Differences between Galois representations in outer-automorphisms of π_1 and those in automorphisms, implied by topology of moduli spaces

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Fix a prime l. Let C be a proper smooth geometrically connected curve over a number field K, and x be its closed point. Let Π denote the pro-l completion of the geometric fundamental group of C with geometric base point over x. We have two non-abelian Galois representations:

$$\rho_A : \mathrm{Gal}_{k(x)} \to \mathrm{Aut}(\Pi), \qquad \rho_O : \mathrm{Gal}_K \to \mathrm{Out}(\Pi).$$

Our question is: in the natural inclusion $\mathrm{Ker}(\rho_A) \subset \mathrm{Ker}(\rho_O) \cap \mathrm{Gal}_{k(x)}$, whether the equality holds or not.

Theorem: Assume that $g \geq 3$, l divides $2g-2$. Then, there are infinitely many pairs (C, K) with the following property. If l does not divide the extension degree $[k(x) : K]$, then $\mathrm{Ker}(\rho_A) \neq (\mathrm{Ker}(\rho_O) \cap \mathrm{Gal}_{k(x)})$ holds.

This is in contrast to the case of the projective line minus three points and its canonical tangential base points, where the equality holds (a result of Deligne and Ihara).

There are two ingredients in the proof: (1) Galois representations often contain the image of the geometric monodromy (namely, the mapping class group) [M-Tamagawa 2000] (2) A topological result [S. Morita 98] [Hain-Reed 2000] on the cohomological obstruction of lifting the outer action of the mapping class group to automorphisms.