# Crystalline representations and Wach modules in the relative case II

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Abstract. We study relative Wach modules generalising our previous works on this subject. Our main result shows a categorical equivalence between relative integral Wach modules and lattices inside relative crystalline representations. Using this result, we deduce a purity statement for relative crystalline representations and provide a criteria for checking the crystallinity of relative $p$-adic representations. Furthermore, we interpret relative Wach modules as modules with $q$-connections. Moreover, for a crystalline representation, we show that its associated Wach module (together with the Nygaard filtration) is the canonical $q$-deformation of the filtered $(\varphi, \partial)$-module associated to the representation.

## 1. Introduction

The study of arithmetic Wach modules and their relationship to crystalline representations is of classical nature, having been taken up in the works of Fontaine [Fon90], Wach [Wac96; Wac97], Colmez [Col99] and Berger [Ber04]. More explicitly, in op. cit. the authors studied the situation of an absolutely unramified extension of $\mathbb{Q}_p$ with perfect residue field. In [Abh21] we defined a similar concept in the relative case, i.e. for certain étale algebras over a formal torus (see §1.5 for precise setup) and showed that such objects give rise to crystalline representations of the fundamental group of the generic fiber. On the other hand, in [Abh23a], we generalised the theory of Wach modules and their relationship to crystalline representations, to the imperfect residue field case. In this article, we combine these two generalisations of the classical theory, to discuss the equivalence in its most natural generality. In addition, we show that Wach modules are $q$-deformations of lattices inside the filtered $(\varphi, \partial)$-module attached to crystalline representations.

Before providing further motivations for our results, let us remark that recent developments in the theory of prismatic $F$-crystals [BS23; DLMS22; GR22] provide a new approach to the classification of lattices inside crystalline representations. These exciting new developments have motivated us in seeking the results of the current paper. However, instead of using the tools from prismatic theory, we employ techniques from the classical theory of $(\varphi, \Gamma)$-modules to obtain our results due to the very nature of the objects studied in this article, i.e. relative Wach modules. Additionally, our proof enables us to provide interesting applications as well, for example, using [Abh23a, Theorem 1.5] and Theorem 1.7, we provide a new criteria for checking crystallinity of $p$-adic representations in the relative case (see Theorem 1.9 and Corollary 1.10). We refer the reader to §1.4 for more details on relation of our results to the prismatic theory.

Our motivation for studying relative Wach modules is twofold, largely stemming from geometry. In [Abh23b], for smooth ($p$-adic formal) schemes, we defined the notion of crystalline syntomic complex with coefficients in global relative Fontaine-Laffaille modules. Moreover, [Abh23b, Theorem 1.15] showed that such a complex is naturally comparable to the complex of $p$-adic nearby cycles of the associated crystalline $\mathbb{Z}_p$-local system on the (rigid analytic) generic fiber of the (formal) scheme. The work in loc. cit. was motivated by the results of [FM87], [Tsu96], [Tsu99] and [CN17], and the proof of [Abh23b, Theorem 1.15] follows via careful computations in the local setting in which relative Wach modules play a pivotal role (see [Abh23b, Corollary 1.12]). To generalise these results beyond the Fontaine-Laffaille case, it is therefore necessary to understand the relationship between crystalline representations of the fundamental group and general relative Wach modules (see Theorem 1.7).

On the other hand, in [BMS19], for smooth $p$-adic formal schemes, the authors defined a prismatic syntomic complex and compared it to the complex of $p$-adic nearby cycles integrally. In the same vein, comparison results beyond the smooth case, have also been obtained in [AMMN22] and [BM23], where

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the latter uses the theory of prismatic cohomology from [BS22]. The aforementioned results were in the case of constant coefficients, and it is natural to ask if [BMS19, Theorem 10.1] could be generalised to non-constant coefficients, i.e. prismatic $F$-crystals. It is reasonable to expect that Wach modules will play an important role in resolving these questions.

Another motivation for considering Wach modules is to construct a deformation of crystalline cohomology, i.e. the functor $\mathbf{D}_{\mathrm{cris}}$ from classical $p$-adic Hodge theory, to better capture mixed characteristic information. In [Fon90, §B.2.3] Fontaine expressed similar expectations which were verified by Berger in [Ber04, Théorème III.4.4] and generalised to finer integral conjectures in [Sch17, §6]. Some conjectures of [Sch17] were resolved by the introduction of prismatic cohomology [BS22]. It is also worth mentioning that the proof of the result comparing prismatic syntomic complex to $p$-adic nearby cycles, i.e. [BMS19, Theorem 10.1], relies on a local computation of prismatic cohomology using the $q$-de Rham complex, i.e. $q$-deformation of the de Rham complex. In addition, the importance of $q$-de Rham cohomology in computation of prismatic cohomology has also been emphasised in [BL22, Example 1.3.3].

In this paper, we interpret Wach modules as $q$-de Rham complexes (see Theorem 1.11). Moreover, we show that such an object is the $q$-deformation of a lattice inside the filtered $(\varphi, \partial)$-module attached to a crystalline representation. In a subsequent work, we will show that in our setting, a relative Wach module can be regarded as the evaluation of a prismatic $F$-crystal over a covering (by a suitable $q$-de Rham prism) of the final object of a certain prismatic topos. Hence, from these apparent tight connections between Wach modules and prismatic $F$-crystals and $p$-adic crystalline representations, we expect these objects to play an important role in the study of $p$-adic nearby cycles of crystalline $\mathbb{Z}_p$-local systems (for smooth formal schemes) and its comparison to prismatic syntomic complex with coefficients.

In summary, within the overarching program sketched above, this paper realises two of our goals (see Theorem 1.7 and Theorem 1.11). Additionally, we provide interesting applications of our results to purity statements in $p$-adic Hodge theory (see Theorem 1.9 and Corollary 1.10).

### 1.1. The arithmetic case.

Let $p$ be a fixed prime number and $\kappa$ a perfect field of characteristic $p$; set $O_F := W(\kappa)$ to be the ring of $p$-typical Witt vectors with coefficients in $\kappa$ and $F := O_F[1/p]$. We fix an algebraic closure $\overline{F}$ of $F$, and let $G_F := \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of $F$. Moreover, set $F_\infty := \cup_n F(\mu_{p^n})$, let $F_\infty^\flat$ denote the tilt of $F_\infty$ (see §1.5) and denote by $\Gamma_F := \mathrm{Gal}(F_\infty/F)$. Furthermore, fix $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ in $O_{F_\infty}^\flat$ and its Teichmüller lift $[\varepsilon]$ in $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty}) := W(O_{F_\infty}^\flat)$, the ring of $p$-typical Witt vectors with coefficients in $O_{F_\infty}^\flat$. Additionally, set $\mu := [\varepsilon] - 1$ and $[p]_q := \varphi(\mu)/\mu$, as elements of $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. Let $\mathbf{A}_F^+ := O_F[\![\mu]\!] \subset \mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$, which is stable under the $(\varphi, \Gamma_F)$-action on $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. We equip $\mathbf{A}_F^+$ with induced structures and note the following:

**Definition 1.1.** A *Wach module* over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$, for some $a, b \in \mathbb{Z}$ with $b \geq a$, is a finite free $\mathbf{A}_F^+$-module $N$ equipped with a continuous and semilinear action of $\Gamma_F$ such that the induced action of $\Gamma_F$ on $N/\mu N$ is trivial. Moreover, there is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ compatible with the respective actions of $\Gamma_F$, and such that $\varphi(\mu^b N) \subset \mu^b N$, the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

Denote by $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$ the category of Wach modules over $\mathbf{A}_F^+$, with morphisms between objects being $\mathbf{A}_F^+$-linear, $\varphi$-equivariant (after inverting $\mu$) and $\Gamma_F$-equivariant morphisms.

Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside $p$-adic crystalline representations of $G_F$. To any $T$ in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_K)$ one can functorially attach the Wach module $\mathbf{N}_F(T)$ over $\mathbf{A}_F^+$ as in [Ber04]. Then, from [Wac96; Col99; Ber04] we have the following:

**Theorem 1.2.** *The Wach module functor induces an equivalence of categories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_F(T),$$

*with a quasi-inverse given as* $N \mapsto (W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_F^+} N)^{\varphi=1}$, *where* $\mathbb{C}_p := \widehat{\overline{F}}$.

**1.2. The relative case.** Let $d \in \mathbb{N}$ and take $X_1, X_2, \ldots, X_d$ to be some indeterminates. We set $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ to be the $p$-adic completion of Laurent polynomial ring $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. Let $R$ denote the $p$-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ with non-empty and geometrically integral special fiber. Set $G_R$ as the étale fundamental group of $R[1/p]$ and $\Gamma_R$ as the Galois group of $R_\infty[1/p]$ over $R[1/p]$, where $R_\infty$ is obtained from $R$ by adjoining to it all $p$-power roots of unity and all $p$-power roots of $X_i$, for each $1 \leq i \leq d$. Then we have $\Gamma_R \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ (see §2 for precise definitions). Set $O_L := (R_{(p)})^\wedge$ as a complete discrete valuation ring with uniformiser $p$, residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and set $L := O_L[1/p]$. Let $G_L$ denote the absolute Galois group of $L$ such that we have a continuous homomorphism $G_L \to G_R$; let $\Gamma_L$ denote the Galois group of $L_\infty$ over $L$, where $L_\infty$ is obtained from $L$ by adjoining to it all $p$-power roots of unity and all $p$-power roots of $X_i$, for each $1 \leq i \leq d$. The continuous homomorphism $G_L \to G_R$ induces a continuous isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. In this setting, we have the theory of crystalline representations of $G_R$ from [Bri08] and the theory of étale $(\varphi, \Gamma)$-modules from [And06; AB08].

**1.2.1. Relative Wach modules.** For $1 \leq i \leq d$, fix $X_i^\flat := (X_i, X_i^{1/p}, \ldots)$ in $R_\infty^\flat$ (the tilt of $R_\infty$) and their Teichmüller lifts $[X_i^\flat]$ in $\mathbf{A}_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat)$. Let $\mathbf{A}_R^+$ denote the $(p, \mu)$-adic completion of the unique extension of the $(p, \mu)$-adic completion of $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$ along the $p$-adically completed étale map $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle \to R$ (see §1.5 and §2.2). The ring $\mathbf{A}_R^+$ is equipped with a Frobenius endomorphism $\varphi$ and a continuous action of $\Gamma_R$; set $\mathbf{A}_L^+$ to be the $(p, \mu)$-adic completion of $(\mathbf{A}_R^+)_{(p,\mu)}$ equipped with an induced Frobenius endomorphism $\varphi$ and a continuous action of $\Gamma_L \xrightarrow{\sim} \Gamma_R$. With this setup, we define the following:

**Definition 1.3.** A *Wach module* over $\mathbf{A}_R^+$ with weights in the interval $[a, b]$, for some $a, b \in \mathbb{Z}$ with $b \geq a$, is a finitely generated $\mathbf{A}_R^+$-module $N$ satisfying the following assumptions:

(1) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on $N$.

(2) $N$ is equipped with a semilinear action of $\Gamma_R$ such that the induced action of $\Gamma_R$ on $N/\mu N$ is trivial.

(3) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ compatible with the respective actions of $\Gamma_R$, and such that $\varphi(\mu^b N) \subset \mu^b N$, the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

Denote by $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q}$ the category of Wach modules over $\mathbf{A}_R^+$, with morphisms between objects being $\mathbf{A}_R^+$-linear, $\varphi$-equivariant (after inverting $\mu$) and $\Gamma_R$-equivariant morphisms.

**Remark 1.4.** The condition (1) in Definition 1.3 is new and relaxes finite projectivity assumption of relative Wach modules in [Abh21, Definition 4.8]. Moreover, condition (1) above is equivalent to the vanishing of local cohomology of $N$ with respect to the ideal $(p, \mu) \subset \mathbf{A}_R^+$ in degree 1 (see Lemma 3.3 and Remark 3.4), in particular, it is equivalent to having $\{p, [p]_q\}$ and $\{[p]_q, p\}$ as regular sequences on $N$ (see Lemma 3.6). Furthermore, one can also show that $N[1/p]$ is finite projective over $\mathbf{A}_R^+[1/p]$ (see Proposition A.1), $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$ (see Proposition 3.11) and $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ (see Lemma 3.5).

**Remark 1.5.** In Definition 1.3, note that in contrast with the arithmetic case in Definition 1.1, we have dropped the assumption on the continuity of the action of $\Gamma_R$ on $N$. However, in Lemma 3.7 we show that the condition (2) in Definition 1.3, i.e. triviality of the action of $\Gamma_R$ on $N/\mu N$, automatically implies that the action of $\Gamma_R$ on $N$ is continuous.

**Remark 1.6.** Definition 1.3 may be adapted to the case of a field, i.e. over the ring $\mathbf{A}_F^+$ (resp. $\mathbf{A}_L^+$). In such cases, from the assumptions of Definition 1.3 it follows that a Wach module over $\mathbf{A}_F^+$ (resp. $\mathbf{A}_L^+$) is necessarily finite free. Indeed, if $N$ is a Wach module over $\mathbf{A}_F^+$ (resp. $\mathbf{A}_L^+$), in the sense of Definition 1.3, then one first observes that $N$ is torsion-free since $N \subset N[1/p]$ and the latter is finite free over $\mathbf{A}_F^+[1/p]$ (resp. $\mathbf{A}_L^+[1/p]$) by [Abh23a, Lemma 2.14]. Then using [Fon90, §B.1.2.4 Proposition] it follows that $N$ is finite free (resp. Lemma 3.5 and [Abh23a, Remark 2.15]). In particular, over $\mathbf{A}_F^+$ (resp. $\mathbf{A}_L^+$), Definition 1.3 is equivalent to Definition 1.1 (resp. [Abh23a, Definition 1.3]).

Set $\mathbf{A}_R := \mathbf{A}_R^+[1/\mu]^\wedge$ as the $p$-adic completion, equipped with the induced Frobenius endomorphism $\varphi$ and the induced continuous action of $\Gamma_R$, and similarly, set $\mathbf{A}_L := \mathbf{A}_L^+[1/\mu]^\wedge$ equipped with the induced Frobenius endomorphism $\varphi$ and the induced continuous action of $\Gamma_L$. Let $T$ be a finite free $\mathbb{Z}_p$-representation of $G_R$, then one can functorially attach to $T$ a finite projective étale $(\varphi, \Gamma_R)$-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$, equipped with a semilinear and continuous action of $\Gamma_R$ and a Frobenius-semilinear operator $\varphi$ commuting with the action of $\Gamma_R$. In fact, the preceding functor induces a categorical equivalence between the category of finite free $\mathbb{Z}_p$-representations of $G_R$ and the category of finite projective étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$ (see [And06, Theorem 7.11]). Additionally, the category of Wach modules over $\mathbf{A}_R^+$ fully faithfully embeds into the latter category, i.e. the category of étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$ (see Proposition 3.15).

**1.2.2. Main results.** Let $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_R)$ denote the category of $\mathbb{Z}_p$-lattices inside $p$-adic crystalline representations of $G_R$. For $T$ a $\mathbb{Z}_p$-lattice inside a $p$-adic crystalline representation of $G_R$, we construct a Wach module $\mathbf{N}_R(T)$ over $\mathbf{A}_R^+$, functorial in $T$, and contained in $\mathbf{D}_R(T)$ (see Theorem 4.1). Our first main result is as follows:

**Theorem 1.7** (Corollary 4.3)**.** *The Wach module functor induces an equivalence of categories*

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_R^+}$$
$$T \longmapsto \mathbf{N}_R(T),$$

*with a quasi-inverse given as* $N \mapsto \mathbf{T}_R(N) := \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} N\big)^{\varphi=1}$.

**Remark 1.8.** In Theorem 1.7, we do not expect the functor $\mathbf{N}_R$ to be an exact equivalence. However, note that after inverting $p$, the Wach module functor induces an exact equivalence between $\otimes$-categories: $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{B}_R^+}$, via $V \mapsto \mathbf{N}_R(V)$, where $\mathbf{B}_R^+ = \mathbf{A}_R^+[1/p]$, and an exact $\otimes$-compatible quasi-inverse functor given as $M \mapsto \mathbf{V}_R(M) := \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} M\big)^{\varphi=1}$ (see Corollary 4.4).

As an application of Theorem 1.7, we obtain the following purity statement:

**Theorem 1.9** (Theorem 4.5)**.** *Let $V$ be a $p$-adic representation of $G_R$. Then $V$ is crystalline as a representation of $G_R$ if and only if it is crystalline as a representation of $G_L$.*

For a $p$-adic representation $V$ of $G_R$, let $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ denote the associated filtered $(\varphi, \partial)$-module over $R[1/p]$ (see [Bri08, §8.2]). We show the following criterion for checking the crystallinity of $V$:

**Corollary 1.10** (Theorem 4.5 & Corollary 4.6)**.** *Let $V$ be a $p$-adic representation of $G_R$. Then $V$ is crystalline if and only if $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. Moreover, under these equivalent conditions, we have a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ of filtered $(\varphi, \partial)$-modules over $L$.*

Important inputs for the proof of Corollary 1.10 are Theorem 1.9 and a careful study of the period rings for the localisation of $\overline{R}$ at its minimal primes above $(p) \subset R$ (see §2.1).

**1.2.3. Strategy for the proof of Theorem 1.7.** The proof of Theorem 1.7 crucially uses analogous results obtained in the imperfect residue field case (see [Abh23a, Theorem 1.5]). Starting with a Wach module $N$ over $\mathbf{A}_R^+$, we use ideas from [Abh21, Theorem 4.25 & Proposition 4.28], the observation that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$ is a Wach module over $\mathbf{A}_L^+$, as well as, [Abh23a, Lemma 3.6 & Theorem 3.12] to establish that $\mathbf{T}_R(N)$ is a $\mathbb{Z}_p$-representation of $G_R$ such that $\mathbf{T}_R(N)[1/p]$ is crystalline (see Theorem 3.27). Conversely, starting with a $\mathbb{Z}_p$-lattice $T$ inside a $p$-adic crystalline representation of $G_R$, we observe that $T[1/p]$ is a $p$-adic crystalline representation of $G_L$, and we use [Abh23a, Theorem 4.1] to obtain a unique Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$. Moreover, from the theory of $(\varphi, \Gamma)$-modules we have an étale $(\varphi, \Gamma_R)$-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$ (see [And06]). We set $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ as an $\mathbf{A}_R^+$-module, where $\mathbf{D}_L(T)$ is the $(\varphi, \Gamma_L)$-module over $\mathbf{A}_L$, associated to $T$. Next, using the compatible Frobenius-semilinear endomorphism $\varphi$ and the continuous action of $\Gamma_L \xrightarrow{\sim} \Gamma_R$ on $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$, we equip the $\mathbf{A}_R^+$-module $\mathbf{N}_R(T)$ with a natural $(\varphi, \Gamma_R)$-action. Finally, we use properties of $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ to show that $\mathbf{N}_R(T)$ is the unique Wach module associated to $T$ (similar to the relative Breuil-Kisin case in [DLMS22]). This completes a sketch of our proof of Theorem 1.7.

**1.3. Wach modules as $q$-deformations.** In §5 we recall the definition of a $q$-connection axiomatically, following [MT20]. Moreover, we show that a Wach module $N$ over $\mathbf{A}_R^+$ can also be seen as a $\varphi$-module equipped with a $q$-connection. More precisely, let $D := O_F[\![\mu]\!]$, and let $\{\gamma_1, \ldots, \gamma_d\}$ be topological generators of the geometric part of $\Gamma_R$, i.e. $\Gamma'_R$ (see §2). Then in Proposition 5.3 we show that the $q$-connection defined as

$$\nabla_q : N \longrightarrow N \otimes_{\mathbf{A}_R^+} \Omega^1_{\mathbf{A}_R^+/D}, \qquad x \longmapsto \textstyle\sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),$$

describes $(N, \nabla_q)$ as a $\varphi$-module with $(p, [p]_q)$-adically quasi-nilpotent $D$-linear flat $q$-connection over $\mathbf{A}_R^+$. We equip $N$ with the Nygaard filtration as in Definition 3.24. Then, it follows that $N/\mu N$ is a $\varphi$-module over $R$ equipped with a $p$-adically quasi-nilpotent flat connection and we further equip it with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $N[1/p]/\mu N[1/p] = (N/\mu N)[1/p]$ with induced structures, in particular, it is a filtered $(\varphi, \partial)$-module over $R[1/p]$.

**Theorem 1.11** (Theorem 5.6)**.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$ and $V := \mathbf{T}_R(N)[1/p]$, the associated crystalline representation from Theorem 1.7. Then we have a natural isomorphism $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ of filtered $(\varphi, \partial)$-modules over $R[1/p]$.*

Note that $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ denotes the filtered $(\varphi, \partial)$-module over $R[1/p]$ associated to $V$ (see [Bri08, §8.2]). Our proof of the theorem follows from computations done for the proof of Theorem 3.27 (building upon ideas developed in [Abh21, Theorem 4.25 & Proposition 4.28] and [Abh23a, Theorem 1.7]).

Finally, let us summarise the relationship between various categories considered in Theorem 1.7 and Theorem 1.11. Recall that $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ is the category of $p$-adic crystalline representations of $G_R$, and let $\mathrm{MF}_R(\varphi, \partial)$ denote the category of filtered $(\varphi, \partial)$-modules over $R[1/p]$. From [Bri08, §8.2] we have a $\otimes$-compatible functor $\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \to \mathrm{MF}_R(\varphi, \partial)$, and let $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ denote its essential image. Then, from [Bri08, Théorème 8.5.1], we have an exact equivalence of $\otimes$-categories $\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \xrightarrow{\sim} \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$, with an exact $\otimes$-compatible quasi-inverse $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}$ (see §2.6). So, Remark 1.8 and Theorem 1.11 can be summarised as follows:

**Corollary 1.12** (Corollary 5.8)**.** *Functors in the following diagram induce exact equivalence of $\otimes$-categories*

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \arrow[rr, shift left, "\mathbf{N}_R"] \arrow[dr, shift left, "\mathcal{O}\mathbf{D}_{\mathrm{cris},R}"] & & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{B}_R^+} \arrow[ll, shift left, "\mathbf{V}_R"] \arrow[dl, "q \mapsto 1"] \\
& \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial). \arrow[ul, shift left, "\mathcal{O}\mathbf{V}_{\mathrm{cris},R}"] &
\end{tikzcd}$$

**1.4. Relation to previous works.** Our first main result, Theorem 1.7, is a generalisation of Theorem 1.2 from [Wac96; Col99; Ber04] and [Abh23a, Theorem 1.5]. That said, the methods of op. cit. do not directly apply to our current sitution. In fact, the proof of Theorem 1.7 uses crucial inputs of results and ideas from [Abh21] and [Abh23a].

Recent developments in the theory of prismatic $F$-crystals in [BS23; DLMS22; GR22] would suggest that there is a categorical equivalence between the category of Wach modules over $\mathbf{A}_R^+$ and (completed/analytic) prismatic $F$-crystals on the absolute prismatic site $(\mathrm{Spf}\, R)_{\Delta}$. From that perspective, Theorem 1.7 could be seen as an analogue of [DLMS22, Theorem 1.2 & Proposition 1.4], i.e. the categorical equivalence between Kisin descent data (up to twisting) and lattices inside crystalline representations of $G_R$. However, a key difference with loc. cit. is that Wach modules admit a natural action of $\Gamma_R$, instead of a prismatic descent datum. Nonetheless, as our base ring $R$ is absolutely unramified (at $p$), the action of $\Gamma_R$ is rich enough to recover a $\mathbb{Z}_p$-lattice inside a $p$-adic crystalline representation of $G_R$, thus establishing Theorem 1.7.

In the current paper, we provide two applications of Theorem 1.7. The first application, i.e. Theorem 1.9 establishes a certain purity statement for crystalline representations. Our result is similar to the purity statement for Hodge-Tate representations in [Tsu11, Theorem 9.1] and rigidity of de Rham local systems in [LZ17, Theorem 1.3]. It should be noted that the purity result in Theorem 1.9 can also be obtained by combining [LZ17, Theorem 1.3] and some unpublished works of Tsuji. Moreover, the result of loc. cit. works for general ramified (at $p$) small base. A similar statement has been obtained in [Moo22, Theorem 1.4] using the results of [DLMS22].

The second application of Theorem 1.7 is given in Corollary 1.10. Our result provides a new criterion for checking the crystallinity of a $p$-adic representation of $G_R$. Note that the analogous statement for de Rham representations is true from the results of [LZ17]. However, our result in the crystalline case is entirely new and uses Theorem 1.9 as an important input. At this point, for expert readers, it is worth mentioning that for general ramified (at $p$) small base, a statement analogous to Corollary 1.10 appears to be true. In particular, we expect that one can deduce the statement using [LZ17, Theorem 1.3], the unpublished results of Tsuji mentioned above and employing arguments similar to our proof of Theorem 4.5.

For our second main result, Theorem 1.11, the motivation for interpreting a Wach module as a $q$-de Rham complex and as the $q$-deformation of crystalline cohomology, i.e. $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$, comes from [Fon90, §B.2.3], [Ber04, Théorème III.4.4] and [Sch17, §6]. In particular, we provide a direct generalisation of [Ber04, Théorème III.4.4], as well as verify expectations put forth in [Abh21, Remark 4.48] and [Abh23a, Remark 1.8] (see Remark 5.7 for the latter).

### 1.5. Setup and notations.

In this section we will describe our setup and fix some notations, which are essentially the same as in [Abh21, §1.4]. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let $p$ be a fixed prime number, $\kappa$ a perfect field of characteristic $p$, $O_F := W(\kappa)$ the ring of $p$-typical Witt vectors with coefficients in $\kappa$. Then $O_F$ is a complete discrete valuation ring with uniformiser $p$ and set $F := O_F[1/p]$ to be the fraction field of $O_F$. Let $\overline{F}$ denote a fixed algebraic closure of $F$ so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of $\kappa$. Furthermore, denote the absolute Galois group of $F$ to be $G_F := \mathrm{Gal}(\overline{F}/F)$.

*Notation.* Let $\Lambda$ be an $I$-adically complete algebra for a finitely generated ideal $I \subset \Lambda$. Let $Z := (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} := (k_1, \ldots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to +\infty$ we will mean that $\sum k_i \to +\infty$. Define

$$\Lambda\langle Z\rangle := \Big\{ \sum_{\mathbf{k}\in\mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \to 0 \text{ } I\text{-adically as } \mathbf{k} \to +\infty \Big\}.$$

We fix $d \in \mathbb{N}$ and let $X := (X_1, X_2, \ldots, X_d)$ be some indeterminates. Let $R$ be the $p$-adic completion of an étale algebra over $R^\square := O_F\langle X, X^{-1}\rangle$, with non-empty geometrically integral special fiber. We fix an algebraic closure $\overline{\mathrm{Frac}(R)}$ of $\mathrm{Frac}(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite $R$-subalgebras $S \subset \overline{\mathrm{Frac}(R)}$, such that $S[1/p]$ is étale over $R[1/p]$. Let $\overline{\eta}$ denote the fixed geometric point of the generic fiber $\mathrm{Spec}\, R[1/p]$ (defined by $\overline{\mathrm{Frac}(R)}$), and let $G_R := \pi_1^{\text{ét}}(\mathrm{Spec}\, R[1/p], \overline{\eta})$ denote the étale fundamental group. We can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}[1/p]$ over the fraction field of $R[1/p]$), i.e. $G_R = \pi_1^{\text{ét}}(\mathrm{Spec}\,(R[1/p]), \overline{\eta}) = \mathrm{Gal}(\overline{R}[1/p]/R[1/p])$. For $k \in \mathbb{N}$, let $\Omega^k_R$ denote the $p$-adic completion of module of $k$-differentials of $R$ relative to $\mathbb{Z}$. Then, we have $\Omega^1_R = \oplus_{i=1}^d R\, d\log X_i$, and $\Omega^k_R = \wedge^k_R \Omega^1_R$.

Let $\varphi$ denote an endomorphism of $R^\square$ which extends the natural Frobenius on $O_F$ by setting $\varphi(X_i) = X_i^p$, for all $1 \leq i \leq d$. The morphism $\varphi : R^\square \to R^\square$ is flat by [Bri08, Lemma 7.1.5], and it is faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset R^\square$. Moreover, using Nakayama Lemma and the fact that the absolute Frobenius on $R^\square/p$ is evidently of degree $p^d$, it easily follows that $\varphi$ on $R^\square$ is finite of degree $p^d$. Recall that the $O_F$-algebra $R$ is given as the $p$-adic completion of an étale algebra $R^\square$, therefore, the Frobenius endomorphism $\varphi$ on $R^\square$ admits a unique extension $\varphi : R \to R$ such that the induced map $\varphi : R/p \to R/p$ is the absolute Frobenius $x \mapsto x^p$ (see [CN17, Proposition 2.1]). Similar to above, again note that the endomorphism $\varphi : R \to R$ is faithfully flat and finite of degree $p^d$.

Let $O_L := (R_{(p)})^\wedge$, where $^\wedge$ denotes the $p$-adic completion. Let $\overline{L}$ denote a fixed algebraic closure of $L$ with ring of integers $O_{\overline{L}}$ such that we have an embedding $\overline{R} \to O_{\overline{L}}$. Then we get a continuous homomorphism $G_L := \mathrm{Gal}(\overline{L}/L) \to G_R$, inducing an isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. The Frobenius on $R$ extends to a unique Frobenius endomorphism $\varphi : O_L \to O_L$, lifting the absolute Frobenius on $O_L/pO_L$ (see [CN17, Proposition 2.1]). Similar to above, $\varphi$ on $O_L$ is faithfully flat and finite of degree $p^d$.

Let $S$ be a commutative ring with $\pi := p^{1/p} \in S$ such that $S$ is $\pi$-adically complete and $\pi$-torsion free, for example, $S = O_{F_\infty}, O_{L_\infty}, O_{\overline{F}}, O_{\overline{L}}, R_\infty, \overline{R}$. Then the tilt of $S$ is defined as $S^\flat := \lim_\varphi S/p$ and the tilt of $S[1/p]$ is defined as $S[1/p]^\flat := S^\flat[1/p^\flat]$, where $p^\flat := (1, p^{1/p}, \ldots) \in S^\flat$ (see [Fon77, Chapitre V, §1.4] and [BMS18, §3]). Finally, consider a $\mathbb{Z}_p$-algebra $A$ equipped with a lift of the absolute Frobenius on $A/p$, i.e. an endomorphism $\varphi : A \to A$ such that $\varphi$ modulo $p$ is the absolute Frobenius. Then for any $A$-module $M$ we write $\varphi^*(M) := A \otimes_{\varphi,A} M$.

**Outline of the paper.** This article consists of four main sections. In §2 we collect relevant results in relative $p$-adic Hodge theory. In §2.1 we consider localisations of $\overline{R}$ at minimal primes above $(p) \subset R$ and study their properties. Then in §2.2, §2.3 & §2.4 we define relative period rings and study their localisations at primes of $\overline{R}$ above $(p) \subset R$. In §2.5 we quickly recall important rings from the theory of relative $(\varphi, \Gamma)$-modules and in §2.6 we recall the relation between $(\varphi, \Gamma)$-module theory and $p$-adic representations, as well as, definition and properties of crystalline representations. The aim of §3 is to define and study properties of a Wach module in the relative case and the associated representation of $G_R$. In §3.1 we first note some technical lemmas and then in §3.2 we define relative Wach modules, study its properties and relate these objects to étale $(\varphi, \Gamma)$-modules (see Proposition 3.15). Furthermore, in §3.3, we functorially attach a $\mathbb{Z}_p$-representation of $G_R$ to a relative Wach module and in §3.4 we show that such representations are closely related to finite $[p]_q$-height representations studied in [Abh21]. In §3.5 we study the Nygaard filtration on relative Wach modules. Finally, in §3.6 we show that the $\mathbb{Z}_p$-representation of $G_R$ associated to a relative Wach module, as in §3.3, is a lattice inside a $p$-adic crystalline representation of $G_R$ (see Theorem 3.27). In §4 we prove our first main result, i.e. Theorem 1.7. Before proving the theorem, we draw some important conclusions from the statement, in particular, in §4.1 we prove Theorem 1.9 and Corollary 1.10. Finally, in §4.2 we construct the promised relative Wach module and prove Theorem 1.7. In §5, we state and prove our second main result, i.e. Theorem 1.11. In §5.1, we recall the formalism on $q$-connections. Then in §5.2, we show that a Wach module can be interpreted as a $\varphi$-module equipped with a $q$-connection (see Proposition 5.3). Finally, using the computations done in the proof of Theorem 3.27, we prove Theorem 1.11.

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## 2. Period rings and $p$-adic representations

We will use the setup and notations from §1.5. Recall that $R$ is the $p$-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ and $O_L := (R_{(p)})^\wedge$. Set $R_\infty := \cup_{i=1}^d R[\mu_{p^\infty}, X_i^{1/p^\infty}]$ and recall that $\overline{R}$ is the union of finite $R$-subalgebras $S$ in a fixed algebraic closure $\overline{\mathrm{Frac}(R)} \supset \overline{F}$, such that $S[1/p]$ is étale over $R[1/p]$. We have (see [Abh21, §2 & §3]),

$$G_R := \mathrm{Gal}(\overline{R}[1/p]/R[1/p]),\ H_R = \mathrm{Gal}(\overline{R}[1/p]/R_\infty[1/p]),$$
$$\Gamma_R := G_R/H_R = \mathrm{Gal}(R_\infty[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,$$
$$\Gamma'_R := \mathrm{Gal}(R_\infty[1/p]/R(\mu_{p^\infty})[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(R(\mu_{p^\infty})[1/p]/R[1/p]) = \Gamma_R/\Gamma'_R \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

We fixed $\overline{L}$ as an algebraic closure of $L := O_L[1/p]$ with ring of integers $O_{\overline{L}}$ and an embedding $\overline{R} \to O_{\overline{L}}$. So, we have a continuous homomorphism of groups $G_L := \mathrm{Gal}(\overline{L}/L) \to G_R$, which induces an isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. For $1 \leq i \leq d$, we fix $X_i^\flat := (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ in $R_\infty^\flat$ and take $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ in $\Gamma_R$ such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of $\Gamma'_R$ satisfying $\gamma_j(X_i^\flat) = \varepsilon X_i^\flat$ if $i = j$ and $X_i^\flat$ otherwise, and $\gamma_0$ is a lift of a topological generator of $\Gamma_R/\Gamma'_R$.

**2.1. Localisation.** Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. The set $\mathcal{S}$ is equipped with a transitive action of $G_R$ (see [Mat89, Theorem 9.3]). For each prime $\mathfrak{p} \in \mathcal{S}$, set $G_R(\mathfrak{p}) := \{g \in G_R \text{ such that } g(\mathfrak{p}) = \mathfrak{p}\}$, i.e. the decomposition group of $G$ at $\mathfrak{p}$. Recall that $O_L = (R_{(p)})^\wedge$ and $L = O_L[1/p]$. For each $\mathfrak{p} \in \mathcal{S}$, let $\overline{L}(\mathfrak{p})$ denote an algebraic closure of $L$ with ring of integers $O_{\overline{L}(\mathfrak{p})}$

containing $(\overline{R})_{\mathfrak{p}}$. Set $\widehat{G}_R(\mathfrak{p}) := \mathrm{Gal}(\overline{L}(\mathfrak{p})/L)$ so that we have a natural homomorphism $\widehat{G}_R(\mathfrak{p}) \to G_R$ which factors as $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p}) \subset G_R$ (see [Bri08, Lemme 3.3.1]). Note that for each $\mathfrak{p} \in \mathcal{S}$, we have a natural embedding $\overline{R} \subset O_{\overline{L}(\mathfrak{p})}$ and hence we have a (non-canonical) isomorphism of Galois groups $\widehat{G}_R(\mathfrak{p}) \xrightarrow{\sim} G_L$.

Now, for each $\mathfrak{p} \in \mathcal{S}$, let $\mathbb{C}_{\mathfrak{p}}^+$ denote the $p$-adic completion of $O_{\overline{L}(\mathfrak{p})}$ and let $\mathbb{C}_{\mathfrak{p}} := \mathrm{Frac}(\mathbb{C}_{\mathfrak{p}}^+)$. Then $\mathbb{C}_{\mathfrak{p}}$ is an algebraically closed valuation field equipped with a continuous action of $\widehat{G}_R(\mathfrak{p})$ and $(\mathbb{C}_{\mathfrak{p}}^+)^{\widehat{G}_R(\mathfrak{p})} = O_L$ (see [Hyo86, Theorem 1]). Furthermore, let $\mathbb{C}^+(\mathfrak{p})$ denote the $p$-adic completion of $(\overline{R})_{\mathfrak{p}}$ and let $\mathbb{C}(\mathfrak{p}) := \mathbb{C}^+(\mathfrak{p})[1/p]$ equipped with a continuous action of $G_R(\mathfrak{p})$.

**Lemma 2.1.** *For each $\mathfrak{p} \in \mathcal{S}$, we have $(\overline{R})_{\mathfrak{p}} \subset \mathbb{C}^+(\mathfrak{p})$ and $(\overline{R})_{\mathfrak{p}} \cap p\mathbb{C}^+(\mathfrak{p}) = p(\overline{R})_{\mathfrak{p}}$. Moreover, $(\overline{R})_{\mathfrak{p}} \cap pO_{\overline{L}(\mathfrak{p})} = p(\overline{R})_{\mathfrak{p}}$.*

*Proof.* The proof is similar to [Bri08, Proposition 2.0.3]. Let $\mathfrak{p} \in \mathcal{S}$ and $x \in (\overline{R})_{\mathfrak{p}}$. Then there exists a finite normal $R$-subalgebra $S \subset \overline{R}$ such that $S[1/p]$ is étale over $R[1/p]$ and $\mathfrak{q} := \mathfrak{p} \cap S$ is a height 1 prime ideal of $S$ with $p \in \mathfrak{q}$ (since $\overline{R}$ is integral over $S$) and $x \in S_{\mathfrak{q}}$. Moreover, $S_{\mathfrak{q}}$ is a 1-dimensional normal noetherian domain, in particular, a discrete valuation ring. Now if the image of $x$ is zero in $\mathbb{C}^+(\mathfrak{p})$, then we have that $x \in p^n(\overline{R})_{\mathfrak{p}} \cap S_{\mathfrak{q}} = p^n S_{\mathfrak{q}}$, for each $n \in \mathbb{N}$, since $S_{\mathfrak{q}}$ is normal. So $x$ must be zero since $S_{\mathfrak{q}}$ is $p$-adically separated. This shows the first claim. For the second claim, let $x = py$ for some $y \in \mathbb{C}^+(\mathfrak{p})$. We have that $y \in S_{\mathfrak{q}}[1/p]$ and we need to show that $y \in S_{\mathfrak{q}}$. Let $\widehat{S}_{\mathfrak{q}}$ denote the completion of $S_{\mathfrak{q}}$ for the valuation (say $v_{\mathfrak{q}}$) described above. Then $\widehat{S}_{\mathfrak{q}}[1/p]$ is a finite separable extension of $L$ and $\widehat{S}_{\mathfrak{q}}$ embeds into $\mathbb{C}_{\mathfrak{p}}^+$. Moreover, the image of $\mathbb{C}^+(\mathfrak{p})$ in $\mathbb{C}_{\mathfrak{p}}$ is contained in $\mathbb{C}_{\mathfrak{p}}^+$, therefore $v_{\mathfrak{q}}(y) \geq 0$, i.e. $y \in S_{\mathfrak{q}}[1/p] \cap \widehat{S}_{\mathfrak{q}} = S_{\mathfrak{q}}$, as desired. Finally, let $x = pz$ for some $z \in O_{\overline{L}(\mathfrak{p})}$. Then similar to above, we have $z \in S_{\mathfrak{q}}[1/p]$ and $v_{\mathfrak{q}}(z) \geq 0$, so $z \in S_{\mathfrak{q}}$. This shows the third claim. ∎

All rings discussed above are $p$-torsion free, so from Lemma 2.1, it easily follows that the inclusion $\mathbb{C}^+(\mathfrak{p}) \subset \mathbb{C}_{\mathfrak{p}}^+$ is compatible with respective actions of $\widehat{G}_R(\mathfrak{p})$, where the action of $\widehat{G}_R(\mathfrak{p})$ on the left-hand term factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. In particular, we get that $\mathbb{C}^+(\mathfrak{p})^{G_R(\mathfrak{p})} = O_L$ (see [Bri08, p. 24]). Now, note that we have natural injective maps $\overline{R} \to (\overline{R})_{\mathfrak{p}} \to O_{\overline{L}(\mathfrak{p})}$. Upon passing to $p$-adic completions and setting $\mathbb{C}^+(\overline{R}) := \widehat{\overline{R}}$, we obtain natural maps $\mathbb{C}^+(\overline{R}) \to \mathbb{C}^+(\mathfrak{p}) \to \mathbb{C}_{\mathfrak{p}}^+$, where the first map need not be injective. However, recall that $\overline{R}$ is a direct limit of finite and normal $R$-algebras, therefore the natural map $\overline{R}/p^n \to \oplus_{\mathfrak{p}\in\mathcal{S}}(\overline{R})_{\mathfrak{p}}/p^n$ is injective. Passing to the limit over $n$, we obtain injective maps

$$\mathbb{C}^+(\overline{R}) \longrightarrow \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p}) \longrightarrow \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}_{\mathfrak{p}}^+. \tag{2.1}$$

Note that in (2.1) the leftmost term admits a natural action of $G_R$, the middle term admits a natural action of $\prod_{\mathfrak{p}\in\mathcal{S}} G_R(\mathfrak{p})$ and the rightmost term admits a natural action of $\prod_{\mathfrak{p}\in\mathcal{S}} \widehat{G}_R(\mathfrak{p})$. The two homomorphisms in (2.1) are compatible with these respective actions. Moreover, from [Bri08, Remarque 3.3.2] the middle term of (2.1) can be equipped with an action of $G_R$ and the left homomorphism in (2.1) is equivariant with respect to this action of $G_R$.

**Remark 2.2.** Note that $\mathbb{C}^+(\mathfrak{p})$ is an $O_L$-algebra for each $\mathfrak{p} \in \mathcal{S}$, so the maps in (2.1) extend to injective maps $O_L \otimes_R \mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}_{\mathfrak{p}}^+$ (see [Bri08, Proposition 3.3.3]).

**Lemma 2.3.** *The $O_L$-algebra $\mathbb{C}^+(\mathfrak{p})$ is perfectoid in the sense of [BMS18, Definition 3.5].*

*Proof.* Note that we have $\pi := p^{1/p} \in \overline{R} \subset (\overline{R})_{\mathfrak{p}} \subset \mathbb{C}^+(\mathfrak{p})$ and $\pi^p = p$ divides $p$. Moreover, it is clear that $\mathbb{C}^+(\mathfrak{p})$ is $\pi$-adically complete. Now, consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^+(\mathfrak{p})/\pi^p \arrow[r, two heads] \arrow[d] & \mathbb{C}^+(\mathfrak{p})/\pi \arrow[r, "\varphi"'] \arrow[d] & \mathbb{C}^+(\mathfrak{p})/\pi^p \arrow[d] \\
\mathbb{C}_{\mathfrak{p}}^+/\pi^p \arrow[r, two heads] & \mathbb{C}_{\mathfrak{p}}^+/\pi \arrow[r, "\sim", "\varphi"'] & \mathbb{C}_{\mathfrak{p}}^+/\pi^p,
\end{tikzcd}$$

where the left and right vertical arrows are injective by Lemma 2.1 and the middle vertical arrow is also injective by an argument similar to the proof of Lemma 2.1. So it follows that the top right horizontal arrow is injective as well. Then, using [BMS18, Lemma 3.9 and Lemma 3.10], we are left to show that $\varphi : \mathbb{C}^+(\mathfrak{p})/p = (\overline{R})_{\mathfrak{p}}/p \to (\overline{R})_{\mathfrak{p}}/p = \mathbb{C}^+(\mathfrak{p})/p$ is surjective. So let $x \in (\overline{R})_{\mathfrak{p}}/p$ and take a lift $y \in (\overline{R})_{\mathfrak{p}}$. Then there exists an $a \in \overline{R} \setminus \mathfrak{p}$ such that $ay \in \overline{R}$. Now, from [Bri08, Proposition 2.0.1], there exists $z, w \in \overline{R}$ such that $ay = z^p + pw$. Moreover, there exists $b \in \overline{R} \setminus \mathfrak{p}$ and $c \in \overline{R}$ such that $a = b^p + pc$. Then we can write $b^p y + pcy = z^p + pw$, or equivalently, $y = (z/b)^p + p(cy + w)/b^p$ with $(z/b)^p \in (\overline{R})_{\mathfrak{p}}$ and $p(cy+w)/b^p \in p(\overline{R})_{\mathfrak{p}}$. Hence, $x = (z/b)^p \mod p(\overline{R})_{\mathfrak{p}}$, proving that $\varphi : (\overline{R})_{\mathfrak{p}}/p \to (\overline{R})_{\mathfrak{p}}/p$ is surjective. ■

**2.2. The period ring $\mathbf{A}_{\text{inf}}$.** In this subsection we will study the relative version of Fonatine's infinitesimal period ring $\mathbf{A}_{\text{inf}}$ to be used in the sequel (see [Abh21, §2 and §3] for details). Let $\mathbf{A}_{\text{inf}}(R_\infty) := W(R_\infty^\flat)$ and $\mathbf{A}_{\text{inf}}(\overline{R}) := W(\overline{R}^\flat)$ admitting the Frobenius on Witt vectors and continuous $G_R$-action (for the weak topology). Moreover, we have $\mathbf{A}_{\text{inf}}(R_\infty) = \mathbf{A}_{\text{inf}}(\overline{R})^{H_R}$ (see [And06, Proposition 7.2]). Let $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$, $\overline{\mu} := \varepsilon - 1 \in O_{F_\infty}^\flat$ and set $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in \mathbf{A}_{\text{inf}}(O_{F_\infty})$. Let $\chi$ denote the $p$-adic cyclotomic character, then for $g \in G_R$, we have $g(1 + \mu) = (1 + \mu)^{\chi(g)}$. Additionally, we have a $G_R$-equivariant surjection $\theta : \mathbf{A}_{\text{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$ and $\operatorname{Ker} \theta = \xi\mathbf{A}_{\text{inf}}(\overline{R})$. The map $\theta$ further induces a $\Gamma_R$-equivariant surjection $\theta : \mathbf{A}_{\text{inf}}(R_\infty) \to \widehat{R}_\infty$.

Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ and for each prime $\mathfrak{p} \in \mathcal{S}$ let $\mathbb{C}_{\mathfrak{p}}$ denote the valuation field described in §2.1 and $\mathbb{C}_{\mathfrak{p}}^+$ its ring of integers. Moreover, from Lemma 2.3, we have that $\mathbb{C}^+(\mathfrak{p})$ is a perfectoid algebra. So we set $\mathbf{A}_{\text{inf}}(\mathbb{C}_{\mathfrak{p}}^+) := W(\mathbb{C}_{\mathfrak{p}}^{+,\flat})$ (resp. $\mathbf{A}_{\text{inf}}(\mathbb{C}^+(\mathfrak{p})) := W(\mathbb{C}^+(\mathfrak{p})^\flat)$) admitting the Frobenius on Witt vectors and continuous $\widehat{G}_R(\mathfrak{p})$-action (resp. $G_R(\mathfrak{p})$-action). Similar to above, we have a $\widehat{G}_R(\mathfrak{p})$-equivariant surjection $\theta : \mathbf{A}_{\text{inf}}(\mathbb{C}_{\mathfrak{p}}^+) \to \mathbb{C}_{\mathfrak{p}}^+$ with $\operatorname{Ker} \theta = \xi\mathbf{A}_{\text{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ (resp. a $G_R(\mathfrak{p})$-equivariant surjection $\theta : \mathbf{A}_{\text{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbb{C}^+(\mathfrak{p})$ with $\operatorname{Ker} \theta = \xi\mathbf{A}_{\text{inf}}(\mathbb{C}^+(\mathfrak{p}))$).

**Lemma 2.4.** *For each $\mathfrak{p} \in \mathcal{S}$ we have $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embeddings $\mathbf{A}_{\text{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\text{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ and $W(\mathbb{C}(\mathfrak{p})^\flat) \to W(\mathbb{C}_{\mathfrak{p}}^\flat)$, where the action of $\widehat{G}_R(\mathfrak{p})$ on left-hand terms factor through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Moreover, we have a $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant identification $\mathbf{A}_{\text{inf}}(\mathbb{C}^+(\mathfrak{p})) = \mathbf{A}_{\text{inf}}(\mathbb{C}_{\mathfrak{p}}^+) \cap W(\mathbb{C}(\mathfrak{p})^\flat)$ as subrings of $W(\mathbb{C}_{\mathfrak{p}}^\flat)$.*

*Proof.* From the discussion before (2.1), we have a $\widehat{G}_R(\mathfrak{p})$-equivariant injective map $\mathbb{C}^+(\mathfrak{p}) \to \mathbb{C}_{\mathfrak{p}}^+$. By applying the tilting functor, we further obtain a $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant commutative diagram of rings

$$\begin{tikzcd} \mathbb{C}^+(\mathfrak{p})^\flat \arrow[r] \arrow[d] & \mathbb{C}_{\mathfrak{p}}^{+,\flat} \arrow[d] \\ \mathbb{C}(\mathfrak{p})^\flat \arrow[r] & \mathbb{C}_{\mathfrak{p}}^\flat, \end{tikzcd} \tag{2.2}$$

where the vertical arrows are injective. Note that the natural map $\mathbb{C}^+(\mathfrak{p})/p = (\overline{R})_{\mathfrak{p}}/p \to O_{\overline{L(\mathfrak{p})}}/p = \mathbb{C}_{\mathfrak{p}}^+/p$ is injective, so by left exactness of $\lim_\varphi$, we obtain that in (2.2) the top horizontal arrow is injective. Moreover, note that $\mathbb{C}(\mathfrak{p})^\flat = \lim_{x \mapsto x^p} \mathbb{C}(\mathfrak{p})$ as a multiplicative monoid, and similarly for $\mathbb{C}_{\mathfrak{p}}^\flat$. Therefore, again by left exactness of lim, it follows that the bottom horizontal arrow in (2.2) is injective. Now, since $\mathbb{C}_{\mathfrak{p}}^\flat$ is a valuation field, let $v_{\mathfrak{p}}^\flat$ denote the normalised valuation on it such that $v_{\mathfrak{p}}^\flat(p^\flat) = 1$. Then we have that $x \in \mathbb{C}_{\mathfrak{p}}^{+,\flat}$ if and only if $v_{\mathfrak{p}}^\flat(x) \geq 0$. Moreover, we have $\mathbb{C}(\mathfrak{p})^\flat = \mathbb{C}^+(\mathfrak{p})^\flat[1/p^\flat]$ and $\mathbb{C}_{\mathfrak{p}}^\flat = \mathbb{C}_{\mathfrak{p}}^{+,\flat}[1/p^\flat]$. From (2.2) and injectivity of its arrows, it now follows that for $x \in \mathbb{C}(\mathfrak{p})^\flat$ we have $x \in \mathbb{C}^+(\mathfrak{p})^\flat$ is and only if $v_{\mathfrak{p}}^\flat(x) \geq 0$. In particular,

$$\mathbb{C}^+(\mathfrak{p})^\flat = \mathbb{C}(\mathfrak{p})^\flat \cap \mathbb{C}_{\mathfrak{p}}^{+,\flat} \subset \mathbb{C}_{\mathfrak{p}}^\flat. \tag{2.3}$$

Furthermore, recall that the $p$-typical Witt vector functor is left exact since it is right adjoint to the forgetful functor from the category of $\delta$-rings to the category of rings (see [Joy85]). Therefore, all maps in the following natural $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant commutative diagram are injective

$$\begin{tikzcd} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \arrow[r] \arrow[d] & \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+) \arrow[d] \\ W(\mathbb{C}(\mathfrak{p})^\flat) \arrow[r] & W(\mathbb{C}_{\mathfrak{p}}^\flat). \end{tikzcd}$$

Hence, from (2.3) it follows that $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) = W(\mathbb{C}(\mathfrak{p})^\flat) \cap \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+) \subset W(\mathbb{C}_{\mathfrak{p}}^\flat)$. ∎

**Remark 2.5.** From Lemma 2.4, the discussion preceding it (see the map $\theta$) and the fact that $\mathbb{C}^+(\mathfrak{p})$ is a subring of $\mathbb{C}_{\mathfrak{p}}^+$, it easily follows that $\xi\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) = \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \cap \xi\mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+) \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$.

**Remark 2.6.** By functoriality of the tilting construction and Witt vector construction, we note that the action of $G_R$ on $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})$ described after (2.1) (see [Bri08, Remarque 3.3.2]), extends to respective natural actions of $G_R$ on $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$.

**Lemma 2.7.** *In the notations described above, we have $(\varphi, G_R)$-equivariant embeddings $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ and $W(\mathbb{C}(\overline{R})^\flat) \to \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$, where right-hand terms are equipped with a $G_R$-action as described in Remark 2.6. Moreover, we have a $(\varphi, G_R)$-equivariant identification $\mathbf{A}_{\mathrm{inf}}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ as subrings of $\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$.*

*Proof.* From (2.1) recall that we have injective maps $\mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} (\overline{R})_{\mathfrak{p}}^\wedge$. By applying the tilting functor, we further obtain a $(\varphi, G_R)$-equivariant commutative diagram:

$$\begin{tikzcd} \mathbb{C}^+(\overline{R})^\flat \arrow[r] \arrow[d] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat \arrow[d] & \\ \mathbb{C}(\overline{R})^\flat \arrow[r] & \Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat\Big)\big[\tfrac{1}{p^\flat}\big] \arrow[r] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}(\mathfrak{p})^\flat, \end{tikzcd} \tag{2.4}$$

where the bottom right horizontal arrow and vertical arrows are injective. From the injectivity of $\overline{R}/p \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})/p$ and left exactness of $\lim_\varphi$, we obtain that in (2.4) the top horizontal arrow is injective and since we have $\mathbb{C}(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat[1/p^\flat]$, it also follows that the bottom left horizontal arrow is injective. Now let $\upsilon_{\mathfrak{p}}^\flat$ denote the valuation on $\mathbb{C}_{\mathfrak{p}}^\flat$ intriduced in the proof of Lemma 2.4. Then under the composition of left vertical and bottom horizontal arrows of (2.4), it follows that for any $x \in \mathbb{C}(\overline{R})^\flat$ we have that $x$ belongs to $\mathbb{C}^+(\overline{R})^\flat$ if and only if $\upsilon_{\mathfrak{p}}^\flat(x) \geq 0$ for each $\mathfrak{p} \in \mathcal{S}$. In particular,

$$\mathbb{C}^+(\overline{R})^\flat = \mathbb{C}(\overline{R})^\flat \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat \subset \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}(\mathfrak{p})^\flat. \tag{2.5}$$

Furthermore, recall that the $p$-typical Witt vector functor is left exact since it is right adjoint to the forgetful functor from the category of $\delta$-rings to the category of rings (see [Joy85]). Therefore, all maps in the following natural $(\varphi, G_R)$-equivariant commutative diagram are injective

$$\begin{tikzcd} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \arrow[r] \arrow[d] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \arrow[d] \\ W(\mathbb{C}(\overline{R})^\flat) \arrow[r] & \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat). \end{tikzcd}$$

Then from (2.5) we obtain $\mathbf{A}_{\mathrm{inf}}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ as subrings of $\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$. ∎

**2.3. de Rham period rings.** In this subsection we will recall the de Rham period rings (see [Abh21, §2.1]). Note that the $\Gamma_R$-equivariant map $\theta : \mathbf{A}_{\mathrm{inf}}(R_\infty) \twoheadrightarrow \widehat{R}_\infty$ described in §2.2 extends to a surjective map $\theta : \mathbf{A}_{\mathrm{inf}}(R_\infty)[1/p] \twoheadrightarrow \widehat{R}_\infty[1/p]$. We set $\mathbf{B}_{\mathrm{dR}}^+(R_\infty) := \lim_n(\mathbf{A}_{\mathrm{inf}}(R_\infty)[1/p])/\xi^n$. Let $t := \log(1+\mu) \in \mathbf{B}_{\mathrm{dR}}^+(R_\infty)$, then $\mathbf{B}_{\mathrm{dR}}^+(R_\infty)$ is $t$-torsion free and we set $\mathbf{B}_{\mathrm{dR}}(R_\infty) := \mathbf{B}_{\mathrm{dR}}^+(R_\infty)[1/t]$. Furthermore, one can define period rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R_\infty)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R_\infty)$. These rings are equipped with a $\Gamma_R$-action, an appropriate extension of the map $\theta$ and a decreasing filtration. Rings with a prefix "$\mathcal{O}$"

are further equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration. One can define variations of these rings over $\overline{R}$ as well.

Next, let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ as in §2.1. Similar to above, for each $\mathfrak{p} \in \mathcal{S}$, we set $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+) := \lim_n(\mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)[1/p])/(\mathrm{Ker}\ \theta)^n$ and $\mathbf{B}_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+) := \mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)[1/t]$ equipped with a $\widehat{G}_R(\mathfrak{p})$-action (resp. $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) := \lim_n(\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))[1/p])/(\mathrm{Ker}\ \theta)^n$ as well as $\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) := \mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))[1/t]$ equipped with a $G_R(\mathfrak{p})$-action), an appropriate extension of the map $\theta$ and a decreasing filtration.

**Lemma 2.8.** *The $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ of Lemma 2.4 extends to a $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$.*

*Proof.* Note that by definition, the $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ induces a $\widehat{G}_R(\mathfrak{p})$-equivariant map $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$. Then from Remark 2.5 and the fact that lim is a left exact functor on the category of abelian groups, we get that the map $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ is injective. The claim now follows since $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$ are $t$-torsion free (see [Bri08, Proposition 5.1.4]). ∎

Moreover, for each $\mathfrak{p} \in \mathcal{S}$ we have big period rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ equipped with an $L$-linear $\widehat{G}_R(\mathfrak{p})$-action (resp. $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ equipped with an $L$-linear $G_R(\mathfrak{p})$-action), an appropriate extension of the map $\theta$, a decreasing filtration and a connection. From [Bri08, §5.2 & §5.3], in particular, from the alternative description of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ (resp. $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$) as a power series ring over $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ (resp. $\mathbf{B}_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$) and using Lemma 2.8, the following is obvious:

**Lemma 2.9.** *The $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ of Lemma 2.8 extends to an $L$-linear $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ compatible with respective filtrations and connections.*

**Remark 2.10.** Recall that product is an exact functor on the category of abelian groups. So the natural embeddings $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, for each $\mathfrak{p} \in \mathcal{S}$, extend to embeddings $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. By an argument similar to [Bri08, Remarque 3.3.2] the products $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ can respectively be equipped with an action of $G_R$, extending the $G_R$-action on $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ (see Remark 2.6), in particular, the embeddings $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ are $G_R$-equivariant.

**Lemma 2.11.** *In the notations described above, we have an $R[1/p]$-linear $G_R$-equivariant embedding $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, where the right-hand term is equipped with a $G_R$-action as described in Remark 2.10. Moreover, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective filtrations and connections.*

*Proof.* Note that from Lemma 2.7 and Remark 2.10, we have $G_R$-equivariant injective maps $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. Then from the definition of $\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$, the preceding maps naturally induce an $R[1/p]$-linear and $G_R$-equivariant commutative diagram:

$$\begin{array}{ccccc} \mathbf{B}_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & \mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})), \end{array} \tag{2.6}$$

where the vertical maps are injective, with the leftmost and rightmost vertical arrows being compatible with respective filtrations and connections for each $\mathfrak{p} \in \mathcal{S}$. We need to show that the top left and bottom left horizontal arrows are injective. But first, let us note that from the explicit description of filtration on $\mathbf{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ in [Bri08, §5.2], it easily follows that compositions of horizontal arrows in (2.6) are compatible with respective filtrations and connections, i.e. for each $k \in \mathbb{Z}$, the respective images of $\mathrm{Fil}^k\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ are contained in $\mathrm{Fil}^k\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ and $\mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, under the composition of horizontal arrows. Similarly, from the explicit description of connection on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ in [Bri08,

§5.3], it easily follows that the composition of bottom horizontal arrows in (2.6) is further compatible with respective connections, for each $\mathfrak{p} \in \mathcal{S}$. Note that the injectivity of the top left horizontal arrow in (2.6) will follow from the injectvity of the the lower horizontal arrow, which we show next (our argument will be similar to [Bri08, Proposition 6.2.6]). Note that the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, for each $\mathfrak{p} \in \mathcal{S}$, is separated. Therefore, it is enough to show that the induced map on grading of the filtration is injective. From [Bri08, Proposition 5.2.7] recall that $\mathrm{gr}^\bullet\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \xrightarrow{\sim} \mathbb{C}^+(\overline{R})[z_1, \ldots, z_d, t^{\pm 1}]$, where $z_i$ denotes the image of $(X_i - [X_i^\flat])/t$ in $\mathrm{gr}^0\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \xrightarrow{\sim} \mathbb{C}^+(\overline{R})[z_1, \ldots, z_d]$. Similarly, we have $\mathrm{gr}^\bullet\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \xrightarrow{\sim} \mathbb{C}^+(\mathfrak{p})[z_1, \ldots, z_d, t^{\pm 1}]$, for each $\mathfrak{p} \in \mathcal{S}$. The claim now follows from injectivity of the natural map $\mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})$ (see (2.1)). This concludes our proof. ∎

**Remark 2.12.** The $G_R$-equivariant embedding $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.11 admits a natural $L$-linear and $G_R$-equivariant extension to an embedding $L\otimes_{R[1/p]}\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. Indeed, this follows from an argument similar to Lemma 2.11 or directly from [Bri08, Proposition 6.2.6]. Furthermore, from Lemma 2.11, it also follows that for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective filtrations and connections, where the left-hand term is equipped with filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and tensor product connection.

**2.4. Crystalline period rings.** In this subsection we will recall crystalline period rings (see [Abh21, §2.2]). We set $\mathbf{A}_{\mathrm{cris}}(R_\infty) := \mathbf{A}_{\mathrm{inf}}(R_\infty)\langle \xi^k/k!, k \in \mathbb{N}\rangle$ and we have $t = \log(1+\mu) \in \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and $\mathbf{A}_{\mathrm{cris}}(R_\infty)$ is $p$-torsion free and $t$-torsion free. So, we set $\mathbf{B}^+_{\mathrm{cris}}(R_\infty) := \mathbf{A}_{\mathrm{cris}}(R_\infty)[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(R_\infty) := \mathbf{B}^+_{\mathrm{cris}}(R_\infty)[1/t]$. Furthermore, one can define period rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$, $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(R_\infty)$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_\infty)$. These rings are equipped with a continuous action of $\Gamma_R$, a Frobenius endomorphism $\varphi$ and a natural extension of the map $\theta$. Rings with a subscript “cris” are equipped with a natural decreasing filtration and rings with a prefix “$\mathcal{O}$” are additionally equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration. Moreover, we have $G_R$-equivariant and filtration compatible natural embeddings $\mathbf{B}_{\mathrm{cris}}(R_\infty) \subset \mathbf{B}_{\mathrm{dR}}(R_\infty)$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_\infty) \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R_\infty)$. One can define variations of these rings over $\overline{R}$ as well. From [MT20, Corollary 4.34] we have a $(\varphi, \Gamma_R)$-equivariant isomorphism $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})^{H_R}$.

As in §2.1, let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. Similar to above, for each $\mathfrak{p} \in \mathcal{S}$, we have rings $\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$, $\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$, $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$ equipped with a $\widehat{G}_R(\mathfrak{p})$-action (resp. $\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$, $\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$, $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ equipped with a $G_R(\mathfrak{p})$-action), an appropriate extension of the map $\theta$, a Frobenius endomorphism $\varphi$, a decreasing filtration and a connection (for rings with prefix “$\mathcal{O}$”). Then, we have the following:

**Lemma 2.13.** *The $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+_{\mathfrak{p}})$ of Lemma 2.4 extends to $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant and filtration compatible embeddings $\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$, where the latter is $L$-linear and also compatible with respective connections.*

*Proof.* By definition, the $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+_{\mathfrak{p}})$ naturall extends to $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant maps $\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$ and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}})$, where the latter is $O_L$-linear and compatible with respective connections. Now consider the following $\widehat{G}_R(\mathfrak{p})$-equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \arrow[r] \arrow[d] & \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \arrow[r] \arrow[d] & \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \arrow[d] \\
\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}}) \arrow[r] & \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\mathbb{C}^+_{\mathfrak{p}}) \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+_{\mathfrak{p}}),
\end{tikzcd}$$

where all horizontal arrows are injective and compatible with respective filtrations and the right vertical arrow is injective and compatible with respective filtrations and connections. Therefore, it follows that the left and middle vertical arrows are injective and compatible with respective filtrations and connections. Finally, the claims for $\mathbf{B}_{\mathrm{cris}}$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ follow by inverting $t$ in the left and middle columns of the diagram. ∎

**Remark 2.14.** From Remark 2.10 it is easy to see that we have injective maps $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, where the first two maps are compatible with respective Frobenii. By an argument similar to [Bri08, Remarque 3.3.2] the products $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ are stable under the $G_R$-action on $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ (see Remark 2.10) and we equip them with the induced action. Then it follows that the injective maps $\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ are $G_R$-equivariant as well.

**Lemma 2.15.** *In the notations described above, we have an $R[1/p]$-linear $(\varphi, G_R)$-equivariant embedding $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$, where the right-hand term is equipped with a $G_R$-action as described in Remark 2.14. Moreover, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective Frobenii, filtrations and connections.*

*Proof.* From Lemma 2.7 and Remark 2.14, we have $(\varphi, G_R)$-equivariant injective maps $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Then from the definition of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$, the preceding maps naturally induce an $R[1/p]$-linear and $(\varphi, G_R)$-equivariant map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. The claim on injectivity of the latter map follows in a manner similar to [Bri08, Proposition 6.2.6]. Indeed, consider the following natural diagram

$$\begin{array}{ccc} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow \\ \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})), \end{array}$$

where the left and right vertical arrows are natural inclusions and the bottom arrow is injective from Lemma 2.11. The diagram commutes since the top and bottom horizontal arrows are defined using the embedding $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.7. In particular, it follows that the top horizontal arrow is injective, proving the first claim. Finally, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is tautologically compatible with respective Frobenii and the claims on filtrations and connections follow from the corresponding claims on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}$ in Lemma 2.11. Hence, the lemma is proved. ∎

**Remark 2.16.** The $(\varphi, G_R)$-equivariant embedding $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.15 admits a natural $L$-linear and $(\varphi, G_R)$-equivariant extension to an embedding $L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Indeed, this follows from an argument similar to Lemma 2.15 or directly from [Bri08, Proposition 6.2.6]. Furthermore, from Lemma 2.15 it also follows that the induced natural map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective Frobenii, filtrations and connections, where the left-hand term is equipped with filtration on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ and tensor product Frobenius and connection.

**2.5. Rings of $(\varphi, \Gamma)$-modules.** Let us fix Teichmüller lifts $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(R_\infty)$, for $1 \leq i \leq d$, and let $A_\square^+$ denote the $(p, \mu)$-adic completion of $O_F[\mu, [X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$. By defininition, there exists a natural embedding $A_\square^+ \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$ and its image is stable under the Witt vector Frobenius endomorphism $\varphi$ and the $\Gamma_R$-action on $\mathbf{A}_{\mathrm{inf}}(R_\infty)$ (see [Abh21, §3]); we equip $A_\square^+$ with induced structures. Furthermore, note that we have an embedding $\iota : R^\square \to A_\square^+$ defined by the map $X_i \mapsto [X_i^\flat]$ and it is easy to see that $\iota$ extends to an isomorphism of rings $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ (enough to check modulo $\mu$ since both source and target are $\mu$-adically complete and $\mu$torsion-free). We extend the Frobenius endomorphism on $R^\square$ to a Frobenius endomorphism $\varphi$ on $R^\square[\![\mu]\!]$ by setting $\varphi(\mu) = (1 + \mu)^p - 1$. Then the Frobenius on $R^\square[\![\mu]\!]$ is finite and faithfully flat of degree $p^{d+1}$ . Moreover, by the preceding discussion, it also follows that the embedding $\iota$ and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ are Frobenius-equivariant.

Let $\mathbf{A}_R^+$ denote the $(p, \mu)$-adic completion of the unique extension of the embedding $A_\square^+ \to \mathbf{A}_{\mathrm{inf}}(R_\infty)$ along the $p$-adically completed étale map $R^\square \to R$ (see [Abh21, §3.3.2] and [CN17, Proposition 2.1]). Then there exists a natural embedding $\mathbf{A}_R^+ \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$ and its image is stable under the Witt vector Frobenius and $\Gamma_R$-action on $\mathbf{A}_{\mathrm{inf}}(R_\infty)$; we equip $\mathbf{A}_R^+$ with induced structures. Furthermore, the embedding $\iota : R^\square \to A_\square^+ \subset \mathbf{A}_R^+$ and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+ \subset \mathbf{A}_R^+$ naturally extend to a unique embedding $\iota : R \to \mathbf{A}_R^+$

and an isomorphism of rings $R[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_R^+$. We extend the Frobenius endomorphism on $R$ to a Frobenius endomorphism $\varphi$ on $R[\![\mu]\!]$ by setting $\varphi(\mu) = (1+\mu)^p-1$. Then the Froebnius on $R[\![\mu]\!]$ is finite and faithfully flat of degree $p^{d+1}$ . Moreover, by the preceding disucssion, it is easy to see that the embedding $\iota$ and the isomorphism $R[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_R^+$ are Frobenius-equivariant. In particular, the induced Frobenius endomorphism $\varphi$ on $\mathbf{A}_R^+$ is finite and faithfully flat of degree $p^{d+1}$ and we have $\varphi^*(\mathbf{A}_R^+) := \mathbf{A}_R^+ \otimes_{\varphi,\mathbf{A}_R^+} \mathbf{A}_R^+ \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_R^+)u_\alpha$, where $u_\alpha := (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$.

Set $\mathbf{A}_R := \mathbf{A}_R^+[1/\mu]^\wedge$ as the $p$-adic completion and note that the Frobenius endomorphism $\varphi$ and the continuous action of $\Gamma_R$ on $\mathbf{A}_R^+$ naturally extend to $\mathbf{A}_R$. Similar to above, the induced Frobenius endomorphism $\varphi$ on $\mathbf{A}_R$ is finite and faithfully flat of degree $p^{d+1}$ and $\varphi^*(\mathbf{A}_R) := \mathbf{A}_R \otimes_{\varphi,\mathbf{A}_R} \mathbf{A}_R \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_R)u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}_R^+)u_\alpha) \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{A}_R) \xleftarrow{\sim} \mathbf{A}_R^+ \otimes_{\varphi,\mathbf{A}_R^+} \mathbf{A}_R$.

Recall that $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})[1/p]$ and we set $\tilde{\mathbf{A}} := W(\mathbb{C}(\overline{R})^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$, equipped with the Frobenius on Witt vectors and a continuous (for the weak topology) action of $G_R$. Moreover, the natural Frobenius and $\Gamma_R$-equivariant embedding $\mathbf{A}_R^+ \subset \mathbf{A}_{\inf}(R_\infty)$ extends to a Frobenius and $\Gamma_R$-equivariant embedding $\mathbf{A}_R \subset \tilde{\mathbf{A}}^{H_R}$ and we set $\mathbf{B}_R := \mathbf{A}_R[1/p]$ equipped with induced Frobenius and $\Gamma_R$-action. Take $\mathbf{A}$ to be the $p$-adic completion of the maximal unramified extension of $\mathbf{A}_R$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under the action of $G_R$ and Frobenius endomorphism on $\tilde{\mathbf{B}}$ and we equip $\mathbf{A}$ and $\mathbf{B}$ with induced structures. Moreover, we have $\mathbf{A}_R = \mathbf{A}^{H_R}$ and $\mathbf{B}_R = \mathbf{B}^{H_R}$. Next, let us set $\mathbf{A}^+ := \mathbf{A}_{\inf}(\overline{R}) \cap \mathbf{A} \subset \tilde{\mathbf{A}}$ and $\mathbf{B}^+ := \mathbf{A}^+[1/p] \subset \mathbf{B}$ and note that these rings are stable under the Frobenius and $G_R$-action on $\mathbf{B}$. Furthemore, we have $\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R}$ and $\mathbf{B}_R^+ = (\mathbf{B}^+)^{H_R}$.

Also note that by identifying the groups $\Gamma_L \xrightarrow{\sim} \Gamma_R$, we have a $(\varphi, \Gamma_L)$-equivariant isomorphism $\mathbf{A}_L^+ \xrightarrow{\sim} ((\mathbf{A}_R^+)_{(p,\mu)})^\wedge$, where $^\wedge$ denotes the $(p, \mu)$-adic completion. The preceding isomorphism extends to an isomorphism $\mathbf{A}_L \xrightarrow{\sim} ((\mathbf{A}_R)_{(p)})^\wedge$, where $^\wedge$ denotes the $p$-adic completion. It is easy to see that we have $\mathbf{A}_R^+ = \mathbf{A}_L^+ \cap \mathbf{A}_R$ as subrings of $\mathbf{A}_L$ and $\mathbf{B}_R^+ := \mathbf{A}_R^+[1/p] = \mathbf{B}_L^+ \cap \mathbf{B}_R$ as subrings of $\mathbf{B}_L$.

**2.6. $p$-adic representations.** Let $T$ be a finite free $\mathbb{Z}_p$-representation of $G_R$. By the theory of étale $(\varphi, \Gamma)$-modules (see [Fon90] and [And06]), one can functorially associate to $T$ a finite projective étale $(\varphi, \Gamma_R)$-module $\mathbf{D}_R(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ over $\mathbf{A}_R$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. Moreover, $\tilde{\mathbf{D}}_R(T) := (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_R} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_R} \otimes_{\mathbf{A}_R} \mathbf{D}_R(T)$ and we have a natural $(\varphi, \Gamma_R)$-equivariant isomorphism

$$\mathbf{A} \otimes_{\mathbf{A}_R} \mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T. \tag{2.7}$$

These constructions are functorial in $\mathbb{Z}_p$-representations and induce an exact equivalence of $\otimes$-categories (see [And06, Theorem 7.11])

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R}^{\text{ét}}, \tag{2.8}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathbf{T}_R(D) := (\mathbf{A} \otimes_{\mathbf{A}_R} D)^{\varphi=1} = (\tilde{\mathbf{A}} \otimes_{\mathbf{A}_R} D)^{\varphi=1}$. Similar statements are also true for $p$-adic representations of $G_R$. Furthermore, let $\mathbf{D}_R^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R}$ be the $(\varphi, \Gamma_R)$-module over $\mathbf{A}_R^+$ associated to $T$ and for $V := T[1/p]$ let $\mathbf{D}_R^+(V) := \mathbf{D}_R^+(T)[1/p]$ be the $(\varphi, \Gamma_R)$-module over $\mathbf{B}_R^+$ associated to $V$.

Let $V$ be a $p$-adic representation of $G_R$. From $p$-adic Hodge theory of $G_R$ (see [Bri08]), one can attach to $V$ a filtered $(\varphi, \partial)$-module over $R[1/p]$ of rank $\leq \dim_{\mathbb{Q}_p} V$ given by the functor

$$\begin{aligned} \mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}(G_R) &\longrightarrow \mathrm{MF}_R(\varphi, \partial) \\ V &\longmapsto (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}. \end{aligned}$$

The representation $V$ is said to be crystalline if the natural map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \rightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ is an isomorphism, in particular, if $V$ is crystalline then $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. Restricting $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ to the category of crystalline representations of $G_R$ and writing $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ for the essential image of restricted functor, we have an exact equivalence of $\otimes$-categories (see [Bri08, Théorème 8.5.1])

$$\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial), \tag{2.9}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}(D) := (\mathrm{Fil}^0(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} D))^{\partial=0,\varphi=1}$. Furthermore, we have a continuous homomorphism $G_L \rightarrow G_R$, i.e. $V$ is also a $p$-adic representation of $G_L$.

Base changing the isomorphism $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ along $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, we obtain a $G_L$-equivariant isomorphism $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$, i.e. $V$ is a crystalline representation of $G_L$. Taking $G_L$-invariants in the preceding isomorphism we further obtain a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ compatible with respective Frobenii, filtrations and connections.

# 3. Relative Wach modules

In this section we will describe relative Wach modules and finite $[p]_q$-height representations of $G_R$ and relate them to crystalline representations. We start by noting some technical lemmas.

## 3.1. Some technical results.

In $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$, let us fix $q := [\varepsilon]$, $\mu := [\varepsilon] - 1 = q - 1$ and $[p]_q := \varphi(\mu)/\mu$.

**Definition 3.1.** Let $N$ be a finitely generated $\mathbf{A}_R^+$-module. The sequence $\{p, \mu\}$ in $\mathbf{A}_R^+$ is said to be *$N$-regular* if $N$ is $p$-torsion free and $N/pN$ is $\mu$-torsion free. Similarly, $\{\mu, p\}$ is $N$-regular if $N$ is $\mu$-torsion free and $N/\mu N$ is $p$-torsion free. The sequence $\{p, \mu\}$ in $\mathbf{A}_R^+$ is said to be *strictly $N$-regular* if both $\{p, \mu\}$ and $\{\mu, p\}$ are $N$-regular.

**Remark 3.2.** In Definition 3.1 note that the sequence $\{p, \mu\}$ is strictly $N$-regular if and only if $N$ is $a$-torsion free for every nonzero element $a$ in the ideal $(p, \mu) \subset \mathbf{A}_R^+$ and $N/\mu N$ is $p$-torsion free. Indeed, the "only if" direction is obvious and for the converse one needs to check that $N/pN$ is $\mu$-torsion free. So let $x \in N$ such that $\mu x = py$ for some $y \in N$; we claim that $x \in pN$. Reducing the preceding equality modulo $p$ and using $(N/\mu N)[p] = 0$, we get that $y = \mu z$ for some $z \in \mathbb{N}$. From $\mu$-torsion freeness of $N$, it follows that $x = pz$, as claimed.

**Lemma 3.3.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module and consider the complex,*

$$\mathscr{C}^\bullet : N \xrightarrow{(p,\mu)} N \oplus N \xrightarrow{(\mu,-p)} N,$$

*where the first map is given by $x \mapsto (px, \mu x)$ and the second map is given by $(x, y) \mapsto \mu x - py$. Then the sequence $\{p, \mu\}$ is strictly $N$-regular if and only if $H^1(\mathscr{C}^\bullet) = 0$. Moreover, under these equivalent conditions $H^0(\mathscr{C}^\bullet) = 0$.*

*Proof.* If $\{p, \mu\}$ is strictly $N$-regular then $(N/p)[\mu] = (N/\mu)[p] = 0$. Therefore, we must have $H^0(\mathscr{C}^\bullet) = H^1(\mathscr{C}^\bullet) = 0$. For the converse, consider the following diagram

$$\begin{tikzcd}
 & N[p,\mu] \arrow[r] \arrow[d] & N[p] \arrow[r, "\mu"] \arrow[d] & N[p] \arrow[r] \arrow[d] & (N/\mu)[p] \arrow[d] & \\
0 \arrow[r] & N[\mu] \arrow[r] \arrow[d, "p"] & N \arrow[r, "\mu"] \arrow[d, "p"] & N \arrow[r] \arrow[d, "p"] & N/\mu \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & N[\mu] \arrow[r] \arrow[d] & N \arrow[r, "\mu"] \arrow[d] & N \arrow[r] \arrow[d] & N/\mu \arrow[r] \arrow[d] & 0 \\
 & (N/p)[\mu] \arrow[r] & N/p \arrow[r, "\mu"] & N/p \arrow[r] & N/(p,\mu). &
\end{tikzcd} \tag{3.1}$$

Since $H^1(\mathscr{C}^\bullet) = 0$, we get that the top right and bottom left corners of (3.1) are zero, i.e. $(N/\mu)[p] = (N/p)[\mu] = 0$. Now let $x \in N[\mu]$, then from the surjectivity of the leftmost vertical arrow from second to third row it follows that there exists $x_1 \in N[\mu]$ such that $x = px_1$. Proceeding by induction it is easy to see that $x \in p^n N[\mu] \subset p^n N$ for all $n \in \mathbb{N}$. But since $N$ is finitely generated over $\mathbf{A}_R^+$, which is $(p, \mu)$-adically complete, it follows that $N$ is $p$-adically separated, i.e. $x = 0$, in particular, $N[\mu] = 0$. A similar argument shows that $N[p] = 0$, in particular, $N[p, \mu] = 0$. This proves both claims in the lemma. ∎

**Remark 3.4.** The complex $\mathscr{C}^\bullet$ in Lemma 3.3 computes local cohomology of $N$ with respect to the ideal $(p, \mu) \subset \mathbf{A}_R^+$ (see [Wei94, Theorem 4.6.8]). So, if we set $Z := V(p, \mu) \subset \mathrm{Spec}\,(\mathbf{A}_R^+) =: X$ as a closed subset, then one also says that $\mathscr{C}^\bullet$ computes $H_Z^i(X, N)$, i.e. cohomology of $X$ with compact support along $Z$ (see [Wei94, Generalization 4.6.2]).

**Lemma 3.5.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module such that $\{p, \mu\}$ is strictly $N$-regular. Then we have $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ as $\mathbf{A}_R^+$-modules. Moreover, $N = N[1/p] \cap N[1/\mu]^\wedge \subset N[1/\mu]^\wedge[1/p]$, where $^\wedge$ denotes the $p$-adic completion.*

*Proof.* Note that from definitions we have $(N/p)[\mu] = (N/\mu)[p] = 0$ and $(N[1/\mu])/p = (N/p)[1/\mu]$. So it follows that $N/p^nN \subset (N/p^n)[1/\mu]$, for all $n \in \mathbb{N}$, and therefore, $N \cap p^nN[1/\mu] = p^nN$. Hence, $N[1/p] \cap N[1/\mu] = N$. Furthermore, since $(N[1/\mu]^\wedge)/p^n = (N[1/\mu])/p^n = (N/p^n)[1/\mu]$, therefore, similar to above we also get that $N \cap p^nN[1/\mu]^\wedge = p^nN$, for all $n \in N$. Hence, $N[1/p] \cap N[1/\mu]^\wedge = N$. ∎

**Lemma 3.6.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module. Then the sequence $\{p, \mu\}$ is strictly $N$-regular if and only if the sequence $\{p, [p]_q\}$ is strictly $N$-regular.*

*Proof.* Let us first assume that the sequence $\{p, \mu\}$ is strictly $N$-regular. Note that we have $[p]_q = \mu^{p-1} \bmod p\mathbf{A}_R^+$, therefore, it follows that $N/p$ is $[p]_q$-torsion free, in particular, the sequence $\{p, [p]_q\}$ is regular on $N$. Moreover, as $[p]_q$ is an element of the ideal $(p, \mu) \subset \mathbf{A}_R^+$, from Remark 3.2 we have that $N$ is $[p]_q$-torsion free. Now considering a diagram similar to (3.1) with $\mu$ replaced by $[p]_q$ and using that $N$ is $p$-torsion free and $N/p$ is $[p]_q$-torsion free, it follows that $N/[p]_q$ is $p$-torsion free, i.e. the sequence $\{p, [p]_q\}$ is strictly $N$-regular. Conversely, assume that the sequence $\{p, [p]_q\}$ is strictly $N$-regular. Then, again as we have $[p]_q = \mu^{p-1} \bmod p\mathbf{A}_R^+$, so from [Sta23, Tag 07DV], it follows that $\{p, \mu\}$ is a regular sequence on $N$. Next, let us note that $\mu^{p-1}$ is an element of the ideal $(p, [p]_q) \subset \mathbf{A}_R^+$, so it follows that $N$ is $\mu^{p-1}$-torsion free, therefore, $\mu$-torsion free. Now considering the diagram (3.1) and using that $N$ is $p$-torsion free and $N/p$ is $\mu$-torsion free, it follows that $N/\mu$ is $p$-torsion free, i.e. the sequence $\{p, \mu\}$ is strictly $N$-regular. Hence, the lemma is proved. ∎

Finally, let us note an important observation for the action of $\Gamma_R$ on $\mathbf{A}_R^+$-modules. Note that the action of $\Gamma_R$ is continuous on $\mathbf{A}_R^+$ for the $(p, \mu)$-adic topology and the induced action of $\Gamma_R$ on $\mathbf{A}_R^+/\mu \xrightarrow{\sim} R$ is trivial. More generally, we claim the following:

**Lemma 3.7.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module equipped with a semilinear action of $\Gamma_R$ such that the induced action of $\Gamma_R$ on $N/\mu N$ is trivial. Then the action of $\Gamma_R$ on $N$ is continuous for the $(p, \mu)$-adic topology.*

*Proof.* Recall that from §2 we have $\Gamma_R \xrightarrow{\sim} \Gamma'_R \rtimes \Gamma_F \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$. Moreover, we fixed $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of $\Gamma'_R$ and $\gamma_0$ in $\Gamma_R$ to be a lift of a topological generator of $\Gamma_R/\Gamma'_R$. Additionally, we may assume that $\chi(\gamma_0) = 1 + pa$, for $p \geq 3$, and $\chi(\gamma_0) = 1 + 4a$, for $p = 2$, where $\chi$ is the $p$-adic cyclotomic character and $a$ is a unit in $\mathbb{Z}_p$. To show that the action of $\Gamma_R$ is continuous on $N$, for the $(p, \mu)$-adic topology, we need to show that for any $x$ in $N$, any $n \geq 1$ and for each $\gamma_i$, there exists an $m \in \mathbb{N}$ such that $\gamma_i^{p^m}(x) = x \bmod (p, \mu)^n$. As the action of $\Gamma_R$ is trivial on $N/\mu N$, let us note that for each $0 \leq i \leq d$, the operators $\nabla_{q,i} := \frac{\gamma_i - 1}{\mu} : N \to N$ are well-defined (see §5.2 for more on such operators). Moreover, note that for any $a$ in $\mathbf{A}_R^+$, $x$ in $N$ and $0 \leq i \leq d$, we have $(\gamma_i - 1)(a \otimes x) = (\gamma_i - 1)a \otimes x + \gamma_i(a) \otimes (\gamma_i - 1)(x)$, and therefore, $\nabla_{q,i}(a \otimes x) = \nabla_{q,i}(a) \otimes x + \gamma_i(a) \otimes \nabla_{q,i}(x)$. Now for $1 \leq i \leq d$, note that $\nabla_{q,i}(\mu) = \mu$, so by setting $m = n$, we get that

$$\gamma_i^{p^n}(x) = (1 + \mu\nabla_{q,i}(x))^{p^n} = x + \textstyle\sum_{k=1}^{p^n} \binom{p^n}{k} \mu^k \nabla_{q,i}^k(x),$$

where the summation in the third term is easily seen to be an element of $(p, \mu)^n N$. Next, let $i = 0$ and using the action of $\gamma_0$ on $\mu$, it is easy to see that $\nabla_{q,0}(\mu) = (1 + \mu)((1 + \mu)^{pa} - 1)/\mu$ is an element of $(p, \mu)\mathbf{A}_F^+$. Then an easy induction on $k \geq 1$ shows that for any $x$ in $N$, we must have that $(\mu\nabla_{q,0})^k(x)$ is an element of $(p, \mu)^k N$. In particular, by setting $m = n$, it follows that we have

$$\gamma_0^{p^n}(x) = (1 + \mu\nabla_{q,0}(x))^{p^n} = x + \textstyle\sum_{k=1}^{p^n} \binom{p^n}{k} (\mu\nabla_{q,0})^k(x),$$

where the summation in the third term is again an element of $(p, \mu)^n N$. Hence, we conclude that the action of $\Gamma_R$ is continuous on $N$. ∎

### 3.2. Wach modules over $\mathbf{A}_R^+$.

We start with the definition of Wach modules.

**Definition 3.8.** A *Wach module* over $\mathbf{A}_R^+$ with weights in the interval $[a, b]$, for some $a, b \in \mathbb{Z}$ with $b \geq a$, is a finitely generated $\mathbf{A}_R^+$-module $N$ satisfying the following assumptions:

(1) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on $N$.

(2) $N$ is equipped with a semilinear action of $\Gamma_R$ such that the induced action of $\Gamma_R$ on $N/\mu N$ is trivial.

(3) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ compatible with respective actions of $\Gamma_R$ such that $\varphi(\mu^b N) \subset \mu^b N$, and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

We define the $[p]_q$*-height* of $N$ to be the largest value of $-a$ for $a \in \mathbb{Z}$ as above. The module $N$ is said to be *effective* if we can take $b = 0$ and $a \leq 0$. A Wach module over $\mathbf{B}_R^+$ is a finitely generated module $M$ equipped with a semilinear action of $\Gamma_R$ and a Frobenius-semilinear operator $\varphi : M[1/\mu] \to M[1/\varphi(\mu)]$ compatible with respective actions of $\Gamma_R$ and such that there exists a $\Gamma_R$-stable and $\varphi$-stable (after inverting $\mu$) $\mathbf{A}_R^+$-submodule $N \subset M$ and equipped with induced $(\varphi, \Gamma_R)$-action $N$ is a Wach module over $\mathbf{A}_R^+$ and $N[1/p] = M$. Denote by $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q}$, the category of Wach modules over $\mathbf{A}_R^+$ with morphisms between objects being $\mathbf{A}_R^+$-linear $\varphi$-equivariant (after inverting $\mu$) and $\Gamma_R$-equivariant morphisms.

**Remark 3.9.** In Definition 3.8, note that from the triviality of the action of $\Gamma_R$ on $N/\mu N$ and Lemma 3.7, it follows that the action of $\Gamma_R$ on $N$ is continuous.

Next, we note some structural properties of Wach modules.

**Lemma 3.10.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module. Then (3) of Definition 3.8 is equivalent to giving an $\mathbf{A}_R^+$-linear and $\Gamma_R$-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] = (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$.*

*Proof.* Suppose $N$ satisfes condition (3) of Definition 3.8. Then, the map $1 \otimes \varphi : \varphi^*(\mu^b N) \to \mu^b N$ induces an isomorphism $1 \otimes \varphi : (\mu^b \varphi^* N)[1/[p]_q] \xrightarrow{\sim} (\mu^b N)[1/[p]_q]$. Hence, we obtain an isomorphism

$$\varphi_N : (\varphi^* N)[1/[p]_q] \underset{\sim}{\xrightarrow{\mu^b}} (\mu^b \varphi^* N)[1/[p]_q] \underset{\sim}{\xrightarrow{1 \otimes \varphi}} (\mu^b N)[1/[p]_q] \underset{\sim}{\xleftarrow{\mu^b}} N[1/[p]_q].$$

Since, $1 \otimes \varphi$ commutes with the action of $\Gamma_R$, we deduce that $\varphi_N$ is $\Gamma_R$-equivariant.

Conversely, suppose that we have an $\mathbf{A}_R^+$-linear $\Gamma_R$-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then note that for some $a, b \in \mathbb{Z}$ with $b \geq a$ we can write $[p]_q^b \varphi_N(\varphi^* N) \subset N \subset [p]_q^a \varphi_N(\varphi^* N)$. So we get an $\mathbf{A}_R^+$-semilinear and $\Gamma_R$-equivariant map as the composition $\varphi : \mu^b N \xrightarrow{\text{can}} \varphi^*(\mu^b N) \xrightarrow{\varphi_N} \mu^b N$. This extends to an $\mathbf{A}_R^+$-semilinear $\Gamma_R$-equivariant map $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ and we have

$$\varphi_N(\varphi^*(\mu^b N)) = \mu^b [p]_q^b \varphi_N(\varphi^* N) \subset \mu^b N \subset [p]_q^{a-b} \varphi_N(\varphi^*(\mu^b N))$$

Then it easily follows that $1 \otimes \varphi = \varphi_N : \varphi^*(\mu^b N) \to \mu^b N$ is injective, its cokernel is killed by $[p]_q^{b-a}$ and it commutes with the action of $\Gamma_R$. Hence, $N$ satisfies condition (3) of Definition 3.8. ∎

**Proposition 3.11.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$. Then $N[1/p]$ is finite projective over $\mathbf{A}_R^+[1/p]$ and $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$.*

*Proof.* For $r \in \mathbb{N}$ large enough, note that the Wach module $\mu^r N(-r)$ is always effective. So without loss of generality, we may assume that $N$ is effective. Then the first claim follows from Lemma 3.10 and Proposition A.1. For the second claim, note that $N$ is $p$-torsion free, so $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$ is a $p$-torsion free étale $(\varphi, \Gamma_R)$-module over $\mathbf{A}_R$, and therefore, finite projective by [And06, Lemma 7.10]. Since $\mathbf{A}_R^+[1/\mu]$ is noetherian, we have $N[1/\mu]^\wedge \xrightarrow{\sim} \mathbf{A}_R \otimes_{\mathbf{A}_R^+[1/\mu]} N[1/\mu] = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$, where $^\wedge$ denotes the $p$-adic completion. Moreover, the natural map $\operatorname{Spec}(\mathbf{A}_R^+[1/\mu]^\wedge) \cup \operatorname{Spec}(\mathbf{A}_R^+[1/\mu, 1/p]) \to \operatorname{Spec}(\mathbf{A}_R^+[1/\mu])$ is a flat cover. Therefore, by faithfully flat descent it follows that $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$. ∎

**Remark 3.12.** Note that the map $\mathrm{Spec}\,(\mathbf{A}_R^+[1/[p]_q]_p^\wedge) \cup \mathrm{Spec}\,(\mathbf{A}_R^+[1/[p]_q, 1/p]) \to \mathrm{Spec}\,(\mathbf{A}_R^+[1/[p]_q])$ is a flat cover and $\mathbf{A}_R^+[1/\mu]_p^\wedge = \mathbf{A}_R^+[1/[p]_q]_p^\wedge$. Now for a Wach module $N$ over $\mathbf{A}_R^+$, we have that the $\mathbf{A}_R^+[1/p]$-module $N[1/p]$ is finite projective and the $\mathbf{A}_R^+[1/\mu]$-module $N[1/\mu]$ is finite projective (see Proposition 3.11). Therefore, by faithfully flat descent, we get that the $\mathbf{A}_R^+[1/[p]_q]$-module $N[1/[p]_q]$ is finite projective. Moreover, from Lemma 3.6 we also have that the sequence $\{p, [p]_q\}$ is strictly $N$-regular and equivalent to condition (1) in Definition 3.8.

**Remark 3.13.** Note that for a Wach module $N$ over $\mathbf{A}_R^+$, we have that $N$ is $p$-torsion free, in particular, $N$ is contained in $N[1/p]$. As $N[1/p]$ is finite projective over $\mathbf{A}_R^+[1/p]$ by Proposition 3.11, therefore, we obtain that $N$ is a torsion free $\mathbf{A}_R^+$-module.

**Lemma 3.14.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$, then we have $N = (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N) \cap (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N) \subset \mathbf{A}_L \otimes_{\mathbf{A}_R^+} N$ as $\mathbf{A}_R^+$-modules.*

*Proof.* Let $N_R := N$, $N_L := \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$ and $D_R := \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$. Note that $N_R[1/p]$ is finite projective over $\mathbf{B}_R^+$, with $N_L[1/p] = \mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} N_R[1/p]$ and $D_R[1/p] = \mathbf{B}_R \otimes_{\mathbf{B}_R^+} N_R[1/p]$, therefore $N_L[1/p] \cap D_R[1/p] = (\mathbf{B}_L^+ \cap \mathbf{B}_R) \otimes_{\mathbf{B}_R^+} N_R[1/p] = N_R[1/p]$. Moreover, we have $N_L \cap D_R \subset N_L[1/p] \cap D_R[1/p] = N_R[1/p]$, and using Lemma 3.5 we see that $N_L \cap D_R = N_L \cap D_R \cap N_R[1/p] = N_R$. ∎

From the proof of Proposition 3.11, it is clear that extending scalars along $\mathbf{A}_R^+ \to \mathbf{A}_R$ induces a functor $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q} \to (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R}^{\text{ét}}$, and we make the following claim:

**Proposition 3.15.** *The following natural functor is fully faithful*

$$\begin{aligned}(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R}^{\text{ét}}\\ N &\longmapsto \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N.\end{aligned}$$

*Proof.* Let $N, N'$ be two Wach modules over $\mathbf{A}_R^+$. Write $N_R := N$, $N_L := \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$, $D_L := \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$ and similarly for $N'$. We need to show that for Wach modules $N_R$ and $N'_R$, we have

$$\mathrm{Hom}_{(\varphi,\Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q}}(N_R, N'_R) \xrightarrow{\sim} \mathrm{Hom}_{(\varphi,\Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R}^{\text{ét}}}(D_R, D'_R) \tag{3.2}$$

Note that $\mathbf{A}_R^+ \to \mathbf{A}_R = \mathbf{A}_R^+[1/\mu]^\wedge$ is injective, in particular, the map in (3.2) is injective. To check that (3.2) is surjective, take an $\mathbf{A}_R$-linear and $(\varphi, \Gamma_R)$-equivariant map $f : D_R \to D'_R$. We need to show that $f(N_R) \subset N'_R$. Base changing $f$ along $\mathbf{A}_R \to \mathbf{A}_L$ and using the isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$ induces an $\mathbf{A}_L$-linear and $(\varphi, \Gamma_L)$-equivariant map $f : D_L \to D'_L$. Then from [Abh23a, Proposition 3.3] we have $f(N_L) \subset N'_L$. Finally, using Lemma 3.14, we get that inside $D'_L$ we have $f(N_R) = f(N_L \cap D_R) = f(N_L) \cap f(D_R) \subset N'_L \cap D'_R = N'_R$, concluding the proof. ∎

Analogous to above, one can define categories $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R^+}^{[p]_q}$ and $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R}^{\text{ét}}$ and a functor from the former to latter by extending scalars along $\mathbf{B}_R^+ \to \mathbf{B}_R$. Then passing to the associated isogeny categories and using Proposition 3.15, we get the following:

**Corollary 3.16.** *The natural functor $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R^+}^{[p]_q} \to (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R}^{\text{ét}}$ is fully faithful.*

### 3.3. $G_R$-representations attached to Wach modules.

Composing the functor in Proposition 3.15 with the equivalence in (2.8), we obtain a fully faithful functor,

$$\begin{aligned}\mathbf{T}_R : (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q} &\longrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(G_R)\\ N &\longmapsto (\mathbf{A} \otimes_{\mathbf{A}_R^+} N)^{\varphi=1} \xrightarrow{\sim} (W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+} N)^{\varphi=1}.\end{aligned} \tag{3.3}$$

**Proposition 3.17.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$ and $T := \mathbf{T}_R(N)$, the associated finite free $\mathbb{Z}_p$-representation of $G_R$. Then we have a natural $G_R$-equivariant comparison isomorphism*

$$\mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbb{Z}_p} T. \tag{3.4}$$

*Additionally, (3.4) is compatible with Frobenius after base change along $\mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \to W(\mathbb{C}(\overline{R})^\flat)$.*

*Proof.* Note that for $T = \mathbf{T}_R(N)$, from the equivalence in (2.8), we have $\mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$ as étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$. Then extending scalars of the isomorphism in (2.7) along $\mathbf{A} \to W(\mathbb{C}(\overline{R})^\flat)$ gives $(\varphi, G_R)$-equivariant isomorphism,

$$W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.5}$$

Nowa, for $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_R(\mu^r N(-r)) = T(-r)$ (the twist $(-r)$ denotes the Tate twist on which $\Gamma_R$ acts via the cyclotomic character). Therefore, we see that it is enough to show the claim for effective Wach modules (see Definition 3.8), in particular, in the rest of the proof we will assume that $N$ is effective.

Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. From §2.1, recall that for each $\mathfrak{p} \in \mathcal{S}$, we have $\overline{L}(\mathfrak{p}) \subset \mathbb{C}_\mathfrak{p}$, an algebraic closure of $L$ containing $(\overline{R})_\mathfrak{p}$, and we have $\widehat{G}_R(\mathfrak{p}) = \mathrm{Gal}(\overline{L}(\mathfrak{p})/L)$. Moreover, we have an isomorphism of groups $\Gamma_L \xrightarrow{\sim} \Gamma_R$ and for each prime $\mathfrak{p} \in \mathcal{S}$, let $\mathbf{A}_L^+(\mathfrak{p})$ denote the base ring for Wach modules in the imperfect residue field case (see [Abh23a, §2.1.2]). To avoid confusion, let us write $N_R := N$ and $N_L(\mathfrak{p}) := \mathbf{A}_L^+(\mathfrak{p}) \otimes_{\mathbf{A}_R^+} N$, in particular, $N_L(\mathfrak{p})$ is a Wach module over $\mathbf{A}_L^+(\mathfrak{p})$ finite free of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. From [Abh23a, Lemma 3.6] note that we have $\widehat{G}_R(\mathfrak{p})$-equivariant inclusions for each $\mathfrak{p} \in \mathcal{S}$,

$$\mu^s \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Z}_p} T. \tag{3.6}$$

Now, note that the $(\varphi, G_R(\mathfrak{p}))$-equivariant composition $\mathbf{A}_R^+ \to W(\mathbb{C}(\overline{R})^\flat) \to W(\mathbb{C}(\mathfrak{p})^\flat)$ naturally factors as the $(\varphi, G_R(\mathfrak{p}))$-equivariant maps $\mathbf{A}_R^+ \to \mathbf{A}_L^+(\mathfrak{p}) \to W(\mathbb{C}(\mathfrak{p})^\flat)$. So, by base changing the $(\varphi, G_R)$-equivariant isomorphism in (3.5) along the $(\varphi, G_R(\mathfrak{p}))$-equivariant map $W(\mathbb{C}(\overline{R})^\flat) \to W(\mathbb{C}(\mathfrak{p})^\flat)$, we obtain a natural $(\varphi, G_R(\mathfrak{p}))$-equivariant isomorphism,

$$W(\mathbb{C}(\mathfrak{p})^\flat) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}(\mathfrak{p})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.7}$$

All terms in (3.6) and (3.7) admit $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding into $W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbb{Z}_p} T$, where the action of $\widehat{G}_R(\mathfrak{p})$ on (3.7) factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Therefore, taking the intersection of (3.6) with (3.7) inside $W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbb{Z}_p} T$ and using Lemma 2.4, for each $\mathfrak{p} \in \mathcal{S}$, we obtain the following $(\varphi, G_R(\mathfrak{p}))$-equivariant inclusions:

$$\mu^s \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T, \tag{3.8}$$

where the middle term can be written as $\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p}) = \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R$.

Now, from Remark 2.6, recall that $\prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ is equipped with an action of $G_R$ and from Lemma 2.7 we have a $(\varphi, G_R)$-equivariant embedding $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$. Then, we can equip $\prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T) = (\prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{\mathbb{Z}_p} T$ with the diagonal action of $(\varphi, G_R)$ and similarly for $\prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_L^+(\mathfrak{p})} N_L(\mathfrak{p})) = \prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R) = (\prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{\mathbf{A}_R^+} N_R$, where the second equality follows from the fact that product is an exact functor on the category of $\mathbf{A}_R^+$-modules and $N_R$ is finitely presented over the noetherian ring $\mathbf{A}_R^+$ (see [Sta23, Tag 059K]). So, taking the product of (3.8) over all $\mathfrak{p} \in \mathcal{S}$, we obtain the following $(\varphi, G_R)$-equivariant inclusions:

$$\mu^s \prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T) \subset \prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R) \subset \prod_{\mathfrak{p} \in \mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T). \tag{3.9}$$

Inverting $\mu$ in (3.9) and from the discussion above we get a $G_R$-equivariant isomorphism

$$\Big(\prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] \xrightarrow{\sim} \Big(\prod_{\mathfrak{p} \in \mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T. \tag{3.10}$$

Furthermore, the $(\varphi, G_R)$-equivariant isomorphism in (3.5) can be written as

$$W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.11}$$

Using Lemma 2.7, all terms in (3.10) and (3.11) admit an embedding into $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbf{A}_R^+} N_R \xrightarrow{\sim} \big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$ compatible with respective actions of $\varphi$ and $G_R$. Note that $N_R[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$ (see Proposition 3.11), so the intersection of the left-hand terms in (3.10) and (3.11), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbf{A}_R^+} N_R$, gives

$$\begin{aligned}\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big)\\ = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N_R,\end{aligned}$$

where the first equality follows from Lemma 2.7. Similarly, the intersection of the right-hand terms in (3.10) and (3.11), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T\Big) = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T,$$

where the equality again follows from Lemma 2.7. Since (3.10) and (3.11) are isomorphisms, we obtain the natural $G_R$-equivariant isomorphism claimed in (3.4) as,

$$\mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbf{A}_R^+} N_R \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbb{Z}_p} T.$$

From the proof, it also follows that the isomorphism above is compatible with Frobenius after base change along $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \to W(\mathbb{C}(\overline{R})^\flat)$. ∎

**Corollary 3.18.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$ and let $T := \mathbf{T}_R(N)$ denote the associated finite free $\mathbb{Z}_p$-representation of $G_R$. Then we have a natural $(\varphi, G_R)$-equivariant comparison isomorphism*

$$\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T.$$

*Additionally, the isomorphism above is compatible with Frobenius after base change along $\mathbf{A}^+[1/\mu] \to \mathbf{A}$.*

*Proof.* Since $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$, taking the intersection of the isomorphism in Proposition 3.17 with the isomorphism in (2.7), inside $\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T$, we obtain a $G_R$-equivariant isomorphism $\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}_R^+[1/\mu]} N[1/\mu] \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T$, as claimed. Moreover, from §2.5, recall that $\mathbf{A}^+ = \mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A} \subset \tilde{\mathbf{A}}$, therefore, from Proposition 3.17 it also follows that the isomorphism above is compatible with Frobenius after base change along $\mathbf{A}^+[1/\mu] \to \mathbf{A}$. ∎

**Proposition 3.19.** *Let $N$ be an effective Wach module over $\mathbf{A}_R^+$ and $T := \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of $G_R$. Then we have $(\varphi, \Gamma_R)$-equivariant inclusions $\mu^s \mathbf{D}_R^+(T) \subset N \subset \mathbf{D}_R^+(T)$ (see §2.6 for notations).*

*Proof.* The proof follows in a manner similar to the proof of Proposition 3.17, so we will freely use the notation of that proof. Inverting $p$ in (3.9) we get $(\varphi, G_R)$-equivariant inclusions

$$\begin{aligned}\mu^s\Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T)\Big)[\tfrac{1}{p}] &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R)\Big)[\tfrac{1}{p}]\\ &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T)\Big)[\tfrac{1}{p}].\end{aligned} \tag{3.12}$$

The last term of (3.12) can be written as $\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V$ and similarly for the first term. Moreover, we have $\prod_{\mathfrak{p}\in\mathcal{S}}(\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R) = (\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{\mathbf{A}_R^+} N_R$, so the middle term of

(3.12) can be written as $\big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\big)[\frac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\frac{1}{p}]$. Furthermore, by inverting $p$ in (3.5), we have the following $(\varphi, G_R)$-equivariant comparison isomorphism:

$$W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V. \tag{3.13}$$

Using Lemma 2.7, we embed all terms in (3.12) and (3.13) inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbf{B}_R^+} N_R[1/p] \xrightarrow{\sim} \big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbb{Q}_p} V$, compatible with respective actions of $\varphi$ and $G_R$. Since $N_R[1/p]$ is finite projective over $\mathbf{B}_R^+$, the intersection of the middle term in (3.12) and the left-hand term in (3.13), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbf{B}_R^+} N_R[1/p]$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}]\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}]\Big) = \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}],$$

where the equality follows from Lemma 2.7. Similarly, the intersection of the right-hand terms in (3.10) and (3.13), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbb{Q}_p} V$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V\Big) = \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V,$$

where the equality again follows from Lemma 2.7. Therefore, from (3.12) and $(\varphi, G_R)$-equivariance of (3.13), we obtain the following $(\varphi, G_R)$-equivariant inclusions

$$\mu^s\big(\mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V\big) \subset \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}] \subset \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V. \tag{3.14}$$

Inverting $p$ in the isomorphism obtained in Corollary 3.18 and by taking its intersection with (3.14), inside $W(\mathbb{C}(\overline{R})^\flat)[1/p] \otimes_{\mathbf{B}_R^+} N_R[1/p] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat)[1/p] \otimes_{\mathbb{Q}_p} V$, we obtain the following $(\varphi, G_R)$-equivariant inclusions

$$\mu^s\big(\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V\big) \subset \mathbf{B}^+ \otimes_{\mathbf{B}_R^+} N_R[\tfrac{1}{p}] \subset \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V.$$

In the preceding equation, by taking $H_R$-invariants and its intersection with $\mathbf{D}_R(T) = N_R[1/\mu]^\wedge$, inside $\mathbf{D}_R(V)$, we obtain $\mu^s\mathbf{D}_R^+(T) \subset N_R \subset \mathbf{D}_R^+(T)$, since $N_R = N_R[1/p] \cap N_R[1/\mu]^\wedge$ from Lemma 3.5 and $\mathbf{D}_R^+(T) = \mathbf{D}_R(T) \cap \mathbf{D}_R^+(V) \subset \mathbf{D}_R(V)$ by definition. Hence, the proposition is proved. ■

### 3.4. Finite $[p]_q$-height representations.

In this section we will generalise the definition of finite $[p]_q$-height representations from [Abh21, Definition 4.9] in the relative case.

**Definition 3.20.** A finite $[p]_q$-height $\mathbb{Z}_p$-representation of $G_R$ is a finite free $\mathbb{Z}_p$-module $T$ admitting a linear and continuous action of $G_R$ such that there exists a finitely generated $\mathbf{A}_R^+$-submodule $\mathbf{N}_R(T) \subset \mathbf{D}_R(T)$, stable under the action of $\Gamma_R$ on $\mathbf{D}_R(T)$, and such that $\mathbf{N}_R(T)$, equipped with the induced actions of $\varphi$ and $\Gamma_R$, satisfies the following:

(1) $\mathbf{N}_R(T)$ is a Wach module in the sense of Definition 3.8.

(2) $\mathbf{A}_R$-linearly extending the inclusion $\mathbf{N}_R(T) \to \mathbf{D}_R(T)$ induces a $(\varphi, \Gamma_R)$-equivariant isomorphism $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$.

The height of $T$ is defined to be the height of $\mathbf{N}_R(T)$. Say that $T$ is *positive* if $\mathbf{N}_R(T)$ is effective.

A finite $[p]_q$-height $p$-adic representation of $G_R$ is a finite dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of $G_R$ such that there exists a $G_R$-stable $\mathbb{Z}_p$-lattice $T \subset V$, with $T$ of finite $[p]_q$-height. We set $\mathbf{N}_R(V) := \mathbf{N}_R(T)[1/p]$ satisfying properties analogous to (1) and (2) above. The height of $V$ is defined to be the height of $T$. Say that $V$ is positive if $\mathbf{N}_R(V)$ is effective.

**Lemma 3.21.** *Let $T$ be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of $G_R$ then the $\mathbf{A}_R^+$-module $\mathbf{N}_R(T)$, associated to $T$ in Definition 3.20, is unique.*

*Proof.* By definition, $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$ and this scalar extension induces a fullly faithful functor in Proposition 3.15. So from (2.8) we obtain the uniqueness of $\mathbf{N}(T)$. Alternatively, the uniqess can also be deduced using Proposition 3.19 and [Abh21, Proposition 4.13]. ■

**Remark 3.22.** Let $V$ be a finite $[p]_q$-height $p$-adic representation of $G_R$ and $T \subset V$ a finite $[p]_q$-height $G_R$-stable $\mathbb{Z}_p$-lattice. Then we have $\mathbf{N}_R(V) = \mathbf{N}_R(T)[1/p]$ and from Proposition 3.19 we get that if $V$ is positive then $\mu^s \mathbf{D}_R^+(V) \subset \mathbf{N}_R(V) \subset \mathbf{D}_R^+(V)$. Moreover, similar to [Abh21, Remark 3.10], we can show that $\mathbf{N}_R(V)$ is unique, in particular, it is independent of choice of the lattice $T$ by Corollary 3.16.

**Remark 3.23.** By the definition of finite $[p]_q$-height representations, Lemma 3.21 and the fully faithful functor in (3.3) it follows that the data of a finite height representation is equivalent to the data of a Wach module.

## 3.5. Nygaard filtration on Wach modules.

In this section we consider the Nygaard filtration on Wach modules as follows:

**Definition 3.24.** Let $N$ be a Wach module over $\mathbf{A}_R^+$. Define a decreasing filtration on $N$ called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as

$$\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}.$$

From the definition it is clear that $N$ is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we define Nygaard filtration on $M := N[1/p]$ and it easily follows that $\mathrm{Fil}^k M = (\mathrm{Fil}^k N)[1/p]$.

The reason for considering the Nygaard filtration as above is the following: note that $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is equipped with a filtration by divided power ideals and the embedding $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$ induces a filtration on $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ given as $\mathrm{Fil}^k \mathbf{A}_{\mathrm{inf}}(\overline{R}) := \xi^k \mathbf{A}_{\mathrm{inf}}(\overline{R})$ for $k \in \mathbb{N}$. We equip $\mathbf{A}^+$ with the induced filtration $\mathrm{Fil}^k \mathbf{A}^+ := \mathbf{A}^+ \cap \mathrm{Fil}^k \mathbf{A}_{\mathrm{inf}}(\overline{R}) \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$.

**Lemma 3.25.** *Let $N$ be an effective Wach module over $\mathbf{A}_R^+$ and let $T := \mathbf{T}_R(N)$ denote the associated $\mathbb{Z}_p$-representation of $G_R$. Then, for $k \in \mathbb{N}$, we have $(\mathrm{Fil}^k \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \cap N = \mathrm{Fil}^k N$.*

*Proof.* Let $V := T[1/p]$, $M := N[1/p]$ and $\mathrm{Fil}^k \mathbf{B}^+ := (\mathrm{Fil}^k \mathbf{A}^+)[1/p]$. Then it is enough to show that $(\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M = \mathrm{Fil}^k M$. Indeed, from Definition 3.24 we have $\mathrm{Fil}^k N := \mathrm{Fil}^k M \cap N = (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M \cap N = (\mathrm{Fil}^k \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \cap N$ since $\mathrm{Fil}^k \mathbf{B}^+ \cap \mathbf{A}^+ = \mathrm{Fil}^k \mathbf{A}^+$. Now the inclusion $\mathrm{Fil}^k M \subset (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)$ is obvious and for the converse it is enough to show $([p]_q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M = [p]_q^k M$. Indeed, if we have $x \in (\mathrm{Fil}^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M$ then $\varphi(x) \in ([p]_q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M = [p]_q^k M$, i.e. $x \in \mathrm{Fil}^k M$. For the reduced claim, the inclusion $[p]_q^k M \subset ([p]_q^k \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap M$ is obvious. To show the converse, let $x \in \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V$ such that $[p]_q^k x \in M$, in particular, $x \in M[1/[p]_q]$. Then it follows that $h(x) = x$ for all $h \in H_R$, i.e. $x \in (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_R} =: \mathbf{D}_R^+(V) = \mathbf{D}_R^+(T)[1/p]$. From Proposition 3.19 recall that $\mu^s \mathbf{D}_R^+(V) \subset M$, where $s$ is the $[p]_q$-height of $N$. So we get that $\mu^s x \in M$, in particular, $x \in M[1/\mu]$. Combining this with the previous observation we get that $x \in M[1/\mu] \cap M[1/[p]_q] \subset \mathbf{B}_R \otimes_{\mathbf{B}_R^+} M$. But from Proposition 3.11 we know that $M$ is finite projective over $\mathbf{B}_R^+$ and note that $\mathbf{B}_R^+ = \mathbf{B}_R^+[1/\mu] \cap \mathbf{B}_R^+[1/[p]_q] \subset \mathbf{B}_R$, since $[p]_q = p \mod \mu \mathbf{B}_R^+$. Hence, it follows that we have $x \in M[1/\mu] \cap M[1/[p]_q] = M$, as desired. ∎

## 3.6. Wach modules are crystalline.

The goal of this subsection is to prove Theorem 3.27. In order to prove our results, we will need auxiliary period rings $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ from [Abh21, §4.3.1]. We briefly recall their definitions. Let $\varpi := \zeta_p - 1$ and set $\mathbf{A}_{R,\varpi}^+ := \mathbf{A}_R^+[\varphi^{-1}(\mu)] \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$, stable under the $(\varphi, \Gamma_R)$-action on the latter. By restricting the map $\theta$ on $\mathbf{A}_{\mathrm{inf}}(R_\infty)$, to $\mathbf{A}_{R,\varpi}^+$ (see §2.2), we obtain a surjective ring homomorphism $\theta : \mathbf{A}_{R,\varpi}^+ \twoheadrightarrow R[\varpi]$. We define $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ to be the $p$-adic completion of the divided power envelope of the map $\theta$ with respect to $\mathrm{Ker}\,\theta$. Furthermore, the map $\theta$ extends $R$-linearly to a surjective ring homomorphism $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+ \twoheadrightarrow R[\varpi]$, given as $x \otimes y \mapsto x\theta(y)$. Similar to above, we define $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ to be the $p$-adic completion of the divided power envelope of the map $\theta_R$ with respect to $\mathrm{Ker}\,\theta_R$. The morphisms $\theta$ and $\theta_R$ naturally extend to respective surjections $\theta : \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$ and $\theta_R : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$. Now, from loc. cit., we have natural inclusions $\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{\mathrm{cris}}(R_\infty)$ and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$, and it is easy to verify that the former rings are stable under respective actions of $\varphi$ and $\Gamma_R$ on the latter rings. Therefore, we equip $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ with induced structures, in particular, a filtration and an $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear connection $\partial_A$ on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration, and it is easy to show that $(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})^{\partial_A=0} = \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Note that the aforementioned filtration on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ coincide with the divided power filtration by Ker $\theta$ and Ker $\theta_R$ respectively (see [Abh21, Remark 4.23]).

**Remark 3.26.** Let us first remark that the ring $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is flat over $\mathbf{A}_R^+$. Indeed, note that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is the $p$-adic completion of a divided power algebra over $\mathbf{A}_{R,\varpi}^+$, given as $\mathbf{A}_{R,\varpi}^+[\xi^k/k!, k \in \mathbb{N}]$, where $\xi = \mu/\varphi^{-1}\mu$. Now, since $(p, \xi)$ is a regular sequence on $\mathbf{A}_{R,\varpi}^+$, therefore, using [BS22, Lemma 2.38 and Lemma 2.43], it follows that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-completely flat over $\mathbf{A}_{R,\varpi}^+$, therefore, flat since $\mathbf{A}_{R,\varpi}^+$ is noetherian (see [Sta23, Tag 0912]). As $\mathbf{A}_{R,\varpi}^+$ is flat over $\mathbf{A}_R^+$, it follows that $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is flat over $\mathbf{A}_R^+$. Furthermore, from [Abh21, Lemma 4.20], note that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is a PD-polynomial algebra over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ in $d$ variables. So again from [BS22, Lemma 2.38 and Lemma 2.43], it follows that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-completely flat over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. As $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is flat over $\mathbf{A}_R^+$, we get that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is $p$-completely flat over $\mathbf{A}_R^+$, hence flat.

Next, note that for any $k \in \mathbb{N}$, the graded quotient $\mathrm{gr}^k(\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) = \mathrm{Fil}^k(\mathbf{A}_{R,\varpi}^{\mathrm{PD}})/\mathrm{Fil}^{k+1}(\mathbf{A}_{R,\varpi}^{\mathrm{PD}})$ is isomorphic to $\xi^{[k]}R[\varpi]$, in particular, we have that $\mathrm{gr}^k(\mathbf{A}_{R,\varpi}^{\mathrm{PD}})$ is a free $R$-module. Now, since $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is a PD-polynomial algebra over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we also get that for any $k \in \mathbb{N}$, the graded quotient $\mathrm{gr}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) = \mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})/\mathrm{Fil}^{k+1}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}})$ is a free $R$-module. Moreover, we have $\mathbf{A}_R^+/\mu \xrightarrow{\sim} R$, so the flat dimension of $R$ as an $\mathbf{A}_R^+$-module is 1, and it follows that the flat dimension of $\mathrm{gr}^k\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as an $\mathbf{A}_R^+$-module is also 1. Since $\mathrm{Fil}^0\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} = \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, therefore, using induction on $k \in \mathbb{N}$, we conclude that $\mathrm{Fil}^k\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is flat as an $\mathbf{A}_R^+$-module.

**Theorem 3.27.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$ and let $T := \mathbf{T}_R(N)$ be the associated finite free $\mathbb{Z}_p$-representation of $G_R$. Then $V := T[1/p]$ is a $p$-adic crystalline representation of $G_R$ and we have a natural isomorphism of $R[1/p]$-modules $(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p])^{\Gamma_R} \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ compatible with the respective Frobenii, filtrations and connections.*

*Proof.* For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and $\mathbf{T}_R(\mu^r N(-r)) = \mathbf{T}_R(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which $\Gamma_R$ acts via $\chi^{-r}$, where $\chi$ is the $p$-adic cyclotomic character). Therefore, it follows that it is enough to show the claim for effective Wach modules. So, in the rest of the proof, we will assume that $N$ is effective. Now, let us set $D_R := (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p])^{\Gamma_R} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, and using Proposition 3.28, we note that $D_R$ is a finite projective $R[1/p]$-module of rank $= \mathrm{rk}_{\mathbf{B}_R^+} N[1/p]$. Moreover, $D_R$ is equipped with the tensor product Frobenius and a filtration defined as $\mathrm{Fil}^k D_R := \big(\sum_{i+j=k} \mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j N[1/p]\big)^{\Gamma_R}$, where $N[1/p]$ is equipped with the Nygaard filtration as in Definition 3.24. Note that the filtration is well-defined, i.e. $\mathrm{Fil}^k D_R$ is an $R[1/p]$-submodule of $D_R$, for each $k \in \mathbb{N}$. Indeed, it is enough to check that $\mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j N[1/p]$ is an $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-submodule of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$, for each $i, j \in \mathbb{N}$. This easily follows from the fact that the $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear composition $\mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j N[1/p] \rightarrow \mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p] \rightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$ is injective, where the first arrow is obtained by tensoring the $\mathbf{A}_R^+$-linear inclusion $\mathrm{Fil}^j N[1/p] \rightarrow N[1/p]$ with the flat $\mathbf{A}_R^+$-module $\mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Remark 3.26) and the second arrow is obtained by tensoring the $\mathbf{A}_R^+$-linear inclusion $\mathrm{Fil}^i\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \rightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ with the flat $\mathbf{A}_R^+$-module $N[1/p]$ (see Proposition 3.11). Next, we note that $D_R$ is equipped with a connection induced from the connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and it satisfies Griffiths transversality with respect to the filtration described above. Using Proposition 3.28, note that we have a natural isomorphism $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$. Now, consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} D_R \arrow[r, "(3.17)", "\sim"'] \arrow[d, "(3.18)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p] \arrow[d, "(3.4)"', "\wr"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd} \tag{3.15}$$

where the left vertical arrow is the extension of the $R[1/p]$-linear injective map $D_R \rightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, from (3.18), along the faithfully flat ring homomorphism $R[1/p] \rightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ (see [Bri08, Thèoréme 6.3.8]), the top horizontal arrow is the extension along $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ of the isomorphism (3.17) in Proposition 3.28, the right vertical arrow is the extension along $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ of the isomorphism in Proposition 3.17 and the bottom horizontal arrow is the natural injective map (see [Bri08, Proposition 8.2.6]). Commutativity of the diagram (3.15) and compatibility of its arrows with the respective actions of $(\varphi, G_R)$ and connections follow from (3.18). Since the top horizontal and right vertical arrows in (3.15) are bijective, we conclude that its left vertical arrow and the bottom horizontal arrow are also bijective. Therefore, $V$ is a $p$-adic crystalline representation of $G_R$, and by taking $G_R$ fixed part of the left vertical arrow in (3.15), we obtain an isomorphism of $R[1/p]$-modules

$$D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \tag{3.16}$$

compatible with the respective Frobenii and connections. Moreover, from [Bri08, Proposition 8.4.3] recall that the bottom horizontal arrow of (3.15) is compatible with the respective filtrations, where the bottom left corner is equipped with the tensor product filtration for the Hodge filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$. So, using the preceding observation and [Abh21, Proposition 4.49], it can be shown that (3.16) is compatible with the respective filtrations. Indeed, the claim has been proven in loc. cit. under the assumption that $\mathbf{N}_R(T)$ becomes free after extension of scalars along a map $\mathbf{A}_R \to \mathbf{A}_{R'}^+$ (in the notation of loc. cit.). This assumption does not always hold in our current case, however, in [Abh21, §4.5.1] we use the preceding assumption only in the proof of [Abh21, Lemma 4.53]. So in our current setting we can replace [Abh21, Lemma 4.53] by Lemma 3.25. Then the results of [Abh21, §4.5.1] remain valid for our setting as well, in particular, we get that (3.16) is compatible with the respective filtrations. This concludes our proof. ■

**Proposition 3.28.** *Let $N$ be an effective Wach module over $\mathbf{A}_R^+$, then $D_R := \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]\big)^{\Gamma_R}$ is a finite projective $R[1/p]$-module of rank $= \mathrm{rk}_{\mathbf{B}_R^+} N[1/p]$ equipped with a Frobenius, a filtration and a connection satisfying Griffiths transversality with respect to the filtration. Moreover, we have a natural comparison isomorphism*

$$\begin{aligned} f : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p] \\ a \otimes b \otimes x &\longmapsto ab \otimes x, \end{aligned} \tag{3.17}$$

*compatible with the respective Frobenii, connections and actions of $\Gamma_R$.*

**Remark 3.29.** In (3.17), the Frobenius on each term is given as $\varphi \otimes \varphi$; the connection on the right-hand term is given as the natural $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear differential operator $\partial \otimes 1$ and on the left-hand term, it is given as $\partial \otimes 1 + 1 \otimes \partial_D$, where $\partial_D$ is the connection on $D_R$; the action of any $g$ in $\Gamma_R$ on the left-hand term is given as $g \otimes 1$ and on the the right-hand term, it is given as $g \otimes g$.

*Proof of Proposition 3.28.* We will adapt the proof of [Abh21, Proposition 4.28]. Recall the following rings from [Abh21, §4.4.1]: for $n \in \mathbb{N}$, we take a $p$-adically complete ring $S_n^{\mathrm{PD}} := \mathbf{A}_R^+\langle \frac{\mu}{p^n}, \frac{\mu^2}{2!p^{2n}}, \ldots, \frac{\mu^k}{k!p^{kn}}, \ldots \rangle$. We have a Frobenius homomorphism $\varphi : S_n^{\mathrm{PD}} \to S_{n-1}^{\mathrm{PD}}$, in particular, $\varphi^n(S_n^{\mathrm{PD}}) \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and the ring $S_n^{\mathrm{PD}}$ is further equipped with a continuous (for the $p$-adic topology) action of $\Gamma_R$. The reader should note that in [Abh21, §4.4.1] we consider a further completion of $S_n^{\mathrm{PD}}$, with respect to certain filtration by PD-ideals, which we denote as $\widehat{S}_n^{\mathrm{PD}}$ in loc. cit. However, such a completion is not strictly necessary and all the proofs of loc. cit. can be carried out without it. In particular, many good properties of $\widehat{S}_n^{\mathrm{PD}}$ restrict to good properties on $S_n^{\mathrm{PD}}$ as well (for example, $(\varphi, \Gamma_R)$-action above).

Let us now consider the $O_F$-linear homomorphism of rings $\iota : R \to S_n^{\mathrm{PD}}$, defined by sending $X_j \mapsto [X_j^\flat]$, for $1 \leq j \leq d$. Using $\iota$ we can define an $O_F$-linear homomorphism of rings $f : R \otimes_{O_F} S_n^{\mathrm{PD}} \to S_n^{\mathrm{PD}}$, sending $a \otimes b \mapsto \iota(a)b$. Let $\mathcal{O}S_n^{\mathrm{PD}}$ denote the $p$-adic completion of the divided power envelope of $R \otimes_{O_F} S_n^{\mathrm{PD}}$, with respect to $\mathrm{Ker}\, f$. The tensor product Frobenius induces $\varphi : \mathcal{O}S_n^{\mathrm{PD}} \to \mathcal{O}S_{n-1}^{\mathrm{PD}}$, such that $\varphi^n(\mathcal{O}S_n^{\mathrm{PD}}) \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, and the action of $\Gamma_R$ extends to a continuous (for the $p$-adic topology) action on $\mathcal{O}S_n^{\mathrm{PD}}$. Moreover, we have a $(\varphi, \Gamma_R)$-equivariant embedding $S_n^{\mathrm{PD}} \subset \mathcal{O}S_n^{\mathrm{PD}}$ and the latter is equipped with a $\Gamma_R$-equivariant $S_n^{\mathrm{PD}}$-linear integrable connection given as the universal continuous $S_n^{\mathrm{PD}}$-linear de Rham differential $d : \mathcal{O}S_n^{\mathrm{PD}} \to \Omega^1_{\mathcal{O}S_n^{\mathrm{PD}}/S_n^{\mathrm{PD}}}$. Furthermore, we have $R = (\mathcal{O}S_n^{\mathrm{PD}})^{\Gamma_R}$ and with $V_j = \frac{X_j \otimes 1}{1 \otimes [X_j^\flat]}$, for $1 \leq j \leq d$, we have the $p$-adically complete divided power ideals of $\mathcal{O}S_n^{\mathrm{PD}}$ as follows:

$$J^{[i]}\mathcal{O}S_n^{\mathrm{PD}} := \Big\langle \tfrac{\mu^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d}(1-V_j)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geq i \Big\rangle.$$

We equip $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N$ with the tensor product Frobenius and the connection on $\mathcal{O}S_n^{\mathrm{PD}}$ induces an $S_n^{\mathrm{PD}}$-linear integrable connection on $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N$. Then $D_n := (\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p])^{\Gamma_R}$ is an $R[1/p]$-module equipped with a Frobenius $\varphi : D_n \to D_{n-1}$ and an integrable connection. In particular, it follows that $\varphi^n(D_n) \subset D_R = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p])^{\Gamma_R} \subset (\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{A}_R^+} N[1/p])^{H_R}$, where we have $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) = \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})^{H_R}$ (see [MT20, Corollary 4.34] for the equality). Let $T := \mathbf{T}_R(N)$ denote the finite free $\mathbb{Z}_p$-representation of $G_R$, associated to $N$, and set $V := T[1/p]$, then we have

$$\begin{aligned} D_R \subset (\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p])^{G_R} &\subset (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p])^{G_R} \\ &\xrightarrow{\sim} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V)^{G_R} = \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V), \end{aligned} \tag{3.18}$$

where the isomorphism follows by taking $G_R$-fixed elements of the isomorphism (3.4) in Proposition 3.17, after extending scalars along $\mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. Since $\varphi^n(D_n) \subset D_R$, or equivalently, the $R[1/p]$-linear map $1 \otimes \varphi^n : R[1/p] \otimes_{\varphi^n, R[1/p]} D_n \to D_R$ is injective, we get that $R[1/p] \otimes_{\varphi^n, R[1/p]} D_n$ is a finitely generated $R[1/p]$-module. Moreover, recall that $\varphi^n : R[1/p] \to R[1/p]$ is finite flat (see §1.5), so it follows that $D_n$ is finitely generated over the source of $\varphi^n$, i.e. $D_n$ is a finitely generated $R[1/p]$-module equipped with an integrable connection, in particular, it is finite projective over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Furthermore, recall that $N[1/p]$ is a finite projective $\mathbf{B}_R^+$-module (see Proposition 3.11), therefore $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$ is a finite projective $\mathcal{O}S_n^{\mathrm{PD}}[1/p]$-module, and from [AGT16, Lemma IV.3.2.2], it follows that $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N$ is $p$-adically complete. Now, for $n \geq 1$, similar to the proof of [Abh21, Lemmas 4.32 & 4.36], it is easy to show that $\log \gamma_i := \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$ converges as a series of operators on $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N$, where $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ are topological generators of $\Gamma_R$ (see §2).

**Lemma 3.30.** *Let $m \geq 1$ (let $m \geq 2$ if $p = 2$), then we have a $\Gamma_R$-equivariant isomorphism via the natural map $a \otimes b \otimes x \mapsto ab \otimes x$:*

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m \xrightarrow{\sim} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]. \tag{3.19}$$

*Proof.* The map in (3.19) is obviously compatible with the respective actions of $\Gamma_R$, so we need to check that it is bijective. Let us first check the injectivity of (3.19). We have a composition of injective homomorphisms $\mathcal{O}S_m^{\mathrm{PD}}[1/p] \xrightarrow{\varphi^m} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. As $D_m$ is finite projective over $R[1/p]$, the map

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{R[1/p]} D_m \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m, \tag{3.20}$$

is injective. Next, we have $V = T[1/p]$ and we consider the following composition,

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m \xrightarrow{1 \otimes \varphi^m} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} D_R \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V). \tag{3.21}$$

As $R[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is faithfully flat (see [Bri08, Théorème 6.3.8]) and (3.18) is injective, so in (3.21), the second map is injective and the first map is injective because $1 \otimes \varphi^m : R[1/p] \otimes_{\varphi^m, R[1/p]} D_m \to D_R$ is injective, in particular, we see that (3.21) is injective. Moreover, since $N[1/p]$ is a finite projective $\mathbf{B}_R^+$-module, therefore, similar to (3.20), it can be shown that the map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p] = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_R^+} N[1/p] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, \mathbf{B}_R^+} N[1/p]$ is injective. Furthermore, from the definition of Wach modules (see Definition 3.8), we have an isomorphism $1 \otimes \varphi : \mathbf{B}_R^+ \otimes_{\varphi, \mathbf{B}_R^+} N[1/p, 1/[p]_q] \xrightarrow{\sim} N[1/p, 1/[p]_q]$. Therefore, iterating it $m$ times and by extending scalars along $\mathbf{B}_R^+ \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, we obtain an isomorphism $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, \mathbf{B}_R^+} N[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p]$, since $[p]_q$ is unit in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. So, from the preceding observations, it follows that the following composition,

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, \mathbf{B}_R^+} N[1/p] \xrightarrow[\sim]{1 \otimes \varphi^m} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p], \tag{3.22}$$

is injective. Let us now consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m \arrow[r, "(3.20)"] \arrow[d, "(3.19)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m \arrow[r, "(3.21)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \arrow[d] \\
\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p] \arrow[r, "(3.22)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p] \arrow[r, "(3.4)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd}$$

where the right vertical arrow is the natural injective map (see [Bri08, Proposition 8.2.6]). From the definitions, it easily follows that the diagram commutes, therefore, we see that the left vertical arrow, i.e. (3.19) is injective.

Next, let us check the surjectivity of the map in (3.19). We define the following operators on $\mathcal{O}N_m^{\mathrm{PD}} := \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$,

$$\partial_i := \begin{cases} -(\log \gamma_0)/t & \text{for } i = 0, \\ (\log \gamma_i)/(tV_i) & \text{for } 1 \leq i \leq d, \end{cases}$$

where $V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}$, for $1 \leq i \leq d$ (see [Abh21, §4.4.2]). Note that for any $g \in \Gamma_R$ and any $x \in \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$, we have $(g-1)(ax) = (g-1)a \cdot x + g(a)(g-1)x$. Then, from the identity $\log(\gamma_i) = \lim_{n \to +\infty}(\gamma_i^{p^n} - 1)/p^n$, it easily follows that the operators $\partial_i$ satisfy the Leibniz rule for all $0 \leq i \leq d$. In particular, the operator $\partial : \mathcal{O}N_m^{\mathrm{PD}} \to \mathcal{O}N_m^{\mathrm{PD}} \otimes_{\mathcal{O}S_m^{\mathrm{PD}}} \Omega^1_{\mathcal{O}S_m^{\mathrm{PD}}/R}$, given by $x \mapsto \partial_0(x)dt + \sum_{i=1}^d \partial_i(x) d[X_i^\flat]$, defines a connection on $\mathcal{O}N_m^{\mathrm{PD}}$. Furthermore, from [Abh21, Lemma 4.38] the operators $\partial_i$ commute with each other, so the connection $\partial$ is integrable and using the finite $[p]_q$-height property of $N$, similar to [Abh21, Lemma 4.39], it is easy to show that $\partial$ is $p$-adically quasi-nilpotent. Now, similar to the proof of [Abh21, Lemma 4.39 & Lemma 4.41], it follows that for $x \in N[1/p]$, the following sum converges in $D_m = (\mathcal{O}N_m^{\mathrm{PD}})^{\Gamma_R} = (\mathcal{O}N_m^{\mathrm{PD}})^{\partial=0}$:

$$y = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \frac{t^{[k_0]}}{p^{mk_0}} (1 - V_1)^{[k_1]} \cdots (1 - V_d)^{[k_d]}. \tag{3.23}$$

Using the construction above we define an $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$-linear transformation $\alpha$ on the finite projective module $\mathcal{O}N_m^{\mathrm{PD}}$ and claim that $\alpha$ is an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$. Indeed, let us first choose a presentation $\mathcal{O}N_m^{\mathrm{PD}} \oplus N' = (\mathcal{O}S_m^{\mathrm{PD}})^r$, for some $r \in \mathbb{N}$. Then, on a chosen basis of $(\mathcal{O}S_m^{\mathrm{PD}})^r$, we can define a linear transformation $\beta$ using (3.23) over $\mathcal{O}N_m^{\mathrm{PD}}$ and the identity on $N'$. Note that the transformation $\beta$ preserves $\mathcal{O}N_m^{\mathrm{PD}}$ and we set $\det \alpha = \det \beta$, which is independent of the chosen presentation (see [Gol61, Proposition 1.2]). Now by an argument similar to the proof of [Abh21, Lemma 4.43], it easily follows that for some $N \in \mathbb{N}$ large enough, one can write $p^N \det \alpha = p^N \det \beta \in 1 + J^{[1]}\mathcal{O}S_m^{\mathrm{PD}}$, in particular, we get that $\det \alpha$ is a unit in $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$, so $\alpha$ defines an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$ (see [Gol61, Proposition 1.3]). Since the formula considered in (3.23) converges in $D_m$, we conclude that the natural map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m \to \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$, is surjective. Hence, (3.19) is bijective, proving the lemma. ■

Recall that $D_R$ is an $R[1/p]$-module equipped with an integrable connection and it is finite over $R[1/p]$ since we have an inclusion $D_R \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ of $R[1/p]$-modules from (3.18). In particular, we see that $D_R$ is a finite projective module over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Moreover, $D_R$ is equipped with a Frobenius-semilinear operator $\varphi$ and a filtration given as $\mathrm{Fil}^k D_R = \big(\sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j N[1/p]\big)^{\Gamma_R}$, where $N[1/p]$ is equipped with the Nygaard filtration of Definition 3.24. Recall that the connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ satisfies Griffiths transversality with respect to the filtration, so the connection on $D_R$, induced from the connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, also satisfies Griffiths transversality with respect to the filtration defined above. Now consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\varphi^m, R} D_m \arrow[r, "1 \otimes \varphi^m"] \arrow[d, "(3.19)"', "\wr"] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R \arrow[d, "(3.17)"'] \\
\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\varphi^m, \mathbf{A}_R^+} N[1/p] \arrow[r, "\sim"] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p],
\end{tikzcd} \tag{3.24}$$

where the left vertical arrow is the extension along $\varphi^m : \mathcal{O}S_m^{\mathrm{PD}} \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ of the isomorphism (3.19) in Lemma 3.30 and the bottom horizontal isomorphism follows from an argument similar to [Abh21, Lemma 4.46]. By the description of the arrows, it follows that the diagram is $(\varphi, \Gamma_R)$-equivariant and commutative. Taking $\Gamma_R$-invariants of the diagram (3.24), we obtain an isomorphism of $R[1/p]$-modules $1 \otimes \varphi^m : R \otimes_{\varphi^m, R[1/p]} D_m \xrightarrow{\sim} D_R$. In particular, it follows that the top horizontal arrow of (3.24) is an isomorphism. Hence, we conclude that the right vertical arrow of (3.24) is bijective as well, in particular, the comparison in (3.17) is an isomorphism compatible with the respective Frobenii, connections and actions of $\Gamma_R$. This finishes our proof. ∎

**Corollary 3.31.** *The $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$linear isomorphism (3.17), in Proposition 3.28, is comaptible with the respective tensor product filtrations.*

*Proof.* Let us first note that, from Theorem 3.27, for each $k \in \mathbb{N}$, we have that $\mathrm{Fil}^k D_R \xrightarrow{\sim} \mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, and $\mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathrm{gr}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ are finite projective $R[1/p]$-modules (see [Bri08, Proposition 8.3.2]). Now, the filtration on the left-hand term of (3.17) is given as $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R) = \sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j D_R$, for each $k \in \mathbb{N}$. The preceding filtration is well-defined, i.e. $\mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j D_R$ is an $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-submodule of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R$, for each $i, j \in \mathbb{N}$. Indeed, for each $j \in \mathbb{N}$, let us consider the following exact sequence of finite projective $R[1/p]$-modules, in particular, flat $R$-modules,

$$0 \longrightarrow \mathrm{Fil}^{j+1} D_R \longrightarrow \mathrm{Fil}^j D_R \longrightarrow \mathrm{gr}^j D_R \longrightarrow 0. \tag{3.25}$$

Extending scalars in (3.25) along the natural map $R \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and by decreasing induction on $j \in \mathbb{N}$, it is easy to see that the natural map $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j D_R \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^0 D_R = \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R$ is injective. Therefore, for any $i + j = r$, it follows that the natural composition $\mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j D_R \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j D_R \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R$ is injective, where the first arrow is induced from the inclusion $\mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and the second arrow is as above. Similarly, the filtration on the right-hand term of (3.17) is given as $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V)) = \sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{N}(V)$. The aforementioned filtration was shown to be well-defined at the beginning of the proof of Theorem 3.27. Finally, using the diagram (3.15), the compatibility of (3.16) with respective filtrations and [Abh21, Corollary 4.54], it easily follows that the isomorphism (3.17), in Proposition 3.28, is compatible with the respective tensor product filtrations. ∎

**Remark 3.32.** Let us make an observation that will be useful for the proof of Theorem 5.6. In the basis $\{d\log(X_1), \ldots, d\log(X_d)\}$ of $\Omega_R^1$, let $\partial_{A,i}$ denote the $i^{\mathrm{th}}$ component of the connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, for $1 \leq i \leq d$, and let $\partial_{D,i}$ denote the induced operator on $D_R$. Moreover, employing arguments similar to [Abh23b, Lemmas 4.12, 5.17 & 5.18], we can show that, for $1 \leq i \leq d$, the operator $\nabla_i = (\log \gamma_i)/t = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$ converges as a series of operators on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$. Now, using (3.23) and the top horizontal arrow in diagram (3.24), we note that for any $x \in N[1/p]$, there exists $w \in D_R$ and $z \in (\mathrm{Fil}^1 \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} N[1/p]$, such that $x = f(w) + z$, where $f$ is the isomorphism in (3.17). Then an easy computation shows that $\nabla_i(x) - f(\partial_{D,i}(w)) = \nabla_i(z) + \partial_{A,i}(z) \in (\mathrm{Fil}^1 \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} N[1/p]$.

## 4. Crystalline implies finite height

The goal of this section is to prove the following claim:

**Theorem 4.1.** *Let $T$ be a finite free $\mathbb{Z}_p$-representation of $G_R$ such that $V := T[1/p]$ is a $p$-adic crystalline representation of $G_R$. Then there exists a unique Wach module $\mathbf{N}_R(T)$ over $\mathbf{A}_R^+$ attached to $T$. In other words, $T$ is of finite $[p]_q$-height.*

*Proof.* For a $p$-adic representation, the property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by $\chi^r$, where $\chi$ is the $p$-adic cyclotomic character and $r \in \mathbb{N}$. Therefore, we can assume that $V$ is positive crystalline. Note that $V$ is also a positive crystalline representation of $G_L$, and therefore, it is also positive and of finite $[p]_q$-height as a $p$-adic representation of $G_L$ (see [Abh23a, Definition 3.7]). In particular, we also get that $T$ is positive and of finite $[p]_q$-height as a $\mathbb{Z}_p$-representation of $G_L$. Moreover, associated to $T$, from loc. cit. we have the Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$ and we set $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ as an $\mathbf{A}_R^+$-module. From Proposition 4.7, the module $\mathbf{N}_R(T)$ satisfies all the axioms of Definition 3.8 and Definition 3.20. Hence, it follows that $\mathbf{N}_R(T)$ is the unique Wach module attached to $T$, or equivalently, $T$ is of finite $[p]_q$-height. ∎

**Remark 4.2.** From Theorem 4.1, note that $T$ is a $\mathbb{Z}_p$-representation of $G_L$ such that $V := T[1/p]$ is crystalline for $G_L$. Then, from [Abh23a, Theorem 4.1] it follows $T$ is of finite $[p]_q$-height as a representation of $G_L$, i.e. there exists a unique Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$ attached to $T$. Moreover, note that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T)$ is also a Wach module over $\mathbf{A}_L^+$ attached to $T$, where we use $\Gamma_L \xrightarrow{\sim} \Gamma_R$. Now, using Proposition 4.12, we have that $\mathbf{A}_L \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{A}_L \otimes_{\mathbf{A}_R} \mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{D}_L(T)$ as étale $(\varphi, \Gamma_L)$-modules over $\mathbf{A}_L$. Hence, by the uniqueness of the Wach module attached to $T$ over $\mathbf{A}_L^+$ in [Abh23a, Lemma 3.9] it follows that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ as $(\varphi, \Gamma_L)$-modules over $\mathbf{A}_L^+$.

### 4.1. Consequences of Theorem 4.1.

Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R)$ denote the category of $\mathbb{Z}_p$-lattices inside $p$-adic crystalline representations of $G_R$. Then, by combining Theorem 3.27 and Theorem 4.1, we obtain the following:

**Corollary 4.3.** *The Wach module functor induces an equivalence of categories*

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R) &\xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q} \\ T &\longmapsto \mathbf{N}_R(T), \end{aligned}$$

*with a quasi-inverse given as* $N \mapsto \mathbf{T}_R(N) := \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} N\big)^{\varphi=1}$.

Passing to associated isogeny categories, we obtain the following:

**Corollary 4.4.** *The Wach module functor induces an exact equivalence of* $\otimes$*-categories* $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R^+}^{[p]_q}$, *via* $V \mapsto \mathbf{N}_R(V)$, *and with an exact* $\otimes$*-compatible quasi-inverse given as* $M \mapsto \mathbf{V}_R(M) := \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} M\big)^{\varphi=1}$.

*Proof.* The equivalence of categories follows from Theorem 4.1. For the rest of the proof, let us remark that for a $p$-adic crystalline representation $V$ of $G_R$, from Proposition 4.7, we have $\mathbf{N}_R(V) = \mathbf{N}_L(V) \cap \mathbf{D}_R(V) \subset \mathbf{D}_L(V)$ as finite projective $(\varphi, \Gamma_R)$-modules over $\mathbf{B}_R^+$. Moreover, from Proposition 4.12 and Remark 4.2, note that $\mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} \mathbf{N}_R(V) \xrightarrow{\sim} \mathbf{N}_L(V)$ and $\mathbf{B}_R \otimes_{\mathbf{B}_R^+} \mathbf{N}_R(V) \xrightarrow{\sim} \mathbf{D}_R(V)$ compatible with resepctive natural actions of $(\varphi, \Gamma_R)$.

Now, let $V_1$ and $V_2$ be two crystalline representations of $G_R$, then $V_1 \otimes_{\mathbb{Q}_p} V_2$ is again crystalline (see [Bri08, Théorèm 8.4.2]). We have

$$\begin{aligned} \mathbf{N}_R(V_1) \otimes_{\mathbf{B}_R^+} \mathbf{N}_R(V_2) &= \mathbf{N}_R(V_1) \otimes_{\mathbf{B}_R^+} (\mathbf{N}_L(V_2) \cap \mathbf{D}_R(V_2)) \\ &= (\mathbf{N}_R(V_1) \otimes_{\mathbf{B}_R^+} \mathbf{N}_L(V_2)) \cap (\mathbf{N}_R(V_1) \otimes_{\mathbf{B}_R^+} \mathbf{D}_R(V_2)) \\ &= (\mathbf{N}_L(V_1) \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V_2)) \cap (\mathbf{D}_R(V_1) \otimes_{\mathbf{B}_L^+} \mathbf{D}_R(V_2)) \\ &= \mathbf{N}_L(V_1 \otimes_{\mathbb{Q}_p} V_2) \cap \mathbf{D}_R(V_1 \otimes_{\mathbb{Q}_p} V_2) = \mathbf{N}_R(V_1 \otimes V_2), \end{aligned}$$

where the first equality follows from the discussion above, the second equality follows since $\mathbf{N}_R(V_1)$ is projective, the third equality again follows from the discussion above and the last equality follows from [Abh23a, Corollary 4.3] and (2.8). This shows the compatibility of $\mathbf{N}_R$ with tensor products. Conversely, let $N_1$ and $N_2$ be two Wach modules over $\mathbf{A}_R^+$ and set $N_3 := (N_1 \otimes_{\mathbf{A}_R^+} N_2)/(p\text{-torsion})$ as a finitely generated $\mathbf{A}_R^+$-module. Then, note that we have $N_3 \subset N_3[1/p] = N_1[1/p] \otimes_{\mathbf{B}_R^+} N_2[1/p]$, where the right-hand term is a projective $\mathbf{B}_R^+$-module. Therefore, $N_3$ is torsion free and by definition $N_3/\mu$ is also $p$-torsion free, in particular, the sequence $\{p, \mu\}$ is strictly $N_3$-regular by Remark 3.2. Furthermore, assumptions for the $(\varphi, \Gamma_R)$-action on $N_3$, as in Definition 3.8, can be verified similar to [Abh21, Proposition 4.14]. So it follows that $N_3$ is a Wach module over $\mathbf{A}_R^+$. Since, $N_3[1/p] = N_1[1/p] \otimes_{\mathbf{B}_R^+} N_2[1/p]$, compatibility of the functor $\mathbf{V}_R$ with tensor products now follows from (2.8).

It remains to show the exactness of $\mathbf{N}_R$ since exactness of the quasi-inverse functor $\mathbf{V}_R$ follows from Proposition 3.15 and the exact equivalence in (2.8). So, let us consider an exact sequence of $p$-adic crystalline representations of $G_R$ as $0 \to V_1 \to V_2 \to V_3 \to 0$, and we wish to show that the sequence

$$0 \longrightarrow \mathbf{N}_R(V_1) \longrightarrow \mathbf{N}_R(V_2) \longrightarrow \mathbf{N}_R(V_3) \longrightarrow 0, \tag{4.1}$$

is exact. Let $T_2 \subset V_2$ be a $G_R$-stable $\mathbb{Z}_p$-lattice, then $T_1 := V_1 \cap T_2 \subset V_2$ is a $G_R$-stable $\mathbb{Z}_p$-lattice inside $V_1$ and set $T_3 := T_2/T_1 \subset V_3$ as a $G_R$-stable $\mathbb{Z}_p$-lattice. By definition, we have Wach modules $\mathbf{N}_R(T_1)$, $\mathbf{N}_R(T_2)$ and $\mathbf{N}_R(T_3)$ and we set $N := \mathbf{N}_R(T_2)/\mathbf{N}_R(T_1)$ as a finitely generated $\mathbf{A}_R^+$-module equipped with a Frobenius $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ and a continuous action of $\Gamma_R$ induced from the corresponding structures on $\mathbf{N}_R(T_2)$. We claim that $N[1/p] \xrightarrow{\sim} \mathbf{N}_R(V_3)$ as $(\varphi, \Gamma_R)$-modules over $\mathbf{B}_R^+$.

Indeed, first recall that $\mathbf{D}_R$ is an exact functor from the category of $\mathbb{Z}_p$-representations of $G_R$ to the category of étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$ (see §2.6). So we get that the natural map $N = \mathbf{N}_R(T_2)/\mathbf{N}_R(T_1) \to \mathbf{N}_R(T_3)$ is injective, and since $\mathbf{A}_R^+ \to \mathbf{A}_R$ is flat, therefore, we have $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{D}_R(T_2)/\mathbf{D}_R(T_1) \xrightarrow{\sim} \mathbf{D}_R(T_3) \xleftarrow{\sim} \mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T_3)$ as étale $(\varphi, \Gamma_R)$-modules over $\mathbf{A}_R$. Moreover, since $\varphi : \mathbf{A}_R^+ \to \mathbf{A}_R^+$ is flat, therefore, using the finite $[p]_q$-height property of $\mathbf{N}_R(T_1)$ and $\mathbf{N}_R(T_2)$, we get that $N$ is of finite $[p]_q$-height, i.e. $1 \otimes \varphi : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. In particular, $N[1/p]$ is finite projective over $\mathbf{B}_R^+$ by Proposition A.1. Next, for $i \in \{1, 2, 3\}$, considering $V_i$ as a $p$-adic crystalline representation of $G_L$, from [Abh23a, Corollary 4.3], we have an exact sequence $0 \to \mathbf{N}_L(V_1) \to \mathbf{N}_L(V_2) \to \mathbf{N}_L(V_3) \to 0$ of Wach modules over $\mathbf{B}_L^+ = \mathbf{A}_L^+[1/p]$. Note that the natural map $\mathbf{B}_R^+ \to \mathbf{B}_L^+$ is flat, so we get that $\mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} N[1/p] \xrightarrow{\sim} \mathbf{N}_L(V_3)$ as $(\varphi, \Gamma_L)$-modules over $\mathbf{B}_L^+$. Moreover, from Remark 4.2, we have that $\mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} \mathbf{N}_R(V_i) \xrightarrow{\sim} \mathbf{N}_L(V_i)$, for $i \in \{1, 2, 3\}$. Now, since $N[1/p]$ is finite projective over $\mathbf{B}_R^+$, therefore, as submodules of $\mathbf{D}_L(V_3)$, we obtain an isomorphism of $(\varphi, \Gamma_R)$-modules over $\mathbf{B}_R^+$ as follows:

$$N[1/p] = (\mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} N[1/p]) \cap (\mathbf{B}_R \otimes_{\mathbf{B}_R^+} N[1/p]) \xrightarrow{\sim} \mathbf{N}_L(V_3) \cap \mathbf{D}_R(V_3) \xleftarrow{\sim} \mathbf{N}_R(V_3).$$

Hence, (4.1) is exact, concluding our proof. ∎

We obtain applications of Theorem 4.1 as follows:

**Theorem 4.5.** *Let $V$ be a $p$-adic representation of $G_R$. Then the following are equivalent:*

(1) *$V$ is crystalline as a representation of $G_R$;*

(2) *$V$ is crystalline as a representation of $G_L$;*

(3) $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$.

*Proof.* Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$, then obviously we have that $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$. Conversely, let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$ and choose a $G_R$-stable $\mathbb{Z}_p$-lattice $T \subset V$ such that $T$ is of finite $[p]_q$-height as a representation of $G_L$. Then, using Proposition 4.7, note that $T$ is of finite $[p]_q$-height as a representation of $G_R$. Therefore, $V = T[1/p]$ is a crystalline representation of $G_R$ by Theorem 3.27. This shows the equivalence of (1) and (2).

Next, if $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$, then $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$ (see §2.6), proving that (1) implies (3). So it remains to show that (3) implies (2). Let $V$ be a $p$-adic representation of $G_R$ such that $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. From [Bri06, §3.3] recall that $V$ is crystalline for $G_L$ if and only if $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \dim_{\mathbb{Q}_p} V$. So we will show that $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \dim_{\mathbb{Q}_p} V$ by constructing a natural isomorphism of $L$-vector spaces $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Since $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \leq \dim_{\mathbb{Q}_p} V$, it is enough to construct a natural $L$-linear injective map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \to \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and the claim would follow by considering $L$-dimensions.

From Remark 2.16, note that we have a natural $(\varphi, G_R)$-equivariant $L$-linear injective map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Tensoring this map with $V$ and considering the diagonal action of $G_R$, we obtain a $(\varphi, G_R)$-equivariant injective map

$$L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V \longrightarrow \Big(\prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))\Big) \otimes_{\mathbb{Q}_p} V = \prod_{\mathfrak{p} \in \mathcal{S}} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V). \tag{4.2}$$

The map in (4.2) further induces a natural map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V$, compatible with the respective Frobenii, filtrations and connections (see Remark 2.16). Now, we take the

$G_R$-invariant part of (4.2) and note that product commutes with the left exact functors, in particular, with taking $G_R$-invariants. So we obtain $\varphi$-equivariant $L$-linear injective maps

$$\begin{aligned} L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) &\longrightarrow \Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V\Big)^{G_R} \\ &= \prod_{\mathfrak{p}\in\mathcal{S}} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R} \\ &\longrightarrow \prod_{\mathfrak{p}\in\mathcal{S}} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}, \end{aligned} \tag{4.3}$$

where note that the last arrow is injective since $G_R(\mathfrak{p}) \subset G_R$ is a subgroup. Moreover, since $G_R$ acts transitively on $\mathcal{S}$, it transitively permutes the components of $\prod_{\mathfrak{p}\in\mathcal{S}}(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}$, i.e. if $0 \neq x \in L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, then its image $(x_\mathfrak{p})_{\mathfrak{p}\in\mathcal{S}}$ under the composition (4.3) satisfies that $x_\mathfrak{p} \neq 0$, for all $\mathfrak{p} \in \mathcal{S}$. Therefore, for each $\mathfrak{p} \in \mathcal{S}$, composing (4.3) with the natural $\varphi$-equivariant $L$-linear projection $\prod_{\mathfrak{p}\in\mathcal{S}}(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})} \to (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}$ gives a natural $\varphi$-equivariant $L$-linear injective map

$$L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \longrightarrow (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}, \tag{4.4}$$

compatible with the respective Frobenii, filtrations and connections (see above and Remark 2.16), where the left-hand term is equipped with the tensor product Frobenius, the filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ and natural connection.

Finally, from Lemma 2.13, recall that we have a natural $L$-linear $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant injective map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$ compatible with respective filtrations and connections and where the $\widehat{G}_R(\mathfrak{p})$-action on the left term factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Tensoring the preceding injective map with $V$, equipping each term with the diagonal action of $\widehat{G}_R(\mathfrak{p})$ and taking $\widehat{G}_R(\mathfrak{p})$-invariants produces a natural $L$-linear injective map $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))\otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})} \to \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, compatible with the respective Frobenii, filtrations and connections. Composing (4.4) with the preceding $L$-linear map gives a natural $L$-linear injective map

$$L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \to \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \tag{4.5}$$

compatible with the respective Frobenii, filtrations and connections. By considering $L$-dimensions, it follows that (4.5) is bijective (see Corollary 4.6 for a stronger statement). Hence, $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \mathrm{rk}_{R[1/p]}\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$, showing that (3) implies (2). This concludes our proof. ■

**Corollary 4.6.** *Let $V$ be a $p$-adic representation of $G_R$. Under the equivalent conditions of Theorem 4.5, we have a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ of filtered $(\varphi, \partial)$-modules over $L$.*

*Proof.* Assume that $V$ is a crystalline representation of $G_R$, so that we have a natural $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-linear isomorphism $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$, compatible with the respective Frobenii, filtrations, connections and $G_R$-actions (see [Bri08, Proposition 8.4.3]). For any $\mathfrak{p} \in \mathcal{S}$, by base changing the preceding isomorphism along the composition $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$, we get a $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$-linear isomorphism

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Q}_p} V, \tag{4.6}$$

compatible with the respective Frobenii, filtrations, connections and $\widehat{G}_R(\mathfrak{p})$-actions. In (4.6), by taking $\widehat{G}_R(\mathfrak{p})$-invariants we get (4.5), i.e. $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, and by construction, the preceding isomorphism is compatible with the respective Frobenii, filtrations and connections. Hence, the claim follows. ■

**4.2. Main ingredients for the proof of Theorem 4.1.** In this subsection, let $T$ be a finite free $\mathbb{Z}_p$-representation of $G_R$ such that $T$ is a finite $[p]_q$-height representation of $G_L$ (see Definition 3.20 and [Abh23a, Definition 3.7]). In particular, we can attach to $T$ a $(\varphi, \Gamma_R)$-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$, as well as, a Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$. Our goal is to prove the following claim:

**Proposition 4.7.** *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ satisfies all the axioms of Definition 3.20. In particular, $T$ is a finite $[p]_q$-height representation of $G_R$.*

*Proof.* It is immediate that $\mathbf{N}_R(T)$ is $p$-torsion free and $\mu$-torsion free. From Lemma 4.8 and its proof, note that $\mathbf{N}_R(T)$ is finitely generated over $\mathbf{A}_R^+$ and we have that $\mathbf{N}_R(T)/p \subset (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$, in particular, $\mathbf{N}_R(T)/p$ is $\mu$-torsion free. Next, from Lemma 4.9, we know that $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. the cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$, where $s$ is the height of $\mathbf{N}_L(T)$. Furthermore, from Proposition 4.12 we have that $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$. Finally, recall that the action of $\Gamma_L$ is trivial on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ and $\Gamma_L \xrightarrow{\sim} \Gamma_R$, so for any $g \in \Gamma_R$, we have $(g-1)\mathbf{N}_L(T) \subset \mu\mathbf{N}_L(T)$. Therefore, we get that $(g-1)\mathbf{N}_R(T) \subset (\mu\mathbf{N}_L(T)) \cap \mathbf{D}_R(T) = \mu\mathbf{N}_R(T)$, so it follows that $\Gamma_R$ acts trivially on $\mathbf{N}_R(T)/\mu\mathbf{N}_R(T)$. This concludes our proof. ∎

**Lemma 4.8.** *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T)$ is finitely generated.*

*Proof.* We first claim that for each $n \in \mathbb{N}_{\geq 1}$, the natural $\mathbf{A}_R^+/p^n$-linear map $\mathbf{N}_R(T)/p^n \to (\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n) \subset \mathbf{D}_L(T)/p^n$ is injective and the intersection $(\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n)$ is a finitely generated $\mathbf{A}_R^+/p^n$-module. Since $\mathbf{N}_R(T)$, $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ are $p$-torsion free, it is enough to show the claim for $n = 1$ and the claim for $n \geq 1$ can be deduced by an easy induction. So we are reduced to showing that $\mathbf{N}_R(T)/p \to (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$ is injective and $(\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p)$ is a finitely generated $\mathbf{A}_R^+/p =: \mathbf{E}_R^+$-module. Note that we have $p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_L(T) \cap \mathbf{D}_R(T) = p\mathbf{D}_R(T)$, so we get that,

$$p\mathbf{N}_R(T) \subset p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \subset p\mathbf{N}_L(T) \cap p\mathbf{D}_R(T) = p\mathbf{N}_R(T),$$

in particular, $\mathbf{N}_R(T)/p \subset \mathbf{N}_L(T)/p$. Moreover, $p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_L(T) \cap \mathbf{N}_L(T) = p\mathbf{N}_L(T)$, so we have,

$$p\mathbf{N}_R(T) \subset p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_R(T) \cap p\mathbf{N}_L(T) = p\mathbf{N}_R(T),$$

in particular, $\mathbf{N}_R(T)/p \subset \mathbf{D}_R(T)/p$.

Next, we will show that $(\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p)$ is a finitely generated $\mathbf{E}_R^+$-module. Assume that $\overline{D}_R := \mathbf{D}_R(T)/p$ is finite free (a priori it is finite projective) of rank $h$ over $\mathbf{E}_R := \mathbf{A}_R/p$. Let $\mathbf{e} = \{e_1, \ldots, e_h\}$ be a basis of $\overline{N}_L := \mathbf{N}_L(T)/p$ over $\mathbf{E}_L^+ := \mathbf{A}_L^+/p$ and $\mathbf{f} = \{f_1, \ldots, f_h\}$ a basis of $\overline{D}_R$ over $\mathbf{E}_R$. Then, for $\mathbf{E}_L := \mathbf{A}_L/p$, we have $\mathbf{f} = A\mathbf{e}$, for some $A := (a_{ij}) \in \mathrm{GL}(h, \mathbf{E}_L)$, and write $A^{-1} = (b_{ij}) \in \mathrm{GL}(h, \mathbf{E}_L)$. Set $M := \oplus_{i=1}^h \mathbf{E}_R^+ f_i$, so that $M[1/\mu] = \overline{D}_R$. Let $x \in M[1/\mu] \cap \overline{N}_L$ and write $x = \sum_{i=1}^h c_i e_i = \sum_{i=1}^h d_i f_i$ with $c_i \in \mathbf{E}_L^+$ and $d_i \in \mathbf{E}_R$, for all $1 \leq i \leq h$. So we obtain that $d_i = \sum_{j=1}^h b_{ji} c_j$, for all $1 \leq i \leq h$. In particular, for some $k$ large enough, we have $d_i \in \mu^{-k}\mathbf{E}_L^+$, for all $1 \leq i \leq h$. Note that $\mu^{-k}\mathbf{E}_L^+ \cap \mathbf{E}_R = \mu^{-k}\mathbf{E}_R^+$, so we obtain that $d_i \in \mu^{-k}\mathbf{E}_R^+$. Hence, $M[1/\mu] \cap \overline{N}_L \subset \mu^{-k}M$, in particular, $M[1/\mu] \cap \overline{N}_L = \overline{D}_R \cap \overline{N}_L$ is finitely generated over $\mathbf{E}_R^+$.

In general, when $\overline{D}_R$ is finite projective, we choose an $\mathbf{E}_R^+$-module $D'$ such that $\overline{D}_R \oplus D' = \mathbf{E}_R^{\oplus k}$, for some $k \in \mathbb{N}$. Let $D'_L := \mathbf{E}_L \otimes_{\mathbf{E}_R} \overline{D}_R$, so we have that $\overline{D}_L \oplus D'_L = \mathbf{E}_L^{\oplus k}$. Note that since $\mathbf{E}_L$ is a field with ring of integers $\mathbf{E}_L^+$, therefore, we can choose a lattice of $D'_L$ over $\mathbf{E}_L^+$, i.e. there exists a free $\mathbf{E}_L^+$-submodule $N'_L \subset D'_L$ such that $N'_L[1/\mu] = D'_L$. So, we get that $\overline{N}_L \oplus N'_L$ is a free $\mathbf{E}_L^+$-module such that $\mathbf{E}_L \otimes_{\mathbf{E}_L^+} (\overline{N}_L \oplus N'_L) = \overline{D}_L \oplus D'_L = \mathbf{E}_L^{\oplus k}$. Inside $\mathbf{E}_L^{\oplus k}$, consider the inclusion of $\mathbf{E}_R^+$-modules

$$(\overline{D}_R \cap \overline{N}_L) \oplus (D' \cap N'_L) = (\overline{D}_R \oplus D') \cap (\overline{N}_L \oplus N'_L) \subset \mathbf{E}_R^{\oplus k} \cap (\overline{N}_L \oplus N'_L). \tag{4.7}$$

Using the conclusion in the free case from the previous paragraph, we get that the last term in (4.7) is a finite $\mathbf{E}_R^+$-module. Hence, $\overline{D}_R \cap \overline{N}_L$ is also a finite $\mathbf{E}_R^+$-module, proving the claim.

To prove the lemma, it remains to show that $\mathbf{N}_R(T)$ is $p$-adically complete. Indeed, from the claim above note that, for all $n \in \mathbb{N}_{\geq 1}$, $\mathbf{N}_R(T)/p^n$ is a finitely generated $\mathbf{A}_R^+/p^n$-module. Since $\mathbf{A}_R^+$ is noetherian, therefore, for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ as in the previous paragraph, we have a presentation $0 \to M_n \to (\mathbf{A}_R^+/p^n)^{\oplus k} \to \mathbf{N}_R(T)/p^n \to 0$, where $M_n$ is a finitely generated $\mathbf{A}_R^+/p^n$-module. By taking a finite presentation of $M_n$ as an $\mathbf{A}_R^+/p^n$-module, it is easy to see that the system $\{M_n\}_{n \in \mathbb{N}_{\geq 1}}$ is Mittag-Leffler. In particular, it follows that $\lim_n \mathbf{N}_R(T)/p^n$ is a finitely generated $\lim_n \mathbf{A}_R^+/p^n = \mathbf{A}_R^+$-module. Now consider the following natural $\mathbf{A}_R^+$-linear maps:

$$\begin{aligned} f : \mathbf{N}_R(T) \longrightarrow \lim_n \mathbf{N}_R(T)/p^n &\longrightarrow \lim_n((\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n)) \\ &\longrightarrow (\lim_n \mathbf{N}_L(T)/p^n) \cap (\lim_n \mathbf{D}_R(T)/p^n) \\ &\xrightarrow{\sim} \mathbf{N}_L(T) \cap \mathbf{D}_R(T) = \mathbf{N}_R(T), \end{aligned}$$

where the first arrow is the natural projection map, the second arrow is injective by the claim proved above, the third arrow is injective by definition and the fourth arrow is bijective since $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ are $p$-adically complete. Chasing an element of $x \in \mathbf{N}_R(T)$ through the composition, we see that $f(x) = x$. Hence, we get that $\mathbf{N}_R(T) \xrightarrow{\sim} \lim_n \mathbf{N}_R(T)/p^n$, in particular, it is a finitely generated $\mathbf{A}_R^+$-module. ∎

**Lemma 4.9.** *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. the cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$, for some $s \in \mathbb{N}$.*

*Proof.* Note that $\varphi : \mathbf{A}_R^+ \to \mathbf{A}_R^+$ is finite and faithfully flat of degree $p^{d+1}$ (see §2.2). Moreover, from §2.2 we have that $\varphi^*(\mathbf{A}_R) \xrightarrow{\sim} \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{A}_R$ and $\varphi^*(\mathbf{A}_L^+) := \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L^+ \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_L^+) u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}_R^+) u_\alpha) \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{A}_L^+) \xleftarrow{\sim} \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{A}_L^+$. Therefore, we also obtain that $\varphi^*(\mathbf{N}_L(T)) := \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_L(T)$ and $\varphi^*(\mathbf{D}_R(T)) := \mathbf{A}_R \otimes_{\varphi, \mathbf{A}_R} \mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{D}_R(T)$. Hence, as $\mathbf{A}_R^+$-submodules of $\varphi^*(\mathbf{D}_L(T))$, we have that

$$\begin{aligned} \varphi^*(\mathbf{N}_R(T)) := \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_R(T) &= \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} (\mathbf{N}_L(T) \cap \mathbf{D}_R(T)) \\ &= (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_L(T)) \cap (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{D}_R(T)) \xrightarrow{\sim} \varphi^*(\mathbf{N}_L(T)) \cap \varphi^*(\mathbf{D}_R(T)). \end{aligned}$$

Since the cokernel of the injective map $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$, for some $s \in \mathbb{N}$ and $(1 \otimes \varphi) : \varphi^*(\mathbf{D}_R(T)) \xrightarrow{\sim} \mathbf{D}_R(T)$, it easily follows that the cokernel of $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$ as well. ∎

Finally, we will show that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$. For $n \in \mathbb{N}_{\geq 1}$, let $N_{R,n} := \mathbf{N}_R(T)/p^n$, $D_{R,n} := \mathbf{D}_R(T)/p^n$, $N_{L,n} := \mathbf{N}_L(T)/p^n$, $D_{L,n} := \mathbf{D}_L(T)/p^n$ and $M_n := N_{L,n} \cap D_{R,n} \subset D_{L,n}$. Then have the following commutative diagram,

$$\begin{tikzcd} M_n \arrow[r, "f_n"] \arrow[d] & M_1 \arrow[d] \\ D_{R,n} \arrow[r, two heads, "f_n"] & D_{R,1}, \end{tikzcd}$$

where the vertical arrows are natural inclusions, the bottom horizontal arrow $f_n$ is the natural projection map and the top arrow is the induced map. We have a similar diagram with the bottom row replaced by $N_{L,n} \twoheadrightarrow N_{L,1}$.

**Lemma 4.10.** *We have the following,*

1. *$M_n$ is a finitely generated $\mathbf{A}_R^+/p^n$-module and $\mathbf{N}_R(T) \xrightarrow{\sim} \lim_n M_n$.*
2. *$M_n$ is of finite $[p]_q$-height $s$, for $s \in \mathbb{N}$ as in Lemma 4.9.*
3. *$M_n[1/\mu] = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} M_n \xrightarrow{\sim} D_{R,n}$ and $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} M_n \xrightarrow{\sim} N_{L,n}$.*

*Proof.* The claim in (1) follows from the proof of Lemma 4.8 and the claim in (2) follows similar to Lemma 4.9. As the maps $\mathbf{A}_R^+ \to \mathbf{A}_R$ and $\mathbf{A}_R^+ \to \mathbf{A}_L^+$ are flat, the last claim follows from the following equalities:

$$\begin{aligned} \mathbf{A}_R \otimes_{\mathbf{A}_R^+} M_n = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N_{L,n}) = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} D_{R,n}) = D_{R,n}, \\ \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} M_n = (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N_{L,n}) = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N_{L,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N_{L,n}) = N_{L,n}. \end{aligned}$$

Hence, the lemma is proved. ∎

Let $\mathcal{S}$ denote the set of $\mathbf{A}_R^+$-submodules $M' \subset M_1$ such that $M'$ is stable under the action of $\varphi$, it is of finite $[p]_q$-height $s$ and $M'[1/\mu] = M_1[1/\mu] = D_{R,1} = \mathbf{D}_R(T)/p$. Set $M^\circ := \cap_{M' \in \mathcal{S}} M' \subset M_1$.

**Lemma 4.11.** *The $\mathbf{A}_R^+$-module $M^\circ$ belongs to $\mathcal{S}$ and $f_n(M_n)$ is also in $\mathcal{S}$, for all $n \in \mathbb{N}_{\geq 1}$.*

*Proof.* The idea of the proof is motivated from [DLMS22, Lemma 4.25]. Let $M'$ be an element of $\mathcal{S}$. For the first claim, we need to show that there exists $r \in \mathbb{N}$ such that $\mu^r M_1 \subset M' \subset M_1$. Let $M'' := M_1/M'$ such that $M'' \neq 0$ and let $k = p(p-1)s \in \mathbb{N}$. Also, let $\varphi^*(M'') := \varphi^*(M_1)/\varphi^*(M')$ and let $1 \otimes \varphi_{M''} : \varphi^*(M'') \to M''$ denote the map induced from $1 \otimes \varphi_M$. Since $M_1$ (resp. $M'$) is of finite $[p]_q$-height $k$ (since $s < k$), we define $\psi_M : M_1 \xrightarrow{\mu^k} \mu^k M_1 \to \varphi^*(M_1)$ (resp. $\psi_{M'} : M' \xrightarrow{\mu^k} \mu^k M' \to \varphi^*(M')$) to be the unique $\mathbf{A}_R^+/p$-linear map such that $\psi_M \circ (1 \otimes \varphi_M) = \mu^k \mathrm{Id}_{\varphi^*_M}$ (resp. $\psi_{M'} \circ (1 \otimes \varphi_{M'}) = \mu^k \mathrm{Id}_{\varphi^*_{M'}}$). Let $\psi_{M''} : M'' \to \varphi^*(M'')$ denote the map induced from $\psi_M$. Now, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(M') & \longrightarrow & \varphi^*(M_1) & \longrightarrow & \varphi^*(M'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi_{M'}} & & \downarrow{\scriptstyle 1\otimes\varphi_{M}} & & \downarrow{\scriptstyle 1\otimes\varphi_{M''}} & & \\
0 & \longrightarrow & M' & \longrightarrow & M_1 & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{M'}} & & \downarrow{\scriptstyle \psi_{M}} & & \downarrow{\scriptstyle \psi_{M''}} & & \\
0 & \longrightarrow & \varphi^*(M') & \longrightarrow & \varphi^*(M_1) & \longrightarrow & \varphi^*(M'') & \longrightarrow & 0.
\end{array}$$

Note that $[p]_q = \mu^{p-1} \mod p$, $\varphi(\mu) = \mu^p \mod p$ and $\varphi([p]_q) = \mu^{p(p-1)} \mod p$. Since $M_1[1/\mu] = M'[1/\mu]$, let $i \in \mathbb{N}_{\geq 1}$ such that $\mu^{pi} M'' = 0$ and $\mu^{p(i-1)} M'' \neq 0$. Let $x \in M''$ such that $\mu^{pi} x \neq 0$ and set $y = 1 \otimes x \in \varphi^*(M'')$. Then $\varphi(\mu^{pi}) y = 1 \otimes \mu^{pi} x = 0$, but $\mu^{p^2(i-1)} y = \varphi(\mu^{p(i-1)}) y = 1 \otimes \mu^{p(i-1)} x \neq 0$. Let $z = (1 \otimes \varphi_{M''}) y \in M''$, then $\mu^{pi} z = 0$. So we have $0 = \psi_{M''}(\mu^{pi} z) = \mu^{pi}(\psi_{M''} \circ (1 \otimes \varphi_{M''}) y) = \mu^{pi+k} y$. Therefore, we get that $pi + k = pi + p(p-1)s > p^2(i-1)$, i.e. $i < s + \frac{p}{p-1}$. Hence, $\mu^{s+1} M'' = 0$. Since the constant obtained is independent of $M'$, we also get that $\mu^{s+1} M_1 \subset M^\circ \subset M_1$ and $M^\circ[1/\mu] = M_1[1/\mu]$.

Next, we will show that $M^\circ$ is of finite height $s$. Let $x \in M^\circ$, so that $x \in M'$ for each $M' \in \mathcal{S}$ and there exists some $y \in \varphi^*(M') \subset \varphi^*(M_1)$ such that $(1 \otimes \varphi) y = [p]_q^s x$. Note that $y$ is unique in $\varphi^*(M_1)$ and since $\varphi : \mathbf{A}_R^+ \to \mathbf{A}_R^+$ is flat, we get that $y \in \cap_{M' \in \mathcal{S}} (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} M') = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} (\cap_{M' \in \mathcal{S}} M') = \varphi^*(M^\circ)$. Therefore, we concldue that $M^\circ \in \mathcal{S}$.

For the second part of the claim note that $M_n[1/\mu] = D_{R,n}$ and $f_n(D_{R,n}) = D_{R,n}/p = \mathbf{D}_R(T)/p$ (see Lemma 4.10). So we get that $f_n(M_n[1/\mu]) = \mathbf{D}_R(T)/p$ and we are left to show that $f_n(M_n)$ is of finite height $s$. Note that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(\text{kernel}) & \longrightarrow & \varphi^*(M_n) & \longrightarrow & \varphi^*(f_n(M_n)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \\
0 & \longrightarrow & \text{kernel} & \longrightarrow & M_n & \xrightarrow{f_n} & f_n(M_n) & \longrightarrow & 0.
\end{array}$$

The rightmost vertical arrow is injective since $f_n(M_n) \subset D_{R,n}$ and the cokernel of the middle vertical arrow is killed by $[p]_q^s$ (see Lemma 4.10). Hence, the cokernel of the rightmost vertical arrow is also killed by $[p]_q^s$. This concludes our proof. ∎

**Proposition 4.12.** *The natural inclusion $\mathbf{N}_R(T) \subset \mathbf{D}_R(T)$ extends to a $(\varphi, \Gamma_R)$-equivariant isomorphism $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$.*

*Proof.* Since everything is $p$-adically complete and $\mathbf{D}_R(T)$ and $\mathbf{N}_R(T)$ are $p$-torsion free, it is enough to show the claim modulo $p$. Recall that we have $\mathbf{N}_R(T)/p \subset M_1 = \mathbf{D}_R(T)/p \cap \mathbf{N}_L(T)/p \subset \mathbf{D}_L(T)/p$ and from Lemma 4.11 we have $M^\circ \subset \mathbf{N}_R(T)/p$. Therefore, we get that $\mathbf{D}_R(T)/p = M^\circ[1/\mu] \subset \mathbf{A}_R/p \otimes_{\mathbf{A}_R^+/p} \mathbf{N}_R(T)/p \subset M_1[1/\mu] = \mathbf{D}_R(T)/p$. ∎

# 5. Wach modules and $q$-connections

In this section we will interpret Wach modules over $\mathbf{A}_R^+$ (resp. $\mathbf{B}_R^+$) as modules with $q$-connection and show that Wach modules over $\mathbf{B}_R^+$ can be seen as $q$-deformation of filtered $(\varphi, \partial)$-modules over $R[1/p]$, coming from $p$-adic crystalline representations of $G_R$ (see Theorem 5.6). For our definitions, we will follow [MT20, §2], with slight modifications.

**5.1. Formalism on $q$-connection.** Let $D$ be a commutative ring and consider a $D$-algebra $A$ equipped with $d$ commuting $D$-algebra automorphisms $\gamma_1 \ldots, \gamma_d$, i.e. an action of $\mathbb{Z}^d$. Moreover, fix an element $q \in D$ such that $q-1$ is a nonzerodivisor of $D$ and $\gamma_i = 1 \bmod (q-1)A$, for all $1 \leq i \leq d$. Assume that we have units $U_1, \ldots, U_d \in A^\times$ such that $\gamma_i(U_j) = qU_j$, if $i = j$ or $U_j$ if $i \neq j$. We fix these choices for the rest of the section.

**Definition 5.1** ([MT20, Definition 2.1])**.** Let $q\Omega^\bullet_{A/D} := \oplus_{k=0}^d q\Omega^k_{A/D}$ be a differential graded $D$-algebra defined as:

- $q\Omega^0_{A/D} := A$ and $q\Omega^1_{A/D}$ is a free left $A$-module on formal basis elements $d\log(U_i)$.
- The right $A$-module structure on $q\Omega^1_{A/D}$ is twisted by the rule $d\log(U_i) \cdot f = \gamma_i(f) d\log(U_i)$.
- $d\log(U_i) d\log(U_j) = -d\log(U_j) d\log(U_i)$ if $i \neq j$ and $0$ if $i = j$.
- The following map is an isomorphism of $A$-modules:
$$\begin{aligned} \oplus_{\mathbf{i} \in I_k} A &\xrightarrow{\sim} q\Omega^k_{A/D} \\ (f_{\mathbf{i}}) &\longmapsto \textstyle\sum_{\mathbf{i} \in I_k} f_{\mathbf{i}} d\log(U_{i_1}) \cdots d\log(U_{i_k}), \end{aligned}$$
where $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ such that $1 \leq i_1 < \cdots < i_k \leq d\}$.
- The $0^{\text{th}}$ differential $d_q : A \to \Omega^1_{A/D}$ is given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f) - f}{q-1} d\log(U_i)$.
- The elements $d\log(U_i) \in q\Omega^1_{D/A}$ are cocycles, for all $1 \leq i \leq d$.

The data $d_q : A \to q\Omega^1_{A/D}$ forms a differential ring over $D$, i.e. $q\Omega^1_{A/D}$ is a $D$-bimodule and $d_q$ is $D$-linear satisfying the Leibniz rule $d_q(fg) = d_q(f)g + f d_q(g)$ (see [And01, §II.1.2.1]).

**Definition 5.2** ([MT20, Definition 2.2])**.** A module with *$q$-connection* over $A$ is a right $A$-module $N$ equipped with a $D$-linear map $\nabla_q : N \to N \otimes_A q\Omega^1_{A/D}$ satisfying the Leibniz rule $\nabla_q(xf) = \nabla_q(x) f + x \otimes d_q(f)$, for all $f \in A$ and $x \in N$. The $q$-connection $\nabla_q$ extends uniquely to a map of graded $D$-modules $\nabla_q : N \otimes_A q\Omega^\bullet_{A/D} \to N \otimes_A q\Omega^{\bullet+1}_{A/D}$ satisfying $\nabla_q((n \otimes \omega) \cdot \omega') = \nabla_q(n \otimes \omega) \cdot \omega' + (-1)^{\deg \omega} (n \otimes \omega) \cdot d_q(\omega')$. The $q$-connection $\nabla_q$ is said to be *flat* or *integrable* if $\nabla_q \circ \nabla_q = 0$.

Now, assume that $D$ is equipped with an endomorphism $\varphi : D \to D$ such that it is a lift of the absolute Frobenius on $D/pD$ and $\varphi(q) = q^p$. Further, assume that $A$ is equipped with a compatible (with $\varphi$ on $D$) endomorphism $\varphi : A \to A$ such that it is a lift of the absolute Frobenius on $A/p$ and commutes with the action of $\gamma_1, \ldots, \gamma_d$ on $A$. The endomorphism $\varphi$ induces an endomorphism $\varphi_\Omega$ on $q\Omega^1_{A/D}$ given as $\varphi_\Omega(\sum_{i=1}^d f_i d\log(U_i)) = [p]_q \sum_{i=1}^d \varphi(f_i) d\log(U_i)$. In particular, from [MT20, Lemma 2.12] the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{d_q} & q\Omega^1_{A/D} \\ \varphi \downarrow & & \downarrow \varphi_\Omega \\ A & \xrightarrow{d_q} & q\Omega^1_{A/D}. \end{array}$$

It follows that given a $q$-connection $(N, \nabla_q)$ we can define the base change via Frobenius, of the $q$-connection, denoted $\varphi^*\nabla_q$ on $\varphi^*N := N \otimes_{A,\varphi} A$, as

$$\begin{aligned}\varphi^*\nabla_q : \varphi^*N &\longrightarrow N \otimes_{A,\varphi} q\Omega^1_{A/D} = \varphi^*N \otimes q\Omega^1_{A/D}\\ x \otimes f &\longmapsto (1 \otimes \varphi_\Omega)(\nabla_q(x)) \cdot f + n \otimes d_q(f).\end{aligned}$$

A $\varphi$-module with $q$-connection is a pair $(N, \nabla_q)$ as above equipped with an $A$-linear isomorphism $\varphi_N : (\varphi^*N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ such that the following diagram commutes:

$$\begin{tikzcd}(\varphi^*N)[1/[p]_q] \arrow[r, "\varphi^*\nabla_q"] \arrow[d, "\varphi_N"'] & (\varphi^*N)[1/[p]_q] \otimes q\Omega^1_{A/D} \arrow[d, "\varphi_N\otimes 1"] \\ N[1/[p]_q] \arrow[r, "\nabla_q"] & N[1/[p]_q] \otimes q\Omega^1_{A/D}.\end{tikzcd} \tag{5.1}$$

**5.2. Wach modules as $q$-deformations.** In this subsection, we take $D := O_F[\![\mu]\!]$, $A := \mathbf{A}_R^+$ equipped with the action of $\Gamma_R$ and $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of $\Gamma'_R$, the geometric part of $\Gamma_R$ (see §2). Then, by setting $q := 1 + \mu$ and $U_i := [X_i^\flat]$, for $1 \leq i \leq d$, we have $\gamma_i = 1 \bmod \mu\mathbf{A}_R^+$, for all $1 \leq i \leq d$. In particular, $\mathbf{A}_R^+$ satisfies the hypotheses of Definition 5.1. Moreover, the Frobenius endomorphism on $\mathbf{A}_R^+$ extends the Frobenius on $D$ given by identity on $\mathbb{Z}_p$ and $\varphi(\mu) = (1 + \mu)^p - 1$. Furthermore, in this case, $q\Omega^1_{\mathbf{A}_R^+/D}$ identifies with $\Omega^1_{\mathbf{A}_R^+/D}$ given as $(p, \mu)$-adic completion of the module of Kähler differentials of $\mathbf{A}_R^+$ with respect to $D$.

Note that we have a Frobenius-equivariant isomorphism of rings $\mathbf{A}_R^+/\mu \xrightarrow{\sim} R$, so from [MT20, Remarks 2.4 & 2.10], reduction modulo $q - 1$ of the differential ring $d_q : \mathbf{A}_R^+ \to \Omega^1_{\mathbf{A}_R^+/D}$ is the usual de Rham differential $d : R \to \Omega^1_R$. Similarly, the reduction modulo $q - 1$ of a module with $q$-connection over $\mathbf{A}_R^+$ (Definition 5.2) is an $R$-module with connection. We say that a $q$-connection is $(p, [p]_q)$*-adically quasi-nilpotent* (equivalently, $(p, q - 1)$*-adically* quasi-nilpotent) if $\nabla_q \bmod q - 1$ is $p$-adically quasi-nilpotent.

**Proposition 5.3.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$. Then the geometric $q$-connection*

$$\begin{aligned}\nabla_q : N &\longrightarrow N \otimes_{\mathbf{A}_R^+} \Omega^1_{\mathbf{A}_R^+/D}\\ x &\longmapsto \sum_{i=1}^{d} \tfrac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),\end{aligned}$$

*describes $(N, \nabla_q)$ as a $\varphi$-module equipped with a $(p, [p]_q)$-adically quasi-nilpotent flat $q$-connection over $\mathbf{A}_R^+$.*

*Proof.* Flatness of the $q$-connection $\nabla_q$ follows from the first part of the proof of [MT20, Proposition 2.6]. Moreover, from Definition 3.8 and Lemma 3.10, note that we have $\varphi \otimes 1 : (N \otimes_{\mathbf{A}_R^+,\varphi} \mathbf{A}_R^+)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. So we get that the pair $(N, \nabla_q)$ is a $\varphi$-module equipped with a $q$-connection over $\mathbf{A}_R^+$. Moreover, since the action of $\varphi$ and $\Gamma'_R$ commute on $N$, therefore, it follows that the corresponding diagram (5.1) is commutative. Now, from the commutativity of the action of $\varphi$ and $\Gamma_R$ and the diagram (5.1), note that we have $\nabla_q \circ \varphi = [p]_q\nabla_q \circ \varphi$. Frurthermore, from the Frobenius finite height condition on $N$, we have that for any $x$ in $N$, there exists $r \in \mathbb{N}$ large enough, such that $[p]_q^r x$ belongs to $\varphi^*(N)$. So, using the relation $\nabla_q \circ \varphi = [p]_q\nabla_q \circ \varphi$ and the fact that $[p]_q = p \bmod q - 1$, we see that $\nabla_q^k([p]_q^r x) \bmod q - 1$ converges $p$-adically to 0 as $k \to +\infty$. Hence, it follows that $\nabla_q^k(x) = [p]_q^{-r}\nabla_q([p]_q^r x)$ modulo $q - 1$ converges $p$-adically to 0, i.e. $\nabla_q$ is $(p, [p]_q)$-adically quasi-nilpotent. This concludes our proof. ∎

**Remark 5.4.** In Proposition 5.3 we call the $q$-connection "geometric" because in the definition we only use the geometric part of $\Gamma_R$, i.e. $\Gamma'_R$.

**Remark 5.5.** From §3.6 recall that we have the ring $\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \subset \mathbf{A}_{\mathrm{cris}}(R_\infty)$ stable under the Frobenius and the action of $\Gamma_R$. For $R = O_F$, we denote the aforementioned ring, i.e. $\mathbf{A}^{\mathrm{PD}}_{F,\varpi}$ by $D^{\mathrm{PD}}$ and for general $R$, we denote it by $A^{\mathrm{PD}} := \mathbf{A}^{\mathrm{PD}}_{R,\varpi}$ (we do not use $D$ and $A$ for these rings to avoid conflict with assumptions at the beginning of this subsection). Then, it is easy to see that the hypotheses of Definition 5.1 are satisfied for $D^{\mathrm{PD}}$, $A^{\mathrm{PD}}$ with $\Gamma_R$-action and $U_i := [X_i^\flat]$. Now, given a Wach module $N$ over $\mathbf{A}_R^+$, similar to Propostion 5.3, one can show that for $N^{\mathrm{PD}} := A^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N$, the $q$-connection

$$\nabla_q : N^{\mathrm{PD}} \longrightarrow N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/D^{\mathrm{PD}}}, \qquad x \longmapsto \textstyle\sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),$$

describes $(N^{\mathrm{PD}}, \nabla_q)$ as a $\varphi$-module equipped with a $p$-adically quasi-nilpotent flat $q$-connection over $A^{\mathrm{PD}}$. Set $\nabla_{q,i} := (\gamma_i - 1)/\mu$, for $1 \leq i \leq d$. Furthermore, employing arguments similar to [Abh23b, Lemmas 4.12, 5.17 & 5.18] we can show that for $1 \leq i \leq d$, the operator $\nabla_i := (\log \gamma_i)/t = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k \frac{(\gamma_i-1)^{k+1}}{k+1}$ converges as a series of operators on $N^{\mathrm{PD}}$. So using the explicit formulas described above, it is easy to see that for any $x \in N$, we have $\nabla_{q,i}(x) - \nabla_i(x) = (\frac{\gamma_i-1}{\mu} - \frac{\log\gamma_i}{t})(x) \in (\mathrm{Fil}^1 A^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} N$, since $t/\mu$ is a unit in $A^{\mathrm{PD}}$ by [Abh21, Lemma 3.14].

We are now ready to state the main result of this section. Let $N$ be a Wach module over $\mathbf{A}_R^+$ equipped with a $q$-connection as in Proposition 5.3 and a Nygaard filtration as in Definition 3.24. Then, from the discussion preceding Proposition 5.3, we note that $N/\mu N$ is a $\varphi$-module over $R$ equipped with a $p$-adically quasi-nilpotent flat connection and a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $N[1/p]/\mu N[1/p] = (N/\mu N)[1/p]$ with the induced structures, in particular, it is a filtered $(\varphi, \partial)$-module over $R[1/p]$.

**Theorem 5.6.** *Let $N$ be a Wach module over $\mathbf{A}_R^+$ and $V := \mathbf{T}_R(N)[1/p]$ the associated crystalline representation from Theorem 3.27. Then we have $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ as filtered $(\varphi, \partial)$-modules over $R[1/p]$.*

*Proof.* For $r \in \mathbb{N}$ large enough, note that the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_R(\mu^r N(-r)) = \mathbf{T}_R(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which $\Gamma_R$ acts via $\chi^{-r}$, where $\chi$ is the $p$-adic cyclotomic character). Therefore, it is enough to show both the claims for effective Wach modules. So we assume that $N$ is effective and set $M := N[1/p]$ equipped with an induced action of $\Gamma_R$, a Frobenius-semilinear operator $\varphi$ and the Nygaard filtration. It follows that the finite projective $R[1/p]$-module $M/\mu$ is equipped with a Frobenius-semilinear operator $\varphi$, induced from $M$. Note that $[p]_q = p \mod \mu\mathbf{A}_R^+$, therefore, we have $1\otimes\varphi : \varphi^*(M/\mu) \xrightarrow{\sim} M/\mu$ . Furthermore, the filtration $\mathrm{Fil}^k(M/\mu)$ is defined to be the image of $\mathrm{Fil}^k M$ under the surjective map $M \twoheadrightarrow M/\mu$. Now, recall that we have a period ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$ equipped with compatible Frobenius endomorphism $\varphi$, $\Gamma_R$-action, filtration and a connection satisfying Griffiths transversality with respect to the filtration (see §3.6). Moreover, from Theorem 3.27, we have $D_R := (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} M)^{\Gamma_R}$ equipped with a Frobenius-semilinear operator $\varphi$, filtration and a connection satisfying Griffiths transversality with respect to the filtration; we have $D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ compatible with the respective Frobenii, filtrations and connections (see (3.16) in Theorem 3.27). So, let us consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mu M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M \arrow[r] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} M \arrow[r] & R[\varpi] \otimes_R (M/\mu) \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_R D_R \arrow[r] \arrow[u, "\wr"] & \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R \arrow[r] \arrow[u, "\wr", "(3.17)"'] & R[\varpi] \otimes_R D_R \arrow[r] \arrow[u, "\wr"] & 0.
\end{tikzcd}$$

Note that $(\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} M) \cap M = (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \cap \mathbf{A}_R^+) \otimes_{\mathbf{A}_R^+} M = \mu M$,Then from the exactness of the second row it follows that the vertical maps from first to second row are natural inclusions. The middle vertical arrow from the third to the second row is the isomorphism (3.17) in Proposition 3.28 from which it can easily be shown that the left vertical arrow is an isomorphism as well, hence, it follows that the right vertical arrow is also an isomorphism. Taking the $\mathrm{Gal}(R[1/p][\varpi]/R[1/p]) = \mathrm{Gal}(F(\zeta_p)/F)$-invariants of the right vertical arrows gives $M/\mu \xleftarrow{\sim} D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, where the last isomorphism is compatible with the respective Frobenii, filtrations and connections as explained in the proof of Theorem 3.27 (see (3.16)).

Note that the isomorphism $D_R \xrightarrow{\sim} M/\mu$ is compatible with the respective Frobenii and we need to check the compatibility between the respective filtrations and connections. Let us compare the filtrations first. In the commutative diagram above, filtration on the middle term of the second row is given by the tensor product filtration, so it is easy to see that under the surjective map from second to third term, the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} M)$ is given as $R[\varpi] \otimes_R \mathrm{Fil}^k(M/\mu)$. Similarly, in the third row, the middle term is equipped with the tensor product filtration, so the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R)$ under the surjective map from the second to the third term is given as $R[\varpi] \otimes_R \mathrm{Fil}^k D_R$. Recall that the isomorphism $f$ in (3.17) of Proposition 3.28 is compatible with the respective filtrations, so we conclude that $R[\varpi] \otimes_R \mathrm{Fil}^k D_R \xrightarrow{\sim} R[\varpi] \otimes_R \mathrm{Fil}^k(M/\mu)$. Now, taking invariants under the natural $\mathrm{Gal}(F(\zeta_p)/F)$-action on both terms of the preceding isomorphism gives $\mathrm{Fil}^k D_R \xrightarrow{\sim} \mathrm{Fil}^k(M/\mu)$, i.e. the isomorphism $D_R \xrightarrow{\sim} M/\mu$ is compatible with the respective filtrations. Next, note that the connection on $M/\mu$ is obtained by first reducing, the $q$-connection $\nabla_q$ on $N$, modulo $\mu = q-1$ and then inverting $p$. On the other hand, the connection $\partial_D$ on $D_R = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} M)^{\Gamma_R}$ is induced from the natural $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$-linear connection on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Let $\nabla_{q,i}$ and $\partial_{D,i}$ respectively denote the $i^{\text{th}}$ component of the $q$-connection on $N$ and the connection on $D_R$. Now let $x \in M$, and note that from Remark 3.32 there exists $w \in \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R D_R$ such that $x = f(w) \bmod (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M$, where $f$ is the isomorphism in (3.17). Then it follows that to check the compatibility of the isomorphism $D_R \xrightarrow{\sim} M/\mu$ with connections, it is enough to show that we have $\nabla_{q,i}(x) - f(\partial_{D,i}(w)) \in (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M$. From Remark 3.32 for $\nabla_i = (\log\gamma_i)/t$, we know that $\nabla_i(x) - f(\partial_{D,i}(w)) \in (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M$. Furthermore, from Remark 5.5 we have that $\nabla_{q,i}(x) - \nabla_i(x) \in (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M$. Upon combining the two, we get that $\nabla_{q,i}(x) - f(\partial_{D_i}(w)) \in (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_R^+} M$, i.e. the isomorphism $D_R \xrightarrow{\sim} M/\mu$ is also compatible with the respective connections. This concludes our proof. ■

**Remark 5.7.** Obvious variation of Theorem 5.6 is also true in the imperfect residue field case. Indeed, for $O_L$, recall that all compatibilities except for the connection part was already proven in [Abh23a, Corollary 3.15]. To verify the compatibility of connections, similar to Proposition 5.3, we can define a $q$-connection over a Wach module over $\mathbf{A}_L^+$. Then, using the results of [Abh23a, §3.3], one obtains an obvious variation of Remark 5.5 over $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$. Finally, proceeding exactly as in the proof of Theorem 5.6 (after replacing each object by analogous object for $L$), we obtain the desired isomorphism of filtered $(\varphi, \partial)$-modules over $L$.

Let us summarise the relationship between various categories considered in (2.9), Corollary 4.4 and Theorem 5.6. Recall that $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$ is the category of $p$-adic crystalline representations of $G_R$ and $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ denotes the essential image of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ restricted to $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$.

**Corollary 5.8.** *Functors in the following diagram induce exact equivalence of ⊗-categories*

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \arrow[rr, shift left, "\mathbf{N}_R"] \arrow[dr, shift left, "\mathcal{O}\mathbf{D}_{\mathrm{cris},R}"] & & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{B}_R^+}^{[p]_q} \arrow[ll, shift left, "\mathbf{V}_R"] \arrow[dl, "q \mapsto 1"] \\
& \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial). \arrow[ul, shift left, "\mathcal{O}\mathbf{V}_{\mathrm{cris},R}"] &
\end{tikzcd}$$

*Proof.* The exact equivalence induced by functors $\mathbf{N}_R$ and $\mathbf{V}_R$ is from Corollary 4.4 and the exact equivalence induced by $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ and $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}$ is from [Bri08, Théorème 8.5.1]. Moreover, from Theorem 5.6, note that for a Wach module $M$ over $\mathbf{B}_R^+$ we have $M/(q-1) = M/\mu \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(\mathbf{V}_R(M))$. Hence, from the preceding exact equivalence of ⊗-categories, it follows that the slanted arrow labelled "$q \mapsto 1$" is also an exact equivalence of ⊗-categories. ■

# A. Structure of $\varphi$-modules

We will use setup and notations from §1.5 and the rings defined in §2.2. Let $q$ be an indeterminate and recall that we have a Frobenius-equivariant isomorphism $R[\![q-1]\!] \xrightarrow{\sim} \mathbf{A}_R^+$, via the map $X_i \mapsto [X_i^\flat]$ and $q \mapsto 1+\mu$. We will show the following structural result:

**Proposition A.1.** *Let $N$ be a finitely generated $\mathbf{A}_R^+$-module and suppose that $N$ is equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p_q]]$. Then $N[1/p]$ is finite projective over $\mathbf{B}_R^+$.*

*Proof.* The proof is essentially the same as [DLMS22, Proposition 4.13]. Compared to loc. cit., the Frobenius endomorphism on $\mathbf{A}_R^+$ and finite height assumption on $N$ are different and we do not assume $N$ to be torsion free. However, one observes that torsion freeness of $N$ is not used in the proof and one can use [Abh23a, Lemma 2.14] and Lemma A.2 instead of [BMS18, Proposition 4.3] and [DLMS22, Lemma 4.12]. ■

**Lemma A.2.** *Let $k$ be a perfect field of characteristic $p$ and $S := W(k)[\![u_1, \ldots, u_m]\!]$ equipped with a Frobenius endomorphism $\varphi$ extending the Witt vector Frobenius on $W(k)$ such that $\varphi(u_i) \in S$ has zero constant term for each $1 \leq i \leq m$. Let $A := S[\![q-1]\!]$ equipped with a Frobenius endomorphism extending the one on $S$ by $\varphi(q) = q^p$ and let $N$ be a finitely generated $A$-module equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then $N[1/p]$ is finite projective over $A[1/p]$.*

*Proof.* The proof is essentially the same as [DLMS22, Lemma 4.12], except for a few changes. One proceeds by induction on $m$. The case $m = 0$ follows from [Abh23a, Lemma 2.14], so let $m \geq 1$. Take $J$ to be the smallest non-zero Fitting ideal of $N$ over $A$. It suffices to show that $JA[1/p] = A[1/p]$. Compatibility of Fitting ideals under base change implies that $JA[1/[p]_q] = \varphi(J)A[1/[p]_q]$ as ideals of $A[1/[p]_q]$, therefore, $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$. Let us assume $JA[1/p] \neq A[1/p]$ and we will show a contradiction.

In our setting, the Frobenius endomorphism on $A$ and the finite height condition are different from [DLMS22, Lemma 4.12]. Therefore, we need some modifications in the arguments of loc. cit.; let us point out the differences in terms of their notations. Let $K = W(k)[1/p]$, fix $\overline{K}$ as an algebraic closure of $K$. Consider the $\overline{K}$-valued points of $\mathrm{Spec}\,(A[1/p]/J)$ and let $Z = \{(|u_1|, \ldots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}\}$ be the corresponding set of $(m+1)$-tuple norms. Define the set $Z' = \{(|u_1|, \ldots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}$ such that $(|\varphi(u_1), \ldots |\varphi(u_m)|, |q^p - 1|) \in Z\}$ and take $\zeta_p - 1$ as the chosen uniformiser. Then, one proceeds as in loc. cit. to show that $JA[1/p] \subset (u_1, \ldots, u_m, q-1)A[1/p]$ and $JA[1/p] \not\subset IA[1/p]$, where $I = (u_1, \ldots, u_m) \subset A[1/p]$.

Finally, consider the Frobenius-equivariant projection $A \to \overline{A} = A/I = W(k)[\![q-1]\!]$ and let $\overline{J} \subset \overline{A}$ denote the image of $J$. Since $JA[1/p] \not\subset IA[1/p]$, we get that $\overline{J} \neq 0$. Moreover, $\overline{JA}[1/p] \neq \overline{A}[1/p]$ since $JA[1/p] \subset (u_1, \ldots, u_m, q-1)A[1/p]$. However, the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$ implies that $(\overline{A}/\overline{J})[1/[p]_q] = (\overline{A}/\varphi(\overline{J}))[1/[p]_q]$, i.e. $\overline{JA}[1/p] = \overline{A}[1/p]$ by inductive hypothesis (see [Abh23a, Lemma 2.14]). This gives a contradiction. Hence, we must have $JA[1/p] = A[1/p]$, thus proving the lemma. ■

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