GLOBAL CAUCHY PROBLEM FOR SEMILINEAR HYPERBOLIC SYSTEMS
WITH NON-LOCAL INTERACTIONS. APPLICATIONS TO DIRAC
EQUATIONS

ALAIN BACHELOT

Abstract. We investigate the global Cauchy problem for a class of semilinear hyperbolic systems
where the interaction can be non local in space and time. We establish global existence theorems
for the initial value problem when the non linearity is dissipative in a weak sense, and satisfies the
causality condition. The argument is abstract and the technique is based on the non-linear resolvent.
We apply these results to get low regularity global solutions of several models for relativistic field
theory: the Dirac-Maxwell-Klein-Gordon system, and the Thirring model on the Minkowski space-
time \( \mathbb{R}^{1+1} \); the Dirac-Klein-Gordon system on Schwarzschild type manifolds, or outside a star
undergoing a gravitational collapse to a black-hole.

I. INTRODUCTION

Semilinear hyperbolic systems with local quadratic interactions arise in various contexts such as,
discrete kinetic theory, wave propagation, etc. Several results concerning the existence of global
solutions are known (see e.g. in one space dimension, Aregba-Driollet, B. Hanouzet [1]). Having in
mind applications to models of relativistic field theory, we investigate the global Cauchy problem
for a class of systems of the following type

\[
\partial_t u(t, x) + A(t, x, \nabla x)u(t, x) = [F(u)](t, x), \quad t \in [0, \infty[, \quad x \in \mathbb{R}^d, \quad u(t, x) \in \mathbb{C}^N,
\]

where the non linearity \( F(u) \) is allowed to be non local both in space and in time: more precisely, \( F \) is merely assumed to be a continuous map on \( \mathbb{C}^0([0, \infty[, L^2(\mathbb{R}^d)) \), and to satisfy the causality
condition:

\[
\forall T > 0, \quad (t \in [0, T] \Rightarrow u(t, .) = 0) \Rightarrow (t \in [0, T] \Rightarrow [F(u)](t, .) = 0).
\]

Then we prove that the global Cauchy problem is well posed in many cases where the \( L^2 \) norm of
local solutions is well controled.

A motivation of this work, is the Dirac-Klein-Gordon system on some lorentzian manifolds \( \mathcal{M} \),

\[
i\gamma^\mu \nabla_\mu \Psi + M \Psi = \kappa \Phi \Psi,
\]

\[
\nabla_\mu \nabla^\mu \Phi + m^2 \Phi = \kappa \Psi^* \gamma^0 \Psi,
\]
especially in the framework of General Relativity, when $\mathcal{M}$ describes a Schwarzschild type Black-Hole, or the gravitational collapse of a star.

The study of global solutions of this important model of relativistic field theory has a long history (see in particular [2], [8] to [15]), however many difficult open problems remain, even for the two dimensional Minkowski space-time $\mathcal{M} = \mathbb{R}^{1+1}$, such as the existence of global solutions with low regularity. This question has recently been investigated by N. Bourrouilles [9] and Y. Fang [17]; they solved the global Cauchy problem for initial data in $L^2(\mathbb{R}_x; \mathbb{C}^2) \times [H^1(\mathbb{R}_x; \mathbb{R}) \times L^2(\mathbb{R}_x; \mathbb{R})]$. The key point of their proofs is a rather surprising fact: if $\Psi \in C^0([0, T]; L^2(\mathbb{R}_x))$ is a solution, then $\Psi^* \gamma^0 \Psi \in L^2([0, T]_t \times \mathbb{R}_x)$ even though $\Psi \notin L^\infty$. This phenomenon, that has already been noted by M. Beals and M. Beznard [8] for local solutions in $\mathbb{R}^{1+3}$, is due to the algebraic properties of the quadratic form $\Psi^* \gamma^0 \Psi$, the so called compatibility condition associated with Lorentz invariance (see [2]), that is close to the null condition of S. Klainerman [24]; we clarify the role of this property. In fact we show that it is not needed to assure global existence of low regularity solutions with initial data in $L^2(\mathbb{R}_x; \mathbb{C}^2) \times [H^s(\mathbb{R}_x; \mathbb{R}) \times H^{s-1}(\mathbb{R}_x; \mathbb{R})], \frac{1}{2} < s \leq 1$. It is only useful to get the $H^s$ regularity of the scalar field $\Phi$. Another interesting case is that of the 1+3 dimensional lorentzian manifolds arising in General Relativity, in particular those describing black-holes. Since the seminal work by Y. Choquet-Bruhat and D. Christodoulou on the massless fields [13], few results have been published concerning global solutions of the non linear spin field equations on black-hole spacetimes (see e.g. [7], [30], [31], [34]). The major difficulty arising in the study of nonlinear massive Dirac equations is that the energy is not positive definite. However, the conservation of the charge of the spinor, that is to say the conservation of the $L^2$ norm of the spinor field, can be used to derive global existence results for arbitrarily large $L^2$ initial data, in 1 + 1 dimension, and also in 1 + 3 dimension when the fields satisfy a condition of symmetry.

The basic idea of our approach is quite natural and simple. The non linearity $\kappa \Phi \Psi$ in the Dirac equation (1.3), that is obviously local when we consider the unknown $(\Psi, \Phi)$, can be considered as a non local nonlinearity with respect to the single unknown $\Psi$, by solving the Klein-Gordon equation (1.4), given the initial data for $\Phi$. In the first step, we solve the Dirac equation with this non local nonlinearity, then we solve the inhomogeneous linear Klein-Gordon equation. The method for solving the non linear equation is very general, so we develop in part II an abstract setting that allows to treat many examples. With the weak assumption of dissipation that we want to consider, it is difficult to get directly an a priori estimate, and the usual fixed point method involving the iterative scheme

$$n\mu \gamma^\mu \Psi = i\Psi, \quad \Phi = 0 \text{ on } \partial \mathcal{M},$$

is not convenient to get global solutions. Instead, we adopt a technique based on the nonlinear resolvent associated with $F$, and the global solution $u$ of (1.1) is obtained as follows: we solve

$$\partial_t u_n(t, x) + Au_n(t, x) = [F(\mu_{n-1})](t, x),$$

then we prove that $T_n \to \infty$, and $u := \lim_n u_n$ is the global solution of (1.1).

In part III we use our abstract results to solve the global Cauchy problem for general conservative semilinear hyperbolic systems. We also establish decay of the local energy. In the last part, we apply these results to get low regularity global solutions of important equations of field theory and general relativity: the Dirac-Maxwell-Klein-Gordon system and the Thirring model on $\mathbb{R}^{1+1}$, and the Dirac-Klein-Gordon system on Schwarzschild type manifolds. Our method is sufficiently general to be able to allow the treatment of moving boundary mixed problems: we consider the interesting case of the Dirac-Klein-Gordon system outside a star outgoing a gravitational collapse.
Global Cauchy Problem for Semilinear Hyperbolic Systems

II. Abstract setting

Let $X^0$ be a Banach space, non necessarily reflexive, and denote its norm by $\| \cdot \|_0$. Given $T^* \in [0, \infty]$, $x \in X^0$, we investigate the Cauchy problem

\begin{equation}
\frac{du(t)}{dt} + A_t u(t) = [F(u)](t), \quad 0 \leq t < T^*,
\end{equation}

\begin{equation}
 u(0) = x.
\end{equation}

Here $A_t$ is the generator of a contraction semi-group for each $t$, and $F$ is a function defined on the space $C^0([0, T^*]; X^0)$, satisfying classical Lipschitz conditions, and the property of causality, i.e. if $u = 0$ on $[0, T]$, then $F(u) = 0$ on $[0, T]$. As examples of non-local in time nonlinearities, with some additional assumption of dissipative type, we shall be able to treat the integral interactions:

\begin{equation}
[F(u)](t) = \int_0^t f(t, s, u(s))ds, \quad f : [0, T^*] \times [0, T^*] \times X^0 \rightarrow X^0,
\end{equation}

and the time-delay nonlinearities:

\begin{equation}
[F(u)](t) = f(t, u(\alpha(t))), \quad \alpha : [0, T^*] \rightarrow [0, T^*], \quad \alpha(t) \leq t.
\end{equation}

We now describe the different assumptions.

As regards the left member of equation (II.1), we assume that the linear problem is well posed. More precisely, $(A_t)_{0 \leq t < T^*}$ is a family of linear densely defined operators on $X^0$ satisfying the following conditions:

**Assumption II.1.** There exists a dense subspace $X^1$ of $X^0$, contained in the domain of $A_t$ for any $t \in [0, T^*]$. $X^1$ is a Banach space for a norm $\| \cdot \|_1$, and there exists a constant $C > 0$ such that $\| x \|_0 \leq C \| x \|_1$ for any $x \in X^1$. The function $A_t$ of $t$ is continuous in the norm of $\mathcal{L}(X^1, X^0)$.

We recall that a contraction propagator on $X^0$, is a family $(U(t, s))_{(s, t) \in \Delta}$ where

\[ \Delta := \{(s, t) \in \mathbb{R}^2; \quad 0 \leq s \leq t < T^* \} \]

and $U(t, s)$ is a map defined from $X^0$ to $X^0$, satisfying:

\begin{equation}
U(s, s) = Id, \quad 0 \leq s \leq r \leq t < T^* \implies U(t, s) = U(t, r)U(r, s),
\end{equation}

\begin{equation}
\forall x \in X^0, \quad \| U(t, s)x \|_0 \leq \| x \|_0,
\end{equation}

\begin{equation}
\forall x \in X^0, \quad U(t, s)x \in C^0(\Delta, X^0).
\end{equation}

**Assumption II.2.** There exists a contraction propagator $(U_0(t, s))_{(s, t) \in \Delta}$, such that for all $(s, t) \in \Delta$, $U_0(t, s)$ is a linear operator on $X^0$, satisfying:

\begin{equation}
U_0(t, s) \in \mathcal{L}(X^1),
\end{equation}

\begin{equation}
\forall x \in X^1, \quad U_0(t, s)x \in C^1(\Delta, X^0) \cap C^0(\Delta, X^1), \quad \left\{ \begin{array}{l}
\partial_t [U_0(t, s)x] = -A_t U_0(t, s)x, \\
\partial_s [U_0(t, s)x] = U_0(t, s)A_s x.
\end{array} \right\}
\end{equation}

Since the seminal works by T. Kato, numerous results assure the existence of the propagator $U_0(t, s)$ when certain conditions on $(A_t)_{0 \leq t < T^*}$ are assumed (see e.g. [23], [37], [38]).

The nonlinear function $F$ can be non-local in time, but satisfies a causality condition and a Lipschitz property on $X^2$. For $T \in [0, T^*]$, we denote

\begin{equation}
X^0_T := C^0([0, T]; X^0), \quad \| u \|_{X^0_T} := \sup_{t \in [0, T]} \| u(t) \|, \quad X^1_T := C^0([0, T]; X^1), \quad X^2_T := C^0([0, T]; X^2).
\end{equation}
Proof of Theorem II.5. The main idea, used by Iannelli [22] for the local nonlinearity \(F(u)(t) = f(u(t))\), consists in solving a sequence of approximate problems:

\[
\frac{du_n(t)}{dt} + A_t u_n(t) = \left[ F\left( (n-F)^{-1}(nu_n) \right) \right](t), \quad 0 \leq t \leq T_n,
\]

(II.17)

\[ u_n(0) = x. \]

(II.18)

We have replaced the nonlinearity \(F(u)\), which is estimated for \(|u| \leq r\), by \(|F(u)| \leq C(r)\) \(|u|\), by \(F(\left( (n-F)^{-1}(nu_n) \right) = -nu + n(n-F)^{-1}(nu)\) with \(n \geq C(r)\). The nice gain is that, the linear part \(-nu\) is dissipative, and the nonlinear part \((n-F)^{-1}(nu)\) is linearly bounded, \(|(n-F)^{-1}(nu)| \leq \|u\|\). Therefore we shall be able to prove that

\[
\|u_n\|_{X^0_T} \leq \|x\|_0,
\]

(II.19)

\[ T_n \to T^*, \quad n \to \infty. \]

(II.20)

The key tool will be the nonlinear resolvent \((n-F)^{-1}\). To construct it, we start with some generalities.

We consider a Banach space \((X, \|\cdot\|)\) and a map \(F\) from \(X\) to \(X\) such that

\[
F(0) = 0,
\]

(II.21)
Lemma II.6. For all \( \lambda > C(r) \), \( \lambda \text{Id} - F \) is injective from \( B'(0, r) := \{ x; \| x \| \leq r \} \) to \( X \) and we have :

\[
B'(0, \lambda r - r C(r)) \subset (\lambda \text{Id} - F) B'(0, r),
\]

and if \( x_j \in B'(0, r) \), \( y_j = (\lambda \text{Id} - F)(x_j) \), we have :

\[
\| x_1 - x_2 \| \leq \frac{1}{\lambda - C(r)} \| y_1 - y_2 \|.
\]

Moreover, if \( F \) satisfies

\[
\forall \lambda > 0, \forall x \in X, \lambda \| x \| \leq \| \lambda x - F(x) \|,
\]

then :

\[
\forall r > 0, \lambda > 2 C(2r) \implies B'(0, \lambda r) \subset (\lambda \text{Id} - F) B'(0, r).
\]

Proof of Lemma II.6. For \( \lambda > C(r) \), the map \( \lambda^{-1} F \) is strictly contracting from \( B'(0, r) \) to \( X \). Hence the contracting mapping principle (see e.g. [36], theorems 1.17 and 1.18), assures that \( \text{Id} - \lambda^{-1} F \) is injective, \( (\text{Id} - \lambda^{-1} F) B'(0, r) \) covers \( B'(0, r - r C(\lambda)^{-1}) \), and the inverse \( (\text{Id} - \lambda^{-1} F)^{-1} \) which is defined on \( (\text{Id} - \lambda^{-1} F) B'(0, r) \), satisfies the Lipschitz condition with constant \( \lambda (\lambda - C(r))^{-1} \).

That proves (II.23) and (II.24). At last, by (II.23), we have for \( \lambda > 2 C(2r) \):

\[
B'(0, \lambda r) \subset B'(0, 2 \lambda r - 2 r C(2r)) \subset (\lambda \text{Id} - F) B'(0, 2r).
\]

Hence given \( y \in B'(0, \lambda r) \) there exists a unique \( x \in B'(0, 2r) \) such that \( y = \lambda x - F(x) \). Now (II.25) implies that \( x \in B'(0, r) \).

Q.E.D.

We introduce the non-linear resolvent :

\[
R^r(\lambda) x := (\lambda \text{Id} - F)^{-1}(\lambda x),
\]

defined for \( \lambda > C(r) \) on the domain

\[
D(R^r(\lambda)) := B' \left( 0, \left( 1 - \frac{C(r)}{\lambda} \right) r \right),
\]

or, when (II.25) is satisfied, for \( \lambda > 2 C(2r) \):

\[
D(R^r(\lambda)) := B'(0, r).
\]

As direct consequences of the previous Lemma, we have :

Lemma II.7. If \( 0 < r_1 < r_2 \) and \( \lambda > C(r_2) \), then \( R^{r_1}(\lambda) x = R^{r_2}(\lambda) x \) for all \( x \in D(R^{r_1}(\lambda)) \cap D(R^{r_2}(\lambda)) \). \( R^r(\lambda) \) is a Lipschitz map from \( D(R^r(\lambda)) \) to \( B'(0, r) \):

\[
\forall x_j \in D(R^r(\lambda)), \quad \| R^r(\lambda) x_1 - R^r(\lambda) x_2 \| \leq \frac{\lambda}{\lambda - C(r)} \| x_1 - x_2 \|.
\]

Moreover, if \( F \) satisfies (II.25), then

\[
\forall x \in B'(0, r), \quad \forall \lambda > 2 C(2r), \quad \| R^r(\lambda) x \| \leq \| x \|.
\]
Taking advantage of the nonlinear resolvant, we now construct an approximation of $F$. For $x \in D(R^r(\lambda))$ we put

$$(II.32)\quad F_\lambda(x) := F(R^r(\lambda)x).$$

The first assertion of the preceding Lemma, assures that given $x \in X$, $R_r(\lambda)x$ is independent of $r$ satisfying

$$(II.33)\quad \| x \| \leq r \left( 1 - \frac{C(r)}{\lambda} \right).$$

Hence definition (II.32) makes sense. As an example, for $X = \mathbb{R}$ and $F(x) = x^2$, we have for $\lambda > 4 \cdot | x |$, $F_\lambda(x) = -\lambda x + 2\lambda \left( 1 + \sqrt{1 - \frac{4x}{\lambda}} \right)^{-1} x$. We show that $F_\lambda \to F$ as $\lambda \to \infty$:

**Lemma II.8.** Given $x \in B'(0, r)$, we have for $\lambda > 2C(2r)$:

$$(II.34)\quad \| F_\lambda(x) - F(x) \| \leq \frac{2r(C(2r))^2}{\lambda}.$$ 

**Proof of Lemma II.8.** For $r = \| x \|$, we have for $\lambda > 2C(2r)$, $B'(0, r) \subset D(R^{2r}(\lambda))$ and $F_\lambda(x) = F(R^{2r}(\lambda)x)$. We have by (II.30):

$$(II.35)\quad \| R^{2r}(\lambda)x \| \leq \frac{\lambda}{\lambda - C(2r)} r \leq 2r,$$

dence we deduce from (II.22):

$$(II.36)\quad \| F_\lambda(x) - F(x) \| \leq C(2r) \| R^{2r}(\lambda)x - x \|.$$ 

We conclude by noting that:

$$(II.37)\quad \| R^{2r}(\lambda)x - x \| = \| \frac{1}{\lambda} F(R^{2r}(\lambda)x) \| \leq \frac{2rC(2r)}{\lambda}.$$ 

Q.E.D.

We now return to the proof of the Theorem. Given $x \in X^1$, $u \in C^1 \left( [0, T^*]; X^0 \right) \cap X^1_{T^*}$ is solution of (II.1), (II.2), iff for all $t \in [0, T^*]$, $u$ is solution of the integral equation:

$$(II.38)\quad u(t) = U_0(t, 0)x + \int_0^t U_0(t, s) [F(u)](s)ds.$$ 

Given $t \in [0, T^*]$, (II.8), (II.9) and the Banach Steinhaus Theorem assure that there exists $K_t \in C^0([0, T^*]; \mathbb{R}^+)$ such that

$$(II.39)\quad 0 \leq s \leq t < T^* \Rightarrow \| U_0(t, s) \|_{L(X^1)} \leq K_t.$$ 

Therefore, thanks to Assumptions II.2 and II.3, the standard arguments based on the Gronwall Lemma, assure that equation (II.38) has at most one solution, and there exists a local solution in $C^1 \left( [0, T_*]; X^0 \right) \cap X^1_{T_*}$ for some $T_* \in [0, T^*]$. We prove that $T_*$ can be taken arbitrarily closed of $T^*$.

First, thanks to (II.12), we remark that when $u_1, u_2 \in X^0_{T^*}$ are equal for $t \in [0, T]$, then $F(u_1) = F(u_2)$ on this interval. Therefore given $u \in X^0_{T^*}$, $t \in [0, T]$, $[F(u)](t)$ is independent of $u \in X^0_{T^*}$ such that $u = \tilde{u}$ on $[0, T]$. Hence we introduce for $T \in [0, T^*]$ :

$$(II.40)\quad F_T : u \in X^0_{T^*} \mapsto (t \in [0, T] \mapsto [F(u)](t)) \in X^0_T,$$

and $u$ is solution of (II.38) on $[0, T]$, iff it is solution of:

$$(II.41)\quad u(t) = U_0(t, 0)x + \int_0^t U_0(t, s) [F_T(u)](s)ds, \quad 0 \leq t \leq T.$$
We now apply the previous results to \( X = X^j_T \), and \( F_T \) that satisfies (II.21) and (II.22) with constant \( C_T(r) \) for \( j = 0, 1 \), and also (II.25) for \( j = 0 \). For \( \lambda > 2C_T(2r) \), we denote \( R_T^r(\lambda) \) the nonlinear resolvent associated with \( F_T \). Then \( R_T^r(\lambda)u \) is well defined in \( X^0_T \) for \( u \in X^0_T \) with \( \| u \|_{X^0_T} \leq r \), and \( R_T^r(\lambda) u \in X^1_T \) if \( \| u \|_{X^0_T} \leq r(1 - C_T(r)/\lambda) \).

The causality is respected by the resolvants:

**Lemma II.9.** Given \( 0 \leq T_1 \leq T_2 \), \( u_j \in X^0_T \), \( \| u_j \|_{X^0_T} \leq r \), if \( u_1(t) = u_2(t) \) for all \( t \in [0, T_1] \), then \( R_{T_1}^r(\lambda) u_1 = R_{T_1}^r(\lambda) u_2 \) on \([0, T_1]\) for any \( \lambda > 2C_{T_2}(2r) \).

**Proof of Lemma II.9.** If \( v_i := R_{T_1}^r(\lambda) u_i \), we have \( \lambda v_1 - F_{T_1}(v_1) = \lambda v_2 - F_{T_2}(v_2) \) on \([0, T_1]\). We denote by \( v_{21} \), the restriction of \( v_2 \) on \([0, T_1]\). Since \( F_{T_1}(v_{21}) = F_{T_2}(v_2) \) on \([0, T_1]\), we deduce that 
\[ \lambda v_1 - F_{T_1}(v_1) = \lambda v_{21} - F_{T_1}(v_{21}) \]
Since \( \lambda - F_{T_1} \) is injective, we conclude that \( v_1 = v_{21} \).

\( Q.E.D. \)

This property allows to define \( R_{T_1}^r(\lambda) \) on \( X^0_T \) for \( T_1 \leq T_2 \) by putting for \( u_1 \in X^0_T \):
\[ R_{T_2}^r(\lambda) u_1 = R_{T_2}^r(\lambda) u_2 \] on \([0, T_1]\),
where \( u_2 \) is any element of \( X^0_{T_2} \) such that \( u_2 = u_1 \) on \([0, T_1]\) and \( \| u_2 \|_{X^0_{T_2}} \leq r \). Given \( x \in X^1 \), we denote \( r := \| x \|_0 \) and for \( T \in [0, T^*], \lambda > 2C_T(2r) \), we consider the integral equation:

(II.42) \[ u_\lambda(t) = U_0(t, 0)x + \int_0^t U_0(t, s) [F_{T, \lambda}(u_\lambda)](s)ds, \quad 0 \leq t \leq T, \]
where we have introduced for \( v \in X^0_T \), \( \| v \|_{X^0_T} \leq r \):

(II.43) \[ F_{T, \lambda}(v) := F_T(\lambda v). \]

**Lemma II.10.** Equation (II.42) has a unique solution in \( C^1([0, T]; X^0) \cap X^1_T \) with \( \| u_\lambda \|_{X^0_T} \leq r \).

**Proof of Lemma II.10.** Assumption II.3 and (II.30) assure that \( F_{T, \lambda} \) is locally Lipschitz. For all \( v_i \in X^j_T \), \( \| v_i \|_{X^j_T} \leq r \), we have:

(II.44) \[ \| F_{T, \lambda}(v_1) - F_{T, \lambda}(v_2) \|_{X^1_T} \leq \frac{\lambda C_T(r)}{\lambda - C_T(r)} \| v_1 - v_2 \|_{X^j_T}, \]
and the uniqueness follows by the Gronwall Lemma.

To get the existence, we remark that \( u_\lambda \in C^1([0, T]; X^0) \cap X^1_T \), \( \| u_\lambda \|_{X^0_T} \leq r \), is solution of (II.42), iff:

(II.45) \[ \frac{du_\lambda(t)}{dt} + A_T u_\lambda(t) = [F_{T, \lambda}(u_\lambda)](t), \quad 0 \leq t \leq T, \quad u_\lambda(0) = x. \]

Since \( F_{T, \lambda}(u_\lambda) = -\lambda u_\lambda + \lambda R_T^r(\lambda) u_\lambda \), we deduce that (II.42) and

(II.46) \[ u_\lambda(t) = e^{-\lambda t} U_0(t, 0)x + \lambda \int_0^t e^{-\lambda(t-s)} U_0(t, s) [R_T^r(\lambda) u_\lambda](s)ds, \quad 0 \leq t \leq T, \]
have the same solution in \( C^1([0, T]; X^0) \cap X^1_T \). To solve this last problem, we choose \( N \in \mathbb{N} \) large enough to have

(II.47) \[ \alpha := \frac{\lambda}{\lambda - C_T(r)} \left(1 - e^{-\lambda T/N}\right) < 1. \]

For any integer \( k, 0 \leq k \leq N \), we put \( T_k := kT/N \). We construct by iteration a sequence \( (u_\lambda^k)_{0 \leq k \leq N}, u_\lambda^k \in X^0_{T_k}, \) such that:

(II.48) \[ u_\lambda^0 = x \in X^0_{T_0}, \]
and for $0 \leq k \leq N - 1$, $u^k_\lambda$ satisfies:

\((\text{II.49})\)

\[
0 \leq t \leq T_k \implies u^{k+1}_\lambda(t) = u^k(t),
\]

\((\text{II.50})\)

\[
T_k \leq t \leq T_{k+1} \Rightarrow u^{k+1}_\lambda(t) = e^{-\lambda(t-T_k)}U_0(t, T_k) u^k_\lambda(T_k) + \lambda \int_{T_k}^t e^{-\lambda(t-s)} U_0(t, s) \left[ R^r_T(\lambda) u^k_\lambda \right] (s) ds.
\]

To justify this construction, we introduce a sequence of closed parts:

\(B'_0 := \{x\}, \ 0 \leq k \leq N - 1, \ B'_{k+1} := \left\{ v \in X^0_T(k+1) : t \in [0, T_k] \Rightarrow v(t) = u^k_\lambda(t), \ \| v \|_{X^0_{T_{k+1}}} \leq r \right\}, \)

and a sequence of maps \(L_k : B'_k \rightarrow X^0_{T_{k+1}}\) given for \(v \in B'_{k+1}, \ 0 \leq k \leq N - 1, \) by:

\[
t \in [0, T_k] \Rightarrow L_{k+1}(v)(t) = u^k_\lambda(t),
\]

\[
t \in [T_k, T_{k+1}] \Rightarrow L_{k+1}(v)(t) = e^{-\lambda(t-T_k)}U_0(t, T_k) u^k_\lambda(T_k) + \lambda \int_{T_k}^t e^{-\lambda(t-s)} U_0(t, s) \left[ R^r_T(\lambda) v \right] (s) ds.
\]

Assume that we have \(u^k_\lambda \in X^0_{T_k}, \ \| u^k_\lambda \|_{X^0_{T_{k+1}}} \leq r.\) Then for \(v \in B'_{k+1}, \ | L_{k+1}(v)(t) | \leq r \) for \(0 \leq t \leq T_k. \) Moreover (II.6) and (II.31) imply:

\[
t \in [T_k, T_{k+1}] \Rightarrow \| L_{k+1}(v)(t) \|_{X^0_{T_{k+1}}} \leq e^{-\lambda(t-T_k)}r + \| v \|_{X^0_{T_{k+1}}} \lambda \int_{T_k}^t e^{-\lambda(s-t)} ds \leq r.
\]

Hence \(L_{k+1}(B'_{k+1}) \subset B'_{k+1}. \) Now given \(v \in B'_{k+1}, \) (II.6) and (II.30) also assure that

\[
\| L_{k+1}(v_1)(t) - L_{k+1}(v_2)(t) \|_{X^0_{T_{k+1}}} \leq \frac{\lambda^2}{\lambda - C_T(r)} \| v_1 - v_2 \|_{X^0_{T_{k+1}}} \int_{T_k}^t e^{-\lambda(s-t)} ds \leq \alpha \| v_1 - v_2 \|_{X^0_{T_{k+1}}}.
\]

Therefore \(L_k\) is a strict contraction on \(B'_k,\) and given \(u^0_\lambda := x,\) the sequence \((u^k_\lambda)_{0 \leq k \leq N}\) is well defined by \(L_k \circ u^k_\lambda = u^k_\lambda \in B'_k.\) By using (II.5) and Lemma II.9, an easy recurrence shows that

\[
0 \leq t \leq T_k \Rightarrow u^k_\lambda(t) = e^{-\lambda T_0(t, 0)x} + \lambda \int_0^t e^{-\lambda(t-s)} U_0(t, s) \left[ R^r_T(\lambda) u^k_\lambda \right] (s) ds,
\]

and by putting

\((\text{II.51})\)

\[
u_\lambda(t) := u^k_\lambda(t) \text{ for } t \in [0, T_k],
\]

we have constructed a solution of (II.46), satisfying

\((\text{II.52})\)

\[\| u_\lambda \|_{X^0_T} \leq r.\]

We now prove that \(u_\lambda \in X^1_T.\) (II.30) and (II.39) imply that for any \(v \in X^1_T, \ \| v \|_{X^1_T} \leq r, \lambda > 2C_T(2r),\) we have:

\[
(s \mapsto U_0(t, s) \left[ R^r_T(\lambda) v \right] (s)) \in X^1_T,
\]

\[
\| U_0(t, .) \left[ R^r_T(\lambda) v_1 \right] - U_0(t, .) \left[ R^r_T(\lambda) v_2 \right] \|_{X^1_T} \leq \frac{\lambda K_1}{\lambda - C_T(r)} \| v_1 - v_2 \|_{X^1_T}.
\]

Therefore the usual methods show that the integral equation (II.46) has a unique maximal solution in \(C^0([0, T_*]; X^1)\) for some \(T_* \in ]0, T],\) and by the principle of continuation, to prove that \(u_\lambda \in X^1_T,\) it is sufficient to establish:

\((\text{II.53})\)

\[
\sup_{0 \leq t < T_*} \| u_\lambda(t) \|_1 < \infty.
\]

We get from (II.30) that for \(0 \leq t < T_*:\)

\[
\| u_\lambda \|_{X^1_T} \leq K_T \| x \|_1 + \frac{\lambda K_T}{\lambda - C_T(r)} \int_0^t \| u_\lambda \|_{X^1_T} ds.
\]
Moreover we have \( u(t) \in C^1 ([0, T]; X^0) \).

We can achieve the proof of the existence of the solution of (II.38). Given \( T \in [0, T^*[, \mu \geq \lambda > 2C_T(2r) \), we estimate for \( 0 \leq t \leq T \):

\[
\| u_\lambda - u_\mu \|_{X^0_T} \leq T \frac{4rT}{\lambda} (C_T(2r))^2 + \frac{\lambda C_T(r)}{\lambda - C_T(r)} \int_0^t \| F_{T, \lambda} (u_\lambda) - F_{T, \mu} (u_\lambda) \|_{X^0_T} \, ds
\]

We use (II.34) and (II.44) to get:

\[
\| u_\lambda - u_\mu \|_{X^0_T} \leq 4rT \frac{C_T(2r)^2}{\lambda} + \frac{\lambda C_T(r)}{\lambda - C_T(r)} \int_0^t \| u_\lambda - u_\mu \|_{X^0_T} \, ds.
\]

Therefore the Gronwall Lemma assures that \( (u_\lambda)_\lambda \) is a Cauchy sequence in \( X^0_T \). Its limit, \( u \), is solution of (II.38) in \( X^0_T \) and satisfies \( \| u \|_{X^2_T} \leq \| x \|_0 \). Since the local Cauchy problem is well posed in \( X^2 \), we know that \( u \in C^0 ([0, T_*]; X^1) \) for some \( T_* \leq T \), and to prove the global existence in \( X^1_T \), it is sufficient to show that \( \| u(t) \|_1 \) does not blow up as \( t \to T_* \). We use (II.11) and (II.39) to estimate:

\[
\| u \|_{X^1_T} \leq K_T \| x \|_1 + K_T C_T(r) \int_0^t \| u \|_{X^1_T} \, ds,
\]

and we conclude by the Gronwall Lemma again. To prove (II.15), (II.16), we use (II.6) or (II.39), and (II.12) to get

\[
\| u_1 - u_2 \|_{X^1_T} \leq K^1_T \| x_1 - x_2 \|_J + K^1_T C_T(\max \| x_i \|_0) \int_0^t \| u_1 - u_2 \|_{X^1_T} \, ds,
\]

with \( K^1_T = K_T \), and we apply the Gronwall Lemma once more.

Q.E.D.

When the nonlinearity is local in time, i.e. \( [F(u)](t) = f(t, u(t)) \) where \( f \) is a map from \([0, T^*] \times X^0 \) to \( X^0 \), the solution of (II.1), (II.2), is given by a contraction propagator.

**Corollary II.11.** Suppose Assumptions II.1 and II.2 are satisfied. Let \( f \) be in \( C^0 ([0, T^*] \times X^0; X^0) \) satisfying for any \( T \in [0, T^*[, j = 0, 1 : \)

\[
0 \leq t \leq T, \ x \in X^2, \ \| x \|_0 \leq r \implies \| f(t, x) \|_j \leq C_T(r) \| x \|_j,
\]

(II.54)

\[
0 \leq t \leq T, \ x_i \in X^j, \ \| x_i \|_0 \leq r \implies \| f(t, x_1) - f(t, x_2) \|_j \leq C_T(r) \| x_1 - x_2 \|_j,
\]

(II.55)

where \( C_T(r) \) is a continuous nondecreasing function of \( T \) and \( r \).

We also assume that

\[
\forall t \in [0, T^*[, \ \forall x \in X^0, \ \forall \lambda > 0, \ \lambda \| x \|_0 \leq \| \lambda x - f(t, x) \|_0 .
\]

Then for any \( x \in X^1 \), the Cauchy problem

\[
\frac{du(t)}{dt} + A_t u(t) = f(t, u(t)), \ 0 \leq t < T^*,
\]

(II.57)

\[
u(0) = x.
\]

has a unique solution \( u \in C^1 ([0, T^*]; X^0) \cap X^1_T \), and there exists a (nonlinear) contraction propagator \( U(t, s) \) on \( X^0 \), such that for any \( (s, t) \in \Delta \)

\[
u(t) = U(t, s)u(s).
\]

Moreover we have

\[
0 \leq s \leq t < T^* \implies \| u(t) \|_0 \leq \| u(s) \|_0,
\]

(II.60)
and there exists a continuous nondecreasing function \( k(t, r) \) such that for \( j = 0, 1 \):

\[
(II.61) \quad \| U(t, 0)x_1 - U(t, 0)x_2 \|_j \leq k(t, \max(\| x_1 \|_0, \| x_2 \|_0)) \| x_1 - x_2 \|_j.
\]

Proof of Corollary II.11. (II.54), (II.55) and (II.56) assure that Assumptions II.3 and II.4 are satisfied, so we can apply Theorem II.5. Now given \( \theta \in [0, T^*] \), we can apply this result to \( A_t^\theta := A_{t+t}, \quad U_0^\theta(t, s) := U_0(\theta + t, \theta + s), \quad F^\theta(u)(t) := f(\theta + t, u(t)) \), with \( 0 \leq s \leq t < T^\theta_\theta := T^* - \theta \), and for these data we denote \( U^\theta(t)x \) the solution of (II.1), (II.2), defined for \( t \in [0, T^\theta_\theta] \). For \( x \in X^1 \), \( 0 \leq s \leq r \leq t \) we put:

\[
s \leq \tau < T^* \Rightarrow u^1(\tau) = U^s(\tau - s)x,
\]

\[
s \leq \tau \leq r \Rightarrow u^2(\tau) = u^1(\tau), \quad r \leq \tau < T^* \Rightarrow u^2(\tau) = U^r(\tau - r)u^1(\tau).
\]

\( u^1(\cdot + s) \) and \( u^2(\cdot + s) \) are solutions of (II.1), (II.2) in \( C^1([0, T^*]; X^0) \cap X^1_{T^*} \). Then the uniqueness implies \( u^1 = u^2 \). Therefore if we introduce for \( 0 \leq s \leq t < T^* \):

\[
U(t, s) := U^s(t - s),
\]

we write \( U(t,s)x = U^s(t - s)x = u^1(t) = u^2(t) = U^r(t - r)u^1(r) = U^r(t - r)U^s(r - s)x = U(t,r)U^s(r,s)x \). Therefore \( U \) is a propagator that solves equation (II.1) with initial data at time \( s \), and (II.60) follows from (II.14).

\[\text{Q.E.D.}\]

We also give a result concerning the strictly conservative systems.

Corollary II.12. In addition to the assumptions of Corollary II.11, suppose the following conditions are satisfied:

\[
(II.62) \quad \text{For any } (s, t) \in \Delta, \quad U_0(t, s) \text{ is one-to-one and onto on } X^0 \text{ and } X^1, \quad \text{and isometric on } X^0.
\]

\[
(II.63) \quad \forall t \in [0, T^*], \quad \forall x \in X^0, \quad \forall \lambda > 0, \quad \lambda \| x \|_0 \leq \| \lambda x + f(t, x) \|_0.
\]

Then for any \( x \in X^1 \), the unique solution \( u \in C^1([0, T^*]; X^0) \cap X^1_{T^*} \) of (II.57), (II.58), satisfies:

\[
(II.64) \quad \forall t \in [0, T^*], \quad \| u(t) \|_0 = \| x \|_0.
\]

Proof of Corollary II.12. Pick \( T \in [0, T^*] \). We define for \( (s, t) \in \Delta^T := \{(s, t); 0 \leq s \leq t < T\} \):

\[
A_t^T := -A_{T-t}, \quad U_0^T(t, s) := [U_0(T - s, T - t)]^{-1}, \quad f^T(t, x) := -f(T - t, x).
\]

The family \( (A_t^T)_{0 \leq t < T} \) satisfies Assumption II.1 and \( U_0^T \) is a contraction propagator. Moreover the open mapping theorem assures that \( U_0^T(t, s) \in \mathcal{L}(X^1) \). Given \( x \in X^1 \), we denote \( y := U_0(T - s, T - t)x \) and we compute:

\[
0 = \frac{d}{dt} [U_0^T(t, s)U_0(T - s, T - t)x] = \left[ \frac{d}{dt} U_0^T(t, s) \right] y + A_t^T U_0^T(t, s)y,
\]

\[
0 = \frac{d}{ds} [U_0^T(t, s)U_0(T - s, T - t)x] = \left[ \frac{d}{ds} U_0^T(t, s) \right] y - U_0^T(t, s)A_t^T y.
\]

Since \( U_0^T(t, s) \) is one-to-one and surjective on \( X^1 \), we deduce that \( U_0^T \) satisfies Assumption II.2 where we replace respectively \( T^*, A_t, U_0 \) by \( T, A_t^T, U_0^T \). Thanks to Corollary II.11, the Cauchy problem:

\[
(II.65) \quad \frac{du^T(t)}{dt} + A_t^T u^T(t) = f^T(t, u^T(t)), \quad 0 \leq t < T,
\]

\[
(II.66) \quad u^T(0) = u(T),
\]
has a unique solution in $u^T \in C^1([0, T]; X^0) \cap C^0([0, T]; X^1)$, and this solution satisfies
\begin{equation}
0 \leq t < T \Rightarrow \| u^T(t) \|_0 \leq\| u(T) \|_0.
\end{equation}

Since $u(T-t)$ is solution of (II.65) and (II.66), we have $u^T(t) = u(T-t)$ and (II.67) implies
\begin{equation}
0 \leq t < T \Rightarrow \| u(T-t) \|_0 \leq\| u(T) \|_0.
\end{equation}

Since $u \in X^0_T$, we can take the limit as $t \to T^-$ in (II.68) and we get (II.64).

\[ \text{Q.E.D.} \]

For the applications, it is interesting to be able to consider the case where the linear problem is solved by a propagator that is not a contraction:
\begin{equation}
\frac{du(t)}{dt} + A_t u(t) + B(t) u(t) = [F(u)](t), \quad 0 \leq t < T^*,
\end{equation}
\begin{equation}
u(0) = x.
\end{equation}

Here the potential $B(t)$ satisfies:

**Assumption II.13.** For any $t \in [0, T^*[$, $j = 0, 1$, $B(t) \in \mathcal{L}(X^j)$, and we have
\begin{equation}
B \in C^0([0, T^*]; \mathcal{L}(X^j)).
\end{equation}

We have to strengthen the condition (II.13) on the nonlinearity.

**Assumption II.14.** For any $t \in [0, T^*[$, $\lambda > 0$, we have for all $u \in X^0_T$ :
\begin{equation}
\lambda \| u(t) \|_0 \leq\| \lambda u(t) - [F(u)](t) \|_0.
\end{equation}

**Theorem II.15.** Suppose Assumptions II.1, II.2, II.3, II.13 and II.14 are satisfied. Then for any $x \in X^1$, there exists a unique $u \in C^1([0, T^*]; X^0) \cap X^1_T$, solution of (II.69), (II.70). Moreover $u$ depends continuously of $x$ : there exists a continuous nondecreasing function $k(t, r)$, such that if $u_i$, $i = 1, 2$, are two solutions with initial data $x_i$, we have for $t \in [0, T^*[$, $j = 0, 1$ :
\begin{equation}
\| u_1(t) - u_2(t) \|_j \leq k(t, \max \| x_i \|_0) \| x_1 - x_2 \|_j.
\end{equation}

**Proof of Theorem II.15.** To prove the existence, we take $T \in [0, T^*[$, and we put
\begin{equation}
b = \beta(T) := \int_0^T \| B(\tau) \|_{\mathcal{L}(X^0)} \, d\tau.
\end{equation}
The equation
\[ \frac{dv(t)}{dt} \quad \text{is solved by a contraction propagator} \quad U_b(t, s) \quad \text{on} \quad X^0, \quad \text{given by} : \]
\begin{equation}
0 \leq s \leq t < T^* \Longrightarrow U_b(t, s) = e^{-b(t-s)} U_0(t, s),
\end{equation}
where $U_0(t, s)$ is the propagator associated to $A_t$ by Assumption II.2, and we have
\begin{equation}
\| U_b(t, s) \|_{\mathcal{L}(X^0)} \leq e^{-b(t-s)}, \quad U_b(t, s) \in \mathcal{L}(X^1).
\end{equation}
Moreover, since $U_0(t, s)x \in C^0(\Delta; X^1)$ for any $x \in X^1$, the Banach-Steinhaus theorem assures that
\begin{equation}
\| U_b(t, s) \|_{\mathcal{L}(X^1)} \in C^0(\Delta).
\end{equation}

We introduce the family :
\begin{equation}
\tilde{A}_t := A_t + B(t) + b.
\end{equation}
\( \dot{A}_t \) satisfies Assumption II.1. The Cauchy problem for the equation

\[
(II.79) \quad \frac{dv(t)}{dt} + \dot{A}_t v = 0,
\]

is solved by the usual way, by considering for \( 0 \leq s \leq t < T^* \), the integral equation

\[
(II.80) \quad v(t) = U_b(t, s)v(s) + \int_s^t U_b(t, \tau) (B(\tau)v(\tau))d\tau.
\]

Thanks to the classic fixed point argument, we construct the solution of this equation, in \( C^0([s, T^*]; X^0) \) if \( v(s) \in X^0 \), in \( C^0([s, T^*]; X^1) \cap C^1([s, T^*]; X^0) \) if \( v(s) \in X^1 \), and we have:

\[
(II.81) \quad \| v(t) \|_2 \leq e^{-b(t-s)} \| v(s) \|_2 + \int_s^t e^{-b(t-\tau)} \| U_b(t, \tau) \|_{\mathcal{L}(X^j)} \| B(\tau) \|_{\mathcal{L}(X^j)} \| v(\tau) \|_2 d\tau.
\]

Then the Gronwall Lemma implies that we have for \( 0 \leq s \leq t \leq T \):

\[ \| v(t) \|_2 \leq \| v(s) \|_2 e^{\beta(t-s)-\beta(T)}, \]

and we conclude that the map \( v(s) \mapsto v(t) \) associated with (II.80) is a contraction propagator \( U(t, s) \) on \( [0, T] \), satisfying Assumption II.2. Now \( u \in C^0([0, T]; X^0) \) is solution of the equation (II.69), (II.70), iff \( v(t, x) := e^{-bt}u(t, x) \) is solution of:

\[
(II.82) \quad \partial_t v(t) + \dot{A}_t v(t) = [F_b(v)](t), \quad 0 < t < T, \quad v(0) = u_0,
\]

with \( [F_b(v)](t) := e^{-bt} [F(e^{bs}u(s))] (t) \). Thanks to Assumptions II.3 and II.14, \( F_b \) satisfies Assumptions II.3 and II.4. Therefore, given \( u \in X^1 \), the existence of \( v \in C^1([0, T]; X^0) \cap C^0([0, T]; X^1) \) follows from Theorem II.5. Since \( \| v(t) \|_2 \leq \| v(0) \|_2 \) for \( 0 \leq t < T \), we deduce that

\[
(II.83) \quad \| u(T) \|_0 \leq e^{\beta(T)} \| u_0 \|_0,
\]

and (II.73) is a consequence of (II.15) and (II.16).

Q.E.D.

III. Semilinear Hyperbolic Systems

We investigate the global Cauchy problem for a Semilinear Hyperbolic System:

\[
(III.1) \quad \partial_t u + \sum_{k=1}^d A^k(t)\partial_{x^k} u + iB(t, x)u = [F(u)](t), \quad 0 < t < T^* \leq \infty, \quad x = (x^1, ...x^d) \in \mathbb{R}^d,
\]

\[
(III.2) \quad u(0, x) = u_0(x).
\]

Here \( u = \{u_1, ...u_N\} \) is a function on \( \mathbb{R} \times \mathbb{R}^d_x \) into \( \mathbb{C}^N \), \( A^k(t) \) and \( B(t, x) \) are square matrices of order \( N \). We distinguish two cases: when the nonlinearity is defined on \( L^2 \), we consider the hermitian systems, and when the nonlinearity is only defined on \( C^0 \), we restrict our attention at the transport type equations. Next, with additional hypotheses, we prove that the local \( L^2 \)-norm tends to zero as \( t \rightarrow T^* = \infty \).

III.1. Symmetric Semilinear Systems. We introduce the Hilbert spaces:

\[
(III.3) \quad X^0 = L^2(\mathbb{R}^d_x, \mathbb{C}^N), \quad X^1 = H^1(\mathbb{R}^d_x, \mathbb{C}^N),
\]

where \( \mathbb{C}^N \) is endowed with the euclidean norm, i.e. the norms are given by:

\[
(III.4) \quad \| u \|_0^2 = \sum_{k=1}^N \| u_k \|_{L^2(\mathbb{R}^d)}^2, \quad \| u \|_{2}^2 := \| u \|_0^2 + \sum_{k=1}^N \| \partial_{x^k} u_k \|_{L^2(\mathbb{R}^d)}^2.
\]

Firstly, we suppose that the coefficients satisfy the following condition of \( C^0 \) regularity.
Assumption III.1. For $k = 1 \ldots d$, $A^k(t)$ is a hermitian matrix of order $N$, with coefficients in $C^0([0, T^*] \times \mathbb{C})$. $B(t, x)$ is a hermitian matrix of order $N$, with coefficients in $C^0([0, T^*]; L^\infty(\mathbb{R}_x^d))$. For each $t \in [0, T^*]$, $B(t, x)$ is an uniformly continuous function of $x$.

Theorem III.2. Suppose Assumption III.1 is satisfied. Let $F$ be a map from $C^0([0, T^*]; X_0^0)$ to $C^0([0, T^*]; X_0^0)$, satisfying the Assumptions II.3 and II.4 for the choice of spaces (III.3). Then for any $u_0 \in X^0$, the Cauchy problem (III.1), (III.2) has a unique solution $u \in C^0([0, T^*]; X^0)$. $u$ continuously depends on $u_0$ and satisfies

$$\forall t \in [0, T^*], \quad \| u(t) \|_{X^0} \leq \| u_0 \|_{X^0}.$$  

If $\nabla_x B \in C^0([0, T^*]; L^\infty(\mathbb{R}_x^d))$ and $u_0 \in X^1$, then $u \in C^1([0, T^*]; X^0) \cap C^0([0, T^*]; X^1)$.

Proof of Theorem III.2. For $n \geq 1$, let $\varphi_n$ be given by:

$$\varphi_n(x) = n^d \varphi \left( \frac{x}{n} \right), \quad \varphi \in C_0^\infty(\mathbb{R}^d), \quad \int \varphi(x) dx = 1.$$  

To establish the continuity of $u_0 \mapsto u$, and the uniqueness, we consider $u^1, u^2 \in C^0([0, T^*]; X^0)$, solution of (III.1). We put $u_n^1(t) := \varphi_n \ast_x u^1(t)$. Since $\varphi_n \ast_x [B(t)u^1(t)]$, $\varphi_n \ast_x [F(u^1)] (t) \in C^0([0, T^*]; X^0)$, $u_n^1$ belongs to $C^0([0, T^*]; X^1) \cap C^1([0, T^*]; X^0)$. Taking the hermitian product in $\mathbb{C}^N$ of (III.1) with $u_n^1 - u_n^2$ and integrating over $[0, t] \times \mathbb{R}_x^d$, we get:

$$\| u_n^1(t) - u_n^2(t) \|_0^2 \leq \| u_n^1(0) - u_n^2(0) \|_0^2 + \int_0^t \| u_n^1(s) - u_n^2(s) \|_0 \| B(s)(u^1(s) - u^2(s)) \|_0 ds + \| u_n^1(s) - u_n^2(s) \|_0 \| [F(u^1)](s) - [F(u^2)](s) \|_0 ds.$$  

Since the sequence $(u_n^1)$ is equibounded and equicontinuous in $X^1_0$, and $u_n^1(t) \to u^1(t)$ in $X^0$ as $n \to \infty$, the Ascoli-Arzelà Theorem assures that $\| u_n^1 - u_n^2 \|_{X^0} \to 0$ as $u^1 - u^2 \|_{X^0}$ as $n \to \infty$. Then using Assumptions II.3 and III.1, we get

$$\| u^1 - u^2 \|_{X^0} \leq \| u^1(0) - u^2(0) \|_0^2 + \int_0^t \| u^1(s) - u^2(s) \|_{X^0}^2 g(s) ds, \quad g \in C^0([0, T^*]).$$  

Therefore the solution continuously depends on the initial data, and when $u^1(0) = u^2(0)$, the Gronwall Lemma implies $u^1 = u^2$.

To get the existence of the solution, we regularize $u_0$ and $B$ by putting

$$u_{0,n} := \varphi_n \ast_x u_0, \quad B_n(t) := \varphi_n \ast_x B(t).$$  

We construct the linear contraction propagator $U_0(t, s)$, associated with $\partial_t u + A(t)u = 0$, $A(t) := \sum_{k=1}^d A^k(t) \partial_x k$, by putting

$$U_0(t, s) := \mathcal{F}^{-1}_\xi \left( R(t, s; \xi) \right) \mathcal{F}_x, \quad \partial_t R(t, x; \xi) = -i \sum_{k=1}^d \xi_k A^k(t) R(t, x; \xi), \quad R(t, t; \xi) = I_d,$$  

where the Fourier transform on $\mathbb{R}^d$ is denoted by $\mathcal{F}$. We easily check that $A(t)$ and $U_0(t, s)$ satisfy Assumptions II.1, II.2. Since $B_n, \nabla_x B_n \in C^0([0, T^*]; L^\infty(\mathbb{R}_x^d; \mathbb{C}^{N \times N}))$, and $B_n^* = B_n$, then $f$ defined by $f(t, u) := -i B_n(t)u$ satisfies (II.54), (II.55), and (II.56). Therefore Corollary II.11 assures that the Cauchy problem for the linear hyperbolic system

$$\partial_t u + \sum_{k=1}^d A^k(t) \partial_x k u + i B_n u = 0,$$  

Global Cauchy Problem for Semilinear Hyperbolic Systems
is solved by a linear contraction propagator $U_{0,n}(t,s)$ satisfying Assumption II.2. Therefore, since $A_t := A(t) + iB_t$ satisfies Assumption II.1, and $u_{0,n} \in X^1$, we may apply Theorem II.5 that assures that there exists a solution $u_n \in C^1([0,T^*[1,X^0])$ of

$$
\begin{cases}
\partial_t u_n + \sum_{k=1}^{d} A^k(t) \partial_{x^k} u_n + iB_n u_n = F(u_n),

u_n(0) = u_{0,n},
\end{cases}
$$

and

(III.9) $\parallel u_n(t) \parallel_0 \leq \parallel u_{0,n} \parallel_0 \leq \parallel u_0 \parallel_0$.

We have:

(III.10) $u_n(t) = U_0(t,0)u_{0,n} + \int_0^t U_0(t,s)(-iB_n(s)u_n(s) + [F(u_n)](s)) \, ds$.

We get from Assumption II.3 and (III.9):

(III.11) $\parallel u_p(t) - u_q(t) \parallel_0 \leq \parallel u_{0,p} - u_{0,q} \parallel_0 + t \parallel B_p - B_q \parallel_{L^\infty([0,t] \times \mathbb{R}^d; \mathbb{C}^{N \times N})} \parallel u_0 \parallel_0$

$\quad + \int_0^t \parallel B_q \parallel_{L^\infty([0,s] \times \mathbb{R}^d; \mathbb{C}^{N \times N})} \parallel u_p(s) - u_q(s) \parallel_0 + C_s(\parallel u_0 \parallel_0) \parallel u_p - u_q \parallel_0 \parallel ds$.

Hence the Gronwall Lemma implies:

(III.12) $\parallel u_p - u_q \parallel_{X^0} \leq \left( \parallel u_{0,p} - u_{0,q} \parallel_0 + t \parallel B_p - B_q \parallel_{L^\infty([0,t] \times \mathbb{R}^d)} \parallel u_0 \parallel_0 \right) e^{t\parallel B_q \parallel_{L^\infty([0,t] \times \mathbb{R}^d)} + tC_s(\parallel u_0 \parallel_0)}$.

Now since $x \mapsto B(t,x)$ is uniformly continuous and bounded, we have:

(III.13) $\forall t \in [0,T^*[1], \parallel B_n(t) - B(t) \parallel_{L^\infty(\mathbb{R}^d)} \longrightarrow 0, n \to \infty$.

On the other hand, since:

$$
\parallel B_n(t) - B_n(s) \parallel_{L^\infty(\mathbb{R}^d)} \leq \parallel B(t) - B(s) \parallel_{L^\infty(\mathbb{R}^d)},
$$

the sequence $B_n$ is equi-continuous in $X^0_1$, and the Ascoli-Arzelà Theorem implies that

(III.14) $\forall t \in [0,T^*[1], \parallel B_n - B \parallel_{L^\infty([0,t] \times \mathbb{R}^d)} \longrightarrow 0, n \to \infty$.

Then we deduce from (III.12) that $u_n$ is a Cauchy sequence in $X^0_1$, that converges to some $u \in X^0_{T^*}$. From (II.12) and (III.14), we get

$$
\forall t \in [0,T^*[1], \parallel B_n u_n - B u \parallel_{X^0} \parallel F(u_n) - F(u) \parallel_{X^0} \longrightarrow 0, n \to \infty,
$$

hence $u$ is solution of (III.1), (III.2), and (III.9) gives (III.5). When $u_0 \in X^1$ and $\nabla_x B \in C^0([0,T^*[1; L^\infty(\mathbb{R}^d)])$, it is not necessary to make the regularization. As above, the linear contraction propagator solving the linear system $\partial_t u + \sum_{k=1}^{d} A^k(t) \partial_{x^k} u + iBu = 0$ is obtained by Corollary II.11, and the existence of solution $u \in C^1([0,T^*[1; X^0]) \cap C^0([0,T^*[1; X^1])$ of (III.1), (III.2), directly follows from Theorem II.5.

Q.E.D.

Assumption III.3. For $k = 1...d$, $A^k(t)$ is a hermitian matrix of order $N$, with coefficients in $C^1([0,T^*[1; \mathbb{C})$, and

(III.15) $\forall \xi \in \mathbb{R}^d \setminus \{0\}, \forall t \in [0,T^*[1, \det \left( \sum_{k=1}^{d} \xi_k A^k(t) \right) \neq 0$.
\( B(t, x) \) is a hermitian matrix of order \( N \), with coefficients \( B_{ij} \in C^1([0, T^*[t]; H^s(\mathbb{R}_x^d))] \), where

\[
\begin{cases}
  s = 0 & \text{if } d = 1, \\
  0 < s & \text{if } d = 2, \\
  s = \frac{d}{2} - 1 & \text{if } d \geq 3.
\end{cases}
\]

(III.16)

Moreover for each \( t \in [0, T^*[t] \), there exists \( \varepsilon > 0 \) such that

\[
(III.17) \quad B_{i,j}(t) \in H^{\frac{d}{2} - 1 + \varepsilon}(\mathbb{R}_x^d).
\]

**Theorem III.4.** Suppose Assumption III.3 is satisfied. Let \( F \) be a map from \( C^0([0, T^*[t]; X^0)) \) to \( C^0([0, T^*[t]; X^0)) \), satisfying the Assumptions II.3 and II.4 for the choice of spaces (III.3). Then for any \( u_0 \in X^0 \), the Cauchy problem (III.1), (III.2) has a unique solution \( u \in C^0([0, T^*[t]; X^0), \) and \( u \) satisfies

\[
(III.18) \quad \forall t \in [0, T^*[t], \| u(t) \|_{X^0} \leq \| u_0 \|_{X^0}.
\]

When \( u_0 \in X^1 \), then \( u \in C^1([0, T^*[t]; X^0) \cap C^0([0, T^*[t]; X^1). \)

**Proof of Theorem III.4.** We recall the result on the continuity of the products of distributions in the Sobolev spaces stated in [6]. If \( u^j \in H^{s_j}(\mathbb{R}_x^d) \), then \( u^1 u^2 \in H^s(\mathbb{R}_x^d) \) with the conditions :

\[
(III.19) \quad 0 \leq s_1 + s_2, \quad \sigma = s_1 \wedge s_2 \wedge (s_1 + s_2 - \frac{d}{2} - \varepsilon), \quad 0 < \varepsilon,
\]

(III.20) \quad 0 < s_1 + s_2, \quad s_j \neq \frac{d}{2}, \quad \sigma = s_1 \wedge s_2 \wedge (s_1 + s_2 - \frac{d}{2}).

We consider the family of differential operators

\[
(III.21) \quad A_t := A^0_t + iB(t), \quad A^0_t := \sum_{k=1}^{d} A^k(t) \partial_{2^k}.
\]

(III.16) and (III.19) assure that \( A_t \) is well defined in \( C^1([0, T^*[t]; \mathcal{L}(X^1, X^0)) \) and satisfies Assumption II.1. Moreover, (III.17) and (III.20) imply that for \( u \in X^1 \), we have:

\[
\| B(t)u \|_0 \leq C_x \| u \|_{H^{s_1-\varepsilon}} \| B(t) \|_{H^{\frac{d}{2}-1+\varepsilon}} \leq C'_x \| u \|_{H^{1-\varepsilon}} \| u \|_0 \| B(t) \|_{H^{\frac{d}{2}-1+\varepsilon}}.
\]

Hence for any \( \alpha > 0 \), there exists \( C_\alpha \) such that

\[
\| B(t)u \|_0 \leq C_\alpha \| u \|_0 + \alpha \| u \|_1.
\]

Now (III.15) assures that \( A^0_t \) is self-adjoint on \( X^0 \) with domain \( X^1 \) and

\[
\| u \|_1 \leq C_t (\| u \|_0 + \| A^0_t u \|_0),
\]

for some \( C_t > 0 \). We deduce that \( B(t) \) is a self-adjoint operator, \( A^0_t \)-bounded, with relative bound strictly smaller than 1. Then the Kato-Rellich Theorem assures that \( iA_t \) is self-adjoint on \( X^0 \) with domain \( X^1 \). We can invoke the Corollary of Theorem 4.4.2 of [37], to conclude that there exists a contraction propagator \( U(t, s) \) satisfying Assumption II.2. Finally, the existence of a solution of the Cauchy problem (III.1), (III.2), follows from Theorem II.5.

The establish the uniqueness in \( C^0([0, T^*[t]; X^0) \) we prove an energy estimate, that is trivial for the smooth solutions, but rather delicate for the weak solutions.
Lemma III.5. For any \( u, f \in C^0([0, T^*]; X^0) \) satisfying

\[
\partial_t u + \sum_{k=1}^{d} A_k(t) \partial_x u + iB(t, x)u = f(t), \quad 0 < t < T^*,
\]

we have:

\[
\| u(t) \|_0 \leq \| u(0) \|_0 + 2 \int_0^t \| u(s) \| \| f(s) \|_0 ds.
\]

Proof of Lemma III.5. If \( A_t \) is the differential operator (III.21) considered as anti-self-adjoint operator on \( X^0 \) with domain \( X^1 \), we denote \( R_t(\lambda) := (\lambda - A_t)^{-1} \) its resolvant for \( \lambda \in \mathbb{R}^* \). The Hille-Phillips Theorem assures that for any \( v \in X^0 \):

\[
\| R_t(\lambda)v \|_0 \leq \frac{1}{|\lambda|} \| v \|_0,
\]

moreover since \( B(t) \) is a self-adjoint operator, \( A_0^0 \)-bounded, with relative bound strictly smaller than 1, (III.15) implies that for any \( T \in [0, T^*] \), there exists \( C_T(\lambda) > 0 \) such that:

\[
\forall t \in [0, T], \quad \| R_t(\lambda)v \|_1 \leq C_T(\lambda) \| v \|_0.
\]

Now we can extend this resolvant to \( X^{-1} := H^{-1}(\mathbb{R}^d; \mathbb{C}^N) \) endowed with its usual norm \( \| \cdot \|_{-1} \), since for any \( v, w \in X^0 \),

\[
| < R_t(\lambda)v, w > | - | < R_t(-\lambda)w, v > | \leq C_T(-\lambda) \| v \|_{-1} \| w \|_0.
\]

We deduce that \( R_t(\lambda) \) can be extended into a bounded operator \( \tilde{R}_t(\lambda) \) from \( X^{-1} \) to \( X^0 \) satisfying for \( v \in X^{-1} \):

\[
\forall t \in [0, T], \quad \| \tilde{R}_t(\lambda)v \|_0 \leq C_T(-\lambda) \| v \|_{-1},
\]

\[
(\lambda - A_t)\tilde{R}_t(\lambda)v = v,
\]

and since \( A_t \) is a bounded operator from \( X^0 \) to \( X^{-1} \) thanks to (III.16), we also have:

\[
\forall v \in X^0, \quad \tilde{R}_t(\lambda)(\lambda - A_t)v = v.
\]

Now we introduce

\[
u_\lambda(t) := \lambda R_t(\lambda)[u(t)].
\]

The Banach-Steinhaus Theorem and the strong resolvant continuity of \( A_t \) assure that \( u_\lambda \in C^0([0, T^*]; X^0) \) and:

\[
\forall t \in [0, T], \quad \sup_{0 \leq s \leq T} \| u_\lambda(t) - u(t) \|_0 \to 0, \quad \lambda \to \infty.
\]

Moreover, writting

\[
u_\lambda(t - s) = \lambda R_t(\lambda)[u(t) - u(s)] + \lambda R_{t-s}(\lambda)(A_t - A_s)R_s(\lambda)[u(s)],
\]

we deduce from (III.25) that \( u_\lambda \in C^0([0, T^*]; X^1) \). Since \( u \) satisfies equation (III.22), and \( A_t \) is bounded from \( X^0 \) to \( X^1 \), then \( u \in C^1([0, T^*]; X^{-1}) \) and we also have:

\[
\partial_t u_\lambda(t) = \lambda \tilde{R}_t(\lambda) \partial_x u(t) + A'_t u_\lambda(t),
\]

with

\[
A'_t := \sum_{k=1}^{d} \frac{dA_k}{dt}(t) \partial_x + i \frac{\partial}{\partial t} B(t).
\]
Using (III.22) and (III.28), we get

\begin{equation}
\partial_t u_\lambda + \sum_{k=1}^{d} A^k(t) \partial_{x_k} u_\lambda + i B(t, x) u_\lambda = \lambda R(t) [f(t)] + A'_i u_\lambda.
\end{equation}

We deduce that \( u_\lambda \in C^0([0, T^*]; X^1) \cap C^1([0, T^*]; X^0) \), and taking the hermitian product in \( \mathbb{C}^N \) of (III.30) with \( u_\lambda \), an integration on \([0, t] \times \mathbb{R}^d_+\) gives with (III.24) :

\begin{equation}
\| u_\lambda(t) \|_0^2 - \| u_\lambda(0) \|_0^2 \leq 2 \int_0^t \| f(s) \|_0 \| u_\lambda(s) \|_0 \, ds.
\end{equation}

Therefore the result follows from (III.29).

Q.E.D.

We can now achieve the proof of the uniqueness. Given \( u^1, u^2 \in C^0([0, T^*]; X^0) \) two solutions of (III.1) with the same initial data \( u_0 \in X^0 \), we put \( u := u^1 - u^2, f(t) := [F(u^1)](t) - [F(u^2)](t) \). (II.12) implies that \( \| f \|_{X^1_t} \leq k(t) \| u^1 - u^2 \|_{X^1_t} \) with \( k \in C^0([0, T^*]) \). We deduce from Lemma III.5 that

\[ \| u \|_{X^1_0}^2 \leq 2 \int_0^t k(s) \| u \|_{X^2_s}^2 \, ds, \]

and the Gronwall Lemma implies \( u = 0 \).

Q.E.D.

### 3.2. Semilinear Transport Equations

Given \( p \in [1, \infty] \), we introduce the Banach spaces \((X^j, \| \cdot \|_j)_{j=0,1}:

\begin{equation}
X^0 := C^0_{\infty} \left( \mathbb{R}^d_+; \mathbb{C}^N \right) \cap L^p \left( \mathbb{R}^d_+; \mathbb{C}^N \right), \quad X^1 := C^1_{\infty} \left( \mathbb{R}^d_+; \mathbb{C}^N \right) \cap W^{1,p} \left( \mathbb{R}^d_+; \mathbb{C}^N \right),
\end{equation}

with

\begin{align*}
C^0_{\infty} := \left\{ u \in C^0_0; \lim_{|x| \to \infty} u = 0 \right\}, \quad C^1_{\infty} := \left\{ u \in C^0_0; \nabla_x u \in C^0_\infty \right\}, \quad W^{1,p} := \left\{ u \in L^p; \nabla u \in L^p \right\},
\end{align*}

\[ \| u \|_0 := \mathcal{N}_\infty \left( \| u \|_{L^\infty(\mathbb{R}^d)} \right)_{1 \leq j \leq N} + \mathcal{N}_p \left( \| u_j \|_{L^p(\mathbb{R}^d)} \right)_{1 \leq j \leq N}, \]

where we have chosen two norms \( \mathcal{N}_\infty \) and \( \mathcal{N}_p \), on \( \mathbb{R}^N \). We suppose that the coefficients satisfy the following:

**Assumption III.6.** For \( k = 1, \ldots, d, t \in [0, T^*], A^k(t) \) is a diagonal square matrix of order \( N \), and \( A \in C^0([0, T^*]; \mathbb{R}^{N \times N}) \).

**Assumption III.7.** \( B(t, x) \) is a square matrix and \( B \in C^0([0, T^*]; C^1 \cap W^{1,\infty}(\mathbb{R}^d_+; \mathbb{C}^{N \times N})) \).

Let \( F \) be a map from \( C^0([0, T^*]; X^0) \) to \( C^0([0, T^*]; X^0) \).

**Assumption III.8.** \( F \) satisfies the Assumptions II.3 and II.4 for the choice of spaces (III.32). Moreover, when \( B \neq 0 \), \( F \) satisfies Assumption II.14.

**Theorem III.9.** Suppose Assumptions III.6, III.7 and III.8 are satisfied. Then for any \( u_0 \in X^0 \), the Cauchy problem (III.1), (III.2) has a unique solution \( u \in C^0([0, T^*]; X^0) \). \( u \) satisfies

\begin{equation}
\forall t \in [0, T^*], \quad \| u(t) \|_{X^0} \leq e^{\beta(t)} \| u_0 \|_{X^0}, \quad \beta(t) := \int_0^t \| B(\tau) \|_{L(X^0)} \, d\tau.
\end{equation}

When \( u_0 \in X^1 \), then \( u \in C^1([0, T^*]; X^0) \cap C^0([0, T^*]; X^1) \).
Proof of Theorem III.9. We denote $A^k(t) = \left( A_j^k(t) \delta_{ij} \right)_{1 \leq i, j \leq N}$, and for $j = 1 \ldots N$,

\begin{equation}
A_j := \left( A_j^1, \ldots, A_j^N \right).
\end{equation}

If $u_1, u_2 \in C^0([0, T^*])$ are two solutions with same initial data $u_0 \in X^0$, then $v_j(t, x) = u_j^1 \left( t, x + \int_0^t A_j(s)ds \right) - u_j^2 \left( t, x + \int_0^t A_j(s)ds \right)$ satisfies

\begin{equation}
\partial_t v_j(t, x) = \left( [F_j(u_1)] - [F_j(u_2)] - i \sum_{h=1}^N B_{j,h}(u_h^1 - u_h^2) \right) \left( t, x + \int_0^t A_j(s)ds \right), \quad v_j(0, x) = 0.
\end{equation}

Since $\| v_h(t) \|_0 = \| u_h^1(t) - u_h^2(t) \|_0$, we deduce from Assumption II.3 that

\begin{equation*}
\| v(t) \|_0 \leq C \int_0^t \sup_{0 \leq \tau \leq s} \| v(\tau) \|_0 \, ds.
\end{equation*}

Hence the Gronwall Lemma implies $v = 0$ and the uniqueness of the solution of (III.1), (III.2), is established. It will be useful for the following Corollary, to remark that we only have used the fact that $B \in C^0([0, T^*]; C^0 \cap L^\infty(\mathbb{R}^d, \mathbb{C}^N \times \mathbb{C}))$.

To prove the existence, we apply Theorem II.15. The hyperbolic system

\begin{equation}
\partial_t v + \sum_{k=1}^d A^k(t) \partial_{x^k} v = 0,
\end{equation}

is solved by a contraction propagator $U_0(t, s)$ on $X^0$, given by:

\begin{equation}
0 \leq s \leq t < T^* \implies v_j(t, x) = v_j \left( s, x - \int_s^t A_j(\tau)d\tau \right), \quad j = 1, \ldots, N,
\end{equation}

and we have

\begin{equation}
\| U_0(t, s) \|_{\mathcal{L}(X^0)} \leq 1, \quad U_0(t, s) \in \mathcal{L} \left( X^1 \right),
\end{equation}

hence Assumptions II.1, II.2 are satisfied. Moreover Assumption III.7 implies Assumption II.13, and Assumption III.8 assures that Assumptions II.3 and II.14 are also satisfies. Therefore, given $u_0 \in X^1$, the existence of the solution $u \in C^1([0, T^*]; X^0) \cap C^0([0, T^*]; X^1)$ follows from Theorem II.15, and (II.83) gives (III.33). Finally the estimate (II.73) allows to get the existence of the solution $u \in C^0([0, T^*]; X^0)$ when $u_0 \in X^0$, by approximating $u_0$ by $u_{0,n} \in X^1$, $u_{0,n} \to u_0$ in $X^0$ as $n \to \infty$.

Q.E.D.

For the applications, it will be useful to be able to consider less regular $B$.

**Assumption III.10.** $B = B_1 + B_2$ where $B_1$ satisfies Assumption III.7 and $B_2 \in C^0([0, T^*]; C^0(\mathbb{R}^d, \mathbb{C}^{N \times N}))$.

**Corollary III.11.** Suppose Assumptions III.6, III.8 and III.10 are satisfied. Then for any $u_0 \in X^0$, the Cauchy problem (III.1), (III.2) has a unique solution $u \in C^0([0, T^*]; X^0)$, moreover $u$ satisfies (III.33).

**Proof of Corollary III.11.** We have established the uniqueness with Assumption III.10, in the proof of the previous Theorem. To show the existence of the solution, we choose a regularizing sequence $\theta_n$ on $\mathbb{R}$, and we put $B^{(n)}(t, x) := B_1(t, x) + \int B_2(t, y)\theta_n(x - y)dy$. Then $B^{(n)}$
satisfies Assumption III.7 and $B^{(n)}$ tends to $B$ in $C^0 \left([0, T^*]; L^\infty(\mathbb{R}^d_+; C^{N\times N})\right)$ as $n \to \infty$, and $\|B^{(n)}(t)\|_{L(X^0)} \leq \|B(t)\|_{L(X^0)}$. Furthermore, Theorem III.9 assures that the Cauchy problem

$$\partial_t u^{(n)}(t) + \sum_{k=1}^d A^k(t) \partial_{x_k} u^{(n)} + iB^{(n)}(t, x) u^{(n)} = [F(u^{(n)})](t), \quad u^{(n)}(0, x) = u_0(x),$$

has a unique solution in $C^0 \left([0, T^*]; X^0\right)$, and $\|u^{(n)}(t)\|_{X^0} \leq e^{\beta(t)} \|u_0\|_{X^0}$. We introduce

$$v_j^{(p,q)}(t, x) = u_j^{(p)} \left( t, x + \int_0^t A_j(s) ds \right) - u_j^{(q)} \left( t, x + \int_0^t A_j(s) ds \right)$$

that satisfies $v_j^{(p,q)}(0, x) = 0$ and :

$$\partial_t v_j^{(p,q)}(t, x) = \left( F_j \left( u^{(p)} \right) - F_j \left( u^{(q)} \right) - i \sum_{h=1}^N B_{j,h}^{(p)} \left( u_h^{(p)} - u_h^{(q)} \right) + \left( B_{j,h}^{(q)} - B_{j,h}^{(p)} \right) u_h^{(q)} \right) \left( t, x + \int_0^t A_j(s) ds \right).$$

Since $\|v_h^{(p,q)}(t)\|_0 = \|u_h^{(p)}(t) - u_h^{(q)}(t)\|_0$, we deduce from Assumption II.3 that

$$\|v^{(p,q)}(t)\|_0 \leq C \left( t e^{\beta(t)} \|B^{(p)} - B^{(q)}\|_{L^\infty([0,t] \times \mathbb{R})} + \int_0^t \sup_{0 \leq \tau \leq s} \|v^{(p,q)}(\tau)\|_0 ds \right).$$

Hence the Gronwall Lemma implies that $\left(u^{(n)}\right)_n$ is a Cauchy sequence in $C^0 \left([0, T^*]; X^0\right)$. If $u := \lim_{n \to \infty} u^{(n)}$, Assumption II.3 implies that $F \left( u^{(n)} \right) \to F(u)$ in $C^0 \left([0, T^*]; X^0\right)$, hence $u$ is solution of the Cauchy problem (III.1), (III.2) and satisfies (III.33).

Q.E.D.

### III.3. Decay of the local energy.

We consider the hyperbolic system (III.1) with $T^* = \infty$, and we assume that the hypotheses of one of the Theorems III.2, III.4, or III.9 with $p = 2$ in (III.32), are satisfied. To get the decay of the local $L^2$-norm, we need the following : 

**Assumption III.12.** For any $h, k \leq d$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $u \in C^0 \left([0, \infty]; X^0\right)$, we have :

\begin{align}
(III.37) \quad & \exists \alpha > 0; \quad \forall t \geq 0, \quad \alpha 1 \leq \sum_{k=1}^d \left( A^k(t) \right)^2, \\
(III.38) \quad & \left[ A^h(t), A^k(t) \right] = 0, \\
(III.39) \quad & \Re \left( u^*(t, x) A^k(t) [F(u)](t, x) \right) = 0, \\
(III.40) \quad & \Re \left( u^*(t, x) [F(u)](t, x) \right) = 0, \\
(III.41) \quad & \sum_{k=1}^d \int_0^\infty \| \frac{d}{dt} A^k(t) \|_{L(C^N)} + \| A^k(t), B(t) \|_{L^\infty(\mathbb{R}_+, C^{N\times N})} dt < \infty.
\end{align}

Here $u^*$ denotes the conjugate transpose of $u$. Condition (III.40) is stronger than (II.13) and leads to the conservation of the energy $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. We establish that no energy is trapped in a compact domain.
Theorem III.13. For any \( u_0 \in X^0 \), \( R > 0 \), the solution \( u \in C^0 \left( [0, \infty]; X^0 \right) \) of (III.1), (III.2) satisfies:

(III.42)
\[
\int_{|x| \leq R} |u(t, x)|^2 \, dx \longrightarrow 0, \ t \to \infty.
\]

Proof of Theorem III.13. Since \( \| u(t) \|_{L^2} = \| u_0 \|_{L^2} \), it is sufficient to treat the case \( u \in C^0 \left( [0, \infty]; X^1 \right) \cap C^0 \left( [0, \infty]; X^1 \right) \). The proof is largely inspired by the paper by Dias and Figueira [16]. We introduce the function \( \chi : z \in \mathbb{R}^* \mapsto \chi(z) = z \mid z \mid^{-1} \min(\mid z \mid, |R|) \). For \( 1 \leq h \leq d \), we take the real part of the bracket in \( \mathbb{C}^N \) of (III.1) and \( \chi(x^h)A^h(t)u \), and integrating in \( x \in \mathbb{R}^d \), \( t \in [0, T] \), taking account of (III.38, (III.39), we get:

\[
\int_0^T \int_{\{x \in \mathbb{R}^d; |x| < R\}} u^*(t, x) \left( A^h(t) \right)^2 u(t, x) \, dx \, dt
\]

\[
= \int_{\mathbb{R}^d} u^*(T, x)A^h(T)u(T, x) \, dx - \int_{\mathbb{R}^d} u_0^*(x)A^h(0)u_0(x) \, dx
\]

\[
- \int_0^T \int_{\mathbb{R}^d} u^*(t, x) \left( \frac{d}{dt} A^h(t) - i \left[ A^h(t), B(t, x) \right] \right) u(t, x) \, dx \, dt.
\]

With (III.37), we deduce that
\[
\int_0^T \int_{\{x \in \mathbb{R}^d; |x| < R\}} \left| u(t, x) \right|^2 \, dx \, dt
\]

\[
\leq \frac{R}{\alpha} \left\| u_0 \right\|_{L^2}^2 \left( \sum_{k=1}^d 2 \sup_{t \geq 0} \| A^k(t) \|_{L(\mathbb{C}^N)} \right)
\]

\[
+ \int_0^T \| \frac{d}{dt} A^k(t) \|_{L(\mathbb{C}^N)} + \| A^k(t), B(t) \|_{L^\infty(\mathbb{R}^d; \mathbb{C}^N \times N)} \, dt.
\]

We conclude by (III.41) that:

(III.43)
\[
\int_T^{T+1} \int_{|x| \leq R} \left| u(t, x) \right|^2 \, dx \, dt \longrightarrow 0, \ T \to \infty.
\]

For \( r > 0 \) we put:

\[
Q_r(t) := \int_{|x| \leq R} \left| u(t, x) \right|^2 \, dx.
\]

By using (III.40), an integration by part gives:

\[
\frac{d}{dt} Q_r(t) = - \sum_{k=1}^d \int_{|x| = r} u^*(t, x) A^k(t)u(t, x) \frac{x^k}{|x|} \, dS_r.
\]

Given \( R > 0 \), we integrate on \([R, R + 1]_t\) and we get:

\[
\frac{d}{dt} \int_R^{R+1} Q_r(t) \, dr = - \sum_{k=1}^d \int_{R \leq |x| \leq R+1} u^*(t, x) A^k(t)u(t, x) \frac{x^k}{|x|} \, dx.
\]

Now given \( \theta \in [T, T + 1] \), we integrate on \([T, \theta]_t\) and we find:

\[
\int_R^{R+1} Q_r(\theta) \, dr - \int_R^{R+1} Q_r(T) \, dr = - \sum_{k=1}^d \int_T^{\theta} \int_{R \leq |x| \leq R+1} u^*(t, x) A^k(t)u(t, x) \frac{x^k}{|x|} \, dx \, dt.
\]
Finally we integrate on $[T, T+1]_\theta$ and we obtain:

$$
\int_R^{R+1} Q_r(T) dr = \int_T^{T+1} \int_R^{R+1} Q_r(\theta) d\theta + \sum_{k=1}^d \int_T^{T+1} \int_{R \leq |x| \leq R+1} u^k(t,x) A^k(t) u(t,x) \frac{x^k}{|x|} dx dt d\theta.
$$

We conclude that:

$$
Q_R(T) \leq \left( 1 + \sum_{k=1}^d \sup_{t \geq 0} \| A^k(t) \|_{L^\infty(C^\infty)} \right) \int_T^{T+1} Q_{R+1}(t) dt.
$$

Therefore the Theorem follows from (III.43).

Q.E.D.

IV. Applications in Fields Theory and General Relativity

We consider several non-linear Dirac equations with mass $M \geq 0$:

(IV.1) $i \gamma^\mu(g) \nabla_\mu \Psi - M \Psi = F(\Psi)$.

The notations are the following. $\nabla_\mu$ are the covariant derivatives on a $(1 + d)$-dimensional $C^2$ manifold $\mathcal{M}$, $d = 1, 3$, endowed with a lorentzian metric with signature $(+, -, \ldots, -)$, $g_{\mu\nu} = \eta_{\mu\nu}$.

(IV.2) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

$\gamma_{(g)}^\mu$, $0 \leq \mu \leq d$, are the Dirac matrices, unique up to a unitary transform, satisfying:

(IV.3) $\gamma_{(g)}^{0x} = \gamma_{(g)}^0$, $\gamma_{(g)}^{jx} = -\gamma_{(g)}^j$, $1 \leq j \leq d$, $\gamma_{(g)}^\mu \gamma_{(g)}^\nu + \gamma_{(g)}^\nu \gamma_{(g)}^\mu = 2 g^{\mu\nu} 1$.

Here $A^*$ denotes the conjugate transpose of any complex matrix $A$. In the case of the Minkowski space-time $\mathbb{R}^{1+d}$, we omit the subscript $(g)$.

We also introduce the matrix arising in the pseudoscalar interaction:

(IV.4) $\gamma^5 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

We make the following choices.

If $d = 1$, $\Psi$ is $\mathbb{C}^2$-valued and for $\mu = 0, 1$:

(IV.5) $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\gamma^5 := \gamma^0 \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then the Dirac equation (IV.1) takes the form:

(IV.6) \[
\begin{cases}
\partial_t \Psi_1 + \partial_x \Psi_2 + iM \Psi_1 = -iF_1, \\
\partial_t \Psi_2 + \partial_x \Psi_1 - iM \Psi_2 = iF_2.
\end{cases}
\]

If $d = 3$, $\Psi$ is $\mathbb{C}^4$-valued, and $\gamma^\mu$ are the $4 \times 4$ matrices of the Pauli-Dirac representation given for $\mu = 0, 1, 2, 3$ by:

(IV.7) $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$, $\gamma^5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$,

where

(IV.8) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Finally we denote

(IV.9) $\Psi := \Psi^* \gamma^0$. 

Then the Dirac equation (IV.1) takes the form:

\[
\begin{cases}
\partial_t \Psi_1 + \partial_x \Psi_2 + iM \Psi_1 = -F_1, \\
\partial_t \Psi_2 + \partial_x \Psi_1 - iM \Psi_2 = F_2.
\end{cases}
\]
IV.1. Dirac-Maxwell system and Dirac-Klein-Gordon equations on $\mathbb{R}^{1+1}$. We consider the global Cauchy problem for the Dirac-Maxwell system and the Dirac-Klein-Gordon system on the Minkowski space-time $\mathbb{R}^{1+1}$. To simultaneously treat these two problems, we investigate the coupled equations:

\begin{align}
(IV.10) & \quad i\gamma^\mu \partial_\mu \Psi + M\Psi = \left[gA^\mu \gamma_\mu + \Phi \left(hI + ik\gamma^5\right)\right]\Psi, \\
(IV.11) & \quad \partial_t^2 A_\mu - \partial^2_x A_\mu = g\bar{\Psi}\gamma_\mu \Psi, \\
(IV.12) & \quad \partial_t^2 \Phi - \partial^2_x \Phi + m^2 \Phi = h\bar{\Psi}\Psi + ik\bar{\Psi}\gamma^5 \Psi, \\
(IV.13) & \quad \Psi(0, x) = \psi(x), \\
(IV.14) & \quad A_\mu(0, x) = f_\mu(x), \quad \partial_t A_\mu(0, x) = g_\mu(x), \\
(IV.15) & \quad \Phi(0, x) = \varphi_0(x), \quad \partial_t \Phi(0, x) = \varphi_1(x).
\end{align}

Here $g, h, k \in \mathbb{R}$ are the coupling constants, $M, m \geq 0$ are the mass of the fields. The cases $gh = 0, k = 0$ have been solved by J.M. Chadam [10] for $\psi \in H^1(\mathbb{R}; C^2), f_\mu, \varphi_0 \in H^1(\mathbb{R}; \mathbb{R}), g_\mu, \varphi_1 \in L^2(\mathbb{R}; \mathbb{R})$. Chadam and Glassey have also investigated several properties of this system in [11], [12]. An important question is the existence of low regularity solutions. N. Bouraveas [9], and Y. Fang [17] have recently solved the Dirac-Klein-Gordon system $g = k = 0$, for $\psi \in L^2(\mathbb{R}; C^2)$, $\varphi_0 \in H^1(\mathbb{R}; \mathbb{R}), \varphi_1 \in L^2(\mathbb{R}; \mathbb{R})$. Their proof is based on the special algebraic property of $\bar{\Psi}\Psi$, the so-called compatibility condition, [2], related to the null condition of S. Klainerman. Since the Dirac-Maxwell system does not satisfy this property, the existence of global solutions for $\psi \in L^2(\mathbb{R}; C^2)$, is an open problem that we solve.

**Theorem IV.1.** For $\psi \in L^2(\mathbb{R}; C^2), f_\mu \in H^s(\mathbb{R}; \mathbb{R}), g_\mu \in H^{s-1}(\mathbb{R}; \mathbb{R}), 1/2 < s, \varphi_0 \in H^s(\mathbb{R}; \mathbb{R}), \varphi_1 \in H^{s-1}(\mathbb{R}; \mathbb{R}), 1/2 < \sigma \leq 1$, the Cauchy problem (IV.10) to (IV.15) has a unique global solution $\Psi \in C^0(\mathbb{R}; L^2(\mathbb{R}; C^2)) \cap H^s(\mathbb{R}; C^2))$, $\Phi \in C^0(\mathbb{R}; H^\sigma(\mathbb{R}; x)) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{R}; x)), A_\mu \in C^0(\mathbb{R}; H^{\frac{1}{2}-\epsilon} \cap L^\infty(\mathbb{R}; x)) \cap C^1(\mathbb{R}; H^{-\frac{1}{2}-\epsilon}(\mathbb{R}; x))$ for all $\epsilon > 0$.

**Proof of Theorem IV.1.** We introduce

\begin{equation}
B := -M\gamma^0 + \gamma^0 \left[g\tilde{A}^\mu \gamma_\mu + \tilde{\Phi} \left(hI + ik\gamma^5\right)\right]
\end{equation}

where $\tilde{A} \in C^0(\mathbb{R}; H^s(\mathbb{R}; x; \mathbb{R})) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R}; x; \mathbb{R}))$ and $\tilde{\Phi} \in C^0(\mathbb{R}; H^\sigma(\mathbb{R}; x; \mathbb{R})) \cap C^1(\mathbb{R}; H^{\sigma-1}(\mathbb{R}; x; \mathbb{R}))$ are the solutions of

\begin{equation}
\partial_t^2 \tilde{A}_\mu - \partial^2_x \tilde{A}_\mu = 0, \quad \tilde{A}_\mu(0, x) = f_\mu(x), \quad \partial_t \tilde{A}_\mu(0, x) = g_\mu(x), \\
\partial_t^2 \tilde{\Phi} - \partial^2_x \tilde{\Phi} + m^2 \tilde{\Phi} = 0, \quad \tilde{\Phi}(0, x) = \varphi_0(x), \quad \partial_t \tilde{\Phi}(0, x) = \varphi_1(x).
\end{equation}

Since $H^\tau(\mathbb{R}) \subset C^0_{\text{loc}}(\mathbb{R})$ for $\tau > 1/2$, $B$ satisfies Assumption III.1.

Now given $\Psi \in C^0(\mathbb{R}; L^2(\mathbb{R}; C^2))$, we introduce

\begin{equation}
\tilde{\Psi}(t, x) = \frac{1}{2} \int_0^t \left[ \int_{x-t+s}^{x+t-s} \bar{\Psi}\gamma^\mu \Psi(s, y)dy \right] ds,
\end{equation}

where $J_0$ is the usual Bessel function. Since $\bar{\Psi}\Psi, \bar{\Psi}\gamma_\mu \Psi, \bar{\Psi}\gamma^5 \Psi \in C^0(\mathbb{R}; L^1(\mathbb{R}; x)), and L^1(\mathbb{R}) \subset H^{-\frac{1}{2}-\epsilon}(\mathbb{R}), 0 < \epsilon, we have $\tilde{A}_\mu, \tilde{\Phi} \in C^0(\mathbb{R}; H^{\frac{1}{2}-\epsilon}(\mathbb{R}; x; \mathbb{R})), and $A_\mu = \tilde{A}_\mu + \tilde{A}_\mu, \Phi = \tilde{\Phi} + \tilde{\Phi}$ are solution of (IV.11), (IV.12), (IV.14) and (IV.15). Furthermore we have $\tilde{A}_\mu, \tilde{\Phi}$
\( C^0([0,\infty) ; L^2(\mathbb{R}_x ; C^2)) \) and the maps \( \Psi \mapsto A_\mu, \Phi \) are locally lipschitz from \( \mathcal{C}^0([-T,T];L^2(\mathbb{R}_x;\mathbb{C}^2)) \) to \( \mathcal{C}^0([-T,T];L^\infty(\mathbb{R}_x)) \) for any \( T > 0 \). When \( \Psi \in \mathcal{C}^0(\mathbb{R}_t;H^1(\mathbb{R}_x;\mathbb{C}^2)) \), since \( H^1(\mathbb{R}) \) is an algebra, \( A_\mu, \Phi \in \mathcal{C}^0(\mathbb{R}_t;H^2(\mathbb{R}_x;\mathbb{R})) \) and the maps \( \Psi \mapsto A_\mu, \Phi \) are locally lipschitz from \( \mathcal{C}^0([-T,T];H^2(\mathbb{R}_x;\mathbb{C}^2)) \) to \( \mathcal{C}^0([-T,T];H^2(\mathbb{R}_x;\mathbb{R})) \).

We introduce
\[
(IV.19) \quad F(\Psi)(t,x) := -i\gamma^0 \left[ gA_\mu(t,x)\gamma_\mu + \Phi(t,x)(hI + ik\gamma^5) \right] \Psi(t,x).
\]
The previous properties show that \( F \) satisfies Assumption II.3. To prove that Assumption II.4 is fulfilled, we easily check using (IV.3), that
\[
(IV.20) \quad \Re(\Psi^*F(\Psi))(t,x) = 0.
\]
Therefore Theorem III.2 assures that the global Cauchy problem
\[
(IV.21) \quad \partial_t \Psi + \gamma^0\gamma^1\partial_x \Psi + iB \Psi = F(\Psi), \quad \Psi(0) = \psi,
\]
has a unique global solution \( \Psi \in \mathcal{C}^0(\mathbb{R}_t;L^2(\mathbb{R}_x;\mathbb{C}^2)) \).

To get the regularity of \( \Phi, \) it sufficient to prove that \( \Phi \in \mathcal{C}^0(\mathbb{R}_t;H^1(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t;L^2(\mathbb{R}_x)) \). Therefore, we have to establish that:
\[
(IV.22) \quad h\tilde{\Psi}\Psi + ik\tilde{\Phi}\gamma^5\Psi \in L^2_{loc}(\mathbb{R}_t;L^2(\mathbb{R}_x)).
\]
This is a consequence of the following result that is related with the compatibility between this quadratic form and the Dirac equation (see [2], [8], [9], [17]).

**Lemma IV.2.** For all \( h, k \in \mathbb{R}, \) there exists a continuous function \( C(T) \) such that for any \( T > 0, \)
\( \Psi, F \in \mathcal{C}^0(\mathbb{R}_t;L^2(\mathbb{R}_x;\mathbb{C}^2)) \) satisfying \( (IV.6) \), we have:
\[
(IV.23) \quad \|h\tilde{\Psi}\Psi + ik\tilde{\Phi}\gamma^5\Psi\|_{L^2([-T,T] \times \mathbb{R})} \leq C(T) \left( \|\Psi(0,x)\|_{L^2(\mathbb{R}_x)} + \|F\|_{L^2([-T,T] \times \mathbb{R})} \right)^2.
\]

**Proof of Lemma IV.2.** We make the change of unknowns:
\[
(IV.24) \quad u_\pm = \Psi_1 \pm \Psi_2.
\]
Then \( \Psi \) is solution of equation (IV.6) if \( u_\pm \) satisfies
\[
(IV.25) \quad \partial_t u_\pm - \partial_x u_\pm + iMu_\pm = -i(F_1 \mp F_2).
\]
Since
\[
\tilde{\Psi}\Psi = \Re(u_+u_+^*) , \quad \Phi^5\Psi = i3(u_+u_-^*),
\]
it is sufficient to prove
\[
(IV.26) \quad \|u_+u_-^*\|_{L^2([-T,T] \times \mathbb{R})} \leq C(T) \left( \|u_+(0,x)\|_{L^2(\mathbb{R}_x)} + \|u_-(0,x)\|_{L^2(\mathbb{R}_x)} + \|(F_1,F_2)\|_{L^2([-T,T] \times \mathbb{R})} \right)^2.
\]
We have:
\[
u_\pm(t,x) = u_\pm(0,x \mp t) + \int_0^t G_\pm(s,x \mp t \mp s)ds, \quad G_\pm := -i(Mu_\pm + F_1 \mp F_2).
\]
Hence, putting \( \xi_\pm = x \pm t, \) we get for \( |t| \leq T \):
\[
|u_+(t,x)u_+^*(t,x)| \leq |u_+(0,\xi_-)u_+^*(0,\xi_+)| + |u_+(0,\xi_-)| \int_{-T}^T |G_-(s,\xi_+ - s)|ds + |u_-(0,\xi_+)| \int_{-T}^T |G_+(s,\xi_- + s)|ds + |u_-(0,\xi_-)| \int_{-T}^T |G_+(s,\xi_- + s)|ds \int_{-T}^T |G_-(\sigma,\xi_- + \sigma)|d\sigma,
\]
where
\[
\xi_\pm = x \pm t, \quad G_\pm(s,z) := -i(Mu_\pm + F_1 \mp F_2).
\]
then
\[
\frac{1}{3} | u_+(t, x) u_-^*(t, x) |^2 \\
\leq | u_+(0, \xi_-) |^2 | u_-^*(0, \xi_+) |^2 + 2T | u_+(0, \xi_-) |^2 \int_{-T}^T | G_-(s, \xi_+ - s) |^2 ds \\
+ 2T | u_-(0, \xi_+) |^2 \int_{-T}^T | G_+(s, \xi_- + s) |^2 ds \\
+ 4T^2 \int_{-T}^T | G_+(s, \xi_- + s) |^2 ds \int_{-T}^T | G_+(\sigma, \xi_+ + \sigma) |^2 d\sigma,
\]
and finally, by integrating on \( \mathbb{R}_x \):
\[
(IV.27) \quad \| u_+ u_-^* \|^2_{L^2([-T,T] \times \mathbb{R})} \leq C(T) \left( \| u_+(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| u_-(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| (G_+, G_-) \|^2_{L^2([-T,T] \times \mathbb{R})} \right)^2.
\]
We evaluate:
\[
\| G_\pm \|^2_{L^2([-T,T] \times \mathbb{R})} \leq M \| u_\mp \|^2_{L^2([-T,T] \times \mathbb{R})} + \| (F_1, F_2) \|^2_{L^2([-T,T] \times \mathbb{R})}.
\]
Since \( u_\pm \) satisfy (IV.25), we may apply Lemma III.5 and from (III.23) we get for \( |t| \leq T \):
\[
\| u_\pm(t, x) \|^2_{L^2(\mathbb{R}_x)} \leq \| u_\pm(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| F_1 \mp F_2 \|^2_{L^2([-T,T] \times \mathbb{R})} ds + \int_0^T \| u_\pm(s, x) \|^2_{L^2(\mathbb{R}_x)} ds.
\]
Then the Gronwall Lemma gives:
\[
\| u_\pm(t, x) \|^2_{L^2(\mathbb{R}_x)} \leq e^T \left( \| u_\pm(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| (F_1, F_2) \|^2_{L^2([-T,T] \times \mathbb{R})} \right), \quad |t| \leq T,
\]
hence
\[
\| G_\pm \|^2_{L^2([-T,T] \times \mathbb{R})} \leq C(T) \left( \| u_+(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| u_-(0, x) \|^2_{L^2(\mathbb{R}_x)} + \| (F_1, F_2) \|^2_{L^2([-T,T] \times \mathbb{R})} \right).
\]
Now (IV.26) follows from (IV.27).
\[
Q.E.D.
\]

IV.2. Thirring model on \( \mathbb{R}^{1+1} \). We consider the global Cauchy problem for the Dirac-Maxwell system on the Minkowski space-time \( \mathbb{R}^{1+1} \), with the Thirring auto-interaction:
\[
(IV.28) \quad i\gamma^\mu \partial_\mu \Psi + M\Psi = (g A^\mu + h \overline{\Psi} \gamma^\mu \gamma_5 \Psi) \gamma_\mu \Psi,
\]
\[
(IV.29) \quad \partial_t^2 A_\mu - \partial_x^2 A_\mu = g \overline{\Psi} \gamma_\mu \Psi,
\]
\[
(IV.30) \quad \Psi(0, x) = \psi(x),
\]
\[
(IV.31) \quad A_\mu(0, x) = f_\mu(x), \quad \partial_\mu A_\mu(0, x) = g_\mu(x),
\]
g, h are are coupling constants. Similar models were studied in [3], [14], [19], [35].

**Theorem IV.3.** For any \( \psi \in L^2 \cap C^0_{\infty}(\mathbb{R}; \mathbb{C}^2) \), \( f_\mu \in H^s(\mathbb{R}; \mathbb{R}) \), \( g_\mu \in H^{s-1}(\mathbb{R}; \mathbb{R}) \), \( \frac{1}{2} < s \leq 2 \), the Cauchy problem (IV.28) to (IV.31) has a unique global solution \( \Psi \in C^0(\mathbb{R}_t; L^2 \cap C^0_{\infty}(\mathbb{R}; \mathbb{C}^2)) \), \( A_\mu \in C^0(\mathbb{R}_t; H^{\frac{1}{2}-\varepsilon} \cap L^\infty(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_x)) \) for all \( \varepsilon > 0 \). In the case of the massless Dirac-Maxwell system, \( M = 0 \), the local charge of the spinor decays:
\[
(IV.32) \quad \forall R > 0, \int_{-R}^R | \Psi(t, x) |^2 dx \longrightarrow 0, \quad |t| \rightarrow \infty.
\]
When \( \psi \in H^1(\mathbb{R}; \mathbb{C}^2) \), then \( \Psi \in C^0(\mathbb{R}_t; H^1(\mathbb{R}; \mathbb{C}^2)) \), \( A_\mu \in C^0(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; H^{s-1}(\mathbb{R}_x)) \).
Proof of Theorem IV.3. With the change of unknown (IV.24), the Dirac equation (IV.28) becomes:

\[(IV.33) \quad \partial_t u + \gamma^0 \partial_x u + iBu = F,\]

with \(u = \ell(u_+, u_-), F = \ell(F_+, F_-), B = B_1 + B_2,\)

\[
\begin{align*}
[F_\pm(u)](t, x) &= -i \left( h \mid u_\pm(t, x) \right)^2 + \frac{q^2}{2} \int_0^t \int_{x-t+s}^{x+s} \left| u_\pm(s, y) \right|^2 dy ds) u_\pm(t, x), \\
B_1 &= M \gamma^5, \\
B_2 &= \frac{q}{2} \left( \tilde{A}_I + \tilde{A}_I^0 \right),
\end{align*}
\]

where \(\tilde{A}_\mu \in C^0(\mathbb{R}_t; H^s(\mathbb{R}_x; \mathbb{R})) \cap C^1(\mathbb{R}_t; H^{s-1}(\mathbb{R}_x; \mathbb{R}))\) are defined by (IV.17). Since \(H^s(\mathbb{R}) \subset C^0(\mathbb{R}),\) Assumption III.10 is satisfied. On the other hand,

\[
\lambda u_\pm(t, x) - [F_\pm(u)](t, x) \mid^2 = \lambda^2 \mid u_\pm(t, x) \mid^2 + \mid [F_\pm(u)](t, x) \mid^2,
\]

and we easily check that Assumption III.8 is also satisfied. Therefore Corollary III.11 assures that there exists a unique solution \(u \in C^0(\mathbb{R}_t; L^2 \cap C^0(\mathbb{R}; C^2))\) of (IV.33) with \(u_\pm(0, x) = \psi_1(x) \pm \psi_2(x).\) Now it is sufficient to put

\[\Psi = \ell(u_+ + u_-, u_+ - u_-),\]

\[A^\mu(t, x) = \tilde{A}_\mu + \tilde{A}_\mu,\]

where \(\tilde{A}_\mu \in C^0(\mathbb{R}_t; H^{\frac{1}{2} - \varepsilon} \cap L^\infty(\mathbb{R}; \mathbb{R})) \cap C^1(\mathbb{R}_t; H^{-\frac{1}{2} - \varepsilon}(\mathbb{R}; \mathbb{R}))\) is defined by (IV.18). For the massless Dirac-Maxwell system, \(M = 0, (III.39)\) is satisfied and (IV.32) follows from Theorem III.13. Finally the local Cauchy problem is well posed in \(H^1\) since we have :

\[\| \Psi(t) \|_{H^1} \leq C \left( \| \psi \|_{H^1} + \left| \int_0^t \| A(s) \|_{L^\infty} + \| \Psi(s) \|_{L^2} \right| ds \right).
\]

Therefore the global solution \(\Psi\) belongs to \(C^0(\mathbb{R}_t; H^1(\mathbb{R}; C^2))\) when \(\psi \in H^1,\) and the regularity of the electromagnetic field follows.

Q.E.D.

IV.3. Dirac-Klein-Gordon system on the Schwarzschild type manifolds. We are concerned with a 3+1 dimensional, spherically symmetric space-time \(\mathbb{R}_t \times I \times S^2_\phi, I\) being a real open interval, that describes a black hole. In this case the metric can be written as:

\[(IV.34) \quad g_{\mu\nu} dx^\mu dx^\nu = G(r) dt^2 - [G(r)]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad \omega = (\theta, \phi) \in [0, \pi] \times [0, 2\pi],\]

where \(G\) is called the lapse function, and satisfies, either \(G \in C^2 \cap W^{2,\infty}[r_0, \infty[), 0 < r_0 < \infty, G(r_0) = 0, 0 < G'(r_0),\) or \(G \in C^2([r_0, r_+)), 0 < r_0 < r_+ < \infty, G(r_0) = G(r_+) = 0, 0 < r_0 < r_+ \Rightarrow 0 < G(r), 0 < G'(r_0), G'(r_+) < 0.\)

Here \(r_0\) is the radius of the Horizon of the Black-Hole, \(r_+\) is the radius of the Cosmological Horizon. These hypotheses are satisfied, for a suitable choice of the physical parameters, in the important case of the (DeSitter-)Schwarzschild and the (DeSitter-)Reissner-Nordström metric given by:

\[(IV.35) \quad G(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2.\]

Here \(0 < M\) and \(Q \in \mathbb{R}\) are the mass and the charge of the black-hole, \(\Lambda \geq 0\) is the cosmological constant (see e.g. [5], [25]). It is convenient to push the horizons away to infinity by means of the tortoise coordinate \(x\) satisfying

\[(IV.36) \quad \frac{dx}{dr} = \frac{1}{G(r)},\]
Since we have:

\begin{align*}
& (IV.40) \quad x = \frac{1}{G'(r_0)} \left\{ \ln | r - r_0 | - \int_{r_0}^{r} \frac{1}{r - r_0} \frac{G'(r)}{G(r)} \, dr \right\}, \\
& (IV.46) \quad R \text{ fields}, \\
& (IV.47) \quad \phi
\end{align*}

We consider the Dirac-Klein-Gordon system on $\mathcal{M}$:

\begin{align*}
& (IV.38) \quad i\gamma_{(\mu)} \nabla_{\mu} \Psi + M \Psi = k \Phi \Psi, \\
& (IV.39) \quad | g |^{-\frac{1}{2}} \partial_{\mu} \left( | g |^{\frac{1}{2}} g^{\mu\nu} \partial_{\nu} \Phi \right) + m^2 \Phi + \xi R \Phi = k \overline{\Phi} \Psi,
\end{align*}

where $\nabla_{\mu}$ are the covariant derivatives of the spinors on $(\mathcal{M}, g)$, $0 \leq M, m$ are the mass of the fields, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature, $\xi \in \mathbb{R}$ is a numerical factor, $k \in \mathbb{R}$ the coupling constant. For the metric $(IV.34)$, these equations take the form:

\begin{align*}
& (IV.40) \quad \left\{ \begin{array}{l}
\gamma_0 \frac{\partial}{\partial t} + \gamma^1 \left( \frac{\partial}{\partial x} + \frac{G'(r)}{4} + \frac{G(r)}{r} \right) + \frac{G^{1/2}(r)}{r} \gamma^2 \left( \frac{\partial}{\partial \theta} + \frac{1}{2 \tan \theta} \right) + \frac{G^{1/2}(r)}{r} \gamma^3 \frac{\partial}{\partial \phi} + i G^{1/2}(r) M \end{array} \right\} \Psi \\
& = i k G^{1/2}(r) \Phi \Psi, \\
& (IV.41) \quad \left\{ \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial x} r^2 \frac{\partial}{\partial x} + G(r) \left( - \frac{\Delta s^2}{r^2} + m^2 + \xi R(r) \right) \right\} \Phi = k \overline{\Phi} \Psi.
\end{align*}

In $(IV.40)$ the Dirac matrices are given by $(IV.7)$. The Ricci scalar is expressed by

\begin{align*}
& (IV.39) \quad R(r) = G''(r) + \frac{4}{r} G'(r) + \frac{2}{r^2} (G(r) - 1).
\end{align*}

It will be convenient to make a change of unknowns by putting for $(t, x, \omega) \in \mathbb{R}_t \times \mathbb{R}_x \times S^2_{\omega}$:

\begin{align*}
& (IV.42) \quad \varphi(t, x, \omega) := r \Phi(t, x, \omega), \\
& (IV.43) \quad \psi(t, x, \omega) := r |G(r)|^{\frac{1}{2}} T_g \Psi(t, x, \omega),
\end{align*}

where $T_g$ is the rotation of Euler angles $(0, \frac{\pi}{2}, \pi)$:

\begin{align*}
& (IV.44) \quad T_g = - \frac{i}{\sqrt{2}} \begin{pmatrix} \sigma_1 + \sigma_3 & 0 \\ 0 & \sigma_1 + \sigma_3 \end{pmatrix} = -(T_g)^{-1}.
\end{align*}

Since we have:

\begin{align*}
& (IV.45) \quad T_g \gamma^0 (T_g)^{-1} = \gamma^0, \quad T_g \gamma^1 (T_g)^{-1} = \gamma^1, \quad T_g \gamma^2 (T_g)^{-1} = - \gamma^2, \quad T_g \gamma^3 (T_g)^{-1} = \gamma^1,
\end{align*}

the Cauchy problem for the Dirac-Klein-Gordon system becomes

\begin{align*}
& (IV.46) \quad \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{G(r)}{r^2} \Delta s^2 + G(r) \left( m^2 - \frac{G'(r)}{r} + \xi R(r) \right) \right) \varphi = k \frac{|G(r)|^{\frac{1}{2}}}{r} \psi, \\
& (IV.47) \quad \psi(0, x, \omega) = \psi_0(x, \omega), \quad \varphi(0, x, \omega) = \varphi_0(x, \omega), \quad \frac{\partial \varphi}{\partial t}(0, x, \omega) = \varphi_1(x, \omega).
\end{align*}

A natural functional framework is $\left[ L^2 \left( \mathbb{R}_x \times S^2_{\omega}, dxd\omega \right) \right]^4$ for the Dirac spinor, and for the scalar field, the Sobolev space $H^s \left( \mathbb{R}_x \times S^2_{\omega} \right)$, $\frac{1}{2} \leq s \leq 1$, defined by interpolation between $L^2$ and $H^1 \left( \mathbb{R}_x \times S^2_{\omega} \right) := \left\{ f \in L^2 \left( \mathbb{R}_x \times S^2_{\omega}, dxd\omega \right); \| \mathbf{H} f \|^2 := \int \| \partial_x f \|^2 + \| \nabla s f \|^2 + \| f \|^2 dxd\omega < \infty \right\}$. 


In these spaces, the local Cauchy problem is a hard open problem. It is solved in a more regular functional framework, for the Kerr space-time, by F. Melnyk [28]. Here, we solve the global Cauchy problem for initial data satisfying a property of symmetry. More precisely, we assume the spherical invariance of the Klein-Gordon field:

\[(IV.48)\]

\[\phi_0(x, \omega) = f_0(x), \quad \phi_1(x, \omega) = f_1(x),\]

and the Dirac spinor has the unique spin-weighted spherical harmonic \(i(T_{\frac{3}{2}, \frac{1}{2}}, T_{\frac{3}{2}, \frac{1}{2}}, T_{\frac{3}{2}, \frac{1}{2}}, T_{\frac{3}{2}, \frac{1}{2}})\) (see [7], [18], [20], [26], [27], [29], [32], [33]):

\[(IV.49)\]

\[\psi_0(x, \omega) = e^{i\phi/2} \begin{pmatrix} u_0(x) \cos \theta/2 \\ iu_0(x) \sin \theta/2 \\ v_0(x) \cos \theta/2 \\ -iv_0(x) \sin \theta/2 \end{pmatrix}.\]

These constraints have already been used in [7] and [15]. Due to the spherical invariance of the equations, this form is conserved by the evolution, hence we look for the solutions of type:

\[(IV.50)\]

\[\varphi(t, x, \omega) = f(t, x), \quad \psi(t, x, \omega) = e^{i\phi/2} \begin{pmatrix} u(t, x) \cos \theta/2 \\ iu(t, x) \sin \theta/2 \\ v(t, x) \cos \theta/2 \\ -iv(t, x) \sin \theta/2 \end{pmatrix}.\]

With this Ansatz, \((\psi, \varphi)\) is solution of the initial Cauchy problem, iff \((u, v, f)\) is solution of:

\[(IV.51)\]

\[\partial_t u + \partial_x v + Qv + iMrQu = ikQfu,\]

\[(IV.52)\]

\[\partial_t v + \partial_x u - Qu - iMQv = -ikQfv,\]

\[(IV.53)\]

\[\partial_t^2 f - \partial_x^2 f + Vf = kQ (|u|^2 - |v|^2),\]

\[(IV.54)\]

\[u(0, x) = u_0(x), \quad v(0, x) = v_0(x),\]

\[(IV.55)\]

\[f(0, x) = f_0(x), \quad \partial_t f(0, x) = f_1(x),\]

where we have put:

\[(IV.56)\]

\[V(x) := G(r) \left( m^2 - \frac{G'(r)}{r} + \xi R(r) \right), \quad Q(x) := \frac{[G(r)]^{\frac{1}{2}}}{r}.\]

**Theorem IV.4.** For all \(u_0, v_0 \in L^2(\mathbb{R}; \mathbb{C}), f_0 \in H^s(\mathbb{R}; \mathbb{R}), f_1 \in H^{s-1}(\mathbb{R}; \mathbb{R}), \frac{1}{2} < s \leq 1,\) the Cauchy problem \((IV.51), (IV.52), (IV.53), (IV.54), (IV.55)\), has a unique solution \(u, v \in C^0(\mathbb{R}; L^2(\mathbb{R}; \mathbb{C})), f \in C^0(\mathbb{R}; H^s(\mathbb{R}; \mathbb{R})) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R}; \mathbb{R})).\) The solution continuously depends on these spaces on the initial data and the charge of the spinor is conserved:

\[(IV.57)\]

\[\int_{-\infty}^{\infty} |u(t, x)|^2 + |v(t, x)|^2 \, dx = \int_{-\infty}^{\infty} |u_0(x)|^2 + |v_0(x)|^2 \, dx.\]

**Proof of Theorem IV.4.** First we consider the Cauchy problem

\[(IV.58)\]

\[\partial_t^2 f - \partial_x^2 f + Vf = g, \quad f(0, \cdot) \in H^s(\mathbb{R}_x), \quad \partial_t f(0, \cdot) \in H^{s-1}(\mathbb{R}_x),\]

with \(g \in L^1_{\text{loc}}(\mathbb{R}_t; H^\sigma(\mathbb{R}_x)), -1 \leq \sigma.\) Since \(V \in W^{1,\infty}(\mathbb{R}_x),\) the map \(f \mapsto Vf\) is well defined and continuous from \(H^s(\mathbb{R}_x)\) to \(H^s(\mathbb{R}_x)\) for \(-1 \leq s \leq 1.\) Hence \(f \in C^0(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; H^{s-1}(\mathbb{R}_x)),\)

\[
\tau = \min(s, \sigma + 1),\]

is solution of \((IV.58)\) if

\[
\left( \begin{array}{c} f(t) \\ \partial_t f(t) \end{array} \right) = U_0(t) \left( \begin{array}{c} f(0) \\ \partial_t f(0) \end{array} \right) + \int_0^t U_0(t - t') \begin{pmatrix} 0 \\ g(t') - Vf(t') \end{pmatrix} \, dt' \]

for
where \( U_0(t) \) is the propagator associated to the free wave operator \( \partial^2_t f - \partial^2_x f = 0 \). Since \( U_0(t) \) is a strongly continuous group on \( H^7 \times H^{r-1} \), this integral equation is easily solved by iteration. Hence the Cauchy problem (IV.58) is well posed and \( f \) satisfies

\[
\| (f(t), \partial_t f(t)) \|_{H^r \times H^{r-1}} \leq C(t) \left( \| (f(0), \partial_t f(0)) \|_{H^r \times H^{r-1}} + \| g \|_{L^1(0,t;H^r)} \right).
\]

Let \( f(0) \) be the solution of (IV.55) and (IV.58) with \( g = 0 \). Since \( f(0) \in C^0(\mathbb{R}; H^s(\mathbb{R})) \) and \( H^s(\mathbb{R}) \subset C^0(\mathbb{R}) \), the matrices

\[
A_t := \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad B(t) := \begin{pmatrix} MrQ - kQf(0)(t) & -iQ \\ iQ & -MrQ + kQf(0)(t) \end{pmatrix},
\]

satisfy Assumption III.1. Now given \( g \in L^1_{loc}(\mathbb{R}; L^1(\mathbb{R})) \), since \( L^1(\mathbb{R}) \subset H^s(\mathbb{R}), \sigma < -\frac{1}{2} \), there exists a unique solution \( f_g \in C^0(\mathbb{R}; H^{s+1}(\mathbb{R})) \cap C^1(\mathbb{R}; H^s(\mathbb{R})) \) of (IV.58) with \( f_g(0, \cdot) = \partial_t f_g(0, \cdot) = 0 \). We remark that \( f_g \) is solution of

\[
f_g(t, x) = \frac{1}{2} \int_0^t \left( \int_{x-t+s}^{x+t-s} kQ(y)g(s, y) - V(y)f_g(s, y)dy \right) ds,
\]

hence

\[
\| f_g(t) \|_{L^\infty} \leq C \left( \| g \|_{L^1([0,t] \times \mathbb{R})} + \int_0^t \| f_g(s) \|_{L^\infty} ds \right).
\]

We conclude by the Gronwall Lemma that \( f_g \in C^0(\mathbb{R}; L^\infty(\mathbb{R})) \), and

\[
\| f_g(t) \|_{L^\infty} \leq C(t) \| g \|_{L^1([0,t] \times \mathbb{R})}.
\]

We deduce from (IV.62), that the map

\[
(u, v) \mapsto f_g, \quad g = kQ \left( |u|^2 - |v|^2 \right),
\]

is a lipchitz continuous map from \( C^0(\mathbb{R}; L^2(\mathbb{R}; \mathbb{C}^2)) \) to \( C^0(\mathbb{R}; L^\infty(\mathbb{R}; \mathbb{C}^2)) \), and by (IV.59), it is also continuous lipschitz from \( C^0(\mathbb{R}; H^1(\mathbb{R}; \mathbb{C}^2)) \) to \( C^0(\mathbb{R}; H^1(\mathbb{R}; \mathbb{C}^2)) \). We introduce the map

\[
U := \begin{pmatrix} u \\ v \end{pmatrix} \mapsto F(U) := ikQf_g \gamma^0 U, \quad g = kQ \left( |u|^2 - |v|^2 \right).
\]

The previous properties of \( (u, v) \mapsto f_g \), show that \( F \) satisfies Assumption II.3 with \( X^0 = L^2(\mathbb{R}; \mathbb{C}^2), X^1 = H^1(\mathbb{R}; \mathbb{C}^2) \). Since \( f \) is real valued and \( \gamma^0 \) is hermitian, we have

\[
\Re(U^*F(U)) = 0,
\]

hence Assumption II.4 is fulfilled. Therefore Theorem III.2 assures that the global Cauchy problem (IV.51), (IV.52), (IV.54), that is equivalent to

\[
\partial_t U + A_t U + iB(t)U = F(U), \quad U(0) = t^\prime(u_0, v_0),
\]

has a unique solution \( U \in C^0(\mathbb{R}^+_t; L^2(\mathbb{R}; \mathbb{C}^2)) \). But since \( A_t \) generates a unitary group, we can repeat the previous arguments to \( -A_t, -B(t), -F(U) \), and we deduce from a time reversal that the solution also exists in \( C^0(\mathbb{R}^-_t; L^2(\mathbb{R}; \mathbb{C}^2)) \), and that the conservation of the charge (IV.57) holds. Finally Lemma IV.2 assures that \( \overline{U}U \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R})) \), then we deduce from (IV.59) that (IV.53), (IV.55) has a solution \( f \in C^0(\mathbb{R}; H^s(\mathbb{R})) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R})) \).
Global Cauchy Problem for Semilinear Hyperbolic Systems

To prove the continuous dependence and the uniqueness, we consider two solutions \((u^j, v^j, f^j)\), \(j = 1, 2\). We get from (IV.57), (IV.59) and (IV.62) that

\[
\| f^1(t) - f^2(t) \|_{L^\infty} \leq \int_0^t \left( C(t) \left( \| f^1(0) - f^2(0) \|_{H^s} \right) + \left( \| U^1(0) \|_{L^2} + \| U^2(0) \|_{L^2} \right) \int_0^t \| U^1(s) - U^2(s) \|_{L^2} \, ds \right),
\]

and Lemma III.5 and (IV.62) give:

\[
\| U^1(t) - U^2(t) \|^2_{L^2} \leq C(t) \left( \| U^1(0) - U^2(0) \|^2_{L^2} + \left( \| U^1(0) \|_{L^2} + \| U^2(0) \|_{L^2} \right) \int_0^t \| U^1(s) - U^2(s) \|^2_{L^2} + \| f^1(s) - f^2(s) \|^2_{L^2} \, ds \right).
\]

We conclude by the Gronwall Lemma that the map \((u_0, v_0, f_0, f_1) \mapsto (u, v, f)\) is Lipschitz continuous, and one-to-one, from \(L^2 \times L^2 \times H^s \times H^{s-1}\) into \(C^0(\mathbb{R}; L^2 \times L^2 \times L^\infty)\). At last, taking account of this result, we obtain by the Lemma IV.2 and (IV.59) with \(\sigma = 0\), the continuity of \((u_0, v_0, f_0, f_1) \mapsto (f, \partial_t f)\) from \(L^2 \times L^2 \times H^s \times H^{s-1}\) into \(C^0(\mathbb{R}; H^s \times H^{s-1})\).

Q.E.D.

IV.4. Dirac-Klein-Gordon system with gravitational collapse. We consider the Dirac-Klein-Gordon system (IV.38), (IV.39), on the manifold

\[
M = \{ t, x, \omega \in \mathbb{R} \times \mathbb{R} \times S^2, \ x \geq z(t) \},
\]

endowed with the metric (IV.34), that describes the gravitational collapse of a spherical star (see [4]). Here \(z(t)\) is the radius of the star at time \(t\), and this function satisfies:

\[
\begin{cases}
  z \in C^2(\mathbb{R}), \\
  \forall t \in \mathbb{R}, \ -1 < \dot{z}(t) \leq 0, \\
  z(t) = -t - Ae^{-2\kappa_0 t} + \zeta(t), \ A > 0, \ \kappa_0 := \frac{1}{2}G'(r_0), \\
  | \zeta(t) | + | \dot{\zeta}(t) | = O(e^{-4\kappa_0 t}), \ t \to \infty, \\
  \forall t \leq 0, \ z(t) = z(0) < 0.
\end{cases}
\]

Following [5], [7], we add a boundary condition of MIT bag type on the spinor and a Dirichlet condition on the scalar field:

\[
\Phi = 0, \ n_\nu \gamma^\mu_{(g)} \Psi = i \Psi, \ on \ \partial M,
\]

where \(n_\nu\) is the outgoing conormal. With the change of unknowns (IV.43), (IV.44), and with the notations (IV.56), the system becomes:

\[
\left( \gamma^0 \frac{\partial}{\partial t} + \gamma^3 \frac{\partial}{\partial x} - Q\gamma^2 \left( \frac{\partial}{\partial \theta} + \frac{1}{2\tan \theta} \right) + \frac{Q}{\sin \theta} \gamma^1 \frac{\partial}{\partial \phi} + iMrQ \right) \psi = ikQ \varphi \psi, \ x > z(t),
\]

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - Q^2 \Delta_{S^2} + V \right) \varphi = kQ \bar{\psi} \psi, \ x > z(t),
\]

\[
\varphi(t, z(t), \omega) = 0, \ (\dot{z}(t)\gamma^0 - \gamma^3) \psi(t, z(t), \omega) = i\sqrt{1 - \dot{z}^2(t)} \psi(t, z(t), \omega),
\]

\[
\psi(0, x, \omega) = \psi_0(x, \omega), \ \varphi(0, x, \omega) = \varphi_0(x, \omega), \ \frac{\partial \varphi}{\partial t}(0, x, \omega) = \varphi_1(x, \omega), \ x > z(0).
\]
With the Anzatz (IV.50) and the notation (IV.56), \((\psi, \varphi)\) is solution of the mixed problem, iff \((u, v, f)\) satisfies:

\[
\begin{align*}
(\text{IV.72}) & \quad \partial_t u + \partial_x v + Qv + iMrQu = ikQfu, \quad t \in \mathbb{R}, \; x > z(t), \\
(\text{IV.73}) & \quad \partial_t v + \partial_x u - Qu - iMrQv = -ikQfv, \quad t \in \mathbb{R}, \; x > z(t), \\
(\text{IV.74}) & \quad \partial_t^2 f - \partial_x^2 f + Vf = kQ \left| u \right|^2 - \left| v \right|^2, \quad t \in \mathbb{R}, \; x > z(t), \\
(\text{IV.75}) & \quad v(t, z(t)) = \left( \dot{z}(t) - i\sqrt{1 - \dot{z}^2(t)} \right) u(t, z(t)) \text{ for almost all } t \in \mathbb{R}, \\
(\text{IV.76}) & \quad f(t, z(t)) = 0, \quad t \in \mathbb{R}, \\
(\text{IV.77}) & \quad u(0, x) = u_0(x), \; v(0, x) = v_0(x), \; x > z(0), \\
(\text{IV.78}) & \quad f(0, x) = f_0(x), \; \partial_t f(0, x) = f_1(x), \; x > z(0).
\end{align*}
\]

To be able to define the boundary conditions, we assume that for any \(t\), the map \(x \mapsto f(t, x)\) belongs to \(H_0^1([z(t), \infty[)) \subset C^2_0([z(t), \infty[)).\) Then (IV.76) makes sense. Now if we also assume that \(t \mapsto \| f(t, \cdot) \|_{H^1} \) is a continuous function of \(t\), then \(f \in L^{\infty}(\{(t, x); \; x > z(t), \; | t | \leq T\})\). Therefore given \(u, v \in L^2(\{(t, x); \; x > z(t), \; | t | \leq T\})\) solutions of (IV.72), (IV.73), we have \((\partial_t \pm \partial_x)(u \pm v) \in L^2(\{(t, x); \; x > z(t), \; | t | \leq T\})\). Since the curve \(\Gamma := \{(t, x = z(t), \; t \in \mathbb{R}\}\) is non characteristic for \(\partial_t \pm \partial_x\), we conclude that the traces of \(u\) and \(v\) are well defined in \(L^2_{\text{loc}}(\Gamma)\). This allows to impose the boundary condition (IV.75) for almost all \(t \in \mathbb{R}\).

To express the regularity with respect to \(t\), and to justify the initial data (IV.77), (IV.78), it will be convenient to put the mixed problem on \(\mathbb{R}_t \times [0, \infty[\) by a \(x\)-translation. For \(g(t, x)\) defined on \(\{(t, x); \; x > z(t)\}\), we associate \(\bar{g}(t, x)\) given on \(\mathbb{R}_t \times [0, \infty[\) by:

\[
\begin{align*}
(\text{IV.79}) & \quad t \in \mathbb{R}, \; x > 0, \; \bar{g}(t, x) = g(t, z(t) + x).
\end{align*}
\]

The global mixed problem is well posed:

**Theorem IV.5.** For all \(u_0, v_0 \in L^2([z(0), \infty[; \mathbb{C}), f_0 \in H_0^1([z(0), \infty[; \mathbb{R}), f_1 \in L^2([z(0), \infty[; \mathbb{R}),\) \(u, v \in C^0(\mathbb{R}_t; L^2([0, \infty[; \mathbb{R})),\) \(f \in C^0(\mathbb{R}_t; H_0^1([0, \infty[)) \cap C^1(\mathbb{R}_t; L^2([0, \infty[; \mathbb{R})).\) The solution depends continuously on the data when the spaces are endowed with the natural topologies. The charge of the spinor is conserved:

\[
\begin{align*}
(\text{IV.80}) & \quad \forall t \in \mathbb{R}, \; \int_{z(t)}^{\infty} \left| u(t, x) \right|^2 + \left| v(t, x) \right|^2 dx = \int_{z(0)}^{\infty} \left| u_0(x) \right|^2 + \left| v_0(x) \right|^2 dx.
\end{align*}
\]

**Proof of Theorem IV.5.** We start by carefully investigating the mixed problem for the linear Dirac and Klein-Gordon equations. Following [4], we need a \(C^1\) function \(\tau(t, x)\), implicitly defined for \(t \in \mathbb{R}, \; x \geq z(t)\) by the equation:

\[
(\text{IV.81}) \quad x - z[\tau(t, x)] = t - \tau(t, x).
\]

We also put:

\[
(\text{IV.82}) \quad \lambda(t) := \dot{z}(t) - i\sqrt{1 - \dot{z}^2(t)}.
\]

Firstly, we consider the mixed problem:

\[
\begin{align*}
(\text{IV.83}) & \quad \partial_t u_\pm + \partial_x u_\pm = f_\pm, \; x > z(t), \; t \in \mathbb{R}, \\
(\text{IV.84}) & \quad (1 - \lambda(t))u_+(t, z(t)) = (1 + \lambda(t))u_-(t, z(t)), \; t \in \mathbb{R} (\text{a.e.}),
\end{align*}
\]

Alain Bachelot
Lemma IV.6. For all \( s \in \mathbb{R} \), \( u^\pm_\infty \in L^2(|z(s), \infty|) \), \( \frac{\partial}{\partial x} u^\pm_\infty \in L^1_{\text{loc}}(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \), there exists unique \( \tilde{u}^\pm_\infty \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \) solution of (IV.83), (IV.84), (IV.85). For almost all \((t, x)\), \( t > s \), this solution has the following representation:

\[
x > z(t) \Rightarrow \tilde{u}^-(t, x) = u^-(s, x + t - s) + \int_s^t f_-(\sigma, x + t - \sigma) d\sigma,
\]

\[
x > z(s) + t - s \Rightarrow \tilde{u}^+(t, x) = u^+(s, x - t + s) + \int_s^t f_+(\sigma, x - t + \sigma) d\sigma,
\]

\[
z(t) < x \leq z(s) + t - s \Rightarrow \tilde{u}^+(t, x) = \int_{\tau(t, x)}^t f_+(\sigma, x - t + \sigma) d\sigma + \frac{1 + \lambda[\tau(t, x)]}{1 - \lambda[\tau(t, x)]} \left( u^-(s, z[\tau(t, x)] + \tau(t, x) - s) + \int_s^\tau f_-(\sigma, z[\tau(t, x)] + \tau(t, x) - \sigma) d\sigma \right).
\]

There exists \( C > 0 \), independent of \( u_\pm, f_\pm \), such for any \( s \leq t \), we have

\[
\| (u^+(t), u^-(t)) \|_{L^2(|z(t), \infty|)} \leq C \left( \| (u^+(s), u^-(s)) \|_{L^2(|z(s), \infty|)} + \int_s^t \| (f_+(\sigma), f_-(\sigma)) \|_{L^2(|z(\sigma), \infty|)} d\sigma \right).
\]

If \( f_\pm = 0 \), the \( L^2 \) norm is conserved:

\[
\int_{z(t)}^\infty \| u^+(t, x) \|^2 + \| u^-(t, x) \|^2 dx = C_{\text{st}}.
\]

If \( f_\pm \in C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^+)) \) and \( u^\pm_\infty(s, \cdot) \in H^1(|z(s), \infty|) \) for some \( s \in \mathbb{R} \), then \( \tilde{u}^\pm_\infty \in C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^+)) \).

Proof of Lemma IV.6. To prove the uniqueness, we consider \( \tilde{u}^\pm_\infty \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \) solution of (IV.83), (IV.84), (IV.85), with \( u^\pm_\infty = 0 \), \( f_\pm = 0 \). Then, for almost all \((t, x), t > s, x > z(t)\), we have

\[
u^-(t, x) = u^-(s, x + t - s) = 0, \quad u^+(t, x) = u^+(\tau, z(\tau)) = \frac{1 + \lambda(\tau)}{1 - \lambda(\tau)} u^-(\tau, z(\tau)) = \frac{1 + \lambda(\tau)}{1 - \lambda(\tau)} u^-(s, z(\tau) + \tau - s) = 0.
\]

Therefore \( u^\pm = 0 \) for \( t > s \). The proof for \( t < s \) is analogous. The representation formulas (IV.86), (IV.87) directly follows from an integration of equations (IV.83) along a characteristic line \( x \pm t = \text{cst.} \), and since

\[
u^+(t, x) = u^+(\tau(t, x), z[\tau(t, x)]) + \int_{z(t, x)}^t f_+(\sigma, x - t + \sigma) d\sigma \quad \text{a.e.},
\]

(IV.88) is consequence of (IV.84) and (IV.86). The other assertions of the Lemma are easily deduced from these representations of the solutions. In particular to get the \( L^2 \)-estimates, we use the change of variables

\[
y = z[\tau(t, x)] + \tau(t, x) - s, \quad dy = \frac{1}{\tau[\tau(t, x)]} + \frac{1}{\tau[\tau(t, x)] - 1} dx.
\]

Q.E.D.

We also have an estimate of null condition type for this mixed problem:
Lemma IV.7. If $u_{\pm}^j$, $j = 1, 2$, are two solutions of (IV.83), (IV.84), with $\widetilde{u}_{\pm}^j (0, \cdot) \in L^2 (\mathbb{R}_+^2)$, $\overline{f}_j \in L^2 \text{loc} \left( \mathbb{R}_t; L^2 (\mathbb{R}_x^+) \right)$, the product $\overline{u}_{\pm}^j \overline{u}_{\pm}^1$ belongs to $L^2 \text{loc} \left( \mathbb{R}_t; L^2 (\mathbb{R}_x^+) \right)$, and there exists $C \in C^0$, independent of $u_{\pm}^j$, such that for any $T > 0$, we have:

$$
\int_{-T}^T \int_{z(t)}^{\infty} |u_{\pm}^j (t, x) u_{\pm}^1 (t, x)|^2 \, dt \, dx \leq C(T) \prod_{j=1,2} \left( \sum_x |\overline{u}_{\pm}^j (0, \cdot)|_{L^2 (\mathbb{R}_+)} + ||f_j||_{L^2([-T,T] \times \mathbb{R}_x)} \right)^2.
$$

Proof of Lemma IV.7. This estimate is straightforwardly obtained with the representation formulas (IV.86), (IV.87) and some suitable changes of variable. We only consider two significant terms appearing in the product, the others are treated by a similar way. For the term

$$
I = \int_0^T \int_{z(0)+t}^{\infty} \int_0^{\tau(t,x)} f_1^2 (\sigma, x-t+\sigma) \, d\sigma \left| \int_0^t f_1^1 (\sigma', x+t-\sigma') \, d\sigma' \right|^2 \, dx \, dt,
$$

we put

$$
X_1 = x + t - \sigma', \quad X_2 = x - t + \sigma,
$$

and we get

$$
I \leq T^2 \|f_1\|_{L^2([0,T] \times \mathbb{R}_+)}^2 \|f_1\|_{L^2([0,T] \times \mathbb{R}_+)}^2.
$$

The most complex term is:

$$
J = \int_0^T \int_{z(t)}^{z(0)+t} \left| \frac{1 + \lambda [\tau(t,x)]}{1 - \lambda [\tau(t,x)]} \int_0^{\tau(t,x)} f_2^2 (\sigma, z [\tau(t,x)] + \tau(t,x) - \sigma \sigma) \, d\sigma \right|^2 \left| \int_0^t f_1^1 (\sigma', x+t-\sigma') \, d\sigma' \right|^2 \, dx \, dt.
$$

We introduce

$$
X_1 = x + t - \sigma', \quad X_2 = z [\tau(t,x)] + \tau(t,x) - \sigma, \quad dX_1 dX_2 = 2 \left| \frac{1 + \lambda [\tau(t,x)]}{1 - \lambda [\tau(t,x)]} \right|^2 \, dt \, dx,
$$

hence we obtain:

$$
J \leq T^2 \|f_1\|_{L^2([0,T] \times \mathbb{R}_+)}^2 \|f_1\|_{L^2([0,T] \times \mathbb{R}_+)}^2.
$$

Q.E.D.

We now consider the mixed Dirichlet Cauchy problem:

$$
\partial_t^2 f - \partial_x^2 f + V(x) f = g, \quad x > z(t),
$$

$$
f(t, z(t)) = 0, \quad t \in \mathbb{R},
$$

$$
f(0, x) = f_0 (x), \quad \partial_x f(0, x) = f_1 (x), \quad x > 0.
$$

Lemma IV.8. For any $f_0 \in H^1_0 (z(0), \infty)$, $f_1 \in L^2 (z(0), \infty)$, $\overline{g} \in L^1 \text{loc} \left( \mathbb{R}_t; L^2 (\mathbb{R}_x^+) \right)$, there exists a unique $f \in C^0 (\mathbb{R}_t; H^1_0 (\mathbb{R}_x^+)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_x^+))$ such that $f$ is solution of (IV.95), (IV.96), (IV.97). There exists $C \in C^0 (\mathbb{R}_+^2)$ such that for any $T > 0$

$$
\sup_{|t| \leq |T|} \| f(t, \cdot) \|_{H^1_0 (z(0), \infty)} + \| \partial_t f(t, \cdot) \|_{L^2 (z(t), \infty)} \leq C(T) \left( \| f_0 \|_{H^1_0 (z(0), \infty)} + \| f_1 \|_{L^2 (z(0), \infty)} + \int_{-T}^T \| g(t, \cdot) \|_{L^2 (z(t), \infty)} \, dt \right).
$$
Moreover we have

\[(IV.99)\]

\[
\sup_{|t| \leq |T|} \| f(t, \cdot) \|_{L^\infty([z(t), \infty])} \leq C(T) \left( \| f_0 \|_{H^1([z(0), \infty])} + \| f_1 \|_{L^2([z(0), \infty])} + \int_{-T}^T \| g(t, \cdot) \|_{L^1([z(t), \infty])} \, dt \right),
\]

and the map \( (f_0, f_1, g) \mapsto f \) can be extended into a bounded linear map from \( H^1_0([z(0), \infty]) \times L^2([z(0), \infty]) \times L^1_{\text{loc}}(\mathbb{R}_t; L^1(\mathbb{R}_x^+)) \) to \( C^0(\mathbb{R}_t; C^0(\mathbb{R}_x^+)) \).

Proof of Lemma IV.8. Following the general results on the second order hyperbolic operators, the mixed problem (IV.95), (IV.96), (IV.97). is well posed for \( f \in H^1_{\text{loc}}(X), g \in L^2_{\text{loc}}(X), f_0 \in H^1_0([z(0), \infty]), f_1 \in L^2([z(0), \infty]), X := \{(t, x) ; x \geq z(t)\} \) (see e.g. [21], Theorem 24.1.1, by noting that it is sufficient in this functional framework \( X \) to be a \( C^2 \) manifold, and \( V \in C^0) \). In particular the uniqueness is well established. To get the estimates (IV.98), (IV.99), we construct the solution when \( V = g = 0 \). In this case, we have for \( s < t \) :\n
\[(IV.100)\]

\[
z(s) + t - s < x \Rightarrow f(t, x) = \frac{1}{2} \left( f(s, x + t - s) + f(s, x - t + s) + \int_{t-s}^{t+s} \partial_t f(s, x + y) \, dy \right).
\]

Hence:

\[(IV.101)\]

\[
\| f(t) \|_{H^1([z(s)+t-s, \infty])} + \| \partial_t f(t) \|_{L^2([z(s)+t-s, \infty])} \\
\leq C(1 + |t - s|) \left( \| f(s) \|_{H^1([z(s), \infty])} + \| \partial_t f(s) \|_{L^2([z(s), \infty])} \right),
\]

and for \( z(s) + t - s < x, 1 \leq p \leq \infty \):

\[(IV.102)\]

\[
\| f(t) \|_{L^p([z(s)+t-s, \infty])} \leq \| f(s) \|_{L^p([z(s), \infty])} + 2^{-\frac{1}{p}} |t - s|^{1 - \frac{1}{p}} \| \partial_t f(s) \|_{L^p([z(s), \infty])}.
\]

Now for \( z(t) < x < z(s) + t - s, s < t \), we obviously have:

\[(IV.103)\]

\[
(\partial_t - \partial_x) f(t, x) = (\partial_t + \partial_x) f(s, x + t - s).
\]

On the other hand we have \((\partial_t - \partial_x) f(t, x) = (\partial_t - \partial_x) f(\tau, x - t + \tau) \) where \( \tau \) is defined by (IV.81). But the boundary condition \( f(t, z(t)) = 0 \) for any \( t \), implies that \( \partial_t f(t, z(t)) + \dot{z}(t) \partial_x f(t, z(t)) = 0 \). We deduce that

\[
(\partial_t + \partial_x) f(t, z(t)) = \frac{\dot{z}(t)}{\dot{z}(t)} \left[ f(t, x + t - s) - f(t, x - t + s) - \int_{z(\tau) + t - s}^{z(\tau) + t - s} \partial_t f(s, x + y) \, dy \right],
\]

so finally:

\[(IV.104)\]

\[
(\partial_t + \partial_x) f(t, x) = \frac{\dot{z}\tau(t, x)}{\dot{z}\tau(t, x)} - \frac{1}{\dot{z}\tau(t, x)} \left[ \partial_t f(s, z(\tau(t, x)) + \tau(t, x) - s) \right].
\]

Then we can express \( f(t, x) \) from (IV.103), (IV.104) and the Dirichlet condition since \( f(t, x) = \int_{z(t)}^{x(t)} \partial_x f(t, x') \, dx' \). We use the change of variable (IV.91) and we get for \( z(t) < x < z(s) + t - s \) :

\[(IV.105)\]

\[
 f(t, x) = \frac{1}{2} \left( f(s, x + t - s) - f(s, x - t + s) + \int_{\tau(t) + t - s}^{\tau(t) + t - s} \partial_t f(s, x+y) \, dy \right).
\]

Since \( x + t - s - (z(\tau) + \tau - s) = 2(t - \tau) \leq 2(t - s) \), we deduce that (IV.102) is true for \( z(t) < x < z(t) + t - s \) again, and finally we obtain for \( 1 \leq p \leq \infty \):

\[(IV.106)\]

\[
\| f(t) \|_{L^\infty([z(t), \infty])} \leq \| f(s) \|_{L^\infty([z(s), \infty])} + 2^{-\frac{1}{p}} |t - s|^{1 - \frac{1}{p}} \| \partial_t f(s) \|_{L^p([z(s), \infty])}.
\]

On the other hand, we deduce from (IV.104) and (IV.91) that

\[(IV.107)\]

\[
\int_{z(t)}^{z(s) + t - s} |(\partial_t - \partial_x) f(t, x)|^2 \, dx = \int_{z(t)}^{z(s) + t - s} \frac{1 + \dot{z}(\tau)}{1 - \dot{z}(\tau)} |(\partial_t + \partial_x) f(t, y)|^2 \, dy,
\]
hence, tacking account of (IV.101), there exists a continuous function \( C(t,s) \) such that:

\[
\| f(t) \|_{H^1([0,\infty])} + \| \partial_t f(t) \|_{L^2([0,\infty])} 
\leq C(t,s) \left( \| f(s) \|_{H^1([0,\infty])} + \| \partial_t f(s) \|_{L^2([0,\infty])} \right).
\]  

(IV.108)

We introduce the propagator \( \tilde{U}_0(t,s) \):

\[
\tilde{U}_0(t,s) \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) = \left( \begin{array}{c} \tilde{f}(t) \\ \partial_t \tilde{f}(t) \end{array} \right),
\]

where \( f \) is the solution of \( \partial^2_t f - \partial^2_x f = 0 \) for \( x > z(t) \), \( f(s) = \varphi \), \( \partial_t f(s) = \psi \), and \( f(.,z(\cdot)) = 0 \). We deduce from (IV.100), (IV.105), (IV.108) that \( \tilde{U}_0(t,s) \) is a strongly continuous propagator on \( H^1_0(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \). Hence, since \( V \in L^{\infty}(\mathbb{R}) \), the integral equation

\[
\begin{align*}
\left( \begin{array}{c} \tilde{f}(t) \\ \partial_t \tilde{f}(t) \end{array} \right) &= \tilde{U}_0(t,0) \left( \begin{array}{c} \tilde{f}_0 \\ \tilde{f}_1 \end{array} \right) + \int_0^t \tilde{U}_0(t,s) \left( \begin{array}{c} 0 \\ - V \tilde{f}(s) + g(s) \end{array} \right) ds
\end{align*}
\]

(IV.110)

can be solved in the usual way, and has a unique solution \( \tilde{f} \in C^0(\mathbb{R}_t; H^1_0(\mathbb{R}^+)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}^+)) \). Therefore the mixed problem (IV.95), (IV.96), (IV.97) is well posed in this space, and the Gronwall lemma assures that \( f \) satisfies (IV.98). Moreover, by using estimate (IV.106) with \( p = 2 \), \( p = 1 \) and \( p = \infty \), we obtain for \( 0 \leq t \):

\[
\| f(t) \|_{L^\infty(\mathbb{R}^+)} \leq C(t) \left( \| f_0 \|_{L^\infty(\mathbb{R}^+)} + \| \tilde{f}_1 \|_{L^2(\mathbb{R}^+)} + \int_0^t \| g(s) \|_{L^1(\mathbb{R}^+)} + \| f(s) \|_{L^\infty(\mathbb{R}^+)} ds \right),
\]

therefore inequality (IV.99) and the last assertion of the Lemma follow.

Q.E.D.

We now return to the proof of the Theorem. To establish the uniqueness, we consider two solutions \( (u^j, v^j, f^j) \), \( j = 1, 2 \), and for simplicity we put \( U^j := (u^j, v^j) \). By applying estimate (IV.89) to \( u_{\pm} = u^2 - u^1 \pm (v^2 - v^1) \), we get for \( t \geq 0 \):

\[
\| U^2(t) - U^1(t) \|_{L^2([0,\infty])} \leq C \left( 1 + \sup_{0 \leq \sigma \leq t} \| U^2(\sigma) \|_{L^2([0,\infty])} + \| f^1(\sigma) \|_{L^\infty([0,\infty])} \right)
\]

(IV.111)

We use inequality (IV.99) to get

\[
\| f^2(t) - f^1(t) \|_{L^\infty([0,\infty])} \leq C(t) \left( 1 + \sup_{0 \leq \sigma \leq t} \| U^1(\sigma) \|_{L^2([0,\infty])} + \sup_{0 \leq \sigma \leq t} \| U^2(\sigma) \|_{L^2([0,\infty])} \right)
\]

(IV.112)
We deduce from (IV.111), (IV.112), that when \( U^1(0) = U^2(0), f_0^1 = f_0^2, f_1^1 = f_1^2 \), we have:

\[
\| U^2(t) - U^1(t) \|_{L^2([z(t), \infty])} + \| f^2(t) - f^1(t) \|_{L^\infty([z(t), \infty])} \\
\leq C'(t) \int_0^t \| U^2(\sigma) - U^1(\sigma) \|_{L^2([z(\sigma), \infty])} + \| f^2(\sigma) - f^1(\sigma) \|_{L^\infty([z(\sigma), \infty])} \, d\sigma,
\]

hence the uniqueness follows from the Gronwall Lemma.

To get the existence of the solution, we shall apply Theorem II.15. Given \( \bar{f}_j \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \), we consider the system

\[
(IV.113) \quad \partial_t u + \partial_x v = f_1, \quad x > z(t),
\]

\[
(IV.114) \quad \partial_t v + \partial_x u = f_2, \quad x > z(t),
\]

\[
(IV.115) \quad v(t, z(t)) = \lambda(t) u(t, z(t)) \quad a.e.
\]

We put

\[
(IV.116) \quad u_1(t, x) := \lambda(t) \bar{u}(t, x), \quad x > 0,
\]

\[
(IV.117) \quad u_2(t, x) := \bar{v}(t, x), \quad x > 0.
\]

Some tedious but elementary calculations show that \( \bar{u}, \bar{v} \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \) are solutions of (IV.113), (IV.114), (IV.115), if \( u_1, u_2 \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \) are solutions of:

\[
(IV.118) \quad \partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + A_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda(t) \bar{f}_1 \\ \bar{f}_2 \end{pmatrix}, \quad x > 0,
\]

\[
(IV.119) \quad u_1(t, 0) = u_2(t, 0), \quad t \in \mathbb{R},
\]

where

\[
(IV.120) \quad A_t := \begin{pmatrix} -\bar{z}(t) \partial_x - i \frac{-\bar{z}(t)}{\sqrt{1-z^2(t)}} & \lambda(t) \partial_x \\ \lambda^*(t) \frac{-\bar{z}(t)}{\sqrt{1-z^2(t)}} & -\bar{z}(t) \partial_x \end{pmatrix}.
\]

We consider \( A_t \) as a differential operator in the sense of the distributions, defined on:

\[
(IV.121) \quad X^0 := L^2(\mathbb{R}_x^+; C^2),
\]

and also as a densely defined operator on \( X^0 \), with domain:

\[
(IV.122) \quad D(A_t) = X^1 := \left\{ (u_1, u_2) \in H^1(\mathbb{R}_x^+; C^2); \quad u_1(0) = u_2(0) \right\}.
\]

Since \( z \in C^2(\mathbb{R}) \), \( A_t \) satisfies Assumption II.1. Now if \( f_1 = f_2 = 0 \), we compute \( u_1, u_2 \) by using (IV.116), (IV.117), and Lemma IV.6 with \( u_{\pm} = u \pm v \). We deduce that Assumption II.2 is also satisfied. Finally, \( u, v \in C^0(\mathbb{R}_t; L^2([0, \infty[)) \) is solution of (IV.72), (IV.73), (IV.75) if \( u_1, u_2 \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^+)) \) are solution of:

\[
(IV.123) \quad \partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + A_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + B(t) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = F \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad x > 0,
\]

with

\[
(IV.124) \quad B(t) := \begin{pmatrix} iM \bar{r}Q & \lambda(t) \bar{Q} \\ -\lambda^*(t) \bar{Q} & -iM \bar{r}Q \end{pmatrix},
\]

\[
(IV.125) \quad F \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := ik \bar{Q} J^0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]
Lemma IV.7 implies that conclude with (IV.98) that (the continuous dependence of the solution with respect to the initial data. and also (IV.111) and (IV.112) with the Gronwall Lemma, assure that (IV.133)

\[ \text{sup} \]  

Then (IV.132) and (IV.99) imply that there exists (IV.131)

and the Klein-Gordon equation has the form:

\[ \text{sup} n \in \mathbb{N} \]  

\[ \text{sup} n \in \mathbb{N} \]  

Then (IV.132) and (IV.99) imply that there exists \( C \in C^0(\mathbb{R}_t) \) such that (IV.133)

and also (IV.111) and (IV.112) with the Gronwall Lemma, assure that \( (\tilde{u}_\pm, \tilde{v}_\pm, \tilde{f}_\pm) \) are Cauchy sequences in \( C^0(\mathbb{R}_t;L^2(\mathbb{R}_x^+)) \times C^0(\mathbb{R}_t;L^2(\mathbb{R}_x^+)) \times C^0(\mathbb{R}_t;L^\infty(\mathbb{R}_x^+)) \). Hence \( \tilde{u}_\pm \) and \( \tilde{f}_\pm \) are Cauchy sequences in \( C^0(\mathbb{R}_t;L^2(\mathbb{R}_x^+)) \). Since

\[ \text{sup} n \in \mathbb{N} \]  

Lemma IV.7 implies that \( k \tilde{Q} \Re \left[ \tilde{u}_\pm \tilde{v}_\pm - \tilde{u}_\pm \tilde{v}_\pm \right] \) is a Cauchy sequence in \( L^2_{\text{loc}}(\mathbb{R}_t;L^2(\mathbb{R}_x^+)) \), and we conclude with (IV.98) that \( (\tilde{f}_\pm, \partial_t \tilde{f}_\pm) \) is a Cauchy sequence in \( C^0(\mathbb{R}_t;H^1_0(\mathbb{R}_x^+)) \times C^1(\mathbb{R}_t;L^2(\mathbb{R}_x^+)) \). Finally the limit \( (u, v, f) \) is the wished solution, and satisfies (IV.80). The same arguments prove the continuous dependence of the solution with respect to the initial data.

Q.E.D.
REFERENCES


Université Bordeaux-1, Institut de Mathématiques, UMR CNRS 5466, F-33405 Talence Cedex
E-mail address: bachelot@math.u-bordeaux1.fr