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Research Paper

Analysis and discontinuous Galerkin approximation of the Boussinesq system with Navier-type boundary condition

Afaf Bouharguane [a], Nour Seloula [b,*]

[a] Institut de Mathématiques de Bordeaux, CNRS, UMR 5251, Université de Bordeaux, 33405, Talence, France
[b] Université Caen Normandie, CNRS, Normandie Univ, LMNO UMR6139, F-14000, CAEN, France

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ABSTRACT

In this work, we analyse and approximate the nonlinear stationary Boussinesq system to model non-isothermal incompressible fluid flows. The model consists of incompressible Navier-Stokes equations coupled with an advection-diffusion equation for the temperature. We impose a Navier-type boundary condition on the velocity and a Dirichlet boundary condition on the temperature. The well-posedness of the continuous problem is achieved by using a fixed-point strategy. We then propose a discontinuous Galerkin scheme that combines an interior penalty method and an upwind discretization. We prove existence and uniqueness results of the discrete solution under a certain regularity assumption of the domain. A priori error estimates in terms of natural energy norms for the velocity, pressure, and temperature are also derived. Finally, numerical experiments are presented to confirm the theoretical convergence rates of the proposed scheme.

1. Introduction

Over the past few decades, the community of numerical analysts of partial differential equations has extensively focused on developing precise and efficient new finite element methods for the Boussinesq problem. This model is well-known for its relevance in engineering sciences as it deals with fluid motion generated by density differences caused by temperature gradients. Mathematically, it is a nonlinear system of PDEs which is composed of the stationary incompressible Navier-Stokes equations for the velocity and pressure variables and a convection-diffusion equation for the temperature variable. Such coupling is done by means of a buoyancy term in the Navier-Stokes equations depending on the temperature, and a convective term in the heat equation that depends on the fluid velocity. The Boussinesq approximation was justified and used to study several applications such as in geophysics to model climate prediction [1], in oceanography for studying oceanic flows [2,3] and also in magnetohydrodynamic flows [4]. Moreover, simulations that combine fluid flow and temperature are invaluable in predicting the performance of physical designs across a wide range of engineering applications. By conducting these simulations before the manufacturing process, it becomes possible to significantly reduce the costs associated with the development of new products [5,6]. From a numerical analysis perspective, numerous discretization strategies have been proposed for the stationary Boussinesq system, mainly under Dirichlet boundary conditions for the velocity field. Early contributions focused on conforming finite element methods and addressed the existence and uniqueness of weak

* Corresponding author.
E-mail addresses: afaf.bouharguane@math.u-bordeaux.fr (A. Bouharguane), nour-elhouda.seloula@unicaen.fr (N. Seloula).

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solutions as well as the well-posedness of the associated discrete problems. In particular, Bernardi et al. [7] established existence and uniqueness results for the continuous problem and proposed a finite element approximation, while Farhloul et al. [8] introduced a mixed formulation for the two-dimensional case and proved convergence properties of the numerical scheme. More recently, Oyarzúa et al. [9] analyzed a finite element method with exactly divergence-free velocity approximations for a generalized Boussinesq model.

For more complex Boussinesq models involving temperature-dependent viscosity and thermal conductivity, several analytical and numerical difficulties arise due to the strong nonlinear coupling between the momentum and heat equations. A number of works have addressed these issues by relying on Galerkin-type approximations combined with fixed-point or compactness arguments. In this direction, Lorca and Boldrini [10] established existence and regularity results for generalized Boussinesq systems by means of spectral approximations, together with uniqueness results under suitable smallness assumptions. Related analytical results in two-dimensional settings and for alternative boundary conditions were later obtained by other authors, see for instance [11]. On the numerical side, conforming finite element discretizations have been proposed to approximate these generalized models. Pérez et al. [12] analyzed a conforming Galerkin method and derived stability and convergence results under assumptions ensuring the control of the nonlinear terms. A different approach was adopted by Oyarzúa and Zúñiga [13], who considered a formulation combining primal and mixed finite element techniques for the fluid and thermal variables, respectively, and proved the existence of discrete solutions together with *a priori* error estimates. The design of exactly divergence-free schemes has received increasing attention. Motivated by this objective, Oyarzúa et al. [9] proposed a divergence-conforming Brezzi-Douglas-Marini finite element method for generalized Boussinesq systems, while Oyarzúa and Serón [14] developed a divergence-conforming discontinuous Galerkin formulation for the stationary problem. In both cases, the authors proved the existence of discrete solutions and established optimal *a priori* error estimates under suitable smallness assumptions.

High-order and discontinuous Galerkin approaches have attracted increasing attention due to their flexibility and local conservation properties. In [15], Ueckermann et al. authors introduced a hybridizable discontinuous Galerkin framework for the time-dependent classical Boussinesq equations, combined with projection techniques and implicit-explicit time integration schemes. In parallel, Gatica et al. [16,17] investigated augmented formulations of Boussinesq systems and derived *a priori* and *a posteriori* error analyses for fully mixed high-order finite element methods. Recently, Colmenares et al. [18] have carried out a detailed analysis of a symmetric interior penalty discontinuous Galerkin method combined with an upwind treatment for the stationary Boussinesq system, subject to Dirichlet-type boundary conditions for the velocity and mixed boundary conditions for the temperature.

Nevertheless, most of the numerical analyses available in the literature are restricted to Dirichlet boundary conditions for the fluid velocity. However, such conditions are known to be inadequate in several physical settings, including thin film flows, perforated domains, and blood flow in vessels. In these contexts, the French engineer Claude Navier [19] proposed Navier or Navier-type boundary conditions, which provide a more realistic modeling framework. A generalization of the Navier condition can be written as:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{and} \quad 2[D(\boldsymbol{u})\boldsymbol{n}]_\tau+\alpha\boldsymbol{u}_\tau=\boldsymbol{0} \quad \text{on } \Gamma, \tag{1}$$

where $\alpha \geq 0$ is the coefficient of friction and

$$D(\boldsymbol{u})_{ij}=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\ 1\leqslant i,j\leqslant 3.$$

The subscript τ denotes the vectorial tangential trace of any vector, defined by $\boldsymbol{v}_\tau := \boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$. Note that, when α tends to infinity, we formally recover the Dirichlet conditions. In the case of a flat boundary and zero friction coefficient α, this Navier condition is equivalent to the following Navier-type conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=0 \quad \text{on } \Gamma, \tag{2}$$

The Navier-type boundary condition (2) is also well suited to formulations in which the vorticity is introduced as an independent unknown. Indeed, setting $\boldsymbol{\omega}=\mathbf{curl}\,\boldsymbol{u}$, the viscous operator can be written, under the incompressibility constraint, as $\mathbf{curl}\,\boldsymbol{\omega}$, while condition (1.2) prescribes directly the tangential trace of $\boldsymbol{\omega}$. Such formulations allow the vorticity to be approximated directly and are naturally adapted to the operators **curl** and div. See, for instance, our previous work on the Oseen problem [20, Section 4]. As a first step for the coupled Boussinesq system, the present work considers a three-field discontinuous Galerkin formulation for the velocity, pressure and temperature. Introducing the vorticity as an additional unknown would lead to a four-field formulation and would require a new stability analysis for the full coupled system. This extension will be considered in future work.

Concerning the study of Boussinesq system with the above boundary conditions, the literature is rather scarce. The idea was first described in [21], where the steady Boussinesq problem with Navier boundary condition (1) was investigated. There, the authors established the existence of weak solutions in the Hilbertian setting and their L^p-regularity. In [22], the authors considered the time-dependent two-dimensional Boussinesq equations with the Navier-type boundary conditions (1) and proved global existence and uniqueness under minimal regularity assumptions on the initial data. To the best of our knowledge, no numerical analysis has yet been devoted to the stationary Boussinesq system subject to Navier-type boundary conditions of the form (2) considered in this work.

In this work, we develop a new Discontinuous Galerkin (DG) method based on an interior penalty DG approach, where the tangential and normal jumps of the discrete vector fields across the edges are introduced, combined with an upwind discretization to approximate the nonlinear stationary Boussinesq system, subject to a Navier-type boundary condition (2) for the velocity and a Dirichlet boundary condition for the temperature. The DG method relies on a fully discontinuous finite element space for both trial and test functions. This lack of continuity offers significant flexibility in meshing, making the method well-suited for handling complex industrial geometries

that require unstructured or non-conforming meshes. These methods are also attractive because they are element-wise conservative, easily implementable, and they are high-order methods [23].

Let Ω be a bounded connected domain of $\mathbb{R}^d$, with $d = 2$ or $d = 3$, with boundary Γ of class $C^{1,1}$, or assume that Ω is convex. Given a body force $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ and a heat source $f \in L^2(\Omega)$, we seek a velocity field $\boldsymbol{u}$, a pressure field p and a temperature field θ satisfying

$$\nu\, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \nabla p - \theta\boldsymbol{g} = 0 \qquad \text{in } \Omega, \tag{3a}$$

$$\operatorname{div}\boldsymbol{u} = 0, \qquad \text{in } \Omega, \tag{3b}$$

$$-\kappa\,\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = f \qquad \text{in } \Omega, \tag{3c}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{3d}$$

$$\theta = 0 \qquad \text{on } \Gamma. \tag{3e}$$

The parameters ν and κ denote, respectively, the kinematic viscosity and the thermal diffusivity of the fluid. For $d = 2$, the third component of $\boldsymbol{n}$ is zero and $\boldsymbol{u} = (u_1, u_2)$. We set

$$\omega = \operatorname{curl}\boldsymbol{u} = \partial_1 u_2 - \partial_2 u_1 \text{ and } \boldsymbol{u}^\perp = (-u_2, u_1).$$

With these conventions, we have

$$\operatorname{curl}\operatorname{curl}\boldsymbol{u} = \operatorname{curl}\omega \text{ and } \operatorname{curl}\boldsymbol{u}\times\boldsymbol{u} = \omega\boldsymbol{u}^\perp.$$

Thus, in dimension two, the momentum Eq. (3a) becomes

$$\nu\operatorname{curl}\omega + \omega\boldsymbol{u}^\perp + \nabla p - \theta\boldsymbol{g} = \boldsymbol{0} \qquad \text{in } \Omega.$$

Setting $\boldsymbol{\tau} = (-n_2, n_1)$, we obtain $\operatorname{curl}\boldsymbol{u}\times\boldsymbol{n} = \omega\boldsymbol{\tau}$. Consequently, the Navier-type boundary condition (3d) becomes

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{and} \qquad \operatorname{curl}\boldsymbol{u} = 0 \qquad \text{on } \Gamma.$$

The incompressibility condition, the temperature equation and the temperature boundary condition keep the same form in dimensions two and three. The analysis below is written mainly with the three-dimensional notation, but the arguments also apply in dimension two with the conventions introduced above. The estimates involving the curl and divergence operators that are required for the continuous problem are available in the two-dimensional setting [24]. In particular, the Sobolev embeddings used in dimension three are replaced by their two-dimensional counterparts. The discontinuous Galerkin analysis is based on integration by parts, trace and inverse inequalities, and standard mesh regularity assumptions, which are valid in both dimensions. The construction of the two-dimensional discontinuous Galerkin formulation is detailed for the Stokes problem in [25, Chapter 10], and the same structure is used here for the coupled Boussinesq system. Therefore, the theoretical framework covers both $d = 2$ and $d = 3$, and in particular the two-dimensional numerical experiments presented in this paper.

The rest of the paper is structured as follows. In Section 2, we introduce the model and establish the well-posedness of the continuous variational problem by applying the Leray-Schauder's fixed-point theorem. A discontinuous Galerkin formulation based on the symmetric interior penalty method and an upwind discretization is presented in Section 3 where the existence and uniqueness of the corresponding approximate solutions are proved. In Section 4, we state and prove the convergence of the method and give *a priori* error estimates for the velocity, pressure and temperature in suitable energy norms. Finally, in Section 5, we provide numerical experiments to confirm the theoretical convergence rates.

2. Analysis of the continuous problem

2.1. Preliminaries and notations

From now on we use the notations $L^p(\Omega)$, $W^{m,p}(\Omega)$ ($m \geq 0$, integer), $W^{s,p}(\Gamma)$ ($s \in \mathbb{R}$), and so on, for the standard Lebesgue space and Sobolev spaces of functions for $p \in [1, +\infty]$. We denote by $L^2_0(\Omega)$ is the space consisting of functions $q \in L^2(\Omega)$ that hold $\int_\Omega q = 0$. We adopt standard notation for the Sobolev spaces $H^s(\Omega)$ and $H^s_0(\Omega)$. In addition, according to the boundary conditions in (3d), and in view of deriving a weak formulation for (3), we introduce the following functional space:

$$\boldsymbol{H}^1_T(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\ ;\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div}\boldsymbol{v} \in L^2(\Omega),\ v\cdot n = 0 \text{ on } \Gamma\right\}. \tag{4}$$

We recall that the space $\boldsymbol{H}^1_T(\Omega)$ is continuously embedded in $\boldsymbol{H}^1(\Omega)$ if Ω is convex or the boundary of Ω is of class $C^{1,1}$ (c.f. [26, Theorem 3.7] and [27, Theorem 3.4]). In addition we recall the following Poincaré inequalities, needed for proving the existence of weak solutions: there exist $C_P > 0$ and $\widetilde{C}_P > 0$ such that [26]

$$\|\phi\|_{H^1(\Omega)} \leq C_P \|\nabla\phi\|_{L^2(\Omega)}, \qquad \forall \phi \in H^1_0(\Omega), \tag{5a}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq \widetilde{C}_P\left(\|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{L^2(\Omega)}\right), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega). \tag{5b}$$

We also recall that the space $\boldsymbol{H}^1(\Omega)$ is compactly embedding in $\boldsymbol{L}^q(\Omega)$ for any exponent $1 \leq q < 6$ and we have

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leq \delta\|\boldsymbol{v}\|_{1,\Omega}, \tag{6}$$

where δ is a positive constant depending on the domain Ω.

Next, we introduce the following Banach space

$$\boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^3(\Omega);\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^{3/2}(\Omega)\} \tag{7}$$

provided with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^{3/2}(\Omega)}.$$

We denote by $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ the space of restrictions to Ω of infinitely differentiable vector-valued functions with compact support in $\mathbb{R}^3$. We shall use the following density result

$$\boldsymbol{\mathcal{D}}(\overline{\Omega})\text{ is dense in }\ \boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega).$$

This density result follows by standard arguments, see for instance, [25, Proposition 1.0.2] and [27, Remark 5.4]. Moreover, we prove the following Green formula to define the tangential traces of functions which belong to $\boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega)$.

Lemma 2.1. *The linear mapping $\gamma\ :\ \boldsymbol{v}\mapsto\boldsymbol{v}\times\boldsymbol{n}|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping still denoted γ from $\boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega)$ to $\boldsymbol{W}^{-\frac{1}{3},3}(\Gamma)$. Moreover, we have the following Green formula: for any $\boldsymbol{v}\in\boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl},\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,3/2}(\Omega)$,*

$$\int_\Omega\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}-\int_\Omega\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,d\boldsymbol{x}=\langle\boldsymbol{v}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma. \tag{8}$$

where $\langle\cdot,\,\cdot\rangle_\Gamma$ denotes the duality between $\boldsymbol{W}^{-\frac{1}{3},3}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)$.

Proof. We first recall the trace result used below. For every $1<r<\infty$, the trace operator

$$\gamma_0\ :\ \boldsymbol{W}^{1,r}(\Omega)\longrightarrow\boldsymbol{W}^{1-1/r,r}(\Gamma) \tag{9}$$

is linear and continuous. Moreover, for every $\boldsymbol{\mu}\in\boldsymbol{W}^{1-1/r,r}(\Gamma)$, there exists a lifting $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,r}(\Omega)$ such that

$$\gamma_0\boldsymbol{\varphi}=\boldsymbol{\mu}$$

and

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,r}(\Omega)}\le C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1-1/r,r}(\Gamma)}.$$

This is the classical trace and lifting theorem for Sobolev spaces, see for instance [28,29].
We first consider $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$. Let $\boldsymbol{\mu}\in\boldsymbol{W}^{1/3,3/2}(\Gamma)$ and choose a lifting $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,3/2}(\Omega)$ such that

$$\gamma_0\boldsymbol{\varphi}=\boldsymbol{\mu},\qquad\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\le C\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1/3,3/2}(\Gamma)}.$$

We define

$$\langle\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\mu}\rangle_\Gamma=\int_\Omega\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}-\int_\Omega\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,d\boldsymbol{x}.$$

This definition does not depend on the chosen lifting.
We now prove the continuity of this functional. Since $\boldsymbol{W}^{1,3/2}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$, Hölder's inequality gives

$$\begin{aligned}|\langle\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\mu}\rangle_\Gamma|&\le\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^3(\Omega)}+\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{3/2}(\Omega)}\\&\le C\|\boldsymbol{v}\|_{\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\\&\le C\|\boldsymbol{v}\|_{\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)}\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{1/3,3/2}(\Gamma)}.\end{aligned}$$

Therefore, we have the following continuity estimate

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/3,3}(\Gamma)}\le C\|\boldsymbol{v}\|_{\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)},\qquad\forall\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega}).$$

We now consider an arbitrary vector field $\boldsymbol{v}\in\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)$. By the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)$, there exists a sequence $(\boldsymbol{v}_k)_k\subset\boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that

$$\boldsymbol{v}_k\longrightarrow\boldsymbol{v}\quad\text{in }\boldsymbol{L}^3(\Omega)\quad\text{and}\quad\mathbf{curl}\,\boldsymbol{v}_k\longrightarrow\mathbf{curl}\,\boldsymbol{v}\quad\text{in }\boldsymbol{L}^{3/2}(\Omega).$$

Equivalently, $\boldsymbol{v}_k\longrightarrow\boldsymbol{v}$ in $\boldsymbol{H}^{3/2}(\mathrm{curl},\Omega)$. The continuity estimate obtained above shows that the tangential traces depend continuously on the vector field in the $\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)$-norm. Since the sequence $(\boldsymbol{v}_k)_k$ converges in this space, the sequence $(\boldsymbol{v}_k\times\boldsymbol{n})_k$ is therefore Cauchy in $\boldsymbol{W}^{-1/3,3}(\Gamma)$ and converges to some $\boldsymbol{\eta}\in\boldsymbol{W}^{-1/3,3}(\Gamma)$. We define $\boldsymbol{v}\times\boldsymbol{n}:=\boldsymbol{\eta}$. The continuity estimate also shows that this limit does not depend on the chosen approximating sequence and that

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/3,3}(\Gamma)}\le C\|\boldsymbol{v}\|_{\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)},$$

for any $\boldsymbol{v}\in\boldsymbol{H}^{3/2}(\mathbf{curl},\Omega)$. Hence, the tangential trace operator extends uniquely and continuously from $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ to $\boldsymbol{H}^{3/2}(\mathrm{curl},\Omega)$. Finally, let $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,3/2}(\Omega)$. For every k, the Green formula gives

$$\int_\Omega\mathbf{curl}\,\boldsymbol{v}_k\cdot\boldsymbol{\varphi}\,d\boldsymbol{x}-\int_\Omega\boldsymbol{v}_k\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,d\boldsymbol{x}=\langle\boldsymbol{v}_k\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_\Gamma.$$

Using the convergences of $\boldsymbol{v}_k$, $\mathbf{curl}\,\boldsymbol{v}_k$, and $\boldsymbol{v}_k \times \boldsymbol{n}$, we may pass to the limit and obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \boldsymbol{\varphi}, d\boldsymbol{x} - \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi}, d\boldsymbol{x} - \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma.$$

This proves the continuity of the extended tangential trace operator and the Green formula for every $\boldsymbol{v} \in \boldsymbol{H}^{3/2}(\mathbf{curl}, \Omega)$. □

2.2. Weak formulation

In this subsection, we give the definition of a weak solution of (3).

Given $\boldsymbol{f} \in L^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$, $(\boldsymbol{u}, p, \theta) \in \boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$ is called a weak solution of the Boussinesq problem (3) if it satisfies: for all $(\boldsymbol{v}, q, \psi) \in \boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$

$$A(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B(\boldsymbol{v}, p) = D(\theta, \boldsymbol{v}), \tag{10a}$$

$$B(\boldsymbol{u}, q) = 0 \tag{10b}$$

$$M(\theta, \psi) + C(\boldsymbol{u}; \theta, \psi) = F(\psi), \tag{10c}$$

where the forms are defined by

$$\begin{aligned}
&A(\boldsymbol{u}, \boldsymbol{v}) = \nu\Big(\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{u})(\operatorname{div} \boldsymbol{v})\, d\boldsymbol{x} \Big);\\
&M(\theta, \psi) = \kappa \int_\Omega \nabla\theta \cdot \nabla\psi\, d\boldsymbol{x}; \qquad D(\theta, \boldsymbol{v}) = \int_\Omega \theta \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x};\\
&O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\mathbf{curl}\,\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, d\boldsymbol{x}; \qquad C(\boldsymbol{u}; \theta, \psi) = \int_\Omega (\boldsymbol{u} \cdot \nabla\theta)\psi\, d\boldsymbol{x};\\
&B(\boldsymbol{v}, q) = -\int_\Omega (\operatorname{div} \boldsymbol{v}) q\, d\boldsymbol{x}; \qquad F(\psi) = \int_\Omega f\psi\, d\boldsymbol{x}.
\end{aligned} \tag{11}$$

We briefly explain how the boundary conditions are incorporated into the weak formulation. The following discussion focuses on the Navier-type boundary condition imposed on the velocity. Unlike the case of a homogeneous Dirichlet condition for the velocity, the test function $\boldsymbol{v}$ does not vanish on Γ and simply satisfies $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ. Consequently, the boundary term resulting from integration by parts does not automatically vanish and must be handled using the condition $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ.

The idea is to first prove the result for smooth functions, for which the formulas for integration by parts are standard, and then extend the result to the relevant Sobolev spaces by density. More specifically, Lemma 4.1 and Corollary 4.2 of [30] ensure that the tangential trace $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}$ is well-defined. On the other hand, the temperature satisfies a standard homogeneous Dirichlet boundary condition. Consequently, the temperature test function ψ vanishes on Γ, and the corresponding boundary term disappears in the usual way.

We describe a fixed point strategy that allows to solve the problem (10). First, we eliminate the pressure from the problem by restricting to the subspace

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega); \quad B(\boldsymbol{v}, q) = 0, \quad \forall q \in L^2_0(\Omega)\}$$

We then consider the equivalent problem: Find $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ such that

$$A(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = D(\theta, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{12a}$$

$$M(\theta, \psi) + C(\boldsymbol{u}; \theta, \psi) = F(\psi), \quad \forall \psi \in H^1_0(\Omega). \tag{12b}$$

The bilinear form B is continuous in $\boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega)$ and satisfies the following inf-sup condition: there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)} \frac{B(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \geqslant \beta \|q\|_{L^2(\Omega)}, \tag{13}$$

which is a direct consequence of its analog with $\boldsymbol{H}^1_T(\Omega)$ replaced by $\boldsymbol{H}^1_0(\Omega)$ (see [31,32]).

As a consequence, it is easy to see that problems (10) and (12) are equivalent. In fact, we have the following standard result [26].

Lemma 2.2. *If $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ is a solution of (12), then there exists $p \in L^2_0(\Omega)$ such that $(\boldsymbol{u}, p, \theta)$ is a solution of (10). Conversely, if $(\boldsymbol{u}, p, \theta) \in \boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$ is a solution of (10), then $\boldsymbol{u} \in \boldsymbol{K}$ and $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ is a solution of (12).*

We construct a map $\mathcal{S}$ by linearizing the term D and the convection terms O and C in problem (12) and show existence of a fixed point of $\mathcal{S}$. Define $\mathcal{S}$ as follows:

$$\begin{aligned}
&\mathcal{S} : \boldsymbol{K} \times H^1_0(\Omega) \to \boldsymbol{K} \times H^1_0(\Omega)\\
&\qquad (\boldsymbol{w}, \phi) \mapsto \mathcal{S}(\boldsymbol{w}, \phi) = (\boldsymbol{u}, \theta),
\end{aligned} \tag{14}$$

where $(\boldsymbol{u},\theta)$ satisfies the linearized problem: Find $(\boldsymbol{u},\theta)\in \boldsymbol{K}\times H_0^1(\Omega)$ such that

$$\begin{aligned} &A(\boldsymbol{u},\boldsymbol{v})+O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})=D(\phi,\boldsymbol{v}),\quad \forall \boldsymbol{v}\in\boldsymbol{K},\\ &M(\theta,\psi)+C(\boldsymbol{w};\theta,\psi)=F(\psi),\quad \forall\psi\in H_0^1(\Omega). \end{aligned} \tag{15}$$

Observe that, thanks to Lemma 2.2, a fixed point $(\boldsymbol{u},\theta)$ of the map $\mathcal{S}$ is the component of a solution $(\boldsymbol{u},p,\theta)$ of problem (10). The well posedness of (10) will be addressed in Section 2.4 by applying the Leray-Schauder theorem. It requires some stability properties of the forms defined above.

2.3. Stability properties of the forms

In this subsection, we show some crucial stability properties of the forms that are used to prove the well-posedness of the problem (10). First, we discuss the coercivity properties of the forms A, M, O and C. We have the following result.

Proposition 2.1. *There exist positive constants α_A and α_M depending on Ω such that*

$$A(\boldsymbol{v},\boldsymbol{v})\geq \nu\alpha_A\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)},\quad \forall\boldsymbol{v}\in\boldsymbol{K}, \tag{16}$$

$$M(\psi,\psi)\geq \kappa\alpha_M\|\psi\|^2_{H^1(\Omega)},\quad \forall\psi\in H_0^1(\Omega), \tag{17}$$

Furthermore, the following results hold:

(i) Let $\boldsymbol{w},\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$. Then we have

$$O(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v})=0. \tag{18}$$

(ii) Let $\boldsymbol{w}\in\boldsymbol{K}$ and $\psi\in H_0^1(\Omega)$. Then we have

$$C(\boldsymbol{w};\psi,\psi)=0 \tag{19}$$

Proof. The coercivity of the forms A and M follows from the Poincaré-Friedrichs inequalities (5a) and (5b). The property in (18) is obvious. It follows from the definition of O and the fact that the vector $\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$. The property in (19) is a consequence of the following well-known skew-symmetry property of the form C with respect to the last two components:

$$C(\boldsymbol{w};\theta,\psi)=-C(\boldsymbol{w};\psi,\theta),\qquad \forall\boldsymbol{w}\in\boldsymbol{K},\ \forall(\theta,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega).$$

□

We also need to study the continuity of the forms involved in the variational formulation (10). We have the following results where the proof is obtained by applying Cauchy–Schwarz's inequality to the different forms.

Proposition 2.2. *There exist positive constants C_A, C_M, C_B and C_F depending on Ω such that*

$$A(\boldsymbol{u},\boldsymbol{v})\leq \nu C_A\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \tag{20}$$

$$M(\theta,\psi)\leq \kappa C_M\|\theta\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}, \tag{21}$$

$$B(\boldsymbol{v},q)\leq C_B\|q\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \tag{22}$$

$$F(\psi)\leq C_F\|f\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)}, \tag{23}$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}^1_T(\Omega)$, $\forall q\in L^2_0(\Omega)$ and $\theta,\psi\in H^1(\Omega)$.

The next results show that the nonlinear forms O, C and D are continuous.

Proposition 2.3. *There exist positive constants C_O, C_C and C_D depending on Ω such that*

$$O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})\leq C_O\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \tag{24}$$

$$C(\boldsymbol{w};\theta,\psi)\leq C_C\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\|\theta\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}, \tag{25}$$

$$D(\theta,\boldsymbol{v})\leq C_D\|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \tag{26}$$

for all $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}^1_T(\Omega)$ and $\theta,\psi\in H^1(\Omega)$.

Proof. Applying Hölder's inequality, we have

$$|O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})|\leq\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}.$$

Estimate (24) follows from the compact imbedding (6) with $q=4$. The proof of the continuity of C and D uses essentially the same techniques as for the form O, so we skip the details. □

2.4. A priori estimates and solvability

In this subsection, we prove the existence and uniqueness of solutions for the problem (10). We will apply the Leray-Schauder theorem, see [33, Theorem 11.3, p. 280]. For this, we must prove that $\mathcal{S}$ is well-defined, a compact operator on $\boldsymbol{K}\times H_0^1(\Omega)$ and that the set

$$\mathcal{Z}=\{(\boldsymbol{u},\theta)\in \boldsymbol{K}\times H_0^1(\Omega) : (\boldsymbol{u},\theta)=\lambda\mathcal{S}(\boldsymbol{u},\theta) \text{ for some } 0\le\lambda\le 1\}$$

is bounded. For the convenience of the subsequent analysis, setting the following product norm on the space $\boldsymbol{K}\times H_0^1(\Omega)$:

$$\|(\boldsymbol{u},\theta)\|=\left(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\theta\|^2_{H^1(\Omega)}\right)^{1/2}.$$

The main result of this subsection is stated as follows.

Theorem 2.1. *Let $\boldsymbol{g}\in \boldsymbol{L}^2(\Omega)$, $f\in L^2(\Omega)$. Then the problem (10) has at least one weak solution $(\boldsymbol{u},p,\theta)\in \boldsymbol{H}^1_T(\Omega)\times L^2_0(\Omega)\times H^1_0(\Omega)$ satisfying*

$$\|(\boldsymbol{u},\theta)\|\le \kappa^{-1}\alpha_M^{-1}C_F\|f\|_{L^2(\Omega)}\left(1+\nu^{-1}\alpha_A^{-1}C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\right). \tag{27}$$

Moreover, if $\min(\nu\alpha_A,\kappa\nu_M)^{-1}K_3<1$ with K_3 the constant given in (55), the solution is unique.

Since solving (10) is equivalent to finding a fixed point of the map $\mathcal{S}$ defined in (14), we start by checking that the map $\mathcal{S}$ is well-defined, i.e. that there exists a unique solution $(\boldsymbol{u},\theta)$ to the linearized problem (15).

Lemma 2.3. *There exists a unique $(\boldsymbol{u},\theta)\in \boldsymbol{K}\times H_0^1(\Omega)$ satisfying the linearized problem (15). In addition, we have the following bound*

$$\|(\boldsymbol{u},\theta)\|\le \alpha_{\mathcal{A}}^{-1}\left(C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|(\boldsymbol{w},\phi)\|+C_F\|f\|_{L^2(\Omega)}\right) \tag{28}$$

Proof. First, we write the mixed system (15) in the following equivalent compact form:

Find $(\boldsymbol{u},\theta)\in \boldsymbol{K}\times H_0^1(\Omega)$ such that:

$$\mathcal{A}((\boldsymbol{w},\phi);(\boldsymbol{u},\theta),(\boldsymbol{v},\psi))=\mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi)),\quad \forall(\boldsymbol{v},\psi)\in \boldsymbol{K}\times H_0^1(\Omega), \tag{29}$$

where the forms $\mathcal{A}$ and $\mathcal{L}$ are defined as

$$\mathcal{A}((\boldsymbol{w},\phi);(\boldsymbol{u},\theta),(\boldsymbol{v},\psi))=A(\boldsymbol{u},\boldsymbol{v})+M(\theta,\psi)+O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})+C(\boldsymbol{w};\theta,\psi), \tag{30}$$

and

$$\mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi))=D(\phi,\boldsymbol{v})+F(\psi). \tag{31}$$

It follows from Proposition 2.1 that

$$\mathcal{A}((\boldsymbol{w},\phi);(\boldsymbol{u},\theta),(\boldsymbol{u},\theta))\ge \nu\alpha_A\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\kappa\alpha_M\|\theta\|^2_{H^1(\Omega)}\ge \alpha_{\mathcal{A}}\|(\boldsymbol{u},\theta)\|^2, \tag{32}$$

where

$$\alpha_{\mathcal{A}}=\min(\nu\alpha_A,\kappa\alpha_M). \tag{33}$$

Moreover, employing the continuity results in (20)-(21), (24)-(25) together with the inequality $(a+b)\le 2^{1/2}(a^2+b^2)^{1/2}$, we can check that the trilinear form $\mathcal{A}$ verifies the estimate:

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{w},\phi);(\boldsymbol{u},\theta),(\boldsymbol{v},\psi))\le |A(\boldsymbol{u},\boldsymbol{v})|+|M(\theta,\psi)|+|O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})|+|C(\boldsymbol{w};\theta,\psi)|,\\
&\le \left(\nu C_A+\kappa C_M+(C_O+C_C)\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\right)\left(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\theta\|^2_{H^1(\Omega)}\right)^{1/2}\left(\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\psi\|_{H^1(\Omega)}\right)^{1/2}\\
&\le C_{\mathcal{A}}\|(\boldsymbol{u},\theta)\|\|(\boldsymbol{v},\psi)\|,
\end{aligned}$$

with $C_{\mathcal{A}}=\left(\nu C_A+\kappa C_M+(C_O+C_C)\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}\right)$. Thus, we showed that $\mathcal{A}$ is coercive and continuous in $\boldsymbol{K}\times H_0^1(\Omega)$. In addition, thanks to (26) and (23), we have

$$\mathcal{L}((\boldsymbol{w},\phi),(\boldsymbol{v},\psi))\le C_D\|\phi\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}+C_F\|f\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)}\le C_{\mathcal{L}}\|(\boldsymbol{v},\psi)\|, \tag{34}$$

with $C_{\mathcal{L}}=C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|(\boldsymbol{w},\phi)\|+C_F\|f\|_{L^2(\Omega)}$. So, the form $\mathcal{L}((\boldsymbol{w},\phi),\cdot)$ is an element of $(\boldsymbol{K}\times H_0^1(\Omega))'$. Therefore, by Lax-Milgram's theorem, there exists a unique $(\boldsymbol{u},\theta)\in \boldsymbol{K}\times H_0^1(\Omega)$ solution of (29) and equivalently to (15). The estimate (28) follows by combining the estimates in (32) and (34). □

Proof of Theorem 2.1. In order to prove the existence of the velocity $\boldsymbol{u}$ and the temperature θ, we proceed in two steps.

Step 1 : Let us show that the operator $\mathcal{S}$ is compact. Suppose $(\boldsymbol{w}_n,\phi_n)\in \boldsymbol{K}\times H_0^1(\Omega)$, $n\in\mathbb{N}$ such that $(\boldsymbol{w}_n,\phi_n)\rightharpoonup(\boldsymbol{w},\phi)$ weakly in $\boldsymbol{K}\times H_0^1(\Omega)$. Define $(\boldsymbol{u}_n,\theta_n)=\mathcal{S}(\boldsymbol{w}_n,\phi_n)$ and $(\boldsymbol{u},\theta)=\mathcal{S}(\boldsymbol{w},\phi)$. We will show that $(\boldsymbol{u}_n,\theta_n)\to(\boldsymbol{u},\theta)$ strongly in $\boldsymbol{K}\times H_0^1(\Omega)$.

We have $(\boldsymbol{u}_n-\boldsymbol{u},\theta_n-\theta)\in \boldsymbol{K}\times H_0^1(\Omega)$ and satisfies the problem:

$$A(\boldsymbol{u}_n-\boldsymbol{u},\boldsymbol{v})+O(\boldsymbol{w}_n;\boldsymbol{u}_n,\boldsymbol{v})-O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})=D(\phi_n-\phi,\boldsymbol{v}),\quad \forall\boldsymbol{v}\in \boldsymbol{K}, \tag{35a}$$

$$M(\theta_n - \theta, \psi) + C(\boldsymbol{w}_n; \theta_n, \psi) - C(\boldsymbol{w}; \theta, \psi) = 0, \quad \forall \psi \in H_0^1(\Omega). \tag{35b}$$

Taking $\boldsymbol{v} = \boldsymbol{u}_n - \boldsymbol{u}$ in (35a), we obtain

$$A(\boldsymbol{u}_n - \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) + O(\boldsymbol{w}_n; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) - O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) = D(\phi_n - \phi, \boldsymbol{u}_n - \boldsymbol{u}) \tag{36}$$

We can split up the nonlinear terms in the left hand side of (36) as

$$O(\boldsymbol{w}_n; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) - O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u}) = O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) - \underbrace{O(\boldsymbol{w}; \boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})}_{=0}.$$

Applying the coercivity of A (c.f. (16)), we obtain

$$\nu \alpha_{_A} \left\| \boldsymbol{u}_n - \boldsymbol{u} \right\|^2_{\boldsymbol{H}^1(\Omega)} \leq O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) + D(\phi_n - \phi, \boldsymbol{u}_n - \boldsymbol{u}). \tag{37}$$

The term $O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})$ on the right-hand side of (37) can be treated as follows. First, by definition and using the vector identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}), \qquad \text{for all } \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3,$$

with $\boldsymbol{a} = \mathbf{curl}(\boldsymbol{w}_n - \boldsymbol{w})$, $\boldsymbol{b} = \boldsymbol{u}_n$ and $\boldsymbol{c} = \boldsymbol{u}_n - \boldsymbol{u}$, we obtain

$$\begin{aligned} O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) &= \int_\Omega \left(\mathbf{curl}(\boldsymbol{w}_n - \boldsymbol{w}) \times \boldsymbol{u}_n \right) \cdot (\boldsymbol{u}_n - \boldsymbol{u}) \, d\boldsymbol{x} \\ &= \int_\Omega \mathbf{curl}(\boldsymbol{w}_n - \boldsymbol{w}) \cdot \left(\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u}) \right) d\boldsymbol{x}. \end{aligned} \tag{38}$$

The next step is to apply Green's formula (8) with $\boldsymbol{v} = \boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})$ and $\boldsymbol{\varphi} = \boldsymbol{w}_n - \boldsymbol{w}$. Before, we need to verify that $\boldsymbol{v} \in \boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$.

Let us first verify that $\boldsymbol{v} = \boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})$ belongs to $\boldsymbol{H}^{,3,\frac{3}{2}}(\mathbf{curl}, \Omega)$ defined in (7). Since $\boldsymbol{u}_n, \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, the continuous embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ implies that $\boldsymbol{u}_n$ and $\boldsymbol{u}_n - \boldsymbol{u}$ belong to $\boldsymbol{L}^6(\Omega)$. Therefore, by Hölder's inequality,

$$\boldsymbol{v} = \boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{L}^3(\Omega).$$

It remains to verify that $\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^{3/2}(\Omega)$. To this end, we use the vector identity

$$\mathbf{curl}(\boldsymbol{a} \times \boldsymbol{b}) = (\operatorname{div} \boldsymbol{b})\boldsymbol{a} - (\operatorname{div} \boldsymbol{a})\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}.$$

Applying this identity with $\boldsymbol{a} = \boldsymbol{u}_n$ and $\boldsymbol{b} = \boldsymbol{u}_n - \boldsymbol{u}$, and using the divergence-free conditions

$$\operatorname{div} \boldsymbol{u}_n = 0 \qquad \text{and} \qquad \operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}) = 0,$$

we obtain, in the sense of distributions,

$$\mathbf{curl} \left(\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u}) \right) = \left((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \nabla \right) \boldsymbol{u}_n - (\boldsymbol{u}_n \cdot \nabla)(\boldsymbol{u}_n - \boldsymbol{u}). \tag{39}$$

We now show that both terms on the right-hand side of (39) belong to $\boldsymbol{L}^{3/2}(\Omega)$. Since $\boldsymbol{u}_n - \boldsymbol{u} \in \boldsymbol{L}^6(\Omega)$ and $\nabla \boldsymbol{u}_n \in \boldsymbol{L}^2(\Omega)$, Hölder's inequality gives $\left((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \nabla \right) \boldsymbol{u}_n \in \boldsymbol{L}^{3/2}(\Omega)$. Similarly, since $\boldsymbol{u}_n \in \boldsymbol{L}^6(\Omega)$ and $\nabla(\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{L}^2(\Omega)$, we have $(\boldsymbol{u}_n \cdot \nabla)(\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{L}^{3/2}(\Omega)$. Consequently, $\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^{3/2}(\Omega)$. Since $\boldsymbol{v} \in \boldsymbol{L}^3(\Omega)$ and $\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^{3/2}(\Omega)$, we obtain that

$$\boldsymbol{v} = \boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl}, \Omega).$$

Moreover, since $\boldsymbol{\varphi} = \boldsymbol{w}_n - \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and Ω is bounded, we have $\boldsymbol{\varphi} = \boldsymbol{w}_n - \boldsymbol{w} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$. By the trace theorem (see (9)), $\boldsymbol{\varphi}_{|\Gamma} \in \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)$. We can now apply the Green formula (8). Indeed, since $\boldsymbol{v} = \boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u}) \in \boldsymbol{H}^{3,\frac{3}{2}}(\mathbf{curl}, \Omega)$, Lemma 2.1 ensures that its tangential trace is well defined and satisfies $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{3},3}(\Gamma)$. Therefore, the duality pairing

$$\langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-\frac{1}{3},3}(\Gamma) \times \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)} = \langle (\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})) \times \boldsymbol{n}, \boldsymbol{w}_n - \boldsymbol{w} \rangle_{\boldsymbol{W}^{-\frac{1}{3},3}(\Gamma) \times \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)}$$

is well defined. Thus, we can rewrite the term O in (38) as:

$$\begin{aligned} O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) &= \int_\Omega \mathbf{curl}(\boldsymbol{w}_n - \boldsymbol{w}) \cdot (\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})) \, d\boldsymbol{x} \\ &= \int_\Omega (\boldsymbol{w}_n - \boldsymbol{w}) \cdot \mathbf{curl}(\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})) \, d\boldsymbol{x} - \underbrace{\left\langle \boldsymbol{w}_n - \boldsymbol{w}, (\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})) \times \boldsymbol{n} \right\rangle}_{=0 \quad \boldsymbol{W}^{-\frac{1}{3},3}(\Gamma) \times \boldsymbol{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma)}. \end{aligned} \tag{40}$$

On the other hand, since $\boldsymbol{u}_n \cdot \boldsymbol{n} = 0$ on Γ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, we also have $(\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{n} = 0$ on Γ. Using the vector identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}),$$

we then obtain, in the sense of traces,

$$\begin{aligned}\big(\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})\big) \times \boldsymbol{n} &= (\boldsymbol{u}_n - \boldsymbol{u})(\boldsymbol{u}_n \cdot \boldsymbol{n}) - \boldsymbol{u}_n\big((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{n}\big) \\ &= \boldsymbol{0} \qquad \text{on } \Gamma.\end{aligned}$$

Thus, the tangential trace of $\boldsymbol{u}_n \times (\boldsymbol{u}_n - \boldsymbol{u})$ in (40) vanishes on Γ.

Using again (39) in (40), we obtain

$$O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}) = \int_\Omega (\boldsymbol{w}_n - \boldsymbol{w}) \cdot \big((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \nabla\big)\boldsymbol{u}_n \, d\boldsymbol{x} - \int_\Omega (\boldsymbol{w}_n - \boldsymbol{w}) \cdot (\boldsymbol{u}_n \cdot \nabla)(\boldsymbol{u}_n - \boldsymbol{u}) \, d\boldsymbol{x}.$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\big|O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})\big| &\le \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \|\nabla \boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{u}_n\|_{\boldsymbol{L}^6(\Omega)} \|\nabla(\boldsymbol{u}_n - \boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Since $\boldsymbol{H}^1(\Omega)$ is continuously embedded into $\boldsymbol{L}^6(\Omega)$, it follows that

$$\big|O(\boldsymbol{w}_n - \boldsymbol{w}; \boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})\big| \le C \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}. \tag{41}$$

Similarly, Hölder's inequality gives

$$\begin{aligned}\big|D(\phi_n - \phi, \boldsymbol{u}_n - \boldsymbol{u})\big| &\le \|\phi_n - \phi\|_{L^3(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \\ &\le C \|\phi_n - \phi\|_{L^3(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned} \tag{42}$$

Substituting (41) and (42) into (37), we obtain

$$\nu \alpha_A \|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega)} + C \|\phi_n - \phi\|_{L^3(\Omega)} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}. \tag{43}$$

We now estimate the difference $\theta_n - \theta$. Taking $\varphi = \theta_n - \theta$ in (35b) and using the coercivity of M, see (17), we obtain

$$\kappa \alpha_M \|\theta_n - \theta\|^2_{H^1(\Omega)} \le C(\boldsymbol{w}; \theta, \theta_n - \theta) - C(\boldsymbol{w}_n; \theta_n, \theta_n - \theta)$$

By the linearity of C with respect to its first two arguments, we have

$$C(\boldsymbol{w}; \theta, \theta_n - \theta) - C(\boldsymbol{w}_n; \theta_n, \theta_n - \theta) = C(\boldsymbol{w} - \boldsymbol{w}_n; \theta_n, \theta_n - \theta) - \underbrace{C(\boldsymbol{w}; \theta_n - \theta, \theta_n - \theta)}_{=0}.$$

Therefore,

$$\kappa \alpha_M \|\theta_n - \theta\|^2_{H^1(\Omega)} \le \big|C(\boldsymbol{w} - \boldsymbol{w}_n; \theta_n, \theta_n - \theta)\big| \le \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\nabla \theta_n\|_{L^2(\Omega)} \|\theta_n - \theta\|_{L^6(\Omega)}.$$

Using again the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\kappa \alpha_M \|\theta_n - \theta\|_{H^1(\Omega)} \le C \|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \|\theta_n\|_{H^1(\Omega)}. \tag{44}$$

Here and in the preceding estimates, $C > 0$ denotes a constant independent of n. Since $(\boldsymbol{w}_n, \phi_n) \rightharpoonup (\boldsymbol{w}, \phi)$ weakly in $\boldsymbol{K} \times H^1_0(\Omega)$, the sequence $\{(\boldsymbol{w}_n, \phi_n)\}_n$ is bounded in $\boldsymbol{K} \times H^1_0(\Omega)$. Moreover, using the compact embeddings $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ and $H^1_0(\Omega) \hookrightarrow L^3(\Omega)$, we obtain

$$\boldsymbol{w}_n \longrightarrow \boldsymbol{w} \quad \text{strongly in } \boldsymbol{L}^3(\Omega) \quad \text{and} \quad \phi_n \longrightarrow \phi \quad \text{strongly in } L^3(\Omega).$$

On the other hand, the a priori estimate (28), applied to $(\boldsymbol{u}_n, \theta_n) = \mathcal{S}(\boldsymbol{w}_n, \phi_n)$, shows that the sequences $\{\boldsymbol{u}_n\}_n$ and $\{\theta_n\}_n$ are bounded in $\boldsymbol{K}$ and $H^1_0(\Omega)$, respectively.
Since $\{\boldsymbol{u}_n\}_n$ is bounded in $\boldsymbol{H}^1(\Omega)$, estimate (43) and the strong convergences

$$\|\boldsymbol{w}_n - \boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)} \longrightarrow 0, \qquad \|\phi_n - \phi\|_{L^3(\Omega)} \longrightarrow 0,$$

the right-hand side of (43) tends to zero. Therefore, $\|\boldsymbol{u}_n - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \longrightarrow 0$. In particular,

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \quad \text{strongly in } \boldsymbol{K}.$$

Similarly, since $\{\theta_n\}_n$ is bounded in $H^1_0(\Omega)$, estimate (44) and the strong convergence of $\boldsymbol{w}_n$ in $\boldsymbol{L}^3(\Omega)$ imply that $\|\theta_n - \theta\|_{H^1(\Omega)} \longrightarrow 0$. Thus,

$$\theta_n \longrightarrow \theta \quad \text{strongly in } H^1_0(\Omega).$$

Consequently,

$$(\boldsymbol{u}_n, \theta_n) \longrightarrow (\boldsymbol{u}, \theta) \quad \text{strongly in } \boldsymbol{K} \times H^1_0(\Omega).$$

We have therefore shown that $\mathcal{S}$ is a compact operator.

Step 2 : We show that there exists a constant $C_* > 0$ such that $\|(\boldsymbol{u}, \theta)\| \le C_*$ for any $(\boldsymbol{u}, \theta) \in \boldsymbol{K} \times H^1_0(\Omega)$ and for any $\lambda \in [0, 1]$ such that $(\boldsymbol{u}, \theta) = \lambda \mathcal{S}(\boldsymbol{u}, \theta)$.

By the definition of $\mathcal{S}$, we have:

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) &= D(\lambda\theta,\boldsymbol{v}) - O(\lambda\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}), \quad \forall \boldsymbol{v}\in\boldsymbol{K},\\ M(\theta,\psi) &= F(\psi) - C(\lambda\boldsymbol{u};\theta,\psi), \quad \forall \psi\in H_0^1(\Omega). \end{aligned} \tag{45}$$

Choosing $(\boldsymbol{v},\psi)=(\boldsymbol{u},\theta)$ and using the properties (18) and (19), we obtain

$$A(\boldsymbol{u},\boldsymbol{u}) = D(\lambda\theta,\boldsymbol{u}) \qquad \text{and} \qquad M(\theta,\theta) = F(\theta) \tag{46}$$

From the coercivity of A and M (c.f. (16) and (17)) and the continuity of D and F (c.f. (26) and (23)), we have immediately that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le \lambda\nu^{-1}\alpha_A^{-1}C_D\|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{and} \quad \|\theta\|_{H^1(\Omega)} \le \kappa^{-1}\alpha_M^{-1}C_F\|f\|_{L^2(\Omega)}.$$

Then, using the fact that $\lambda\le 1$, it follows

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le K_1(\nu,\kappa,\boldsymbol{g},f) \quad \text{and} \quad \|\theta\|_{H^1(\Omega)} \le K_2(\kappa,f), \tag{47}$$

where $K_1(\nu,\kappa,\boldsymbol{g},f)$ and $K_2(\kappa,f)$ are the following positive constants independent of $(\boldsymbol{u},\theta)$ and λ:

$$K_1(\nu,\kappa,\boldsymbol{g},f) = \nu^{-1}\alpha_A^{-1}C_D K_2(\kappa,f)\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \quad \text{and} \quad K_2(\kappa,f) = \kappa^{-1}\alpha_M^{-1}C_F\|f\|_{L^2(\Omega)}. \tag{48}$$

It follows from (47) that $\|(\boldsymbol{u},\theta)\|\le C_*$ with $C_* = K_1(\nu,\kappa,\boldsymbol{g},f)+K_2(\kappa,f)$ is a positive constant independent of $(\boldsymbol{u},\theta)$ and λ. Finally, by Leray-Schauder fixed point theorem, there exists a fixed point $(\boldsymbol{u},\theta)\in\boldsymbol{K}\times H_0^1(\Omega)$ for the operator $\mathcal{S}$ which is a solution of (12). The estimate (27) follows directly from (47) and (48).

Now, we show the uniqueness of the solution of (10). Suppose that $(\boldsymbol{u}_1,\theta_1,p_1)$ and $(\boldsymbol{u}_2,\theta_2,p_2)$ are two solutions of (10). We set $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$, $\theta=\theta_1-\theta_2$, $p=p_1-p_2$ and we want to prove that $\boldsymbol{u}=\boldsymbol{0}$, $\theta=0$ and $p=0$. Since $(\boldsymbol{u}_1,\theta_1)$ and $(\boldsymbol{u}_2,\theta_2)$ are solutions of (12), substituting $(\boldsymbol{u}_1,\theta_1)$ and $(\boldsymbol{u}_2,\theta_2)$ in (12) and subtracting corresponding equations, we obtain

$$A(\boldsymbol{u},\boldsymbol{v}) = -O(\boldsymbol{u};\boldsymbol{u}_1,\boldsymbol{v}) + O(\boldsymbol{u}_2;\boldsymbol{u},\boldsymbol{v}) + D(\theta,\boldsymbol{v}), \quad \forall\boldsymbol{v}\in\boldsymbol{K}, \tag{49a}$$

$$M(\theta,\psi) = C(\boldsymbol{u};\theta_2,\psi) - C(\boldsymbol{u}_2;\theta,\psi), \quad \forall\psi\in H_0^1(\Omega). \tag{49b}$$

Choosing $\boldsymbol{v}=\boldsymbol{u}$ in (49a) and $\psi=\theta$ in (49b), summing up the two relations, we obtain:

$$A(\boldsymbol{u},\boldsymbol{u}) + M(\theta,\theta) = -O(\boldsymbol{u};\boldsymbol{u}_1,\boldsymbol{u}) + C(\boldsymbol{u};\theta_2,\theta) + D(\theta,\boldsymbol{u}).$$

Applying Proposition (2.1), we obtain

$$\min(\nu\alpha_A,\kappa\nu_M)\big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big) \le |O(\boldsymbol{u};\boldsymbol{u}_1,\boldsymbol{u})| + |C(\boldsymbol{u};\theta_2,\theta)| + |D(\theta,\boldsymbol{u})| \tag{50}$$

Let us bound the nonlinear terms on the right hand side of (50). Using the continuity of the form O from (24) and the estimate (47) for $\boldsymbol{u}_1$, we obtain:

$$\begin{aligned} |O(\boldsymbol{u};\boldsymbol{u}_1,\boldsymbol{u})| &\le C_O\|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}\\ &\le C_O K_1(\nu,\kappa,\boldsymbol{g},f)\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}. \end{aligned} \tag{51}$$

After using the continuity of the form C from (25), the Hölder inequality and the estimate (47) for θ_2, we derive:

$$\begin{aligned} |C(\boldsymbol{u};\theta_2,\theta)| &\le C_C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\theta_2\|_{H^1(\Omega)}\|\theta\|_{H^1(\Omega)}\\ &\le \frac{1}{2}C_C K_2(\kappa,f)\big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big) \end{aligned} \tag{52}$$

Similarly, using the continuity of the form D from (26) and the Hölder inequality, we obtain

$$\begin{aligned} |D(\theta,\boldsymbol{u})| &\le C_D\|\theta\|_{H^1(\Omega)}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\\ &\le \frac{1}{2}C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big) \end{aligned} \tag{53}$$

Collecting the above inequalities into (50), we derive

$$\begin{aligned} &\min(\nu\alpha_A,\kappa\nu_M)\big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big)\\ &\le \max\Big(C_O\|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)}, \frac{1}{2}C_C K_2(\kappa,f), \frac{1}{2}C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\Big)\big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\theta\|^2_{H^1(\Omega)}\big) \end{aligned} \tag{54}$$

Setting

$$K_3 = \max\Big(C_O K_1(\nu,\kappa,\boldsymbol{g},f), \frac{1}{2}C_C K_2(\kappa,f), \frac{1}{2}C_D\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\Big), \tag{55}$$

this implies that if $\min(\nu\alpha_A,\kappa\alpha_M)^{-1}K_3<1$, we have $\boldsymbol{u}=\boldsymbol{0}$ and $\theta=0$.

Next, since $(\boldsymbol{u}_1,\theta_1,p_1)$ and $(\boldsymbol{u}_2,\theta_2,p_2)$ are two solutions of (10), it follows that

$$B(\boldsymbol{v},p_1-p_2) = D(\theta,\boldsymbol{v}) - A(\boldsymbol{u},\boldsymbol{v}) - O(\boldsymbol{u};\boldsymbol{u}_1,\boldsymbol{v}) + O(\boldsymbol{u}_2;\boldsymbol{u},\boldsymbol{v}) = 0.$$

We then conclude by using the inf-sup condition in (13) that $p_1-p_2=0$. This completes the proof. □

3. The DG finite element approximation

In this section, we introduce a mixed DG approximation for the Boussinesq system (3). We provide the solvability and stability of the discrete scheme. To this end, let us introduce some notations. We assume the domain Ω is discretized by a discrete family of conforming meshes $\mathcal{T}_h$ made of tetrahedra. The index h is indicative of the mesh size h which is defined as $h = \max_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. The family is supposed to be regular in Ciarlet's sense [34], i.e. there exists $\varsigma > 0$ independent of h such that the ratio

$$\frac{h_T}{\rho_T} \leq \varsigma, \quad \forall T \in \mathcal{T}_h, \tag{56}$$

where ρ_T is the diameter of the inscribed ball in T. We shall use the assumption (56) throughout this work. Let us denote by $\mathcal{F}_h^I$ the set of internal faces and by $\mathcal{F}_h^\Gamma$ the set of external faces on Γ. We set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma$. Throughout the paper, e denotes a generic mesh interface, namely an edge in dimension two and a face in dimension three. We denote by h_e the diameter of each e. Let T^+ and T^- be two adjacent elements of $\mathcal{T}_h$ and let $\boldsymbol{n}^+$ (respectively $\boldsymbol{n}^-$) be the outward unit normal vector on ∂T^+ (respectively ∂T^-). For a vector field $\boldsymbol{u}$, we denote by $\boldsymbol{u}^\pm$ the trace of $\boldsymbol{u}$ from the interior of $T^\pm$. We define jumps

$$[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}^+ \times \boldsymbol{v}^+ + \boldsymbol{n}^- \times \boldsymbol{v}^-, \qquad [\![\boldsymbol{v}]\!]_N := \boldsymbol{v}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{v}^- \cdot \boldsymbol{n}^-, \quad [\![q]\!] := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-,$$

and averages

$$\{\!\{\boldsymbol{v}\}\!\} := \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad \{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-),$$

and adopt the convention that for boundary faces $e \in \mathcal{F}_h^\Gamma$, we set $[\![\boldsymbol{v}]\!]_T = \boldsymbol{n} \times \boldsymbol{v}$, $[\![\boldsymbol{v}]\!]_N = \boldsymbol{v} \cdot \boldsymbol{n}$, $[\![q]\!] = q\boldsymbol{n}$, $\{\!\{\boldsymbol{v}\}\!\} = \boldsymbol{v}$ and $\{\!\{q\}\!\} = q$. Let $\mathcal{P}_k(T)$ denotes the space of polynomials of total degree at most k on T with $k = 1, 2$ or 3. The corresponding vector-valued function space is denoted by $\boldsymbol{\mathcal{P}}_k(T)$.

3.1. Defining discrete problem

Now, we introduce the following finite element spaces which respectively approximate $\boldsymbol{u}$, θ and p:

$$\boldsymbol{X}_h := \left\{ \boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T),\, \forall T \in \mathcal{T}_h \right\},$$
$$Z_h := \left\{ \psi_h \in L^2(\Omega);\, \psi_{h|T} \in \mathcal{P}_k(T),\, \forall T \in \mathcal{T}_h \right\},$$
$$Q_h := \left\{ q_h \in L^2_0(\Omega);\, q_{h|T} \in \mathcal{P}_{k-1}(T),\, \forall T \in \mathcal{T}_h \right\}.$$

We denote by $\boldsymbol{W}_h$ the product space $\boldsymbol{X}_h \times Z_h$. The norm $\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)}$ is defined by

$$\|\cdot\|^2_{\boldsymbol{L}^2(\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\cdot\|^2_{0,T}, \quad \text{for any } T \in \mathcal{T}_h,$$

with $\|\cdot\|_{0,T} = \|\cdot\|_{\boldsymbol{L}^2(T)}$. Similarly, we use the notation $\|\cdot\|_{0,e} = \|\cdot\|_{\boldsymbol{L}^2(e)}$ for any $e \in \mathcal{F}_h$.
The mixed DG scheme reads: Find $((\boldsymbol{u}_h, \theta_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + B_h(\boldsymbol{v}_h, p_h) = D_h(\theta_h, \boldsymbol{v}_h), \tag{57a}$$
$$B_h(\boldsymbol{u}_h, q_h) = 0, \tag{57b}$$
$$M_h(\theta_h, \psi_h) + C_h(\boldsymbol{u}_h; \theta_h, \psi_h) = F_h(\psi_h), \tag{57c}$$

for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$.
The bilinear form A_h is defined by

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) :=\ & \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{u}_h)(\operatorname{div} \boldsymbol{v}_h)\, d\boldsymbol{x} \\ & - \nu \sum_{e\in\mathcal{F}_h^I} \left(\int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \right) \\ & - \nu \sum_{e\in\mathcal{F}_h} \left(\int_e \{\!\{\operatorname{div} \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \int_e \{\!\{\operatorname{div} \boldsymbol{v}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \right) \\ & + \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds \end{aligned}$$

where $\sigma > 0$ is a stabilization parameter that will be chosen large enough to ensure the coercivity of the bilinear form A_h.

We define the convective term in the Navier-Stokes equations with:

$$O_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{u}_h)\cdot \boldsymbol{v}_h \, d\boldsymbol{x} \tag{58}$$

The divergence constraint on the velocity is represented by B_h:

$$B_h(\boldsymbol{v}_h, q_h) := -\sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{v}_h) q_h \, d\boldsymbol{x} + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{ q_h \}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds. \tag{59}$$

The coupling form D_h is defined by

$$D_h(\theta_h, \boldsymbol{v}_h) = \sum_{T\in\mathcal{T}_h} \int_T (\theta_h \boldsymbol{g})\cdot \boldsymbol{v}_h \, d\boldsymbol{x}$$

The bilinear form M_h is defined by:

$$\begin{aligned} M_h(\theta_h, \psi_h) := {} & \kappa \sum_{T\in\mathcal{T}_h} \int_T \nabla\theta_h \cdot \nabla\psi_h \, d\boldsymbol{x} - \kappa \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{ \nabla\theta_h \}\!\} \cdot [\![\psi_h]\!] \, ds + \int_e \{\!\{ \nabla\psi_h \}\!\} \cdot [\![\theta_h]\!] \, ds \Big) \\ & + \sum_{e\in\mathcal{F}_h} \frac{\kappa\gamma}{h_e} \int_e [\![\theta_h]\!] \cdot [\![\psi_h]\!] \, ds \end{aligned}$$

with $\gamma > 0$ is a stabilization parameter that will be chosen large enough to ensure the coercivity of the bilinear form M_h.

We use the upwinding of Lesaint-Raviart [35] to discretize the convection term $\boldsymbol{u}\cdot\nabla\theta$. So, the coupling form C_h using upwind fluxes is defined by :

$$\begin{aligned} C_h(\boldsymbol{w}_h; \theta_h, \psi_h) := {} & \sum_{T\in\mathcal{T}_h} \Big(\int_T (\boldsymbol{w}_h \cdot \nabla\theta_h)\psi_h \, d\boldsymbol{x} + \int_{\partial T_-(\boldsymbol{w}_h)} |\{\!\{ \boldsymbol{w}_h \}\!\} \cdot \boldsymbol{n}_T| (\theta_h^{int} - \theta_h^{ext}) \psi_h^{int} \, d\boldsymbol{s} \Big), \\ & + \frac{1}{2} \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{w}_h)\theta_h \cdot \psi_h \, d\boldsymbol{x} - \frac{1}{2} \sum_{e\in\mathcal{F}_h} \int_e [\![\boldsymbol{w}_h]\!]_N \{\!\{ \theta_h \cdot \psi_h \}\!\} \, d\boldsymbol{s}, \end{aligned} \tag{60}$$

where θ_h^{int}, ψ_h^{int} (respectively θ_h^{ext}) refers to the trace of θ_h and ψ_h (respectively of θ_h) on a side of T taken from the interior of T (respectively taken from the exterior of T on that side). ∂T_- denotes the inflow boundary of T defined by

$$\partial T_-(\boldsymbol{w}_h) = \{ \boldsymbol{x} \in \partial T : \ \{\!\{ \boldsymbol{w}_h \}\!\} \cdot \boldsymbol{n} < 0 \}.$$

Finally, the form F_h is defined by

$$F_h(\psi_h) = \int_\Omega f \psi_h \, d\boldsymbol{x}.$$

3.2. Preliminary results

We need to introduce the following semi-norms

$$|\boldsymbol{v}_h|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

and

$$\begin{aligned} |||\boldsymbol{v}_h|||^2 &:= |\boldsymbol{v}_h|_{1,h}^2 + \textstyle\sum_{e\in\mathcal{F}_h} \frac{\sigma}{h_e} \big\| [\![\boldsymbol{v}_h]\!]_N \big\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma}{h_e} \big\| [\![\boldsymbol{v}_h]\!]_T \big\|_{0,e}^2, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \\ \|\psi_h\|_h^2 &:= \textstyle\sum_{T\in\mathcal{T}_h} \|\nabla\psi_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e} \big\| [\![\psi_h]\!] \big\|_{0,e}^2, \qquad \forall \psi_h \in Z_h. \\ \|q_h\|_{Q_h}^2 &:= \|q_h\|_{0,\Omega}^2, \qquad \forall q_h \in Q_h. \end{aligned}$$

Lemma 3.1. *For σ and γ large enough, there exists positive constants α_{A_h}, α_{M_h} independent of h such that:*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant \alpha_{A_h} \nu |||\boldsymbol{u}_h|||^2, \qquad \forall \boldsymbol{u}_h \in \boldsymbol{X}_h, \tag{61a}$$

$$M_h(\theta_h, \theta_h) \geqslant \alpha_{M_h} \kappa \|\theta_h\|_h^2, \qquad \forall \theta_h \in Z_h. \tag{61b}$$

Proof. For the proof of (61a), we use similar arguments in [36]. Let $\boldsymbol{u}_h \in \boldsymbol{X}_h$, we have:

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = \nu \sum_{T\in\mathcal{T}_h} (\|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 + \|\operatorname{div} \boldsymbol{u}_h\|_{0,T}^2)$$

$$- 2\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\{\mathbf{curl}\, \boldsymbol{u}_h\}\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds - 2\nu \sum_{e\in\mathcal{F}_h} \int_e \{\{\operatorname{div} \boldsymbol{u}_h\}\} [\![\boldsymbol{u}_h]\!]_N \, ds \tag{62}$$
$$+ \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2.$$

Using the Cauchy-Schwarz inequality, we have:

$$2\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\{\mathbf{curl}\, \boldsymbol{u}_h\}\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \le 2C_1\nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 \Big)^{\frac{1}{2}}$$
$$\le 2C_1\nu |||\boldsymbol{u}_h||| \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 \Big)^{\frac{1}{2}}.$$

In the same way, we get

$$2\nu \sum_{e\in\mathcal{F}_h} \int_e \{\{\operatorname{div} \boldsymbol{u}_h\}\} [\![\boldsymbol{u}_h]\!]_N \, ds \le 2C_2\nu |||\boldsymbol{u}_h||| \Big(\sum_{e\in\mathcal{F}_h} h_e^{-1} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \Big)^{\frac{1}{2}},$$

where C_1 and C_2 are a positive constants independent of h.
Applying now Young's inequality, we obtain

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \;\ge\; \nu\Big(1 - \frac{C}{\sigma}\Big) |||\boldsymbol{u}_h|||^2,$$

with $C = C_1 + C_2$. Therefore, the estimate (61a) follows by assuming that $\sigma > C$.
For the proof of (61b), we refer the reader to [35,37]
□

We will need the following inf-sup condition on B_h, where the proof can be found in [25,36].

Lemma 3.2.
There exists $\beta > 0$ only depending on Ω such that :

$$\inf_{q_h\in Q_h} \sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, q_h)}{|||\boldsymbol{v}_h||| \|q_h\|_{Q_h}} \geqslant \beta. \tag{63}$$

Lemma 3.3.
Let $\boldsymbol{u}_h, \boldsymbol{v}_h \in X_h$, $\theta_h, \psi_h \in Z_h$ and $p_h \in Q_h$. Then, we have :

$$|A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant C_{A_h} \nu \, |||\boldsymbol{u}_h||| \; |||\boldsymbol{v}_h|||, \tag{64a}$$
$$|M_h(\theta_h, \psi_h)| \leqslant C_{M_h} \kappa \, \|\theta_h\|_h \; \|\psi_h\|_h, \tag{64b}$$
$$|B_h(\boldsymbol{v}_h, p_h)| \leqslant C_{B_h} |||\boldsymbol{v}_h||| \|p_h\|_{0,\Omega}, \tag{64c}$$
$$|F_h(\psi_h)| \leqslant C_{F_h} \|f\|_{L^2(\Omega)} \|\psi_h\|_h, \tag{64d}$$

where $C_{A_h}, C_{M_h}, C_{B_h}$ and C_{F_h} are positive constants independent of h.

Proof. The proof of the continuity properties (64b), (64c) can be found in [14,35,37] and the continuity of A_h and F_h follows from the Cauchy-Schwarz inequality; see [36]. □

To prove the solvability of our numerical scheme, the main difficulty lies in establishing the continuity of the trilinear form arising from the Navier-Stokes convection term O_h (see definition in (58)) and the coupling term C_h (see definition in (60)) on the discrete spaces. We then use the following two discrete inequalities.
We first recall the following standard discrete Sobolev inequality for discontinuous finite element spaces (see for example [35]) :

$$\forall \psi_h \in Z_h, \quad \|\psi_h\|_{L^6(\Omega)} \le C \|\psi_h\|_h, \tag{65}$$

where the constant $C > 0$ is independent of h. This inequality does not remain valid when the norm $\|\cdot\|_h$ is replaced by $|||\cdot|||$ without supposing more regularity on the boundary Γ. Indeed, when Γ is Lipschitz polyhedral, we only have the following discrete Poincaré-Friedrichs inequality (see [38, Lemma 3.1])

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{L^2(\Omega)} \le C |||\boldsymbol{v}_h|||. \tag{66}$$

Therefore, when the boundary Γ is of class $C^{2,1}$, an extension of the previous inequality is proved in [39, Lemma 3.4.2] for the $\boldsymbol{L}^p$ norm with $p \in (1,6]$:

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)} \leq C|||\boldsymbol{v}_h|||. \tag{67}$$

The discrete Sobolev inequality (67) plays an important role in the stability of our DG scheme. Indeed, with $p = 4$, this inequality allows to apply an $L^4 - L^2 - L^4$ argument to bound the terms on T in O_h and C_h. Therefore, starting now, we suppose that the domain Ω has a boundary of class $C^{2,1}$.

Lemma 3.4. *We have*

$$\forall \theta_h \in Z_h, \forall \boldsymbol{v}_h \in \boldsymbol{X}_h \quad |D_h(\theta_h, \boldsymbol{v}_h)| \leqslant C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\theta_h\|_h \, |||\boldsymbol{v}_h|||, \tag{68a}$$

$$\forall \boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad |O_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant C_{O_h} |||\boldsymbol{w}_h||| \, |||\boldsymbol{u}_h||| \, |||\boldsymbol{v}_h||| \tag{68b}$$

$$\forall \boldsymbol{w}_h \in \boldsymbol{X}_h, \forall \theta_h, \psi_h \in Z_h, \quad |C_h(\boldsymbol{w}_h; \theta_h, \psi_h)| \leqslant C_{C_h} |||\boldsymbol{w}_h||| \, \|\theta_h\|_h \, \|\psi_h\|_h \tag{68c}$$

where C_{D_h}, C_{O_h} and C_{C_h} are positive constants independent of h.

Proof. Using Hölder's inequality, we can easily obtain

$$|D_h(\theta_h, \boldsymbol{v}_h)| \leq \|g\|_{0,\Omega} \|\theta_h\|_{L^6(\Omega)} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^3(\Omega)}. \tag{69}$$

Thanks to (65) and to (67) with $p = 3$, we have (68a).

Applying Cauchy-Schwarz inequality, we obtain

$$|O_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| = \Big| \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \Big| \leq 2 \|\mathbf{curl}\, \boldsymbol{w}_h\|_{\boldsymbol{L}^2(\mathcal{T}_h)} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^6(\Omega)}.$$

By virtue of (67), the estimate (68b) follows immediately.

Now, applying the Sobolev embedding (67) with $p = 4$ and the trace inequalities (see for instance [37, Proposition 4.2]), we get (68c). □

We define the space

$$\boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_h; \quad B_h(\boldsymbol{v}_h, q_h) = 0, \quad \forall q_h \in Q_h\}$$

and the discrete energy norm on $\boldsymbol{K}_h \times Z_h$:

$$\|(\boldsymbol{u}_h, \theta_h)\|_h = \left(|||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \right)^{1/2}.$$

It is well-known (see, e.g., [35,40]) that

$$C_h(\boldsymbol{w}_h; \theta_h, \theta_h) := \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T_-} |\{\boldsymbol{w}_h\} \cdot \boldsymbol{n}_T| \|\theta_h^{int} - \theta_h^{ext}\|^2 \, d\boldsymbol{s} + \frac{1}{2} \int_{\Gamma_+} |\boldsymbol{w}_h \cdot \boldsymbol{n}_T| \|\theta_h\|^2 \, d\boldsymbol{s} \geq 0. \tag{70}$$

Moreover, since the vector $\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{v}_h$ is orthogonal to $\boldsymbol{v}_h$, it is clear that for any $\boldsymbol{w}_h$, $\boldsymbol{v}_h$ in $\boldsymbol{X}_h$:

$$O_h(\boldsymbol{w}_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w}_h \times \boldsymbol{v}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} = 0, \tag{71}$$

To establish the well posedness of the DG scheme, we first eliminate the pressure from the problem by restricting ourselves to the space $\boldsymbol{K}_h$. So, we consider the following problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = D_h(\theta_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h, \tag{72a}$$

$$M_h(\theta_h, \psi_h) + C_h(\boldsymbol{u}_h; \theta_h, \psi_h) = F_h(\psi_h), \quad \forall \psi_h \in Z_h. \tag{72b}$$

We define the operator $\mathcal{S}_h$ by:

$$\mathcal{S}_h : \boldsymbol{K}_h \times Z_h \to \boldsymbol{K}_h \times Z_h$$
$$(\boldsymbol{w}_h, \phi_h) \mapsto \mathcal{S}_h(\boldsymbol{w}_h, \phi_h) = (\boldsymbol{u}_h, \theta_h),$$

where $(\boldsymbol{u}_h, \theta_h)$ is the solution of the linearized problem: Find $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ such that

$$A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + O_h(\boldsymbol{w}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = D_h(\phi_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h, \tag{73a}$$

$$M_h(\theta_h, \psi_h) + C_h(\boldsymbol{w}_h; \theta_h, \psi_h) = F_h(\psi_h), \quad \forall \psi_h \in Z_h. \tag{73b}$$

Then, the problem (72a)-(72b) can be rewritten equivalently as the following fixed-point problem:

$$\text{Find } (\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h \quad \text{such that} \quad \mathcal{S}_h(\boldsymbol{u}_h, \theta_h) = (\boldsymbol{u}_h, \theta_h). \tag{74}$$

3.3. Solvability of the fixed-point problem

In this subsection, we proceed analogously to Section 2.4 and prove the well-posedness of problem (72) by means of a fixed point argument. We apply the Banach fixed-point theorem to prove the existence and the uniqueness of $(\boldsymbol{u}_h, \theta_h)$. The existence and uniqueness of p_h then follow from the discrete inf-sup condition (63).
We begin by the following result, where we show that the operator $\mathcal{S}_h$ is well defined.

Lemma 3.5. *Assuming the stabilization parameters σ and γ sufficiently large, there exists a unique solution $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ for the linearized problem (73). Moreover, we have the following estimates:*

$$|||\boldsymbol{u}_h||| \leq \alpha_{A_h}^{-1} \nu^{-1} C_{D_h} \|\phi_h\|_h \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \qquad \text{and} \qquad \|\theta_h\|_h \leq \alpha_{M_h}^{-1} \kappa^{-1} C_{F_h} \|f\|_{L^2(\Omega)}. \tag{75}$$

Proof. Since (73a)-(73b) consists of a linear system, it suffices to establish the uniqueness. Assuming that the data are homogeneous $f = 0$ and $\boldsymbol{g} = \boldsymbol{0}$, we can choose $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$ as test functions. This leads to

$$\alpha_{A_h} \nu |||\boldsymbol{u}_h|||^2 + \alpha_{M_h} \kappa \|\theta_h\|_h^2 = 0,$$

implying that $\boldsymbol{u}_h = \boldsymbol{0}$ and $\theta_h = 0$. The estimates in (75) follow as a consequence of the coercivity of A_h and M_h in (61) together with the continuity of D_h and F_h in (68a) and (64d). □

Theorem 3.1. *Assume that*

$$M := \max\left(\sqrt{2}\Big(\frac{C_{O_h} C_1(\nu, \kappa, f, \boldsymbol{g})}{\alpha_{A_h} \nu} + \frac{C_{C_h} C_2(\kappa, f)}{\alpha_{M_h} \kappa}\Big), \sqrt{2}\frac{C_{D_h}}{\alpha_{A_h} \nu}\|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\right) < 1 \tag{76}$$

Then, the problem (72) has a unique solution $(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h$ satisfying

$$\nu |||\boldsymbol{u}_h|||^2 + \kappa \|\theta_h\|_h^2 \leq \alpha_{M_h}^{-2} C_{F_h}^2 \|f\|_{L^2(\Omega)}^2 \big(\kappa^{-2} \alpha_{A_h}^{-2} C_{D_h}^2 \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}^2 + 1\big) \tag{77}$$

Proof. The well-posedness of (72) follows by using the Banach fixed-point theorem and showing that operator $\mathcal{S}_h$ has a unique fixed-point. Setting $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$ in (72) and applying the coercivity of the forms A_h and M_h, the continuity of the form D_h and F_h, we obtain that

$$|||\boldsymbol{u}_h||| \leq \alpha_{A_h}^{-1} \nu^{-1} C_{D_h} \|\theta_h\|_h \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \qquad \text{and} \qquad \|\theta_h\|_h \leq \alpha_{M_h}^{-1} \kappa^{-1} C_{F_h} \|f\|_{L^2(\Omega)}$$

and then,

$$|||\boldsymbol{u}_h||| \leq C_1(\nu, \kappa, f, \boldsymbol{g}) \qquad \text{and} \qquad \|\theta_h\|_h \leq C_2(\kappa, f), \tag{78}$$

where

$$C_1(\nu, \kappa, f, \boldsymbol{g}) = \frac{C_{F_h} C_{D_h}}{\nu \kappa \alpha_{A_h} \alpha_{M_h}} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|f\|_{L^2(\Omega)} \qquad \text{and} \qquad C_2(\kappa, f) = \frac{C_{F_h}}{\kappa \alpha_{M_h}} \|f\|_{L^2(\Omega)}. \tag{79}$$

Motivated by these estimates, we define the compact and convex subset B_R of $\boldsymbol{K}_h \times Z_h$:

$$B_R = \Big\{(\boldsymbol{u}_h, \theta_h) \in \boldsymbol{K}_h \times Z_h \ : \ \|(\boldsymbol{u}_h, \theta_h)\|_h \leq R\Big\},$$

where $R = \alpha_{M_h}^{-1} \kappa^{-1} C_{F_h} \|f\|_{L^2(\Omega)} \big(\alpha_{A_h}^{-1} \nu^{-1} C_{D_h} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} + 1\big)$. Let us check that $\mathcal{S}_h$ maps B_R into itself. For a given $(\boldsymbol{w}_h, \phi_h) \in B_R$, we set $(\boldsymbol{v}_h, \psi_h) = \mathcal{S}(\boldsymbol{w}_h, \phi_h)$. Thanks to the estimate (75), $(\boldsymbol{v}_h, \psi_h)$ satisfies

$$|||\boldsymbol{v}_h||| \leq \alpha_{A_h}^{-1} \nu^{-1} C_{D_h} \|\phi_h\|_h \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \qquad \text{and} \qquad \|\psi_h\|_h \leq \alpha_{M_h}^{-1} \kappa^{-1} C_{F_h} \|f\|_{L^2(\Omega)}. \tag{80}$$

Combining (80) with the fact that $(\boldsymbol{w}_h, \phi_h) \in B_R$, we obtain that $(\boldsymbol{v}_h, \psi_h)$ belongs also to B_R. Now, we show that the operator $\mathcal{S}_h$ is a contraction on B_R. To that end, let $(\boldsymbol{u}_h^1, \theta_h^1)$, $(\boldsymbol{u}_h^2, \theta_h^2) \in B_R$ and $(\boldsymbol{u}_h^1, \theta_h^1) = \mathcal{S}(\boldsymbol{w}_1, \phi_1)$, $(\boldsymbol{u}_h^2, \theta_h^2) = \mathcal{S}(\boldsymbol{w}_2, \phi_2)$. Setting $\boldsymbol{u}_h := \boldsymbol{u}_h^1 - \boldsymbol{u}_h^2$, $\theta_h := \theta_h^1 - \theta_h^2$, by definition, we deduce that

$$\begin{aligned} &A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + M_h(\theta_h, \phi_h) + O_h(\boldsymbol{w}_1; \boldsymbol{u}_h, \boldsymbol{v}_h) + C_h(\boldsymbol{w}_1; \theta_h, \psi_h) \\ &= -O_h(\boldsymbol{w}_1 - \boldsymbol{w}_2; \boldsymbol{u}_h^2, \boldsymbol{v}_h) - C_h(\boldsymbol{w}_1 - \boldsymbol{w}_2; \theta_h^2, \psi_h) + D_h(\phi_1 - \phi_2, \boldsymbol{v}_h). \end{aligned}$$

Taking $(\boldsymbol{v}_h, \psi_h) = (\boldsymbol{u}_h, \theta_h)$, using (61), (68), (70) and (71), we obtain

$$\begin{aligned} |||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \leq{}& \alpha_{A_h}^{-1} \nu^{-1} C_{O_h} |||\boldsymbol{w}_1 - \boldsymbol{w}_2||| \, |||\boldsymbol{u}_h^2||| \, |||\boldsymbol{u}_h||| + \alpha_{M_h}^{-1} \kappa^{-1} C_{C_h} |||\boldsymbol{w}_1 - \boldsymbol{w}_2||| \Big\|\theta_h^2\Big\|_h \|\theta_h\|_h \\ &+ \alpha_{A_h}^{-1} \nu^{-1} C_{D_h} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\phi_1 - \phi_2\|_h |||\boldsymbol{u}_h||| \end{aligned}$$

Applying Young's inequality and using the fact that $(\boldsymbol{u}_h^2, \theta_h^2) \in B_R$, we derive that

$$|||\boldsymbol{u}_h|||^2 + \|\theta_h\|_h^2 \leq 2\Big(\alpha_{A_h}^{-2} \nu^{-2} C_{O_h}^2 \, C_1^2(\nu, \kappa, f, \boldsymbol{g}) + \alpha_{M_h}^{-2} \kappa^{-2} C_{C_h}^2 \, C_2^2(\kappa, f)\Big) |||\boldsymbol{w}_1 - \boldsymbol{w}_2|||^2$$

$$+\ 2\alpha_{A_h}^{-2}\nu^{-2}C_{D_h}^2\|g\|_{L^2(\Omega)}^2\|\phi_1-\phi_2\|_h^2,$$

and then

$$\|(u_h,\theta_h)\|_h \le M\|(w_1,\phi_1)-(w_2,\phi_2)\|_h,$$

with

$$M=\max\left(\sqrt{2}\left(\alpha_{A_h}^{-1}\nu^{-1}C_{O_h}C_1(\nu,\kappa,f,g)+\alpha_{M_h}^{-1}\kappa^{-1}C_{C_h}C_2(\kappa,f)\right),\sqrt{2}\alpha_{A_h}^{-1}\nu^{-1}C_{D_h}\|g\|_{L^2(\Omega)}\right). \tag{81}$$

So, if $M<1$ that is, if the smallness condition (76) is satisfied, the mapping $\mathcal{S}_h$ is a contraction. Consequently, since $\mathcal{S}_h$ maps B_R into itself and is a contraction under condition (3.21), the Banach fixed-point theorem ensures the existence of a unique fixed point in B_R, which is the solution of problem (72). Besides, the stability bound for (u_h,θ_h) follows immediately. □

Now, in order to recover the pressure, we have the following Corollary.

Corollary 3.1. *Let (u_h,θ_h) the solution of the problem (72) given by Theorem 3.1. Then, there exists a unique $p_h\in Q_h$ such that $(u_h,\theta_h,p_h)\in X_h\times Z_h\times Q_h$ is the solution of (57). Moreover, we have the following estimate:*

$$\|p_h\|_{0,\Omega}\le C_3(\nu,\kappa,f,g) \tag{82}$$

where

$$C_3(\nu,\kappa,f,g)=\beta^{-1}C_{A_h}C_1(\nu,\kappa,f,g)+\beta^{-1}C_{O_h}C_1^2(\nu,\kappa,f,g)+\beta^{-1}C_{D_h}C_2(\kappa,f)\|g\|_{L^2(\Omega)} \tag{83}$$

and $C_1(\nu,\kappa,f,g)$, $C_2(\kappa,f)$ are the constants in (78).

Proof. Analogously to the continuous case (see Lemma (2.2)), due to the inf-sup condition (63) and the continuity properties of A_h, O_h and D_h, here we also obtain that (57) and (72) are equivalent. Indeed, the pressure is uniquely solvable by the following problem

$$B_h(v_h,p_h)=-A_h(u_h,v_h)-O_h(u_h;u_h,v_h)+D_h(\theta_h,v_h). \tag{84}$$

Then, it suffices to prove the estimate (82) to conclude the proof. To do that, using the inf sup-condition (63) and the continuity of the forms A_h, O_h and D_h, we obtain

$$\begin{aligned}\beta\|p_h\|_{Q_h} &\le \sup_{v_h\in X_h}\frac{B_h(v_h,p_h)}{|||v_h|||}\\ &= \sup_{v_h\in X_h}\left\{\frac{-A_h(u_h,v_h)-O_h(u_h;u_h,v_h)+D_h(\theta_h,v_h)}{|||v_h|||}\right\}\\ &\le C_{A_h}|||u_h|||+C_{O_h}|||u_h|||^2+C_{D_h}\|g\|_{L^2(\Omega)}\|\theta_h\|_h\end{aligned}$$

Recalling that $(u_h,\theta_h)\in B_R$ and satisfies (78), then we obtain the estimate (82) from the latter estimate which concludes the proof. □

4. Error analysis

Let us begin by introducing an approximation result for the space X_h (see [41]). For $k=1,2,3$, there exists a continuous interpolation operator I_h defined from $H^1(\Omega)$ to X_h such that, for all $T\in\mathcal{T}_h$ and $e\in\mathcal{F}_h$:

$$\forall v\in H^1(\Omega),\ \ \forall q_h\in\mathcal{P}_{k-1}(T),\ \int_T q_h\,\mathrm{div}(I_h(v)-v)\,dx=0, \tag{85}$$

$$\forall v\in H_T^1(\Omega),\ \forall e\in\mathcal{F}_h,\ \forall q_h\in\mathcal{P}_{k-1}(e),\ \ \int_e q_h\left[\left[I_h(v)\right]\right]_N ds=0. \tag{86}$$

Moreover, for $s\in[1,k+1]$ the following interpolation estimate holds:

$$\forall v\in H^s(\Omega),\quad |I_h(v)-v|_{1,T}\leqslant Ch_T^{s-1}|v|_{s,\Delta_T}, \tag{87}$$

where Δ_T is a suitable macro-element containing T.

We use the L^2-projection of degree $k-1$ onto Q_h to approximate the pressure p. So there exists an approximation $\Pi_Q p\in Q_h$ (see [35]), defined on each $T\in\mathcal{T}_h$ by

$$\forall q\in\mathcal{P}_{k-1}(T),\quad \int_T q(p-\Pi_Q p)\,dx=0. \tag{88}$$

and satisfy the following approximation property for every integer $s\in[0,k]$:

$$\|p-\Pi_Q p\|_{0,T}\le Ch_T^s|p|_{s,T},\qquad \forall p\in H^s(\Omega)\cap L_0^2(\Omega). \tag{89}$$

To approximate the temperature θ, we use the following approximation result (see [42, Theorem 2.6] or [34]):

Let Π_Z the projection operator such that $\Pi_Z\theta=\theta$ for all $\theta\in\mathcal{P}^k(T)$. Then we have

$$|\theta-\Pi_Z\theta|_{1,T}\le Ch_T^k|\theta|_{k+1,T},\qquad \forall\theta\in H^{k+1}(\Omega). \tag{90}$$

We finally prove the following theorem:

Theorem 4.1. *Let $(\boldsymbol{u},\theta,p)\in \boldsymbol{H}^{k+1}(\Omega)\times H^{k+1}(\Omega)\times H^{k}(\Omega)$ be a solution of the continuous problem (3). Let $(\boldsymbol{u}_h,\theta_h,p_h)$ be the solution in $\boldsymbol{X}_h\times Z_h\times Q_h$ of the DG problem (57). We assume that*

$$\frac{C_{O_h}K_1(\nu,\kappa,\boldsymbol{g},f)}{\nu\alpha_{A_h}}+\frac{C_{D_h}C_{C_h}K_2(\kappa,f)\|\boldsymbol{g}\|_{0,\Omega}}{\nu\alpha_{A_h}\kappa\alpha_{M_h}}\leq\frac{1}{2}, \tag{91}$$

where $K_1(\nu,\kappa,\boldsymbol{g},f)$ is a positive constant depending on $\nu,\kappa,\boldsymbol{g},f$.
Then, there exists a positive constant $C>0$, independent of h, such that, the following estimates hold true:

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||+\|\theta-\theta_h\|_h \leqslant Ch^k\Big(|\boldsymbol{u}|_{k+1,\Omega}+|\theta|_{k+1,\Omega}+|p|_{k,\Omega}\Big) \tag{92}$$

$$\|p-p_h\|_{0,\Omega}\leq Ch^k\Big(|\boldsymbol{u}|_{k+1,\Omega}+|\theta|_{k+1,\Omega}+|p|_{k,\Omega}\Big) \tag{93}$$

Proof. The proof of this theorem proceeds by decomposing the velocity, temperature, and pressure errors as follows:

$$\boldsymbol{e}_u=\boldsymbol{\eta}_u+\boldsymbol{\chi}_u,\quad e_\theta=\eta_\theta+\chi_\theta,\quad e_p=\eta_p+\chi_p, \tag{94}$$

where

$$\begin{aligned}\boldsymbol{\eta}_u&:=\boldsymbol{u}-I_h\boldsymbol{u},\quad &\boldsymbol{\chi}_u&:=I_h\boldsymbol{u}-\boldsymbol{u}_h\\ \eta_\theta&:=\theta-\Pi_Z\theta,\quad &\chi_\theta&:=\Pi_Z\theta-\theta_h\\ \eta_p&:=p-\Pi_Q p,\quad &\chi_p&:=\Pi_Q p-p_h\end{aligned}$$

Using the error decomposition (94), the triangle inequality and the approximation properties of the operators I_h, Π_Q and Π_Z given in (87), (89) and (90) respectively, we obtain

$$|||\boldsymbol{e}_u|||\leq Ch^k|\boldsymbol{u}|_{k+1,\Omega}+|||\boldsymbol{\chi}_u|||, \tag{95a}$$
$$\|e_\theta\|_h\leq Ch^k|\theta|_{k+1,\Omega}+\|\chi_\theta\|_h, \tag{95b}$$
$$\|e_p\|_{0,\Omega}\leq Ch^k|p|_{k,\Omega}+\|\chi_p\|_{0,\Omega}, \tag{95c}$$

To estimate the error it is therefore sufficient to estimate the terms $|||\boldsymbol{\chi}_u|||$, $\|\chi_\theta\|_h$ and $\|\chi_p\|_{0,\Omega}$.

We begin by proving the error on $(\boldsymbol{u},\theta)$. We note that the solution $(\boldsymbol{u},p,\theta)$ of the continuous problem (3) satisfies

$$\begin{aligned}A_h(\boldsymbol{u},\boldsymbol{v}_h)+O_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)+B_h(\boldsymbol{v}_h,p)&=D_h(\theta,\boldsymbol{v}_h),\\ B_h(\boldsymbol{u},q_h)&=0,\\ M_h(\theta,\psi_h)+C_h(\boldsymbol{u};\theta,\psi_h)&=F_h(\psi_h),\end{aligned}$$

for all $(\boldsymbol{v}_h,q_h,\psi_h)\in\boldsymbol{X}_h\times Q_h\times Z_h$. Subtracting the above equations from the discrete formulation (57), we obtain the error equations

$$A_h(\boldsymbol{e}_u,\boldsymbol{v}_h)+O_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}_h)-O_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)+B_h(\boldsymbol{v}_h,e_p)=D_h(e_\theta,\boldsymbol{v}_h), \tag{96a}$$
$$B_h(\boldsymbol{e}_u,q_h)=0, \tag{96b}$$
$$M_h(e_\theta,\psi_h)+C_h(\boldsymbol{u};\theta,\psi_h)-C_h(\boldsymbol{u}_h;\theta_h,\psi_h)=0, \tag{96c}$$

for all $(\boldsymbol{v}_h,q_h,\psi_h)\in\boldsymbol{X}_h\times Q_h\times Z_h$, which gives using (94) and (96a) with $\boldsymbol{v}_h=\boldsymbol{\chi}_u$

$$A_h(\boldsymbol{\chi}_u,\boldsymbol{\chi}_u)+O_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\chi}_u)-O_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\chi}_u)+B_h(\boldsymbol{\chi}_u,e_p)=D_h(e_\theta,\boldsymbol{\chi}_u)-A_h(\boldsymbol{\eta}_u,\boldsymbol{\chi}_u).$$

Using again the decomposition (94), we have

$$\begin{aligned}O_h(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\chi}_u)-O_h(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{\chi}_u)&=O_h(\boldsymbol{e}_u;\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{u}_h;\boldsymbol{e}_u,\boldsymbol{\chi}_u)\\ &=O_h(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{\chi}_u;\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{u}_h;\boldsymbol{\eta}_u,\boldsymbol{\chi}_u)+\underbrace{O_h(\boldsymbol{u}_h;\boldsymbol{\chi}_u,\boldsymbol{\chi}_u)}_{=0},\end{aligned}$$

Observe that from the definition of I_h in (85)-(86), we have $B_h(\boldsymbol{\chi}_u,\chi_p)=0$ and then

$$B_h(\boldsymbol{\chi}_u,e_p)=B_h(\boldsymbol{\chi}_u,\eta_p)+\underbrace{B_h(\boldsymbol{\chi}_u,\chi_p)}_{=0}$$

Hence, we obtain

$$\begin{aligned}A_h(\boldsymbol{\chi}_u,\boldsymbol{\chi}_u)=\;&D_h(e_\theta,\boldsymbol{\chi}_u)-A_h(\boldsymbol{\eta}_u,\boldsymbol{\chi}_u)-B_h(\boldsymbol{\chi}_u,\eta_p)-O_h(\boldsymbol{\eta}_u;\boldsymbol{u},\boldsymbol{\chi}_u)\\ &-O_h(\boldsymbol{\chi}_u;\boldsymbol{u},\boldsymbol{\chi}_u)-O_h(\boldsymbol{u}_h;\boldsymbol{\eta}_u,\boldsymbol{\chi}_u).\end{aligned}$$

Using Lemma 3.1, Lemma 3.3 and Lemma 3.4, we have

$$\nu\,\alpha_{A_h}|||\boldsymbol{\chi}_u|||^2\leq C_{D_h}\|\boldsymbol{g}\|_{0,\Omega}\|e_\theta\|_h\,|||\boldsymbol{\chi}_u|||+\nu C_{A_h}|||\boldsymbol{\eta}_u|||\,|||\boldsymbol{\chi}_u|||+C_{B_h}\|\eta_p\|_{0,\Omega}|||\boldsymbol{\chi}_u|||+C_{O_h}|||\boldsymbol{\eta}_u|||\,|||\boldsymbol{u}|||\,|||\boldsymbol{\chi}_u|||$$

$$+ C_{O_h} |||\boldsymbol{u}||| \, |||\boldsymbol{\chi_u}|||^2 + C_{O_h} |||\boldsymbol{u}_h||| \, |||\boldsymbol{\eta_u}||| \, |||\boldsymbol{\chi_u}|||.$$

Now, we recall that thanks to the *a priori* estimates (47) and (78), we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq K_1(\nu, \kappa, \boldsymbol{g}, f) \quad \text{and} \quad |||\boldsymbol{u}_h||| \leq C_1(\nu, \kappa, \boldsymbol{g}, f),$$

which implies

$$\begin{aligned}\Big(\nu\alpha_{A_h} - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f)\Big)|||\boldsymbol{\chi_u}||| \; &\leq C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\chi_\theta\|_h + C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\eta_\theta\|_{0,\Omega} + \nu C_{A_h} |||\boldsymbol{\eta_u}||| \\ &\quad + C_{B_h} \|\eta_p\|_{0,\Omega} + C_{O_h} \big(K_1(\nu, \kappa, \boldsymbol{g}, f) + C_1(\nu, \kappa, \boldsymbol{g}, f)\big)|||\boldsymbol{\eta_u}|||.\end{aligned}$$

So, we have

$$\Big(\nu\alpha_{A_h} - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f)\Big)|||\boldsymbol{\chi_u}||| \leq C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\chi_\theta\|_h + C_* \tag{97}$$

with $C_* = C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|\eta_\theta\|_{0,\Omega} + \nu C_{A_h} |||\boldsymbol{\eta_u}||| + C_{B_h} \|\eta_p\|_{0,\Omega} + C_{O_h} \big(K_1(\nu, \kappa, \boldsymbol{g}, f) + C_1(\nu, \kappa, \boldsymbol{g}, f)\big)|||\boldsymbol{\eta_u}|||$.
To estimate $\|\chi_\theta\|_h$, we use (96c) with $\psi_h = \chi_\theta$

$$M_h(e_\theta, \chi_\theta) + C_h(\boldsymbol{u}, \theta, \chi_\theta) - C_h(\boldsymbol{u}_h, \theta_h, \chi_\theta) = 0,$$

which gives using the error decompositions (94) and the positivity of the form C_h (70):

$$M_h(\chi_\theta, \chi_\theta) + \underbrace{C_h(\boldsymbol{u}_h; \chi_\theta, \chi_\theta)}_{\geq 0} = -M_h(\eta_\theta, \chi_\theta) - C_h(\boldsymbol{\eta_u}; \theta, \chi_\theta) - C_h(\boldsymbol{\chi_u}; \theta, \chi_\theta) - C_h(\boldsymbol{u}_h; \eta_\theta, \chi_\theta).$$

Using again the *a priori* estimates for $\boldsymbol{u}_h$ (78) and for θ (47), together with the continuity of the forms M_h and C_h (see (64b) and (68c)), we obtain:

$$\begin{aligned}\kappa\, \alpha_{M_h} \|\chi_\theta\|_h &\leq C_{M_h} \|\eta_\theta\|_h + C_{C_h} |||\boldsymbol{\eta_u}||| \, \|\theta\|_h + C_{C_h} |||\boldsymbol{\chi_u}||| \, \|\theta\|_h + C_{C_h} |||\boldsymbol{u}_h||| \, \|\eta_\theta\|_h \\ &\leq C_{M_h} \|\eta_\theta\|_h + C_{C_h} K_2(\kappa, f)\big(|||\boldsymbol{\eta_u}||| + |||\boldsymbol{\chi_u}|||\big) + C_{C_h} C_1(\nu, \kappa, \boldsymbol{g}, f) \|\eta_\theta\|_h \\ &\leq C_{C_h} K_2(\kappa, f)|||\boldsymbol{\chi_u}||| + \widetilde{C}_*,\end{aligned} \tag{98}$$

with

$$\widetilde{C}_* = C_{M_h} \|\eta_\theta\|_h + C_{C_h} K_2(\kappa, f)|||\boldsymbol{\eta_u}||| + C_{C_h} C_1(\nu, \kappa, \boldsymbol{g}, f) \|\eta_\theta\|_h.$$

Combining the estimates (97) and (98), we obtain

$$\Big(\nu\alpha_A - C_{O_h} K_1(\nu, \kappa, \boldsymbol{g}, f) - \frac{C_{D_h} C_{C_h} K_2(\kappa, f) \|\boldsymbol{g}\|_{0,\Omega}}{\kappa \alpha_{M_h}}\Big)|||\boldsymbol{\chi_u}||| \leq \frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa \alpha_{M_h}} \widetilde{C}_* + C_*,$$

which gives by using assumption (91)

$$\frac{1}{2}\nu\alpha_A |||\boldsymbol{\chi_u}||| \leq \frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa \alpha_{M_h}} \widetilde{C}_* + C_*. \tag{99}$$

Replacing (99) in (98) gives

$$\kappa\, \alpha_{M_h} \|\chi_\theta\|_h \leq \frac{2 C_{C_h} K_2(\kappa, f)}{\nu \alpha_A} \Big(\frac{C_{D_h} \|\boldsymbol{g}\|_{0,\Omega}}{\kappa \alpha_{M_h}} \widetilde{C}_* + C_*\Big) + \widetilde{C}_* \tag{100}$$

The error estimate for the discrete velocity $\boldsymbol{u}_h$ and the temperature θ_h (92) now follows from (89) and (90), together with (99) and (100)

To prove the error estimate for the pressure (93), we use the inf-sup condition (63), the continuity property of B_h (64c) and the fact that $\chi_p = e_p - \eta_p$ to write

$$\begin{aligned}\beta \|\chi_p\|_{0,\Omega} &\leq \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, \chi_p)}{|||\boldsymbol{v}_h|||} = \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} -\Big(\frac{B_h(\boldsymbol{v}_h, -e_p) + B_h(\boldsymbol{v}_h, \eta_p)}{|||\boldsymbol{v}_h|||}\Big) \\ &\leq C_{B_h} \|\eta_p\|_{0,\Omega} + \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, -e_p)}{|||\boldsymbol{v}_h|||}.\end{aligned} \tag{101}$$

To bound the term $B_h(\boldsymbol{v}_h, -e_p)$, we begin by rewriting Eq. (96a), adding and subtracting suitable terms so as to obtain:

$$\begin{aligned}B_h(\boldsymbol{v}_h, -e_p) &= A_h(\boldsymbol{e_u}, \boldsymbol{v}_h) + O_h(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}_h) - O_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - D_h(e_\theta, \boldsymbol{v}_h) \\ &= A_h(\boldsymbol{e_u}, \boldsymbol{v}_h) + O_h(\boldsymbol{e_u}; \boldsymbol{u}, \boldsymbol{v}_h) + O_h(\boldsymbol{u}_h; \boldsymbol{e_u}, \boldsymbol{v}_h) - D_h(e_\theta, \boldsymbol{v}_h)\end{aligned}$$

From the continuity properties of A_h, O_h and D_h (see (64a), (68b) and (68a)), we have

$$|B_h(\boldsymbol{v}_h, -e_p)| \leq \Big(C_{A_h} + C_{O_h} \|\boldsymbol{u}\|_{1,\Omega} + C_{O_h} |||\boldsymbol{u}_h|||\Big)|||\boldsymbol{e_u}||| \, |||\boldsymbol{v}_h||| + C_{D_h} \|\boldsymbol{g}\|_{0,\Omega} \|e_\theta\|_h |||\boldsymbol{v}_h|||$$

By replacing this last bound into (101) and using the *a priori* estimate for $\boldsymbol{u}$ and $\boldsymbol{u}_h$, we get

$$\beta\left\|\chi_p\right\|_{0,\Omega} \leq C_{B_h}\left\|\eta_p\right\|_{0,\Omega} + \left(C_{A_h} + C_{O_h}K_1(\nu,\kappa,\boldsymbol{g},f) + C_{O_h}C_1(\nu,\kappa,\boldsymbol{g},f)\right)|||e_u||| + C_{D_h}\|\boldsymbol{g}\|_{0,\Omega}\|e_\theta\|_h.$$

Using estimates (99) and (100) together with (95) implies the desired estimate (93) for the error on the pressure. □

5. Numerical experiments

In this section, we present numerical experiments designed to validate the *a priori* error estimates established in Theorem 4.1. To assess the convergence behavior, we consider a smooth manufactured solution to the Boussinesq system (3) on the unit square domain $\Omega = (0,1)^2$:

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \sin(\pi x)^2\sin(\pi y)^2\cos(\pi y) \\ -\frac{1}{3}\sin(2\pi x)\sin(\pi y)^3 \end{pmatrix}, \quad p(x,y) = x^4 - y^4, \quad \theta(x,y) = \sin(\pi x)\sin(\pi y)$$

The exact velocity satisfies the impermeability condition, having zero normal component on the entire boundary. The vorticity corresponding to this exact solution is employed to impose a consistent non-homogeneous vorticity boundary trace, whereas the temperature is prescribed to be homogeneous on $\partial\Omega$. We consider the model parameters $\kappa = 1$, $\mathbf{g} = (0,-1)^t$, and several choices of the viscosity parameter $\nu = 1, 0.1, 0.01$.

All numerical simulations are performed in two dimensions using FEniCS. The nonlinear problem is solved by a Picard iteration, and at each step the resulting linear system is solved using PETSc with a direct LU factorization.

We measure the errors on a sequence of successively refined grids and compute the convergence rates for the velocity, pressure and temperature using:

$$e_h(\mathbf{u}) := |||u - u_h|||, \quad e_h(p) := \|p - p_h\|_{0,\Omega}, \quad e_h(\theta) := \|\theta - \theta_h\|_h$$

$$r(u) := \frac{\log(e_h(u)/e_{h'}(u))}{\log(h/h')}, \quad r(p) := \frac{\log(e_h(p)/e_{h'}(p))}{\log(h/h')}, \quad r(\theta) := \frac{\log(e_h(\theta)/e_{h'}(\theta))}{\log(h/h')}$$

where h and h' denote the size of two consecutive meshes.

For $k = 1$ and $k = 2$, the numerical results are obtained with $\sigma = 10$, $\gamma = 10$ and $\nu = 1$. For $k = 3$, the choice $\sigma = 10$ does not give the expected convergence. This is not in contradiction with Lemma 3.1, which requires σ to be sufficiently large to ensure the coercivity of the bilinear form A_h and, consequently, the well-posedness of the discrete problem. The required value of σ may depend on the polynomial degree. We therefore take $\sigma = 100$ and $\gamma = 10$ for $k = 3$. The results presented in Table 1 show the expected convergence rate $\mathcal{O}(h^k)$ for the velocity in the energy norm $|||\cdot|||$, for the pressure in the L^2-norm and for the temperature in the norm $\|\cdot\|_h$.

We also study the effect of the viscosity parameter ν. Table 2 reports the errors and convergence rates obtained for several values of ν, with the polynomial degree fixed at $k = 1$. The results confirm the expected convergence rates.

Table 1
Convergence history for the DG approximation for three polynomial degrees.

h	$\|\|\|u-u_h\|\|\|$	rate	$\|p-p_h\|_{0,\Omega}$	rate	$\|\theta-\theta_h\|_h$	rate
$k=1$						
1.768e-01	6.401e-01	–	4.630e-01	–	4.215e-01	–
8.839e-02	3.444e-01	0.89	3.347e-01	0.47	2.105e-01	1.00
4.419e-02	1.774e-01	0.96	1.931e-01	0.79	1.049e-01	1.00
2.210e-02	8.948e-02	0.99	1.008e-01	0.94	5.232e-02	1.00
1.473e-02	5.976e-02	1.00	6.777e-02	0.98	3.485e-02	1.00
$k=2$						
1.768e-01	1.274e-01	–	5.927e-02	–	3.152e-02	–
8.839e-02	3.291e-02	1.95	1.377e-02	2.11	7.838e-03	2.01
4.419e-02	8.247e-03	2.00	3.327e-03	2.05	1.950e-03	2.01
2.210e-02	2.061e-03	2.00	8.209e-04	2.02	4.862e-04	2.00
1.473e-02	9.160e-04	2.00	3.648e-04	2.00	2.159e-04	2.00
$k=3$						
3.536e-01	1.124e-01	–	1.670e-01	–	1.291e-02	–
2.357e-01	3.690e-02	2.75	6.180e-02	2.45	3.800e-03	3.02
1.768e-01	1.640e-02	2.82	2.995e-02	2.52	1.594e-03	3.02
8.839e-02	2.197e-03	2.90	4.553e-03	2.72	1.968e-04	3.02
5.893e-02	6.627e-04	2.96	1.416e-03	2.88	5.804e-05	3.01
4.419e-02	2.834e-04	2.95	6.096e-04	2.93	2.447e-05	3.00

Table 2
Convergence history for the DG approximation for different values of the viscosity parameter.

h	$\|\|\|u-u_h\|\|\|$	rate	$\|p-p_h\|_{0,\Omega}$	rate	$\|\theta-\theta_h\|_h$	rate
$\nu = 0.1$						
8.839e-02	1.969e-01	–	2.131e-01	–	2.105e-01	–
4.419e-02	1.207e-01	0.71	1.626e-01	0.39	1.049e-01	1.00
2.210e-02	6.530e-02	0.89	9.575e-02	0.76	5.232e-02	1.00
1.473e-02	4.429e-02	0.96	6.620e-02	0.91	3.485e-02	1.00
$\nu = 0.01$						
8.839e-02	7.252e-02	–	3.847e-02	–	2.105e-01	–
4.419e-02	4.259e-02	0.77	2.258e-02	0.77	1.049e-01	1.00
2.210e-02	2.261e-02	0.91	1.212e-02	0.90	5.232e-02	1.00
1.473e-02	1.528e-02	0.97	8.210e-03	0.96	3.485e-02	1.00
$\nu = 0.001$						
8.839e-02	1.062e-01	–	3.009e-02	–	2.105e-01	–
4.419e-02	5.628e-02	0.92	1.484e-02	1.02	1.049e-01	1.00
2.210e-02	2.920e-02	0.95	7.716e-03	0.94	5.234e-02	1.00
1.473e-02	2.007e-02	0.93	6.073e-03	0.92	3.492e-02	1.00

CRediT authorship contribution statement

Afaf Bouharguane: Writing – review & editing, Validation, Software, Methodology, Formal analysis; **Nour Seloula:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Formal analysis, Conceptualization.

References

[1] R. Dmowska, B. Saltzman, Advances in Geophysics, Academic press, 1991.
[2] T.J. McDougall, R.J. Greatbatch, Y. Lu, On conservation equations in oceanography: how accurate are boussinesq ocean models?, J. Phys. Oceanogr. 32 (5) (2002) 1574–1584. https://doi.org/10.1175/1520-0485(2002)032<1574:OCEIOH>2.0.CO;2
[3] A. Novotny, M. Ruzicka, G. Thater, Rigorous derivation of the anelastic approximation to the Oberbeck–Boussinesq equations?, Asymptot. Anal. 75 (2011) 93–123. https://content.iospress.com/articles/asymptotic-analysis/asy1056
[4] L. Consiglieri, Š. Nečasová, J. Sokolowski, New approach to the incompressible Maxwell Boussinesq approximation: existence, uniqueness and shape sensitivity, J. Differ. Equ. 249 (12) (2010) 3052–3080. https://doi.org/10.1016/j.jde.2010.09.029
[5] C. Popa, D. Ospir, S. Fohanno, C. Chereches, Numerical simulation of dynamical aspects of natural convection flow in a double-skin facade, Energy Build. 50 (2012) 229–233.
[6] C. Suárez, P. Joubert, J. Molina, F. Sánchez, Heat transfer and mass flow correlations for ventilated facades, Energy Build. 34 (12) (2011) 3696–3703.
[7] C. Bernardi, B. Métivet, B. Pernaud-Thomas, Couplage des équations de Navier-Stokes et de la chaleur : le modèle et son approximation par éléments finis, M2AN - Modélisation mathématique et analyse numérique 29 (7) (1995) 871–921. http://archive.numdam.org/item/M2AN_1995__29_7_871_0/
[8] M. Farhloul, S. Nicaise, L. Paquet, A refined mixed finite element method for the Boussinesq equations in polygonal domains, IMA J. Numer. Anal. 21 (2) (2001) 525–551. https://academic.oup.com/imajna/article-pdf/21/2/525/2349828/210525.pdf
[9] R. Oyarzúa, T. Qin, D. Schötzau, An exactly divergence-free finite element method for a generalized Boussinesq problem, IMA J. Numer. Anal. 34 (3) (2013) 1104–1135. https://doi.org/10.1093/imanum/drt043
[10] S.A. Lorca, J.L. Boldrini, Stationary solutions for generalized Boussinesq models, J. Differ. Equ. 124 (1996) 389–406.
[11] C.E. Pérez, J.-M. Thomas, S. Blancher, R. Creff, The steady Navier–Stokes/energy system with temperature-dependent viscosity. I. Analysis of the continuous problem, Int. J. Numer. Methods Fluids 56 (2008) 63–89.
[12] C.E. Pérez, J.-M. Thomas, S. Blancher, R. Creff, The steady Navier–Stokes/energy system with temperature-dependent viscosity. II. The discrete problem and numerical experiments, Int. J. Numer. Methods Fluids 56 (2008) 91–114.
[13] R. Oyarzúa, P. Zúñiga, Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters, J. Comput. Appl. Math. 323 (2017) 71–94.
[14] R. Oyarzúa, M. Serón, A Divergence-Conforming DG-mixed finite element method for the stationary Boussinesq problem, J. Sci. Comput. 85 (1) (2020).
[15] M.P. Ueckermann, P.F.J. Lermusiaux, Hybridizable discontinuous Galerkin projection methods for Navier–Stokes and Boussinesq equations, J. Comput. Phys. 306 (2016) 390–421.
[16] J.A. Almonacid, G.N. Gatica, R. Oyarzúa, A mixed-primal finite element method for the Boussinesq problem with temperature-dependent viscosity, Calcolo 55 (2018) 36.
[17] J.A. Almonacid, G.N. Gatica, R. Oyarzúa, A posteriori error analysis of a mixed-primal finite element method for the Boussinesq problem with temperature-dependent viscosity, J. Sci. Comput. 78 (2019) 887–917.
[18] E. Colmenares, R. Oyarzúa, F. Piña, A discontinuous Galerkin method for the stationary Boussinesq system, Comput. Methods Appl. Math. 22 (4) (2022) 797–820.
[19] C. Navier, Memoire sur les lois du mouvement des fluides, Memoires de l'Academie Royale des Sciences de l'Institut de France, 1823.
[20] V. Anaya, A. Bouharguane, D. Mora, C. Reales, R. Ruiz-Baier, N. Seloula, H. Torres, Analysis and approximation of a Vorticity–Velocity–Pressure formulation for the oseen equations, J. Sci. Comput. 80 (2019) 1577–1606. https://doi.org/10.1007/s10915-019-00990-7
[21] P. Acevedo Tapia, Theorie Lp pour le systéme de Boussinesq, Ph.D. thesis, Université de Pau et des Pays de l'Adour, Universidad de Chile, 2015.
[22] W. Hu, Y. Wang, J. Wu, B. Xiao, J. Yuan, Partially dissipative 2D Boussinesq equations with Navier type boundary conditions, Phys. D: Nonlinear Phenom. 376–377 (2018) 39–48.
[23] B. Cockburn, G.E. Karniadakis, C.W. Shu (Eds.), Discontinuous Galerkin Methods: Theory, Computation and Applications, vol. 11 of *Lecture Notes in Computational Science and Engineering*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2000. https://doi.org/10.1007/978-3-642-59721-3
[24] C. Amrouche, E. Wehbe, MHD flow with Navier type boundary conditions, Appl. Math. Lett. 134 (2022) 108361. https://doi.org/10.1016/j.aml.2022.108361
[25] N. Seloula, Mathematical Analysis and Numerical Approximation of the Stokes and Navier-Stokes Equations with Non Standard Boundary Conditions, Ph.D. thesis, Université de Pau et des Pays de l'Adour, 2010.
[26] V. Girault, P.A. Raviart, Finite Element Methods for Navier-Stokes equations, Springer-Verlag, New York-Heidelberg, 1986.
[27] C. Amrouche, N.E.H. Seloula, L^p-theory for vector potentials and Sobolev's inequalities for vector fields. Application to the Stokes equations with pressure boundary conditions, Math. Models Methods Appl. Sci. 23 (1) (2013) 37–92. https://doi.org/10.1142/S0218202512500455

[28] P. Grisvard, Elliptic Problems in Nonsmooth Domains, vol. 24 of *Monographs and Studies in Mathematics*, Pitman Advanced Publishing Program, Boston, 1985.
[29] J.-L. Lions, E. Magenes, Probl'emes aux limites non homog'enes et applications, vol. 1 of *Travaux et recherches math'ematiques*, Dunod, Paris, 1968.
[30] C. Amrouche, N. Seloula, On the stokes equations with the Navier-type boundary conditions, Differ. Equ. Appl. 3 (4) (2011) 581–607. https://doi.org/10.7153/dea-03-36
[31] C. Bernardi, C. Canuto, Y. Maday, Spectral approximations of the Stokes equations with boundary conditions on the pressure, IAM J. Numer. Anal. 28 (1991) 333–362.
[32] C. Bernardi, T. Chacón Rebollo, R. Lewandowski, F. Murat, A model for two coupled turbulent fluids. Part I: analysis of the system, in: D. Cioranescu, J.-L. Lions eds. Collège de France Seminar XIV, North–Holland, Amsterdam, 2002, pp. 69–102.
[33] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001.
[34] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, Vol. 4, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.
[35] V. Girault, B. Rivière, M.F. Wheeler, A discontinuous Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems, Math. Comp. 74 (249) (2005) 53–84.
[36] J. Poirier, N. Seloula, Discontinuous Galerkin method for the incompressible Magnetohydrodynamic system with Navier type boundary condition, preprint (2023).
[37] B. Cockburn, G. Kanschat, D. Schotzau, A locally conservative LDG method for the incompressible Navier-Stokes equations, Math. Comput. 74 (251) (2004) 1067–1095.
[38] A. Zaghdani, C. Daveau, Two new discrete inequalities of Poincaré-Friedrichs on discontinuous spaces for Maxwell's equations, C. R. Math. Acad. Sci. Paris 342 (1) (2006) 29–32.
[39] J. Poirier, Etude mathématique et approximation numérique des équations de la magnetohydrodynamique avec plusieurs types de conditions aux limites, Ph.D. thesis, Université de Caen Normandie, 2021.
[40] V. Girault, B. Rivière, M.F. Wheeler, A splitting method using discontinuous Galerkin for the transient incompressible Navier-Stokes equations, ESAIM. Math. Model. Numer. Anal. 38 (6) (2005) 1115–1147. https://doi.org/10.1051/m2an:2005048
[41] V. Girault, Incompressible finite element methods for Navier-Stokes equations with nonstandard boundary conditions in R^3, Math. Comput. 51 (183) (1988) 55–74. https://doi.org/10.2307/2008579
[42] B. Rivière, Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation, vol. 35 of *Frontiers in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008. https://doi.org/10.1137/1.9780898717440