Finite height representations and syntomic complex

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Finite height representations

Let $p \geq 3$ a prime number, κ a perfect field of characteristic p with $W = W(\kappa)$, $F = \mathrm{Fr}(W)$, $\bar{F}$ = algebraic closure of F and $G_F = \mathrm{Gal}(\bar{F}/F)$.

Fontaine's theory: $V = p$-adic representation of G_F.

Crystalline	$\mathbf{B}_{\mathrm{cris}}(O_{\bar{F}})$-admissible
Finite height	Wach module $\mathbf{N}(V)$ exists

Theorem 1 (Wach, Colmez, Berger)

A p-adic representation of G_F is crystalline if and only if it is of finite height.

Let $R = W\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ (for simplicity), $\overline{\mathrm{Fr}(R)} \supset \bar{F}$ fixed algebraic closure with $\bar{R}$ the integral closure of R in the maximal unramified extension of $R[\frac{1}{p}] \subset \overline{\mathrm{Fr}(R)}$ and $G_R = \mathrm{Gal}\big(\bar{R}[\frac{1}{p}]/R[\frac{1}{p}]\big)$.

Andreatta-Brinon theory: $V = p$-adic representation of G_R.

Crystalline	$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\bar{R})$-admissible
Finite height	Wach module $\mathbf{N}(V)$ exists

Theorem 2 (A.)

Let V be a finite height representation of G_R, then

1. V is crystalline.
2. $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+} \mathbf{N}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ for V positive.

The converse statement, i.e. crystalline implies finite height, as well as, analogous integral results appear to be true (work in progress).

Syntomic complex and Galois cohomology

Let K/F finite ramified O_K = ring of integers, ϖ = uniformizer, $S = O_K \otimes_W R$ and $G_S = \mathrm{Gal}\big(\bar{R}[\frac{1}{p}]/S[\frac{1}{p}]\big)$.

$R_\varpi^+ = (p, X_0)$-adic completion of $W[X_0, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$,
$R_\varpi^{\mathrm{PD}} = p$-adic completion of $R_\varpi^+\big[\frac{E(X_0)^k}{k!}\big]_{k \in \mathbb{N}}$.
Here $E(X_0)$ generates the kernel of $R_\varpi^+ \twoheadrightarrow S$ via $X_0 \mapsto \varpi$.

$$\mathrm{Syn}(S, r) := \mathrm{Cone}\Big(\mathrm{Fil}^r \Omega^\bullet_{R_\varpi^{\mathrm{PD}}} \xrightarrow{p^r - p^\bullet \varphi} \Omega^\bullet_{R_\varpi^{\mathrm{PD}}}\Big)[-1].$$

Theorem 3 (Colmez-Nizioł)

The Fontaine-Messing period maps

$$\tau_{\leq r}\mathrm{Syn}(S, r) \longrightarrow \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, \mathbb{Z}_p(r)),$$
$$\tau_{\leq r}\mathrm{Syn}(S, r)_n \longrightarrow \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, \mathbb{Z}/p^n(r))$$
$$\xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma\big((\mathrm{Sp}\, S[\tfrac{1}{p}])_{\text{ét}}, \mathbb{Z}/p^n(r)\big),$$

are p^N-quasi-isomorphisms for $N = N(K, p, r) \in \mathbb{N}$.

Let $K = F(\zeta_{p^m})$ for $m \geq 1$, V = positive finite height representation of G_R of height s, $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice such that $\mathbf{N}(T)$ is sufficiently "nice" and gives rise to a sufficiently "nice" filtered (φ, ∂)-module M over R such that $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} : \mathrm{Fil}^r M_\varpi^{\mathrm{PD}} \to \mathrm{Fil}^{r-1} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}} \to \cdots.$$

$$\mathrm{Syn}(S, M, r) := \mathrm{Cone}\Big(\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} \xrightarrow{p^r - p^\bullet \varphi} \mathscr{D}^\bullet_{S,M}\Big)[-1].$$

Theorem 4 (A.)

Let $r \in \mathbb{N}$ such that $r \geq s + 1$. Then the Fontaine-Messing period maps

$$\tau_{\leq r-s-1}\mathrm{Syn}(S, M, r) \xrightarrow{\sim} \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$
$$\tau_{\leq r-s-1}\mathrm{Syn}(S, M, r)_n \xrightarrow{\sim} \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)),$$

are p^N-quasi-isomorphisms for $N = N(T, p, m, r) \in \mathbb{N}$.

p-adic nearby cycles

Let $\mathfrak{X}/W$ a proper and smooth (p-adic formal) scheme, X its (rigid) generic fiber, $\mathfrak{X}_\kappa$ its special fiber with $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ and $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ natural maps between sites.

$\mathscr{S}_n(r)_{\mathfrak{X}} :=$ Syntomic sheaf modulo p^n on $\mathfrak{X}_{\kappa,\text{ét}}$.

Theorem 5 (Kato, Kurihara, Tsuji, Colmez-Nizioł)

For $0 \leq k \leq r$, the map

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{H}^k(\mathscr{S}_n(r)_{\mathfrak{X}}) \longrightarrow i^* \mathrm{R}^k j_* \mathbb{Z}/p^n(r)'_X,$$

is a p^N-isomorphism for $N = N(F, p, r) \in \mathbb{N}$.

Let $\mathcal{M}$ be a Fontaine-Laffaille module over $\mathfrak{X}$ of level $s \in [0, p-2]$ such that there exists an affine covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ with $\mathfrak{U}_i$ "small" and $\mathcal{M}_{\mathfrak{U}_i}$ is a free relative Fontaine-Laffaille module. Let $\mathbb{L}$ denote the associated étale $\mathbb{Z}_p$-local system on the generic fiber X.

Theorem 6 (A.)

For $0 \leq k \leq r - s - 1$ the Fontaine-Messing period map

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{H}^k\big(\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}\big) \longrightarrow i^* \mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X,$$

is a p^N-isomorphism for $N = N(p, r, s) \in \mathbb{N}$.

References

[1] P.Colmez, W. Nizioł. *Syntomic complexes and p-adic nearby cycles.* Inventiones mathematicae, 208(1):1-108, 2017.

[2] Abhinandan, *Crystalline representations and Wach modules in the relative case.* Preprint, 2021.

[3] Abhinandan, *Syntomic complex and p-adic nearby cycles.* Preprint, 2022.