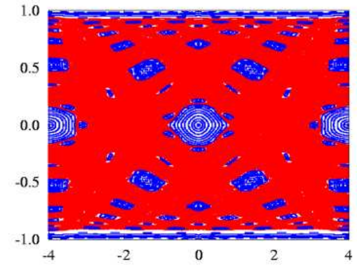

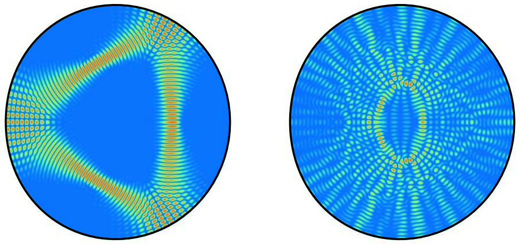

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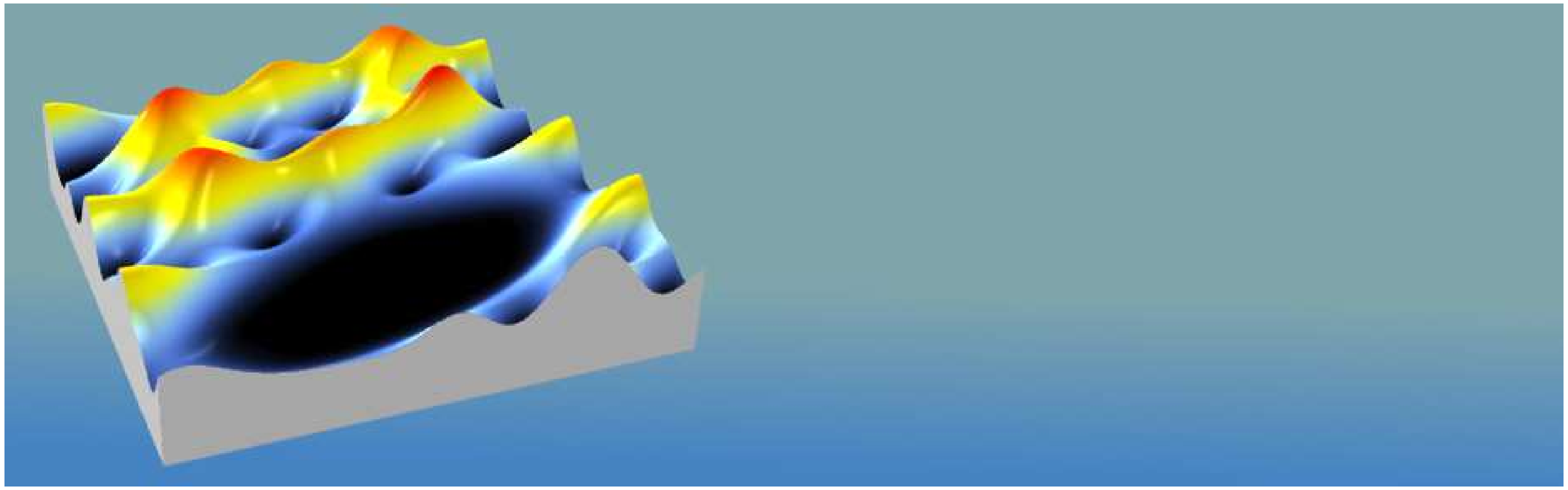
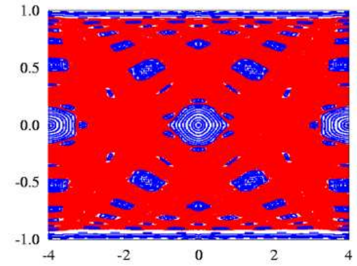
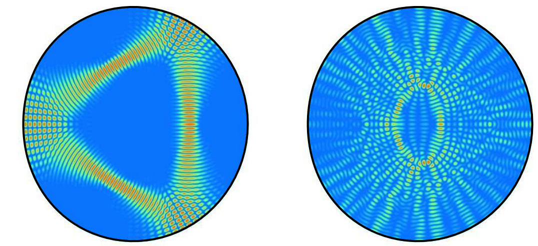
  If violated: Flooding of regular islands



Regular and chaotic states





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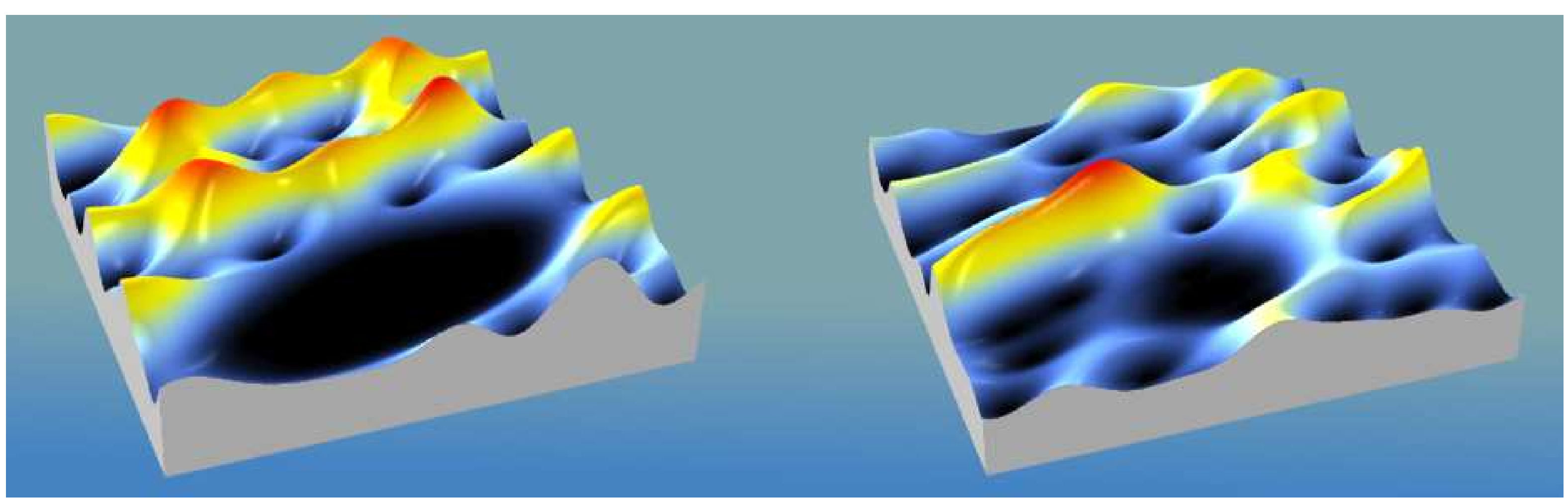
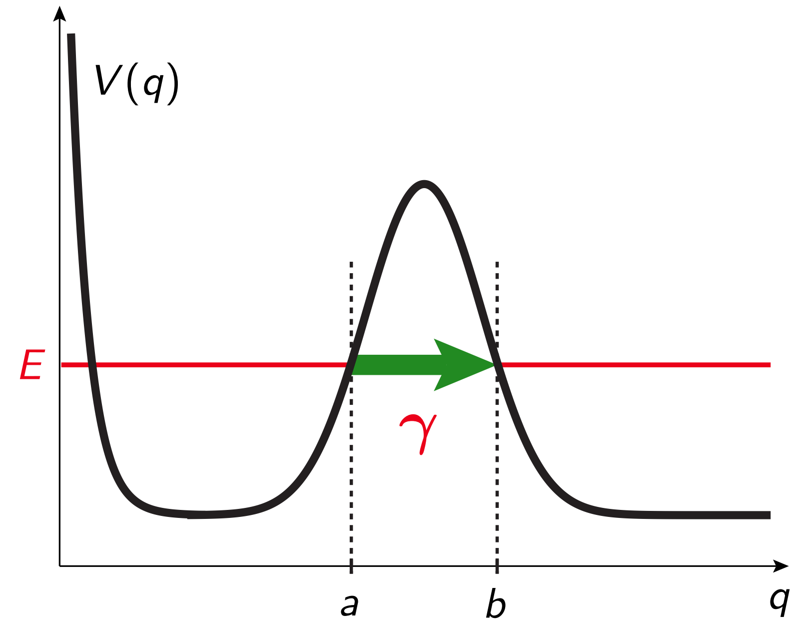


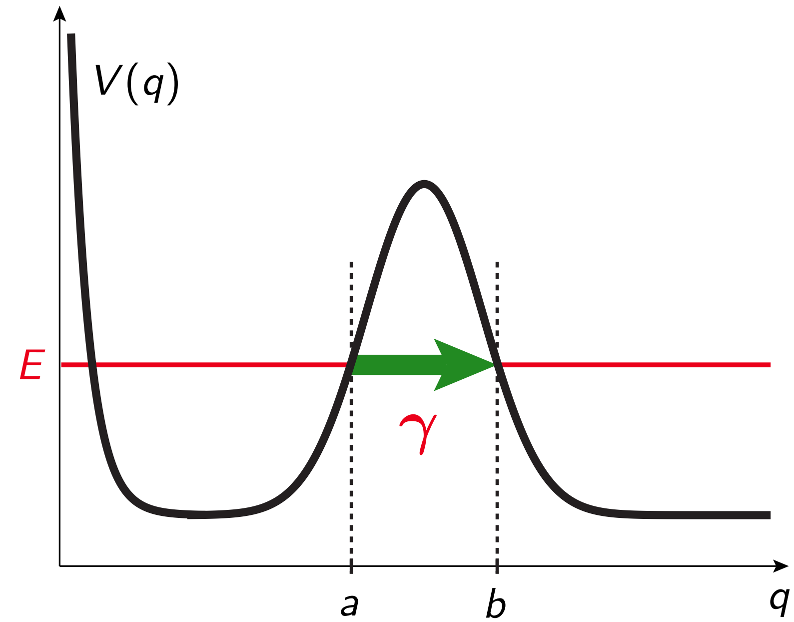
1D barrier tunneling



$$\gamma \propto \exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(V(q)-E)}\,\mathrm{d}q\right)$$

Tunneling

1D barrier tunneling



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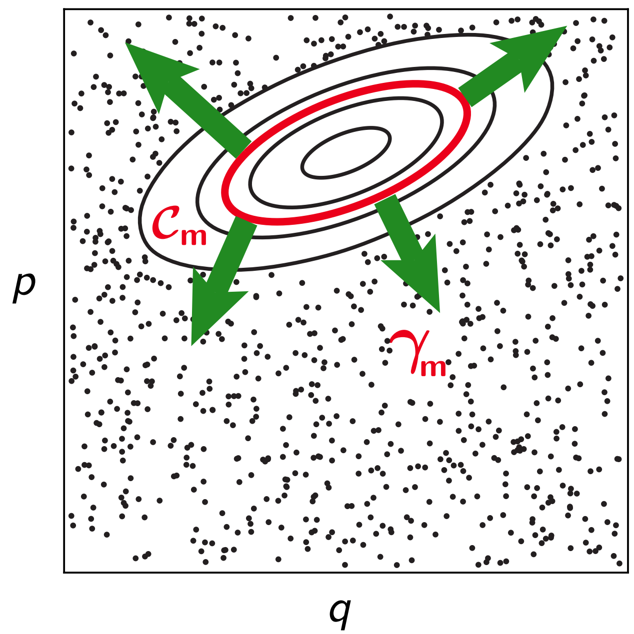
"Dynamical tunneling"

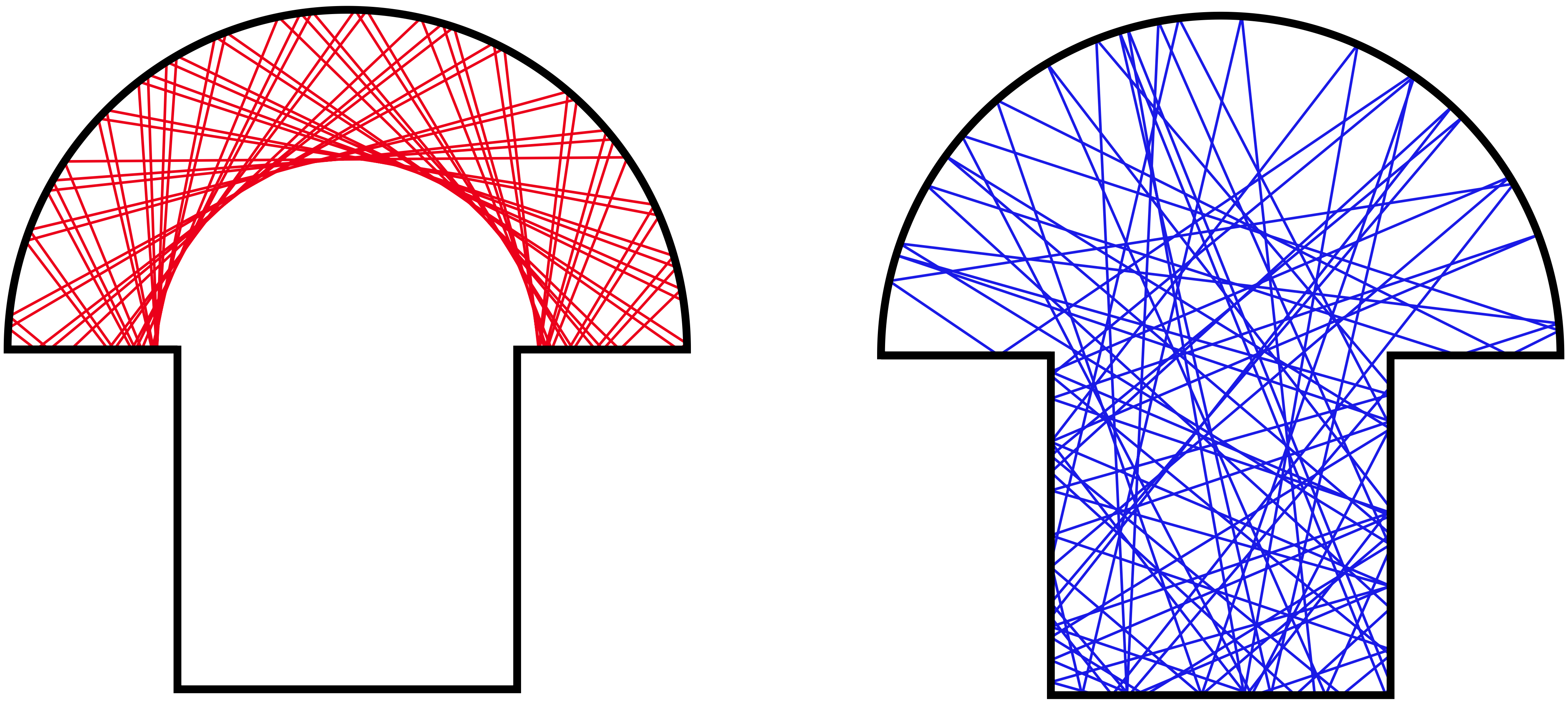
[Davis, Heller 1981]



$$\gamma \propto ?$$

quantitative theory for
regular to chaotic tunneling?

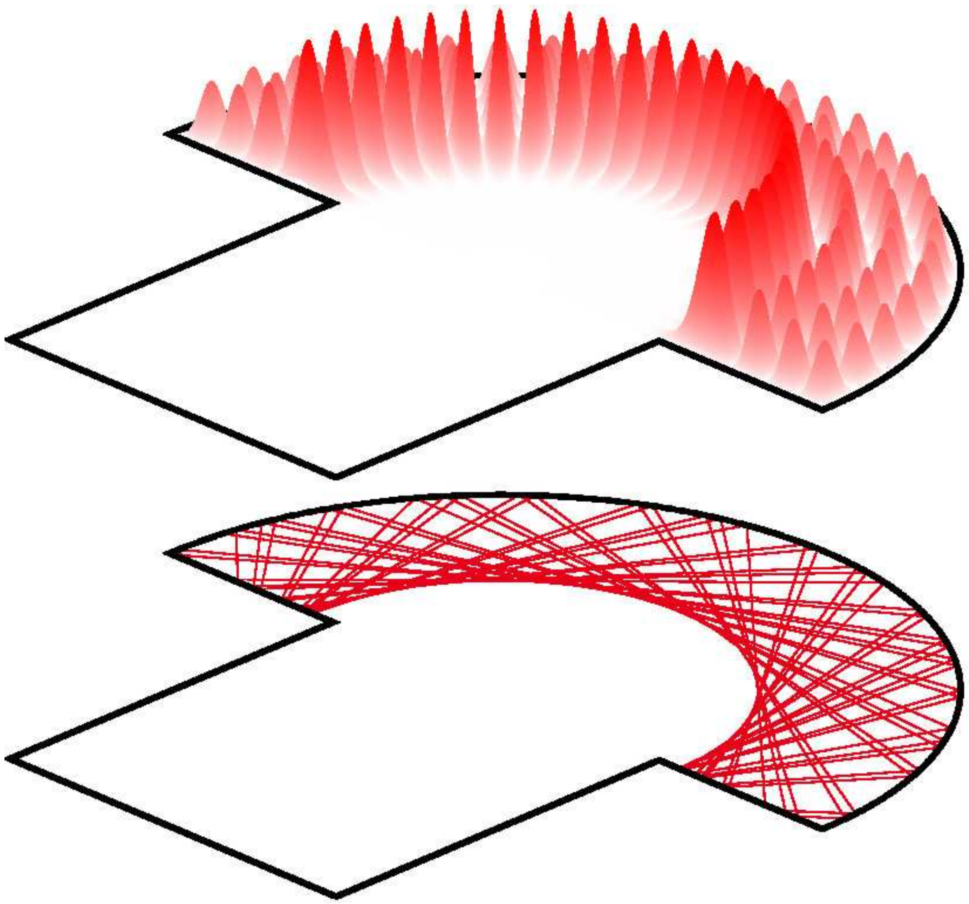
Mushroom Billiard - Classical Dynamics

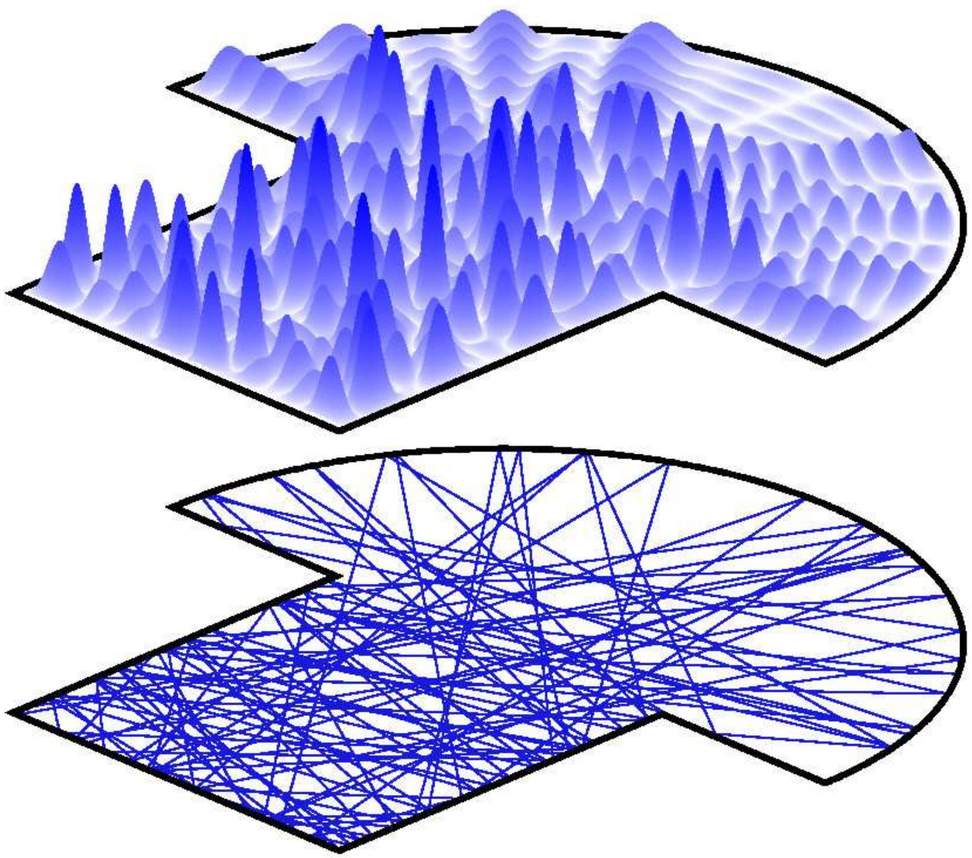


regular chaotic

Bunimovich, Chaos (2001)
Altmann et al, Chaos (2005)
Dietz et al., JPA (2007)
Vidmar et al., JPA (2007)
Barnett, Betcke, Chaos (2007)

Mushroom billiard – quantum eigenstates

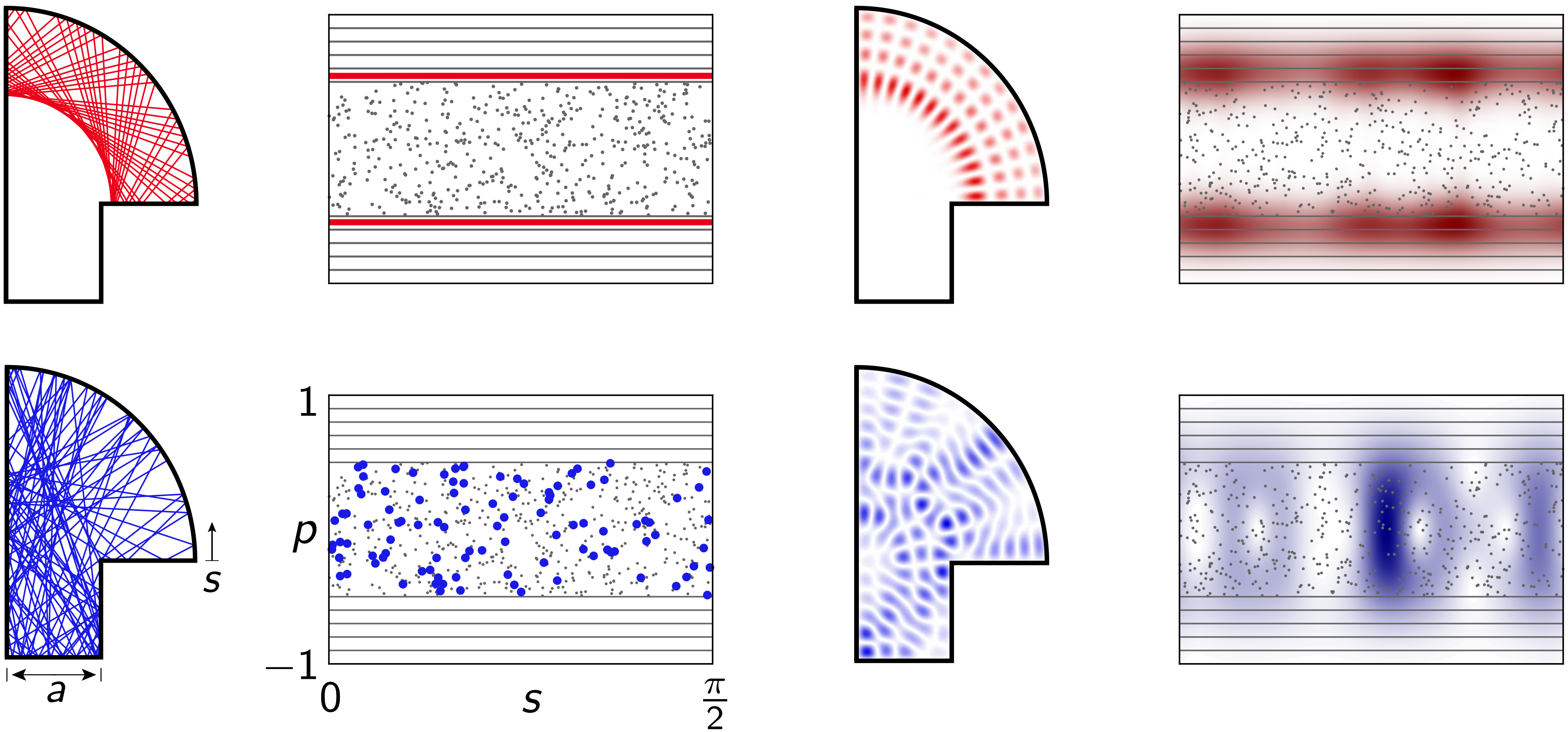




regular eigenstate
lives in cap

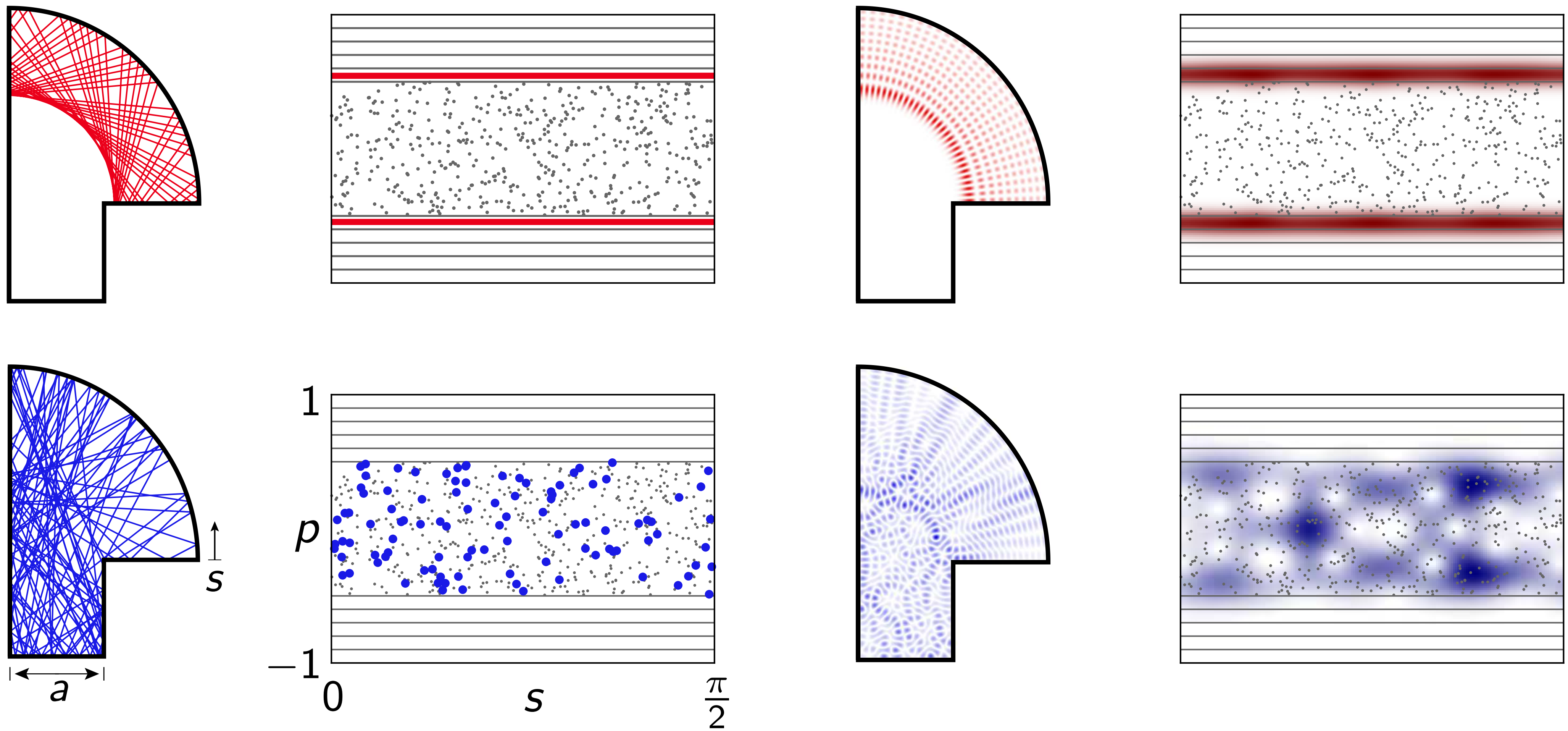
chaotic eigenstate
lives in stem and cap

Mushroom billiard – phase space



Properties:
- sharply divided regular and chaotic dynamics
- no resonance chains inside the regular region

Mushroom billiard – phase space



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Mushroom billiard – Dynamics

initial condition: regular torus p_0

classical

$$p(t) = p_0$$

always regular

Mushroom billiard – Dynamics

initial condition: regular torus p_0

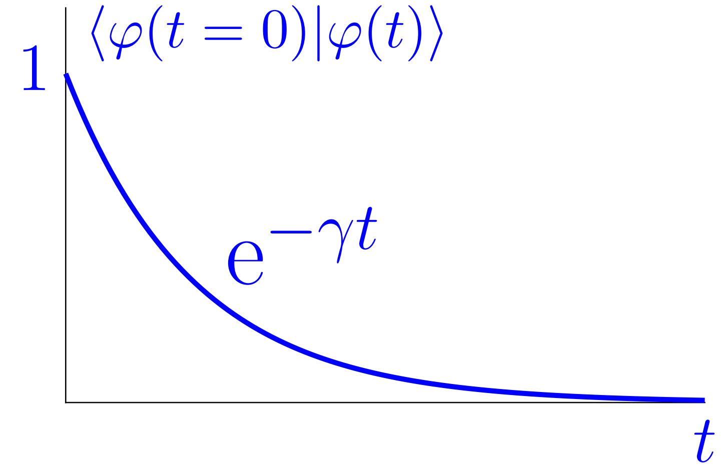
classical

$$p(t) = p_0$$

always regular

quantum

$$|\varphi(t=0)\rangle = \text{circle eigenstate}$$



tunneling rate γ
into the chaotic region

Tunneling: theoretical description

Fictitious regular system leads to [AB, Ketzmerick, Löck, Schilling, PRL 2008]

$$\gamma = \frac{2\pi}{\hbar} \Sigma_E |v|^2 \rho_{\text{ch}}$$

and $$v = \langle \psi_{\text{ch}} | H - H_{\text{reg}} | \psi_{\text{reg}} \rangle$$

Thus

$$\gamma = \frac{2\pi}{\hbar} \rho_{\text{ch}} \langle \psi_{\text{reg}} | H - H_{\text{reg}} \underbrace{\Sigma_E | \psi_{\text{ch}} \rangle \langle \psi_{\text{ch}} |}\, H - H_{\text{reg}} | \psi_{\text{reg}} \rangle$$

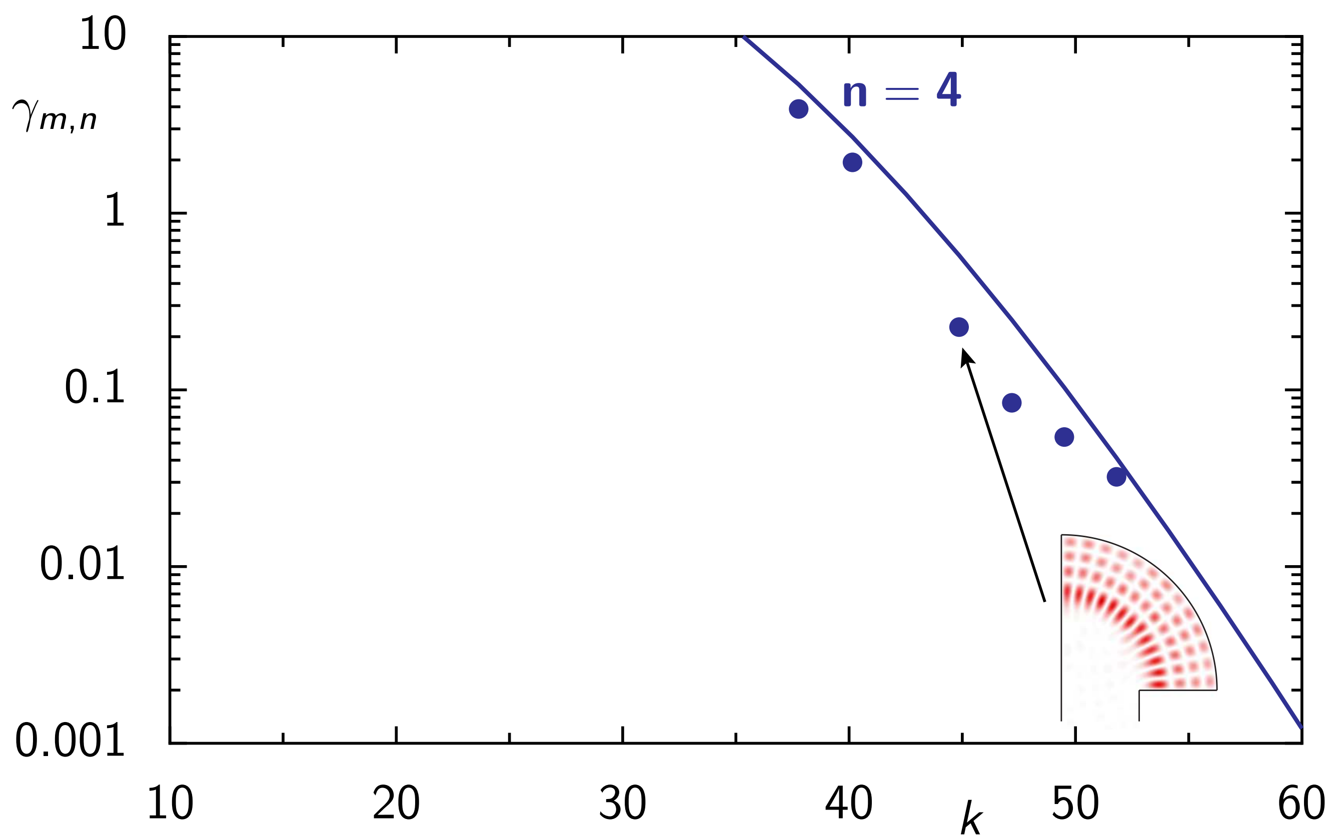
averaged projector
on chaotic region
$\Rightarrow \gamma$ independent of chaotic properties

...

Mushroom billiard — Experiment

with: Robnik, Vidmar (Maribor), Höhmann, Kuhl, Stöckmann (Marburg), PRL 2008

– vary stem height
– determine size of avoided crossings
– average matrix element $\Sigma_E |v|^2$



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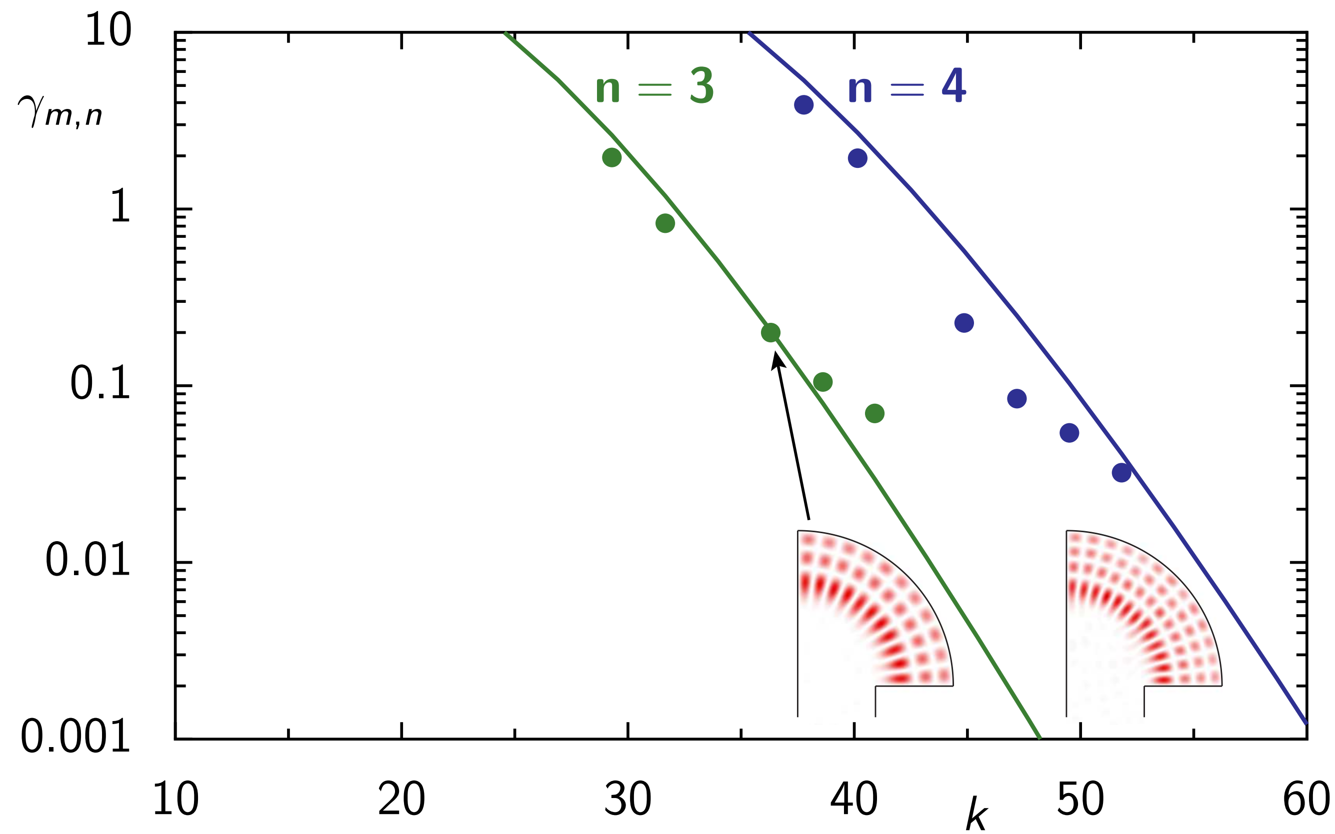

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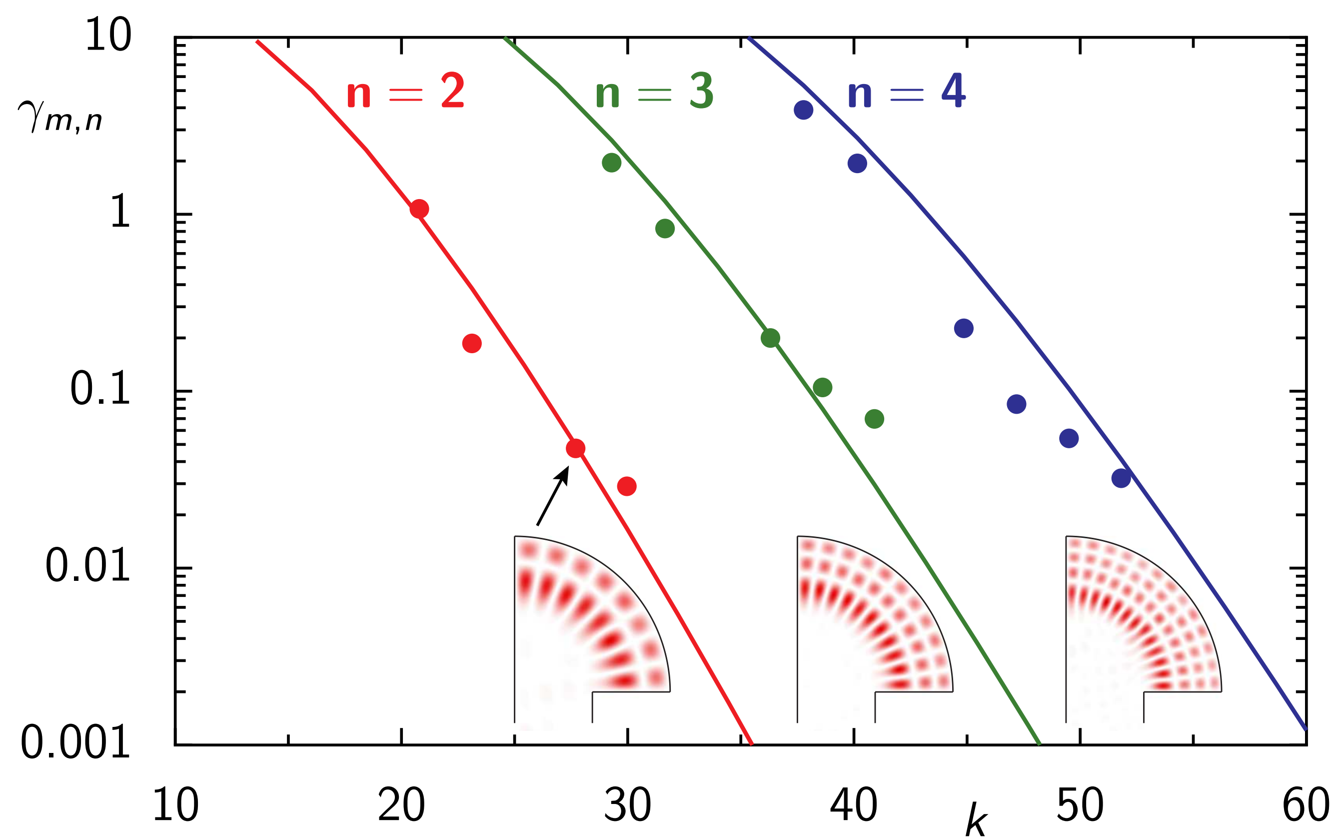

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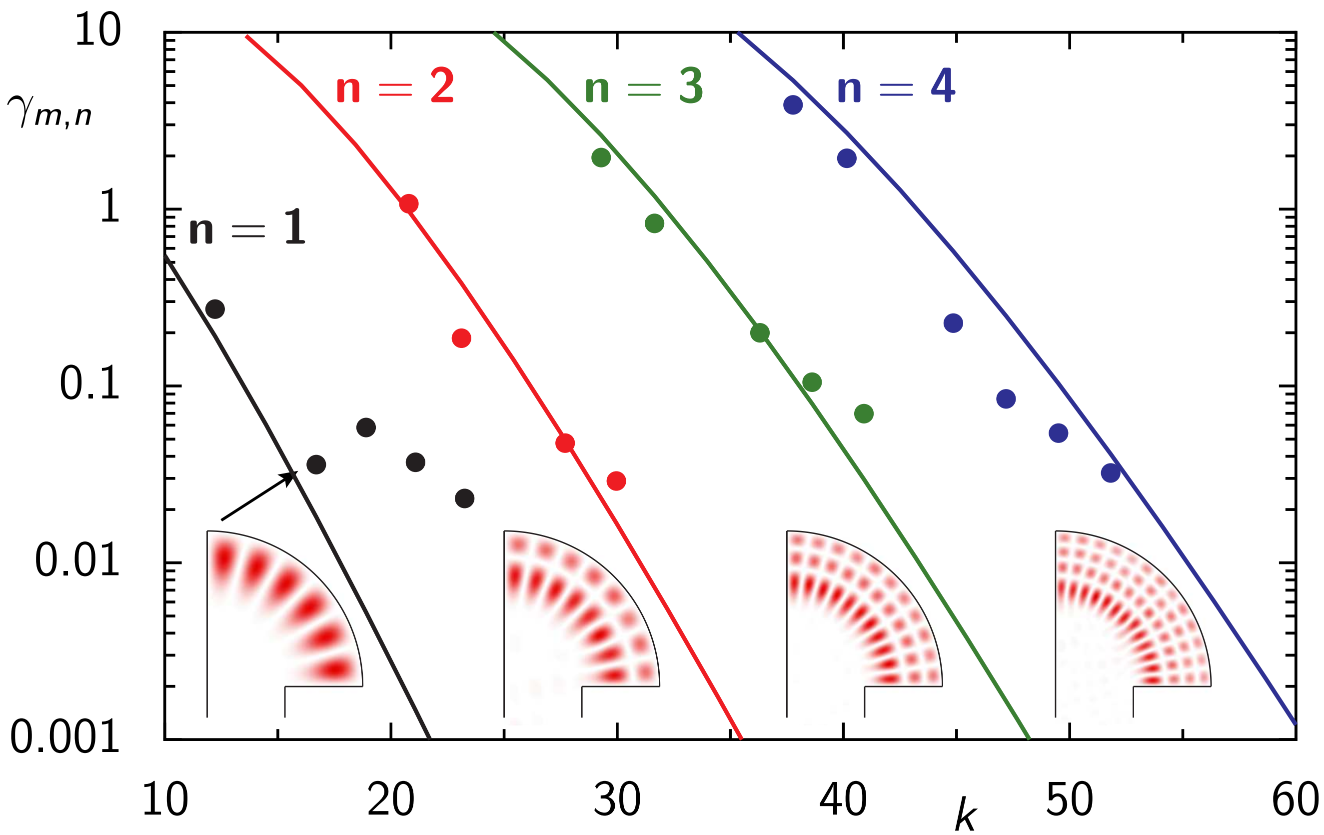
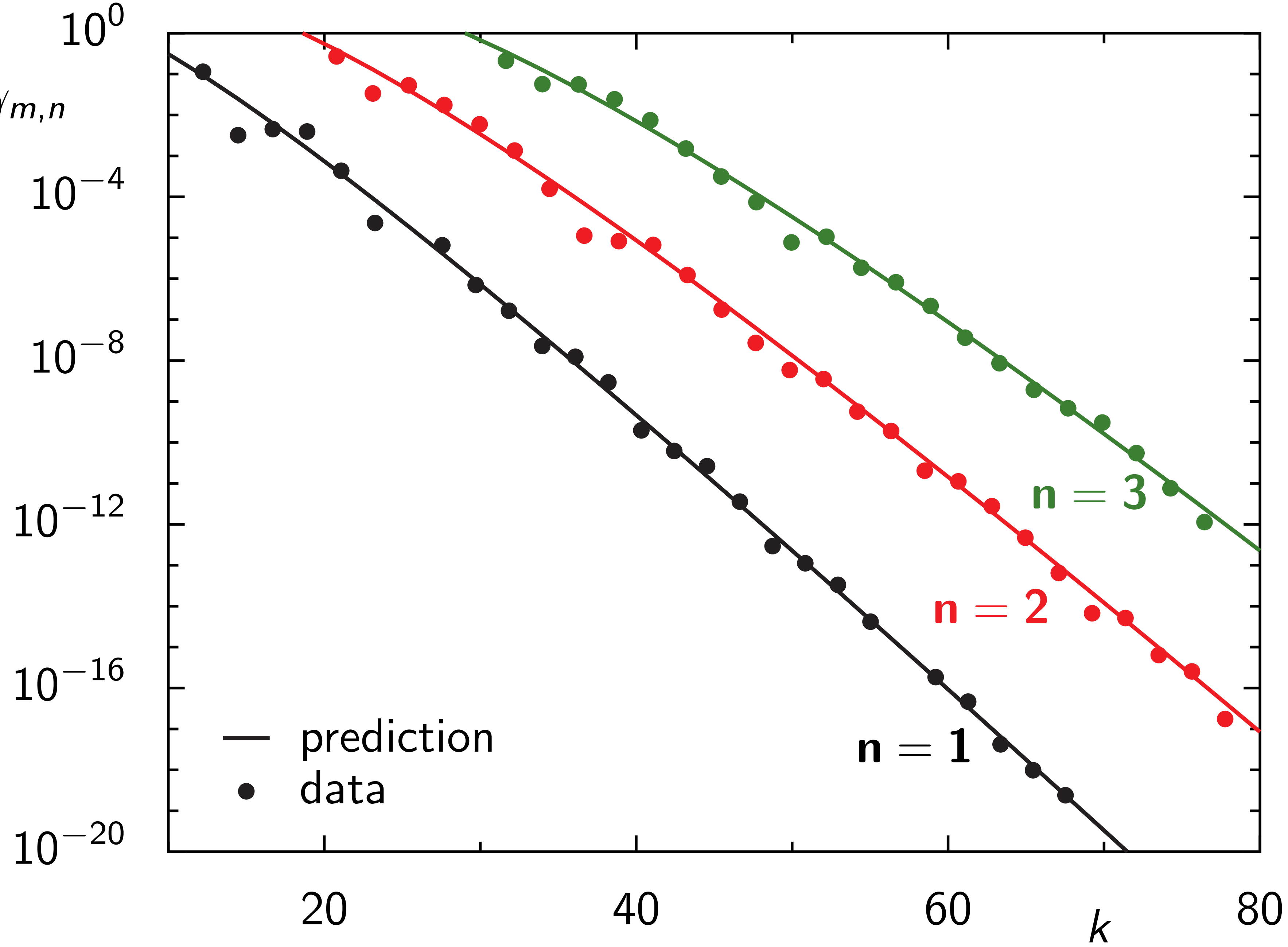


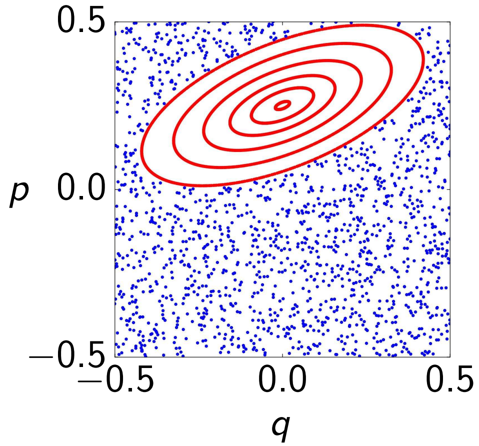
Mushroom billiard — Numerics (method: Barnett/Betcke)



– agreement over 18 orders of magnitude

– no fit parameter

Example 1



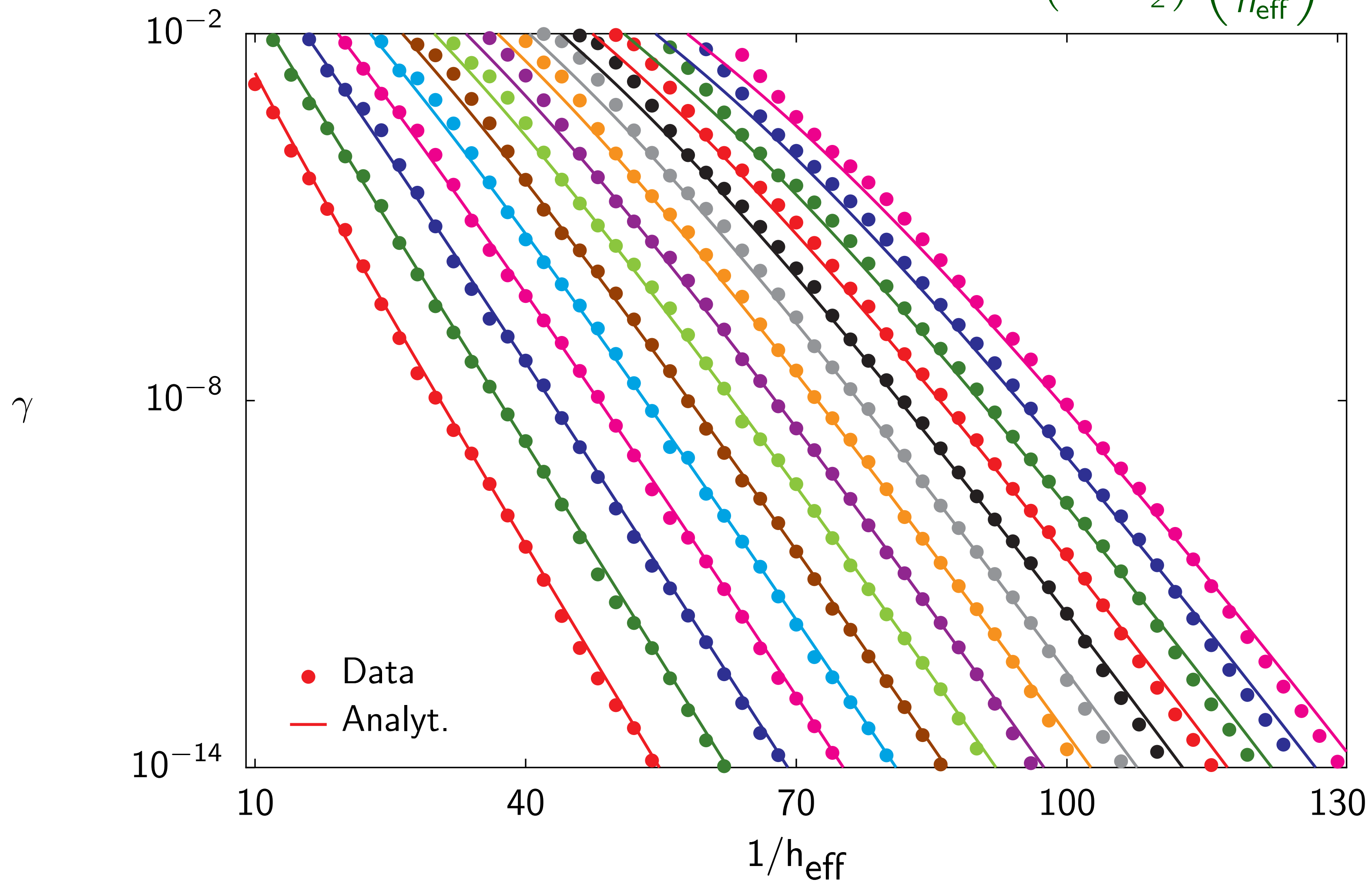
– elliptic island

– constant rotation number

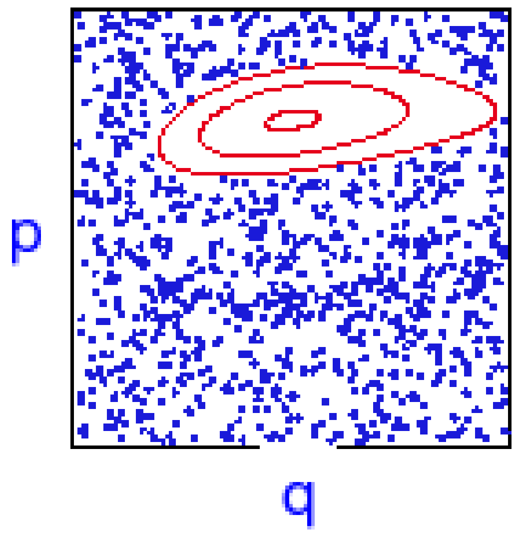
– narrow hierarchical region

$$\gamma_m = c\,\frac{h_{\mathrm{eff}}}{\sqrt{1-\alpha_m}}\exp\left(-2\,\frac{A}{h_{\mathrm{eff}}}\left[\sqrt{1-\alpha_m}-\alpha_m \ln\left(\frac{1+\sqrt{1-\alpha_m}}{\sqrt{\alpha_m}}\right)\right]\right)$$

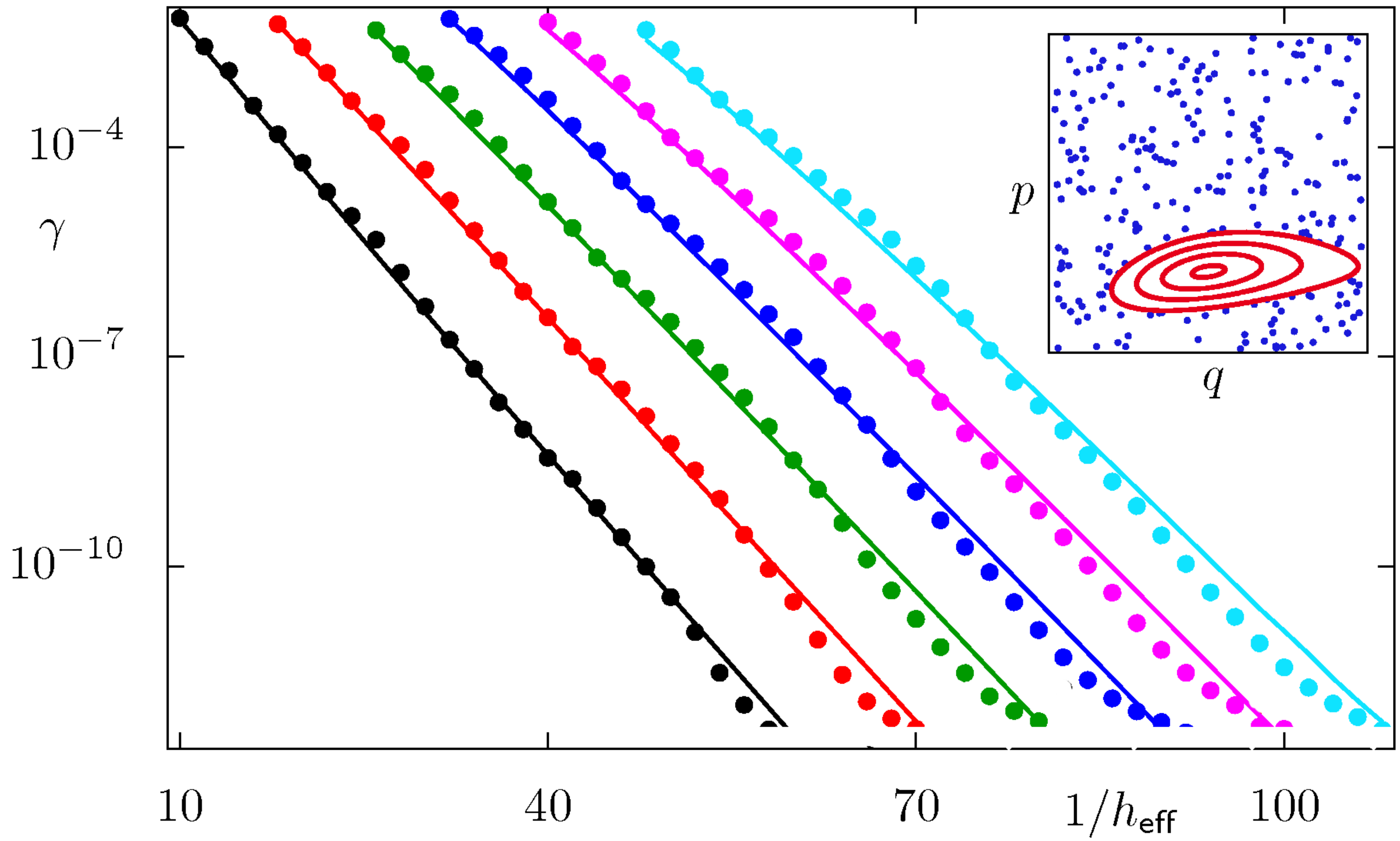
A : island area $\qquad \alpha_m = \left(m+\frac{1}{2}\right)\left(\frac{A}{h_{\mathrm{eff}}}\right)^{-1}$



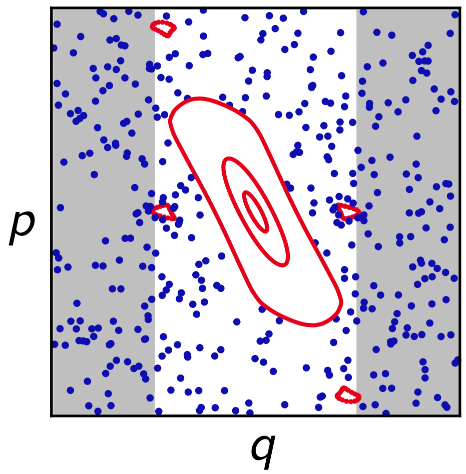
Example 2: deformed island



- deformed island
- Narrow hierarchical region



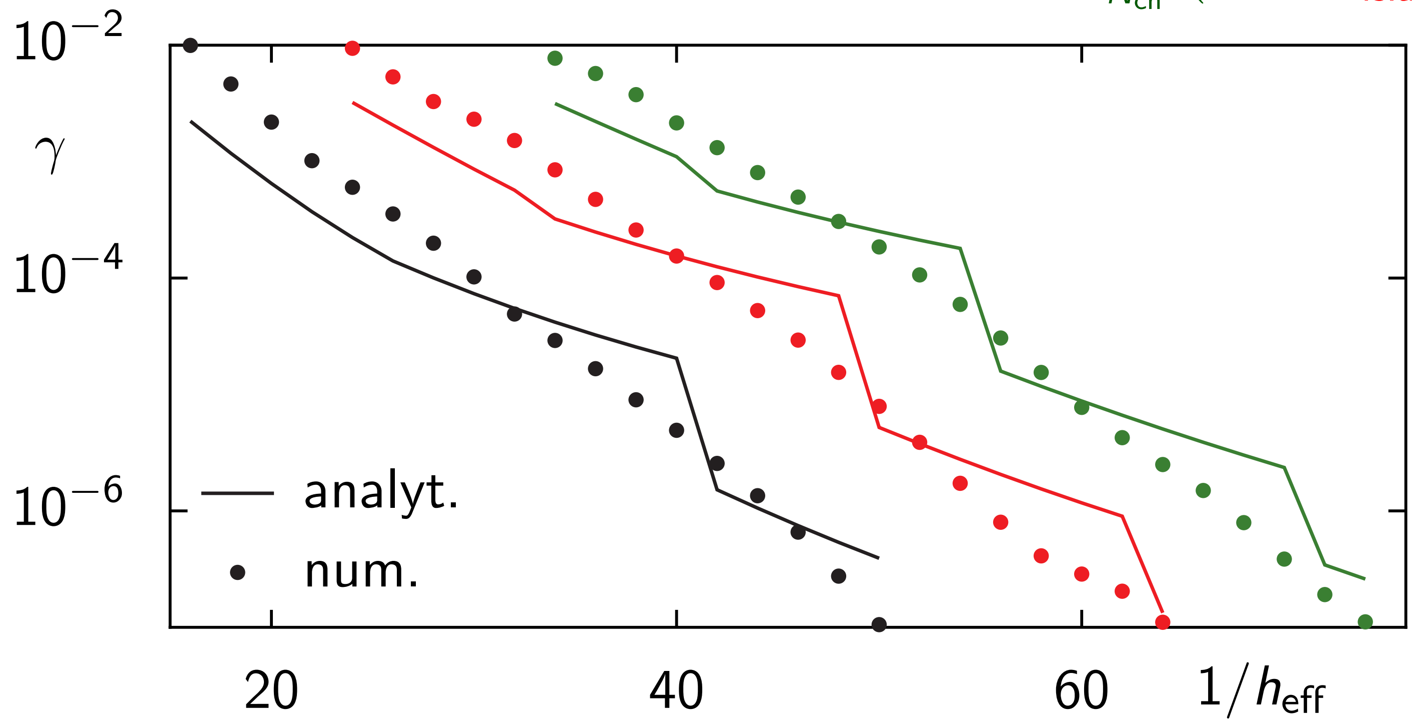
Example 3: Standard map



$$q' = q + p$$
$$p' = p + K \sin q' \qquad (\mathrm{mod}\ 2\pi)$$

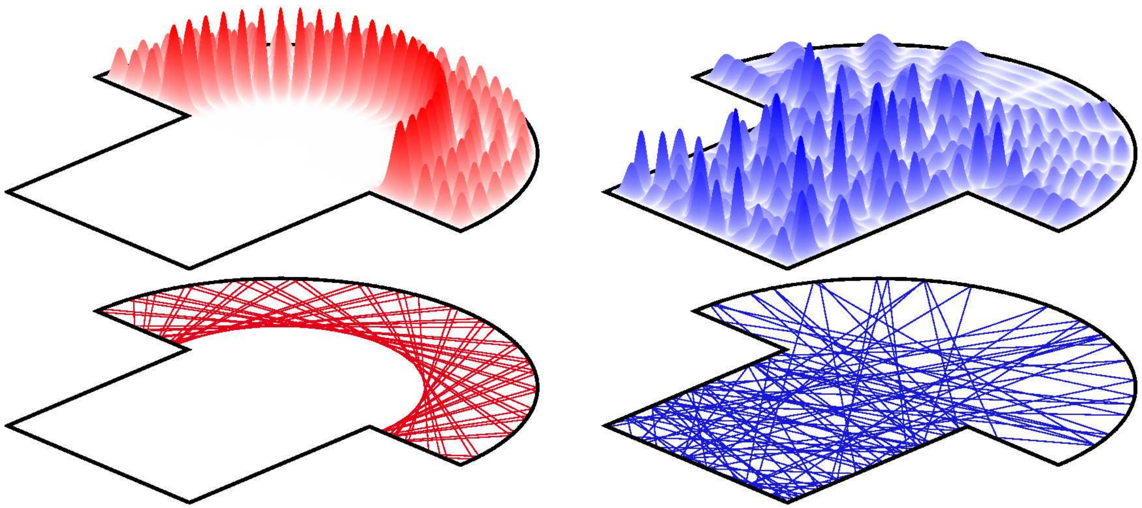
$$K = 2.9$$

$$|\psi_{\mathrm{ch}}\rangle\langle\psi_{\mathrm{ch}}| \approx \frac{1}{N_{\mathrm{ch}}}\left(\mathbf{1} - \sum_{\mathrm{island}} |\psi_{\mathrm{reg}}\rangle\langle\psi_{\mathrm{reg}}|\right)$$

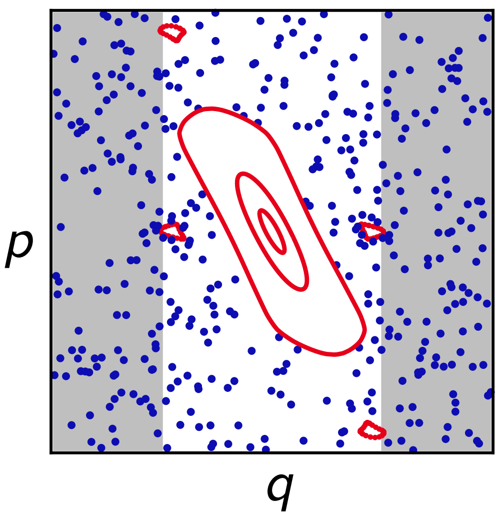


Summary IV: Prediction for tunneling rates

mushroom billiard

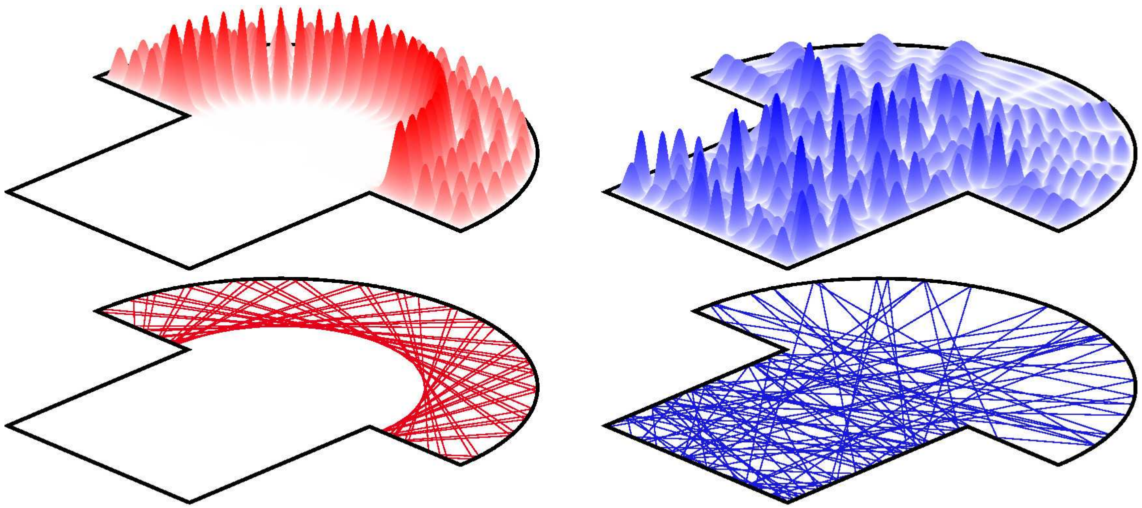


standard map

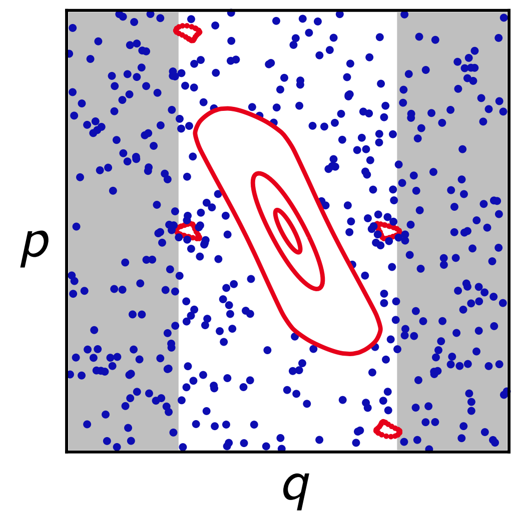


Summary IV: Prediction for tunneling rates

mushroom billiard



standard map

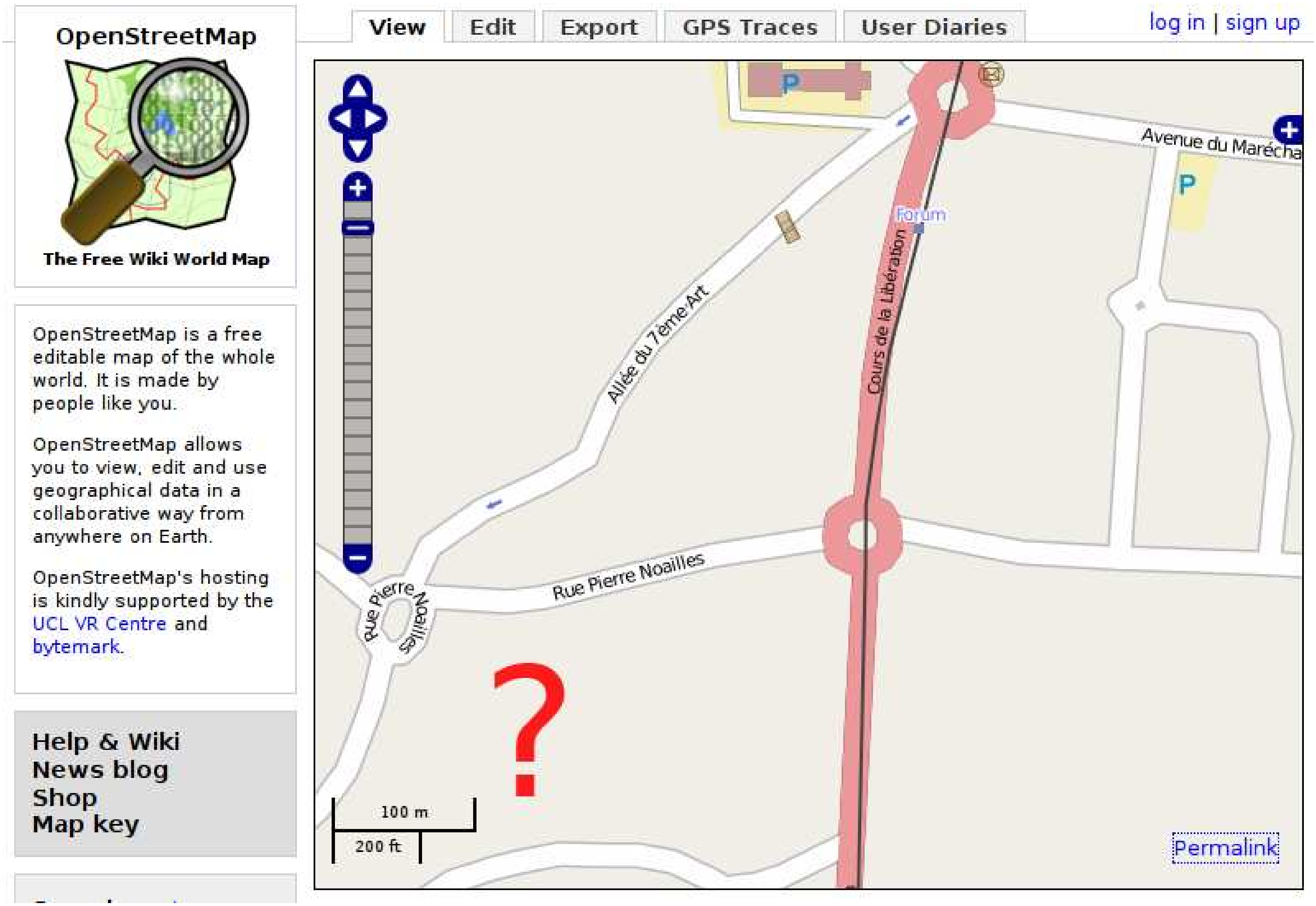


Open questions

- generic billiards?
- general answer: which properties of the island?
- apply to level spacing statistics $P(s)$ at small s [Vidmar et al. 2007]
- apply to microlasers (e.g. annular billiard)

BTW: the campus needs more mapping ... ;-)

http://www.openstreetmap.org/?lat=44.81087&lon=-0.5914&zoom=17&layers=B000FTF



You can improve that: see
http://wiki.openstreetmap.org/wiki/