Quasi-weak equivalences in complicial exact categories (joint work with Satoshi Mochizuki)

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K-theories

- Quillen K-theory
- Waldhausen K-theory
- K-theory for triangulated categories
- Negative K-theory
- Negative Waldhausen K-theory

Quasi-isomorphism

$\mathscr{A}$: abelian

Quasi-isomorphism

$\mathcal{A}$: abelian category
$\mathbf{Ch}(\mathcal{A})$: category of chain complexes

$$\cdots \to x^{k-1} \xrightarrow{d^{k-1}} x^k \xrightarrow{d^k} x^{k+1} \xrightarrow{d^{k+1}} \cdots \quad \text{in } \mathcal{A}, \quad d^{k+1} \circ d^k = 0.$$

A chain map $f^\bullet : x^\bullet \to y^\bullet$ is a **quasi-isomorphism**
$\overset{\text{def}}{\Longleftrightarrow}$ $f^\bullet$ induces isomorphisms $H^k(x^\bullet) \xrightarrow{\simeq} H^k(y^\bullet)$ in $\mathcal{A}$ for $\forall k$

Higher derived category

$\mathscr{A}$: abelian category

$$\text{isomorphism in } \mathscr{A} \rightsquigarrow \text{quasi-isomorphism in } \mathbf{Ch}(\mathscr{A})$$
$$\Downarrow$$
$$D(\mathscr{A}): \text{derived category}$$

Higher derived category

$\mathcal{E}$: exact category (later)

$$\text{isomorphism in } \mathcal{E} \rightsquigarrow \text{quasi-isomorphism in } \mathbf{Ch}(\mathcal{E})$$
$$\Downarrow$$
$$\boldsymbol{D}(\mathcal{E})\text{: derived category}$$

Higher derived category

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$$\Downarrow$$
$$\boldsymbol{D}(\mathcal{E})\text{: derived category}$$

Aim of this talk:

quasi-isomorphism in $\mathbf{Ch}(\mathcal{E}) \rightsquigarrow$ in $\mathbf{Ch}(\mathbf{Ch}(\mathcal{E}))$

Inductively, we obtain the **higher derived category** $\boldsymbol{D}_n(\mathcal{E})$

Higher derived category

$\mathcal{E}$: exact category (later)

$$\text{isomorphism in } \mathcal{E} \rightsquigarrow \text{quasi-isomorphism in } \mathbf{Ch}(\mathcal{E})$$
$$\Downarrow$$
$$\boldsymbol{D}(\mathcal{E})\text{: derived category}$$

Aim of this talk:

$$\text{quasi-isomorphism in } \mathbf{Ch}(\mathcal{E}) \rightsquigarrow \text{quasi}^2\text{-isomorphism in } \mathbf{Ch}(\mathbf{Ch}(\mathcal{E}))$$
$$\Downarrow$$
$$\boldsymbol{D}_2(\mathcal{E})\text{: 2nd derived category}$$

Inductively, we obtain the **higher derived category** $\boldsymbol{D}_n(\mathcal{E})$

Main Theorem

Theorem

$\mathcal{E}$: *exact category*

For any $n > 0$,

1. $K_{-n}(\mathcal{E}) \xrightarrow{\simeq} K_0(D_n(\mathcal{E})^{\sim})$,
2. $K_{-n}(\mathcal{E}) = 0 \iff D_n(\mathcal{E}) = D_n(\mathcal{E})^{\sim}$: *idempotent complete.*

$D_n(\mathcal{E})^{\sim}$: idempotent completion (or pseudo-abelianization)
i.e., Adding the image & the kernel of $\forall e^2 = e : x \to x \in D_n(\mathcal{E})$

For a triangulated category $\mathcal{T}$,

$$K_0(\mathcal{T}) := \frac{\langle [x] \mid x \in \mathcal{T} \rangle}{[y] = [x] + [z] \text{ if } x \to y \to z \xrightarrow{+1}}$$

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i.e., Adding the image & the kernel of $\forall e^2 = e : x \to x \in D_n(\mathcal{E})$

Hence we can say

"$K_{-n}(\mathcal{E})$ *is the obstruction for* $D_n(\mathcal{E})$ *to be idempotent complete*"

Exact category

$\mathcal{E} = (\mathcal{E}, \mathfrak{e})$: exact category $\overset{\text{def}}{\iff}$

- $\mathcal{E}$:additive category,
- $\mathfrak{e} = \{x \rightarrowtail y \twoheadrightarrow z\}$ family of sequences in $\mathcal{E}$ satisfying several axioms. (e.g., $x \rightarrowtail x \oplus z \twoheadrightarrow z \in \mathfrak{e}$)

$x \rightarrowtail y \twoheadrightarrow z \in \mathfrak{e}$ is said to be **admissible exact**.

Example

1. Abelian category with short exact sequences,
2. $\mathbf{Vect}(X)$ for a scheme X, $\mathbf{Proj}(R)$ for a ring R,
3. $\mathbf{Ch}(\mathcal{E})$ for an exact category $\mathcal{E}$.

Exact category with weak equivalences

$w \subset \mathbf{Mor}\,(\mathcal{E})$: class of **weak equivalences** $\overset{\text{def}}{\Longleftrightarrow}$

- $\mathbf{isom} := \{$ isomorphisms in $\mathcal{E}\} \subset w$,
- closed under extensions,
- closed under push-outs along $\rightarrowtail$ and pull-backs along $\twoheadrightarrow$,
- If 2 out of f, g and $g \circ f$ are in w, then so is the third.

Example

1. $\mathbf{isom}$ in $\mathcal{E}$
2. $q\mathbf{isom} := \{$ quasi-isomorphism $\}$ in $\mathbf{Ch}(\mathcal{E})$

Waldhausen K-theory

$(\mathcal{E}, w)$: exact category with weak equivalences
Waldhausen (1985) introduced

$$K_n(\mathcal{E}, w) := \pi_n(K(\mathcal{E}, w)) \quad (\forall n \geq 0)$$

- $K_n(\mathcal{E}, \mathrm{isom}) = K_n(\mathcal{E})$ (Quillen K-theory),
  In particular,

$$K_0(\mathcal{E}) = \frac{\langle [x] \mid x \in \mathcal{E} \rangle}{[y] = [x] + [z] \text{ if } x \rightarrowtail y \twoheadrightarrow z}$$

- $x \overset{\sim}{\to} y \in w \Longrightarrow [x] = [y]$ in $K_0(\mathcal{E}, w)$.

Examples

$K_n(R) = K_n(\mathbf{Proj}(R))$ (R : ring),
$K_n(X) = K_n(\mathbf{Vect}(X))$ (X : scheme)

Example

$K_0(R) \simeq \mathbb{Z}$ (R : local ring),
$K_0(R) \simeq \mathbb{Z} \oplus \mathrm{Cl}(R)$ (R : Dedekind ring),
$K_1(K) = K^\times$ (K : field),
$K_2(\mathbb{F}) = 0$ ($\mathbb{F}$: finite field).

Localization sequences

R : Dedekind ring

$$\oplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(K) \to 0$$

Localization sequences

$\boldsymbol{R}$ **:** Dedekind ring

$$\oplus_{\mathfrak{p}} K_1(R/\mathfrak{p}) \to K_1(R) \to K_1(K) \to$$
$$\oplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(K) \to 0$$

Localization sequences

R : Dedekind ring

$$K_2(R) \to K_2(K) \to \bigoplus_{\mathfrak{p}} K_1(R/\mathfrak{p}) \to K_1(R) \to K_1(K) \to$$
$$\bigoplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(K) \to 0$$

Localization sequences

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$$\bigoplus_{\mathfrak{p}} K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(K) \to 0$$

X : regular scheme $\supset$ U : open

$$\cdots \to K_{n-1}(X \smallsetminus U) \to K_n(X) \to K_n(U) \to K_{n-1}(X \smallsetminus U) \to \cdots$$
$$\to K_1(U) \to K_0(X \smallsetminus U) \to K_0(X) \to K_0(U) \to 0$$

These sequences justify the definition of K_n

Negative K-theory

However, if the scheme X is not regular,

$$K_0(X) \to K_0(U)$$

may not be surjective.

This indicates the existence of K-groups in negative degrees.

In fact, Bass (1968) defined $K_n(R)$ ($n \leq 1$) for ring R based on

$$0 \to K_1(R) \to K_1(R[T]) \oplus K_1(R[T^{-1}]) \to K_1(R[T, T^{-1}]) \to K_0(R) \to 0$$

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$$0 \to K_0(R) \to K_0(R[T]) \oplus K_0(R[T^{-1}]) \to K_0(R[T, T^{-1}]) \to K_{-1}(R) \to 0$$

Negative K-theory

Schlichting (2004) defined

$$\mathbb{K}_n(\mathcal{E}) \quad (n \in \mathbb{Z})$$

for an exact category $\mathcal{E}$, with slight modification:

$$\mathbb{K}_n(\mathcal{E}) = K_n(\mathcal{E}) \quad (n > 0), \quad \mathbb{K}_0(\mathcal{E}) = K_0(\mathcal{E}^{\sim}).$$

We also have

$$\mathbb{K}_n(\mathcal{E}, w) \quad (n \in \mathbb{Z})$$

for a complicial exact category with weak equivalences $(\mathcal{E}, w)$ (e.g., $(\mathrm{Ch}(\mathcal{E}), q\mathrm{isom})$)

Quasi-weak equivalences

We construct

weak equivalences in $\mathcal{E} \rightsquigarrow$ quasi-weak equivalences in $\mathbf{Ch}^{\#}(\mathcal{E})$
$(\sharp \in \{b, +, -, \emptyset\})$

$(\mathcal{E}, w)$: exact category with weak equivalences
$\mathcal{E}^w := \{x \in \mathcal{E} \mid x \overset{\sim}{\to} 0 \in w\}$,
$\pi : \mathbf{Ch}^{\#}(\mathcal{E}) \to D^{\#}(\mathcal{E}) := q\mathrm{isom}^{-1}\, \mathbf{Ch}^{\#}(\mathcal{E})$.

$$qw := \{f \in \mathbf{Ch}^{\#}(\mathcal{E}) \mid \pi(\mathrm{Cone}(f)) \in \pi(\mathbf{Ch}^{b}(\mathcal{E}^w))\}$$

$f \in qw \iff \mathrm{Cone}(f)$ is quasi-isomorphic to an object in $\mathbf{Ch}^{b}(\mathcal{E}^w)$

Gillet-Waldhausen again

Easy to show

- $\boldsymbol{q}\mathbf{isom} \subset \boldsymbol{qw}$,
- If $\boldsymbol{w} = \mathbf{isom}$, then $\boldsymbol{qw} = \boldsymbol{q}\mathbf{isom}$.

Recall that

Theorem (Gillet-Waldhausen Theorem)

$\mathcal{E} \hookrightarrow \mathbf{Ch}^b(\mathcal{E})$ *induces* $\mathbb{K}_n(\mathcal{E}, \mathbf{isom}) \xrightarrow{\simeq} \mathbb{K}_n(\mathbf{Ch}^b(\mathcal{E}), \boldsymbol{q}\mathbf{isom}) \quad (n \in \mathbb{Z})$.

We have a variant of the Gillet-Waldhausen theorem:

Theorem

If $\mathcal{E}$ *is complicial,* $\mathcal{E} \hookrightarrow \mathbf{Ch}^b(\mathcal{E})$ *induces*
$\mathbb{K}_n(\mathcal{E}, \boldsymbol{w}) \xrightarrow{\simeq} \mathbb{K}_n(\mathbf{Ch}^b(\mathcal{E}), \boldsymbol{qw}) \quad (n \in \mathbb{Z})$.

Proof of the Gillet-Waldhausen Theorem

By the localization theorem,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{K}_n(\mathcal{E}^w) & \longrightarrow & \mathbb{K}_n(\mathcal{E}) & \longrightarrow & \mathbb{K}_n(\mathcal{E}, w) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathbb{K}_n(\mathrm{Ch}^b(\mathcal{E})^{qw}, q\mathrm{isom}) & \to & \mathbb{K}_n(\mathrm{Ch}^b(\mathcal{E}), q\mathrm{isom}) & \to & \mathbb{K}_n(\mathrm{Ch}^b(\mathcal{E}), qw) & \to & \cdots . \end{array}$$

From the Gillet-Waldhausen theorem,

$$\mathbb{K}_n(\mathcal{E}^w) \xrightarrow{\simeq} \mathbb{K}_n(\mathrm{Ch}^b(\mathcal{E}^w), q\mathrm{isom}) \simeq \mathbb{K}_n(\mathrm{Ch}^b(\mathcal{E})^{qw}, q\mathrm{isom}).$$

The last isomorphism follows from the following equivalence of categories

$$q\mathrm{isom}^{-1}\,\mathrm{Ch}^b(\mathcal{E})^{qw} \simeq q\mathrm{isom}^{-1}\,\mathrm{Ch}^b(\mathcal{E}^w).$$

This theorem justifies our definition qw.

Higher derived category

$$(\mathcal{E}, w) \quad \rightsquigarrow \quad (\mathrm{Ch}^{\#}(\mathcal{E}), qw)$$
$$\Downarrow$$
$$D^{\#}(\mathcal{E}, w) := qw^{-1}\,\mathrm{Ch}^{\#}(\mathcal{E})$$: derived category

Starting from an exact category $\mathcal{E}$, we obtain

$$D^{\#}(\mathcal{E}) = D^{\#}(\mathcal{E}, \mathrm{isom}),$$
$$D^{\#}_2(\mathcal{E}) := D^{\#}(\mathrm{Ch}^{\#}(\mathcal{E}), q\mathrm{isom}),$$
$$D^{\#}_3(\mathcal{E}) := D^{\#}(\mathrm{Ch}^{\#}\,\mathrm{Ch}^{\#}(\mathcal{E}), qq\mathrm{isom}), \ldots$$

Main Theorem

Theorem

$\mathcal{E}$: *exact category*

For any $n > 0$,

1. $\mathbb{K}_{-n}(\mathcal{E}) \xrightarrow{\simeq} \mathbb{K}_0(D_n(\mathcal{E})) = K_0(D_n(\mathcal{E})^{\sim})$,
2. $\mathbb{K}_{-n}(\mathcal{E}) = 0 \iff D_n(\mathcal{E}) = D_n(\mathcal{E})^{\sim}$: *idempotent complete.*

Only a few calculations are known in the negative K-groups.
The motivation of our works is to study some conjectures for negative K-groups to be trivial:

Proof

Conjectures

Conjecture (Weibel's K-dimensional conjecture)

For any noetherian scheme X of Krull dimension d, $\mathbb{K}_{-n}(X) = 0$ for $n > d$.

Conjecture (Schlichting conjecture)

For any small abelian category $\mathcal{A}$, $\mathbb{K}_{-n}(\mathcal{A}) = 0$ for $n > 0$.

Conjecture (Hsiang conjecture)

For any finitely presented group G, $\mathbb{K}_{-n}(\mathbb{Z}G) = 0$ for $n > 1$.

We show $\mathbb{K}_{-2}(\mathcal{E}) \xrightarrow{\simeq} \mathbb{K}_0(D_2(\mathcal{E}))$. Note that $D_2^{\#}(\mathcal{E}) = D^{\#}(\mathrm{Ch}^{\#}(\mathcal{E}), q\mathrm{isom})$.
The following diagram of the derived category

$$\begin{array}{ccc} D_2^b(\mathcal{E}) & \longrightarrow & D_2^+(\mathcal{E}) \\ \downarrow & & \downarrow \\ D_2^-(\mathcal{E}) & \longrightarrow & D_2(\mathcal{E}). \end{array}$$

yields

$$\begin{aligned} \cdots &\to \mathbb{K}_{-1}(\mathrm{Ch}^+(\mathcal{E}), q\mathrm{isom}) \oplus \mathbb{K}_{-1}(\mathrm{Ch}^-(\mathcal{E}), q\mathrm{isom}) \to \mathbb{K}_{-1}(\mathrm{Ch}(\mathcal{E}), q\mathrm{isom}) \\ &\to \mathbb{K}_{-2}(\mathrm{Ch}^b(\mathcal{E}), q\mathrm{isom}) \to \mathbb{K}_{-2}(\mathrm{Ch}^+(\mathcal{E}), q\mathrm{isom}) \oplus \mathbb{K}_{-2}(\mathrm{Ch}^-(\mathcal{E}), q\mathrm{isom}) \\ &\to \mathbb{K}_{-2}(\mathrm{Ch}(\mathcal{E}), q\mathrm{isom}) \to \cdots . \end{aligned}$$

By the Eilenberg swindle, we have

$$\mathbb{K}_{-n}(\mathrm{Ch}^+(\mathcal{E}), q\mathrm{isom}) = \mathbb{K}_{-n}(\mathrm{Ch}^-(\mathcal{E}), q\mathrm{isom}) = 0.$$

Hence we have

$$\mathbb{K}_{-2}(\mathrm{Ch}^b(\mathcal{E}), q\mathrm{isom}) \simeq \mathbb{K}_{-1}(\mathrm{Ch}(\mathcal{E}), q\mathrm{isom}).$$

In general, we obtain

$$\mathbb{K}_{-n}(\mathrm{Ch}^b(\mathcal{E}), qw) \simeq \mathbb{K}_{-n+1}(\mathrm{Ch}(\mathcal{E}), qw).$$

Finally,

$$\begin{aligned}\mathbb{K}_{-2}(\mathcal{E}) &\simeq \mathbb{K}_{-2}(\mathrm{Ch}^b(\mathcal{E}), q\mathrm{isom}) \\ &\simeq \mathbb{K}_{-1}(\mathrm{Ch}(\mathcal{E}), q\mathrm{isom}) \\ &\simeq \mathbb{K}_{-1}(\mathrm{Ch}^b(\mathrm{Ch}(\mathcal{E})), q^2\mathrm{isom}) \\ &\simeq \mathbb{K}_{0}(\mathrm{Ch}(\mathrm{Ch}(\mathcal{E})), q^2\mathrm{isom}) \simeq \mathbb{K}_{0}(D_2(\mathcal{E})).\end{aligned}$$

For the 2nd statement (⇒):
We use the Thomason's classification theorem:

$$\{H < K_0(D_n(\mathcal{E})^{\sim})\} \overset{1:1}{\longleftrightarrow} \{\mathcal{T} \subset D_n(\mathcal{E})^{\sim} : \text{ dense}\}$$

Note that $D_n(\mathcal{E}) \subset D_n(\mathcal{E})^{\sim}$ is dense.
(dense = ∀ object in $D_n(\mathcal{E})^{\sim}$ is a direct summand of an object of $D_n(\mathcal{E})$ and fully faithful)

If we assume that $K_0(D_n(\mathcal{E})^{\sim}) = 0$, we have $D_n(\mathcal{E})^{\sim} = D_n(\mathcal{E})$.