GALOIS CATEGORIES

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1. Forewords

These notes describe the formalism of Galois categories and fundamental groups, as introduced by A. Grothendieck in [SGA1, Chap. V]. This formalism stems from Galois theory for topological covers and can be regarded as the natural categorical generalization of it. But, far beyond providing a uniform setting for the preexisting Galois theories as those of topological covers and field extensions, this formalism gave rise to the construction and theory of the étale fundamental group of schemes - one of the major achievements of modern algebraic geometry.

In section 2, we give the axiomatic definition of a Galois category and state the main theorem which asserts that a Galois category is a category equivalent to the category of finite discrete Π-sets for some profinite group Π. In section 3, we carry out in details the proof of the main theorem. In section 4, we show that there is a natural equivalence of categories between the category of profinite groups and the category of Galois categories pointed with fibre functors. This gives a powerful dictionary to translate properties of a functor between two pointed Galois categories in terms of properties of the corresponding morphism of profinite groups (and conversely). In section 5 we define the category of étale covers of a connected scheme and prove that it is a Galois category. In section 6, we apply the formalism of section 4 to describe the étale fundamental groups of specific classes of schemes such as abelian varieties or normal schemes. The short section 7 is devoted to geometrically connected schemes of finite type over fields. These schemes have the property that their fundamental group decomposes into a geometric part and an arithmetic part. But the interplay between those two parts remains mysterious and is at the source of some of the most standard conjectures about fundamental groups such as anabelian conjectures or the section conjecture. The four last sections are devoted to the study of the geometric part namely, the fundamental group of a connected scheme of finite type over an algebraically closed field. In section 8, we state the main G.A.G.A. theorem, which describes what occurs over the complex numbers (and, basically, over any algebraically closed field of characteristic 0). In section 9, we construct the specialization morphism from the étale fundamental group of the geometric generic fibre to the étale fundamental of the geometric special fibre of a scheme proper, smooth and geometrically connected over a trait and show that it is an epimorphism. We improve this result in section 10, by showing that, in the smooth case, the specialization epimorphism induces an isomorphism on the prime-to-p completions (where p denotes the residue characteristic of the closed point). In the concluding section 11, we apply the theory of specialization to show that the étale fundamental group of a connected proper scheme over an algebraically closed field is topologically finitely generated. In the appendix, we gather some results (without proof) from descent theory that are needed in the proofs of some of the elaborate theorems presented here.

The main source and guideline for these notes was [SGA1] but for several parts of the exposition, I am also indebted to [Mur67]. In particular, though the case of schemes is only considered there, I could extract a consequent part of sections 3 and 4 from this source (complemented with proposition 3.3, which is a categorical version of a scheme-theoretic result of J.-P. Serre). I also resorted to [Mur67] for section 9. Another source is the first synthetic section of [Mi80], which I used for classical results on étale morphisms in subsection 5.10 and normal schemes in subsection 6.4. Also, at some points, I mention famous conjectures (some of which were proved recently) on étale fundamental groups, such as Abhyankar's conjecture, anabelian conjectures or the section conjecture. For this, I am indebted to the survey expositions in [Sz09] and [Sz10].

Among other introductions to étale fundamental groups (avoiding the language of Galois categories), I should mention the proceedings of the conference *Courbes semi-stables et groupe fondamental en géométrie algébrique* held in Luminy in 1998 [BLR00] and, in particular, the elementary self-contained introductory article of A. Mézard [Me00] as well as the nice book of T. Szamuely [Sz09], which emphazises the parallel story of topological covers, field theory and schemes - especially curves.

The main contribution of these notes to the existing introductory litterature on étale fundamental groups is that we priviligiate the categorical setting to the 'incarnated ones' (as exposed in [Me00] and [Sz09]). In particular, we provide detailed proofs of all the categorical statements in sections 3 and 4. To our knowledge, such statements are only available in the original sources [SGA1] and [Mur67] and, there, their proofs are only sketched. Privilegiating the categorical setting is not only a matter of taste but stems from the conviction that elementary category theory, which is only involved in Galois categories, is much simpler than (even elementary) scheme theory.

Concerning scheme theory, there is nothing new in the material presented here but we tried to make the exposition both concise and exhaustive so that it becomes accessible to graduate students in algebraic geometry. In section 5, 6, 7 and 10, we provide detailed proofs. Sections 8, 9 and 11 require more elaborate tools. In section 8, we only provide the minimal material to understand the statement of the main G.A.G.A. theorem but in sections 9 and 11 we state the main theorems involved and, relying on them, give detailed sketches of proof.

For sections 2 to 4 only some familiarity with the language of categories and the notion of profinite groups are required. For sections 5 to 7, the reader has to be familiar with the basics of commutative algebra as in [AM69] and the theory of schemes as in [Hart77, Chap. II]. As mentioned, sections 8 to 11 rely on difficult theorems but to understand their statements, only a little more knowledge of the theory of schemes is needed - say as in [Hart77, Chap. III].

2. Galois Categories

2.1. **Definition and elementary properties.** Given a category $\mathcal{C}$ and two objects $X,\ Y\in\mathcal{C}$, we will use the following notation:

$$\begin{array}{ll} \mathrm{Hom}_{\mathcal{C}}(X,Y) & : \text{Set of morphisms from } X \text{ to } Y \text{ in } \mathcal{C}\\ \mathrm{Isom}_{\mathcal{C}}(X,Y) & : \text{Set of isomorphisms from } X \text{ to } Y \text{ in } \mathcal{C}\\ \mathrm{Aut}_{\mathcal{C}}(X) & := \mathrm{Isom}_{\mathcal{C}}(X,X) \end{array}$$

A morphism $u:X\to Y$ in a category $\mathcal{C}$ is a *strict epimorphism* if the fibre product $X\times_{u,Y,u}X$ exists in $\mathcal{C}$ and for any object Z in $\mathcal{C}$, the map $u\circ:\mathrm{Hom}_{\mathcal{C}}(Y,Z)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$ is injective and induces a bijection onto the set of all morphism $\psi:X\to Z$ in $\mathcal{C}$ such that $\psi\circ p_1=\psi\circ p_2$, where $p_i:X\times_{u,Y,u}X\to X$ denotes the ith projection, $i=1,2$.

Let $FSets$ denote the category of finite sets.

Définition 2.1. *A **Galois category** is a category $\mathcal{C}$ such that there exists a covariant functor $F:\mathcal{C}\to FSets$ satisfying the following axioms:*

(1) $\mathcal{C}$ has a final object $e_{\mathcal{C}}$ and finite fibre products exist in $\mathcal{C}$.
(2) Finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$.
(3) Any morphism $u:Y\to X$ in $\mathcal{C}$ factors as $Y\overset{u'}{\to}X'\overset{u''}{\to}X$, where u' is a strict epimorphism and u'' is a monomorphism which is an isomorphism onto a direct summand of X.
(4) F sends final objects to final objects and commutes with fibre products.
(5) F commutes with finite coproducts and categorical quotients by finite groups of automorphisms and sends strict epimorphisms to strict epimorphisms.
(6) Let $u:Y\to X$ be a morphism in $\mathcal{C}$, then $F(u)$ is an isomorphism if and only if u is an isomorphism.

Remark 2.2. As the coproduct over the emptyset $\emptyset$ is always an initial object, it follows from axiom (2) that $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$.

2.1.1. *Equivalent formulations of axioms (1), (2), (4), (5):*

(1) is equivalent to:

(1)' Finite projective limits exist in $\mathcal{C}$.

(2) is implied by:

(2)' Finite inductive limits exist in $\mathcal{C}$.

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories admitting finite projective limits (resp. finite inductive limits). A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be *right exact* (resp. *left exact*) if it commutes with finite projective limits (resp. with finite inductive limits). Then, (4) is equivalent to:

(4)' F is right exact

and (5) is implied by:

(5)' F is left exact.

It will follow from theorem 2.8 that (1)-(6) are equivalent to (1), (2)', (3), (4), (5)' and (6).

2.1.2. *Unicity in axiom (3):*

Lemma 2.3. *The decomposition $Y \overset{u'}{\to} X' \overset{u''}{\to} X$ in axiom (3) is unique in the sense that for any two such decompositions $Y \overset{u'_i}{\to} X'_i \overset{u''_i}{\to} X = X'_i \sqcup X''_i$, $i = 1, 2$ there exists a (necessarily) unique isomorphism $\omega : X'_1 \tilde{\to} X'_2$ such that $\omega \circ u'_1 = u'_2$ and $u''_2 \circ \omega = u''_1$.*

Proof. From the injectivity of $- \circ u' : \mathrm{Hom}_{\mathcal{C}}(X', X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X)$, any such decomposition $Y \overset{u'}{\to} X' \overset{u''}{\to} X$ is entirely determined by u, u'. Let $Y \overset{u'_i}{\to} X'_i \overset{u''_i}{\to} X = X'_i \sqcup X''_i$, $i = 1, 2$ be two such decompositions. Since $u = u''_1 \circ u'_1$ one gets:

$$u''_2 \circ (u'_2 \circ p_1) = u \circ p_1 = u \circ p_2 = u''_2 \circ (u'_2 \circ p_2),$$

where $p_i : Y \times_{u'_1, X'_1, u'_1} Y \to Y$ denotes the ith projection, $i = 1, 2$. As $u''_2 : X'_2 \hookrightarrow X$ is a monomorphism, this implies that $u'_2 \circ p_1 = u'_2 \circ p_2$ and, as $u'_1 : Y \to X'_1$ is a strict epimorphism, this in turn implies that $u'_2 : Y \to X'_2$ lies in the image of $u'_1 \circ - : \mathrm{Hom}_{\mathcal{C}}(X'_1, X'_2) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X'_2)$ hence can be written in $\mathcal{C}$ as $u'_2 : Y \overset{u'_1}{\to} X'_1 \overset{\phi}{\to} X'_2$. From axiom (6), to prove that $\phi : X'_1 \tilde{\to} X'_2$ is an isomorphism in $\mathcal{C}$, it is enough to prove that $F(\phi) : F(X'_1) \twoheadrightarrow F(X'_2)$ is bijective. But $F(\phi) : F(X'_1) \twoheadrightarrow F(X'_2)$ is surjective since $F(u'_2)$ is, hence bijective since $|F(X'_1)| = |F(X'_2)| = |F(u)(F(Y))|$. □

2.1.3. *Artinian property.* It follows from axiom (4) that a Galois category is always artinian. More precisely, one has the following elementary categorical lemma.

Lemma 2.4. *Let $\mathcal{C}$ be a category which admits finite fibre products and let $u : X \to Y$ be a morphism in $\mathcal{C}$. Then $u : X \to Y$ is a monomorphism if and only if the first projection $p_1 : X \times_Y X \to X$ is an isomorphism. In particular,*

1. *A functor that commutes with fibre products sends monomorphisms to monomorphisms.*
2. *If $u : X \to Y$ is both a monomorphism and a strict epimorphism then $u : X \to Y$ is an isomorphism.*

Proof. Let $\Delta_{X|Y} : X \to X \times_{u,Y,u} X$ denote the diagonal morphism. By definition, $p_1 \circ \Delta_{X|Y} = Id_X$ so, if $p_1 : X \times_Y X \to X$ is an isomorphism, its inverse is automatically $\Delta_{X|Y} : X \to X \times_Y X$. Assume first that $u : X \to Y$ is a monomorphism. Then, from $u \circ p_1 = u \circ p_2$, one deduces that $p_1 = p_2$. But, then, $p_1 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1$ and:

$$p_2 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1 = p_2.$$

So, from the uniqueness in the universal property of the fibre product, one gets $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$. Conversely, assume that $p_1 : X \times_Y X \tilde{\to} X$ is an isomorphism. Then, for any morphisms $f, g : W \to X$ in $\mathcal{C}$ such that $u \circ f = u \circ g$ there exists a unique morphism $(f, g) : W \to X \times_Y X$ such that $p_1 \circ (f, g) = f$ and $p_2 \circ (f, g) = g$. From the former equality, one obtains that $(f, g) = \Delta_{X|Y} \circ f$ and, from the latter, that $g = p_2 \circ (f, g) = p_2 \circ \Delta_{X|Y} \circ f = f$.

Assertion (1) follows straightforwardly from the fact that functors send isomorphisms to isomorphisms. It remains to prove assertion (2). Since $u : X \to Y$ is a strict epimorphism, the map $u\circ : \mathrm{Hom}_{\mathcal{C}}(Y, X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ induces a bijection onto the set of all morphisms $v : X \to X$ such that $v \circ p_1 = v \circ p_2$, where $p_i : X \times_Y X \to X$ is the ith projection, $i = 1, 2$. But since $u : X \to Y$ is also a monomorphism, the first projection $p_1 : X \times_Y X \tilde{\to} X$ is an isomorphism with inverse $\Delta_{X|Y} : X \to X \times_Y X$. So $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$, which yields:

$$\begin{aligned} p_2 \circ \Delta_{X|Y} \circ p_1 &= p_2 \\ &= Id_X \circ p_1 \quad = p_1. \end{aligned}$$

Thus $p_1 = p_2$, which implies that $u\circ : \mathrm{Hom}_{\mathcal{C}}(Y, X) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ is bijective. In particular, there exists $v : Y \to X$ such that $u \circ v = Id_Y$. But, then, $u \circ v \circ u = u = u \circ Id_X$ whence $v \circ u = Id_X$. $\square$

Corollary 2.5. *A Galois category $\mathcal{C}$ is artinian.*

Proof. Let $F : \mathcal{C} \to FSets$ be a fibre functor for $\mathcal{C}$ and consider a decreasing sequence

$$\cdots \overset{t_{n+1}}{\hookrightarrow} T_n \overset{t_n}{\hookrightarrow} \cdots \overset{t_2}{\hookrightarrow} T_1 \overset{t_1}{\hookrightarrow} T_0$$

of monomorphisms in $\mathcal{C}$. We want to show that $t_{n+1} : T_{n+1} \hookrightarrow T_n$ is an isomorphism for $n \gg 0$. From axiom (6), it is enough to show that $F(t_{n+1}) : F(T_{n+1}) \to F(T_n)$ is an isomorphism for $n \gg 0$. But it follows from lemma 2.4 (1) and axiom (4) that $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is a monomorphism and, since $F(T_0)$ is finite, the monomorphism $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is actually an isomorphism for $n \gg 0$. $\square$

2.1.4. *A reinforcement of axiom (6).* Combining axioms (3), (4) and (6), one also obtains that $F : \mathcal{C} \to FSets$ is "conservative" for strict epimorphisms, monomorphisms, final and initial objects:

Lemma 2.6.

(1) If $u : Y \to X$ is a morphism in $\mathcal{C}$ then $F(u)$ is an epimorphism (resp. a monomorphism) if and only if u is a strict epimorphism (resp. a monomorphism).
(2) For any $X_0 \in \mathcal{C}$, one has:
- $F(X_0) = \emptyset$ if and only if $X_0 = \emptyset_{\mathcal{C}}$;
- $F(X_0) = *$ if and only if $X_0 = e_{\mathcal{C}}$, where $*$ denotes the final object in $FSets$.

Proof.

(1) The "if" implication for epimorphism follows from axiom (4) and the "if" implication for monomorphism from lemma 2.4 (1) and axiom (4).
We now prove the "only if" implications. From axiom (3), any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \overset{u'}{\to} X' \overset{u''}{\to} X$, where u' is a strict epimorphism and u'' is a monomorphism which is an isomorphism onto a direct summand of X. So, if $F(u)$ is an epimorphism then $F(u'')$ is an epimorphism as well. But from the "if" implication, $F(u'')$ is also a monomorphism hence an isomorphism since we are in the category $FSets$. So u'' is an isomorphism by axiom (6). The proof for monomorphism is exactly the same.

(2) We first consider the case of initial objects. By definition of an initial object, for any $X \in \mathcal{C}$ there is exactly one morphism from $\emptyset_{\mathcal{C}}$ to X in $\mathcal{C}$; denote it by $u_X : \emptyset_{\mathcal{C}} \to X$.
Assume first that $F(X_0) = \emptyset$. Since, for any finite set E, there is a morphism from E to $\emptyset$ in $FSets$ if and only if $E = \emptyset$ and since $F(u_{X_0})$ is a morphism from $F(\emptyset_{\mathcal{C}})$ to $F(X_0) = \emptyset$ in $FSets$, one has $F(\emptyset_{\mathcal{C}}) = \emptyset$. But this forces $F(u_{X_0}) = Id_{\emptyset}$. In particular, $F(u_{X_0})$ is an isomorphism hence, by axiom (6) so is u_{X_0}.
Assume now that $X_0 = \emptyset_{\mathcal{C}}$. For any object $X \in \mathcal{C}$, one has a canonical isomorphism $(u_X, Id_X) : \emptyset_{\mathcal{C}} \sqcup X \tilde{\to} X$ (with inverse the canonical morphism $i_X : X \tilde{\to} \emptyset_{\mathcal{C}} \sqcup X$) thus $F((u_X, Id_X)) : F(\emptyset_{\mathcal{C}} \sqcup X) \tilde{\to} F(X)$ is again an isomorphism. But, it follows from axiom (5) that $F(\emptyset_{\mathcal{C}} \sqcup X) \simeq F(\emptyset_{\mathcal{C}}) \sqcup F(X)$, which forces $|F(\emptyset_{\mathcal{C}})| = 0$ hence $F(\emptyset_{\mathcal{C}}) = \emptyset$.
We consider now the case of final object. The fact that $F(e_{\mathcal{C}}) = *$ follows from axiom (4). Conversely, by definition of a final object, for any $X \in \mathcal{C}$ there is exactly one morphism from X to $e_{\mathcal{C}}$ in $\mathcal{C}$; denote

it by $v_X : X \to e_{\mathcal{C}}$. So, $F(X) = *$ forces $F(v_X) : * \to *$ is the identity which, by axiom (6), implies that $v_X : X \tilde{\to} e_{\mathcal{C}}$ is an isomorphism. $\square$

2.2. **Main theorem.** Given a Galois category $\mathcal{C}$, a functor $F : \mathcal{C} \to FSets$ satisfying axioms (4), (5), (6) is called a *fibre functor for* $\mathcal{C}$. Given a fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$, the *fundamental group of* $\mathcal{C}$ *with base point* F is the group - denoted by $\pi_1(\mathcal{C}; F)$ - of automorphisms of the functor $F : \mathcal{C} \to FSets$.

Also, given two fibre functors $F_i : \mathcal{C} \to FSets$ for $\mathcal{C}$, $i = 1, 2$ the *set of paths from* F_1 *to* F_2 *in* $\mathcal{C}$ is the set - denoted by $\pi_1(\mathcal{C}; F_1, F_2) := \mathrm{Isom}_{Fct}(F_1, F_2)$ - of isomorphisms of functors from $F_1 : \mathcal{C} \to FSets$ to $F_2 : \mathcal{C} \to FSets$.

Example 2.7.

(1) For any connected, locally arcwise connected and locally simply connected topological space B, let $\mathcal{C}_B^{top}$ denote the category of finite topological covers of B. Then $\mathcal{C}_B$ is Galois with fibre functors the usual "fibre at b" functors, $b \in B$:

$$\begin{array}{rlcl} F_b : & \mathcal{C}_B^{top} & \to & FSets \\ & f : X \to B & \to & f^{-1}(b) \end{array}.$$

Let $\pi_1^{top}(B; b)$ denote the topological fundamental group of B with base point b and group law defined as follows. For any $\gamma, \gamma' \in \pi_1^{top}(B; b)$ with representatives $c_\gamma, c_{\gamma'} : [0,1] \to B$ we define $\gamma' \cdot \gamma$ to be the homotopy class of:

$$\begin{array}{rlcl} c_{\gamma'} \circ c_\gamma : & [0,1] & \to & B \\ & 0 \leq t \leq \frac{1}{2} & \to & c_\gamma(2t) \\ & \frac{1}{2} \leq t \leq 1 & \to & c_{\gamma'}(2t-1) \end{array}$$

Then, with this convention, one has:

$$\pi_1(\mathcal{C}_B^{top}; F_b) = \widehat{\pi_1^{top}(B; b)}$$

(where $\widehat{(-)}$ denotes the profinite completion).

(2) For any profinite group Π, let $\mathcal{C}(\Pi)$ denote the category of finite (discrete) sets with continuous left Π-action. Then $\mathcal{C}(\Pi)$ is Galois with fibre functor the forgetful functor $For : \mathcal{C}(\Pi) \to FSets$. And, in that case:

$$\pi_1(\mathcal{C}(\Pi); For) = \Pi.$$

Example 2.7 (3) is actually the typical example of Galois categories. Indeed, the fundamental group $\pi_1(\mathcal{C}; F)$ is equipped with a natural structure of profinite group. For this, set:

$$\Pi := \prod_{X \in Ob(\mathcal{C})} \mathrm{Aut}_{FSets}(F(X))$$

and endow Π with the product topology of the discrete topologies, which gives it a structure of profinite group. Considering the monomorphism of groups:

$$\begin{array}{lcl} \pi_1(\mathcal{C}; F) & \hookrightarrow & \Pi \\ \theta & \mapsto & (\theta(X))_{X \in Ob(\mathcal{C})} \end{array}$$

the group $\pi_1(\mathcal{C}; F)$ can be identified with the intersection of all:

$$\mathcal{C}_\phi := \{(\sigma_X)_{X \in Ob(\mathcal{C})} \in \Pi \mid \sigma_X \circ F(\phi) = F(\phi) \circ \sigma_Y\},$$

where $\phi : Y \to X$ describes the set of all morphisms in $\mathcal{C}$. By definition of the product topology, the $\mathcal{C}_\phi$ are closed. So $\pi_1(\mathcal{C}; F)$ is closed as well and, equipped with the topology induced from the product topology on Π, it becomes a profinite group.

By definition of this topology, a fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$ factors as:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "F"'] & FSets \\ \mathcal{C}(\pi_1(\mathcal{C}; F)) \arrow[ur, "For"'] & \end{tikzcd}$$

Theorem 2.8. (Main theorem) *Let $\mathcal{C}$ be a Galois category. Then:*

1. *Any fibre functor $F : \mathcal{C} \to FSets$ induces an equivalence of categories $F : \mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}; F))$.*
2. *For any two fibre functors $F_i : \mathcal{C} \to FSets$, $i = 1, 2$, the set of paths $\pi_1(\mathcal{C}; F_1, F_2)$ is non-empty. The profinite group $\pi_1(\mathcal{C}; F_1)$ is noncanonically isomorphic to $\pi_1(\mathcal{C}; F_2)$ with an isomorphism that is canonical up to inner automorphisms. In particular, the abelianization $\pi_1(\mathcal{C}; F)^{ab}$ of $\pi_1(\mathcal{C}; F)$ does not depend on F up to canonical isomorphism.*

3. Proof of the main theorem

Given a category $\mathcal{C}$ and $X,\ Y \in \mathcal{C}$, we will say that X *dominates* Y *in* $\mathcal{C}$ - and write $X \geq Y$ - if there is at least one morphism from X to Y in $\mathcal{C}$.

From now on, let $\mathcal{C}$ be a Galois category and let $F : \mathcal{C} \to FSets$ be a fibre functor for $\mathcal{C}$.

3.1. **The pointed category associated with $\mathcal{C}$, F.**

We define the *pointed category associated with* $\mathcal{C}$ *and* F to be the category $\mathcal{C}^{pt}$ whose objects are pairs (X, ζ) with $X \in \mathcal{C}$ and $\zeta \in F(X)$ and whose morphisms from (X_1, ζ_1) to (X_2, ζ_2) are the morphisms $u : X_1 \to X_2$ in $\mathcal{C}$ such that $F(u)(\zeta_1) = \zeta_2$.
There is a natural forgetful functor:

$$For : \mathcal{C}^{pt} \to \mathcal{C}$$

and a 1-to-1 correspondence between sections of $For : Ob(\mathcal{C}^{pt}) \to Ob(\mathcal{C})$ and families:

$$\underline{\zeta} = (\zeta_X)_{X \in Ob(\mathcal{C})} \in \prod_{X \in Ob(\mathcal{C})} F(X).$$

The idea behind the notion of pointed categories is to replace the original category $\mathcal{C}$ by a category $\mathcal{C}^{pt}$ with more objects but *less* morphisms between objects.

Let $\mathcal{C}_o \subset \mathcal{C}$ denote the full subcategory of connected objects (see subsection 3.2.1) and let $\mathcal{G} \subset \mathcal{C}_o$ denote the full subcategory of Galois objects (see subsection 3.2.2). Then, it turns out that:

- For any two objects X, Y in $\mathcal{G}$ such that $X \geq Y$ and for any $\zeta_X \in F(X)$, $\zeta_Y \in F(Y)$ there is exactly one morphism from (X, ζ_X) to (Y, ζ_Y) in $\mathcal{G}^{pt}$;
- For any two objects $X, Y \in \mathcal{G}$ there exists an object $Z \in \mathcal{G}$ such that $Z \geq X$ and $Z \geq Y$.

As a result, any section $\underline{\zeta}$ of $For : Ob(\mathcal{C}^{pt}) \to Ob(\mathcal{C})$ endows $Ob(\mathcal{G})$ with a structure of projective system, that we denote by $\mathcal{G}^{\underline{\zeta}}$.

The two remarkable facts concerning $\mathcal{G}^{\underline{\zeta}}$ are:

1. Any object in $\mathcal{C}_o^{pt}$ is dominated by an object in $\mathcal{G}^{\underline{\zeta}}$ (see proposition 3.3);
2. Given any object $X \in \mathcal{G}$, if we replace $\mathcal{C}$ by the full subcategory $\mathcal{C}^X \subset \mathcal{C}$ whose objects are the objects in $\mathcal{C}$ whose connected components are all dominated by X and $F : \mathcal{C} \to FSets$ by its restriction $F^X : \mathcal{C}^X \to FSets$ to $\mathcal{C}^X$ then (see propositon 3.5),
   (a) the evaluation morphism: $ev_{\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{C}^X} \to F^X$ is an isomorphism;
   (b) $\mathcal{C}^X$ is a Galois category with fibre functor $F^X : \mathcal{C}^X \to FSets$ for which theorem 2.8 holds.

(1) provides a well-defined morphism of functors:

$$ev_{\underline{\zeta}} : \varinjlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Hom}_{\mathcal{C}}(X, -) \to F$$

and it will follow from (2) (a) that this is an isomorphism. But, then, the proof of theorem 2.8 follows easily by combining (1) and (2) (b). Furthermore, this will give a natural description of $\pi_1(\mathcal{C}; F)$ as:

$$(\varprojlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Aut}_{\mathcal{C}}(X))^{op}.$$

3.2. **Connected and Galois objects.**

3.2.1. *Connected objects.* An object $X \in \mathcal{C}$ is *connected* if it cannot be written as a coproduct $X = X_1 \sqcup X_2$ with $X_i \neq \emptyset_{\mathcal{C}}$, $i = 1, 2$. We gather below elementary properties of connected objects.

Proposition 3.1. (Minimality and connected components) *An object $X_0 \in \mathcal{C}$ is connected if and only if for any $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ any monomorphism from X to X_0 in $\mathcal{C}$ is automatically an isomorphism.*
In particular, any object $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ can be written as:

$$X = \bigsqcup_{i=1}^{r} X_i,$$

with $X_i \in \mathcal{C}$ connected, $X_i \neq \emptyset_{\mathcal{C}}$, $i = 1, \dots, r$ and this decomposition is unique (up to permutation). We say that the X_i, $i = 1, \dots, r$ are the connected components of X.

Proof. We prove first the "only if" implication. Write $X_0 = X_0' \sqcup X_0''$ and assume, for instance, that $X_0' \neq \emptyset_{\mathcal{C}}$. From lemma 2.6 (1), the canonical morphism $i_{X_0'} : X_0' \to X_0$ is a monomorphism hence automatically an isomorphism, which forces $F(X_0'') = \emptyset$ hence $X_0'' = \emptyset_{\mathcal{C}}$ by lemma 2.6 (2).

We prove now the "if" implication. Assume that $X_0 \neq \emptyset_{\mathcal{C}}$ is connected and let $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$. By axiom (3), any monomorphism $i : X \hookrightarrow X_0$ in $\mathcal{C}$ factors as $X \xrightarrow{i'} X_0' \xrightarrow{i''} X_0 = X_0' \sqcup X_0''$ with $i' : X \to X_0'$ a strict epimorphism and $i'' : X_0' \to X_0$ a monomorphism inducing an isomorphism onto X_0'. Since X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. But if $X_0' = \emptyset_{\mathcal{C}}$ then $F(X) = \emptyset$, which, by lemma 2.6 (2), forces $X = \emptyset_{\mathcal{C}}$ and contradicts our assumption. So $X_0'' = \emptyset_{\mathcal{C}}$ and $i'' : X_0' \hookrightarrow X_0$ is an isomorphism. But, then, $i : X \hookrightarrow X_0$ is both a monomorphism and a strict epimorphism hence an isomorphism by lemma 2.4.

As for the last assertion, since $\mathcal{C}$ is artinian, for any $X \in \mathcal{C}$, $X \neq \emptyset$, there exists $X_1 \in \mathcal{C}$ connected, $X_1 \neq \emptyset_{\mathcal{C}}$ and a monomorphism $i_1 : X_1 \hookrightarrow X$. If i_1 is an isomorphism then X is connected. Else, from axiom (3), i_1 factors as $X_1 \xrightarrow{i_1'} X' \xrightarrow{i_1''} X = X' \sqcup X''$ with i_1' a strict epimorphism and i_1'' a monomorphism inducing an isomorphism onto X'. Since i_1 and i_1'' are monomorphism, i_1' is a monomorphism as well hence an isomorphism, by lemma 2.4 (2). We then iterate the argument on X''. By axiom (5), this process terminates after at most $|F(X)|$ steps. So we obtain a decomposition:

$$X = \bigsqcup_{i=1}^{r} X_i$$

as a coproduct of finitely many non-initial connected objects, which proves the existence. For the unicity, assume that we have another such decomposition:

$$X = \bigsqcup_{i=1}^{s} Y_i.$$

For $1 \leq i \leq r$, let $1 \leq \sigma(i) \leq s$ such that $F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$. Then consider:

$$\begin{tikzcd}
X_i \arrow[r, hook, "i_{X_i}"] & X \\
X_i \times_X Y_{\sigma(i)} \arrow[u, "p"] \arrow[r, "q"'] \arrow[ur, phantom, "\square"] & Y_{\sigma(i)} \arrow[u, hook, "i_{Y_{\sigma(i)}}"']
\end{tikzcd}$$

Since i_{X_i} is a monomorphism, q is a monomorphism as well. Also, by axiom (4) one has $F(X_i \times_X Y_{\sigma(i)}) = F(X_i) \cap F(Y_{\sigma(i)})$, which is nonempty by definition of $\sigma(i)$. So, from lemma 2.6 (1), one has $X_i \times_X Y_{\sigma(i)} \neq \emptyset_{\mathcal{C}}$ and, since $Y_{\sigma(i)}$ is connected and q is a monomorphism, q is an isomorphism. Similarly, p is an isomorphism. $\square$

Proposition 3.2. (Morphisms from and to connected objects)

(1) (Rigidity) *For any $X_0 \in \mathcal{C}$ connected, $X_0 \neq \emptyset_{\mathcal{C}}$, for any $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ and for any $\zeta_0 \in F(X_0)$, $\zeta \in F(X)$, there is at most one morphism from (X_0, ζ_0) to (X, ζ) in $\mathcal{C}^{pt}$;*

(2) (Domination by connected objects). *For any $(X_i, \zeta_i) \in \mathcal{C}^{pt}$, $i = 1, \dots, r$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that $(X_0, \zeta_0) \geq (X_i, \zeta_i)$ in $\mathcal{C}^{pt}$, $i = 1, \dots, r$.*
In particular, for any $X \in \mathcal{C}$, there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that the evaluation

map:

$$\begin{array}{rccc} ev_{\zeta_0}: & \mathrm{Hom}_{\mathcal{C}}(X_0, X) & \tilde{\rightarrow} & F(X) \\ & u : X_0 \to X & \mapsto & F(u)(\zeta_0) \end{array}$$

is bijective.

(3) *(i) If $X_0 \in \mathcal{C}$ is connected then any morphism $u : X \to X_0$ in $\mathcal{C}$ is a strict epimorphism;*
(ii) If $u : X_0 \to X$ is a strict epimorphism in $\mathcal{C}$ and if X_0 is connected then X is also connected;
(iii) If $X_0 \in \mathcal{C}$ is connected then any endomorphism $u : X_0 \to X_0$ in $\mathcal{C}$ is automatically an automorphism.

Proof.

(1) It follows from axiom (1) that the equalizer $\ker(u_1, u_2) \xrightarrow{i} X$ of any two morphisms $u_i : X \to Y$, $i = 1, 2$ in $\mathcal{C}$ exists in $\mathcal{C}$. So, let $u_i : (X_0, \zeta_0) \to (X, \zeta)$ be two morphisms in $\mathcal{C}^{pt}$, $i = 1, 2$ and consider their equalizer $\ker(u_1, u_2) \xrightarrow{i} X_0$ in $\mathcal{C}$. From axiom (4), $F(\ker(u_1, u_2)) \xrightarrow{F(i)} F(X_0)$ is the equalizer of $F(u_i) : F(X_0) \to F(X)$, $i = 1, 2$ in $FSets$. But by assumption, $\zeta_0 \in \ker(F(u_1), F(u_2)) = F(\ker(u_1, u_2))$ so, in particular, $F(\ker(u_1, u_2)) \neq \emptyset$ and it follows from lemma 2.6 (2) that $\ker(u_1, u_2) \neq \emptyset_{\mathcal{C}}$. Since an equalizer is always a monomorphism, it follows then from proposition 3.1 that $i : \ker(u_1, u_2) \tilde{\rightarrow} X_0$ is an isomorphism that is, $u_1 = u_2$.

(2) Take $X_0 := X_1 \times \cdots \times X_r$, $\zeta_0 := (\zeta_1, \ldots, \zeta_r) \in F(X_1) \times \cdots \times F(X_r) = F(X_1 \times \cdots \times X_r)$ (by axiom (4)). The ith projection $pr_i : X_0 \to X_i$ then induces a morphism from (X_0, ζ_0) to (X_i, ζ_i) in $\mathcal{C}^{pt}$, $i = 1, \ldots, r$. So, it is enough to prove that for any $(X, \zeta) \in \mathcal{C}^{pt}$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with X_0 connected such that $(X_0, \zeta_0) \geq (X, \zeta)$ in $\mathcal{C}^{pt}$. If $X \in \mathcal{C}$ is connected then $Id : (X, \zeta) \to (X, \zeta)$ works. Else, write:

$$X = \bigsqcup_{i=1}^{r} X_i$$

as the coproduct of its connected components and let $i_{X_i} : X_i \hookrightarrow X$ denote the canonical monomorphism, $i = 1, \ldots, r$. Then, from axiom (2) one gets:

$$F(X) = \bigsqcup_{i=1}^{r} F(X_i)$$

hence, there exists a unique $1 \leq i \leq r$ such that $\zeta \in F(X_i)$ and $i_{X_i} : (X_i, \zeta) \hookrightarrow (X, \zeta)$ works.

(3) (i) It follows from axiom (3) that $u : X \to X_0$ factors as $X \xrightarrow{u'} X_0' \xrightarrow{u''} X_0' \sqcup X_0'' = X_0$, where u' is a strict epimorphism and u'' is a monomorphism inducing an isomorphism onto X_0'. Furthermore, $X \neq \emptyset_{\mathcal{C}}$ forces $X_0' \neq \emptyset_{\mathcal{C}}$ thus, since X_0 is connected, $X_0'' = \emptyset_{\mathcal{C}}$ hence $u'' : X_0' \tilde{\rightarrow} X_0$ is an isomorphism.
(ii) From axiom (6), it is enough to prove that $F(u) : F(X_0) \tilde{\rightarrow} F(X_0)$ is an isomorphism. But as $F(X_0)$ is finite, it is actually enough to prove that $F(u) : F(X_0) \twoheadrightarrow F(X_0)$ is an epimorphism. By axiom (3) write $u : X_0 \to X_0$ as $X_0 \xrightarrow{u'} X_0' \xrightarrow{u''} X_0 = X_0' \sqcup X_0''$ with $u' : X_0 \to X_0'$ a strict epimorphism and $u'' : X_0' \to X_0$ a monomorphism inducing an isomorphism onto X_0'. Since X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. The former implies $X_0 = \emptyset_{\mathcal{C}}$ and then the claim is straightforward. The latter implies $X_0 = X_0'$ thus $u'' : X_0' \to X_0$ is an isomorphism and $u : X_0 \to X_0$ is a strict epimorphism so the conclusion follows from axiom (4).
(iii) If $X_0 = \emptyset_{\mathcal{C}}$, the claim is straightforward. Else, write $X = X' \sqcup X''$ in $\mathcal{C}$ with $X' \neq \emptyset_{\mathcal{C}}$ and let $i_{X'} : X' \hookrightarrow X$ denote the canonical monomorphism. Fix $\zeta' \in F(X')$ and $\zeta_0 \in F(X_0)$ such that $F(u)(\zeta_0) = \zeta'$. From (2), there exist $(X_0', \zeta_0') \in \mathcal{C}^{pt}$ with X_0' connected and morphisms $p : (X_0', \zeta_0') \to (X_0, \zeta_0)$ and $q : (X_0', \zeta_0') \to (X', \zeta')$ in $\mathcal{C}^{pt}$. From (3) (i) the morphism $p : X_0' \to X_0$ is automatically a strict epimorphism, so $u \circ p : X_0' \to X$ is also a strict epimorphism. From (1), one has: $u \circ p = i_{X'} \circ q$. So $i_{X'} \circ q$ is a strict epimorphism and, in particular, $F(X) = F(X')$, which forces $F(X'') = \emptyset$ hence, $X'' = \emptyset_{\mathcal{C}}$ by lemma 2.6 (2). $\square$

3.2.2. *Galois objects.* It follows from proposition 3.2 (1) and (3) (iii) that for any connected object $X_0 \in \mathcal{C}$, $X_0 \neq \emptyset_{\mathcal{C}}$ and for any $\zeta_0 \in F(X_0)$, the evaluation map:

$$\begin{array}{cccc} ev_{\zeta_0}: & \mathrm{Aut}_{\mathcal{C}}(X_0) & \hookrightarrow & F(X_0) \\ & u : X_0 \tilde{\rightarrow} X_0 & \mapsto & F(u)(\zeta_0) \end{array}$$

is injective. A connected object X_0 in $\mathcal{C}$ is *Galois in* $\mathcal{C}$ if for any $\zeta_0 \in F(X_0)$ the evaluation map ev_{ζ_0} : $\mathrm{Aut}_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$ is bijective. This is equivalent to one of the following:

(1) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$;
(2) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$;
(3) $|\mathrm{Aut}_{\mathcal{C}}(X_0)| = |F(X_0)|$;
(4) $X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)$ is final in $\mathcal{C}$.

The equivalence of (1), (2) and (3) follows from the fact hat $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply on $F(X_0)$. It follows from lemma 2.6 (2) that (4) is equivalent to $F(X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)) = *$. But, from axiom (5), this is also equivalent to $F(X_0)/\mathrm{Aut}_{\mathcal{C}}(X_0) = *$, which is (1). Note that (4) shows that the notion of Galois object does not depend on the fibre functor $F : \mathcal{C} \to FSets$.

Proposition 3.3. (Galois closure) *For any $X \in \mathcal{C}$ connected, there exists $\hat{X} \in \mathcal{C}$ Galois dominating X in $\mathcal{C}$ and minimal among the Galois objects dominating X in $\mathcal{C}$.*

Proof. From lemma 3.2 (2) there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that the evaluation map $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \tilde{\rightarrow} F(X)$ is bijective. Write:

$$\mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{u_1, \dots, u_n\}$$

Set $\zeta_i := F(u_i)(\zeta_0)$, $i = 1, \dots, n$ and let $pr_i : X^n \to X$ denote the ith projection, $i = 1, \dots, n$. By the universal property of product, there exists a unique morphism $\pi := (u_1, \dots, u_n) : X_0 \to X^n$ such that $pr_i \circ \pi = u_i$, $i = 1, \dots, n$.

By axiom (3), one can decompose $\pi : X_0 \to X^n$ as $X_0 \xrightarrow{\pi'} \hat{X} \xrightarrow{\pi''} X^n = \hat{X} \sqcup \hat{X}'$ with π' a strict epimorphism and π'' a monomorphism inducing an isomorphism onto $\hat{X}$. We claim that $\hat{X}$ is Galois and is minimal for morphisms from Galois objects to X.

It follows from lemma 3.2 (3) (ii) that $\hat{X}$ is connected in $\mathcal{C}$. Set $\hat{\zeta}_0 := F(\pi')(\zeta_0) = (\zeta_1, \dots, \zeta_n) \in F(\hat{X})$; we are to prove that the evaluation map $ev_{\hat{\zeta}_0} : \mathrm{Aut}_{\mathcal{C}}(\hat{X}) \to F(\hat{X})$ is surjective that is, for any $\zeta \in F(\hat{X})$ there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$. From proposition 3.2 (2) there exists $(\tilde{X}_0, \tilde{\zeta}_0) \in \mathcal{C}^{pt}$ with $\tilde{X}_0 \in \mathcal{C}$ connected such that $(\tilde{X}_0, \tilde{\zeta}_0) \geq (X_0, \zeta_0)$ and $(\tilde{X}_0, \tilde{\zeta}_0) \geq (\hat{X}, \zeta)$, $\zeta \in F(\hat{X})$ in $\mathcal{C}^{pt}$. So, up to replacing (X_0, ζ_0) with $(\tilde{X}_0, \tilde{\zeta}_0)$, we may assume that there are morphisms $\rho_\zeta : (X_0, \zeta_0) \to (\hat{X}, \zeta)$ in $\mathcal{C}^{pt}$, $\zeta \in F(\hat{X})$. So, on the one hand, one can write $F(\omega)(\hat{\zeta}_0) = F(\omega \circ \pi')(\zeta_0)$ and, on the other hand, $\zeta = F(\rho_\zeta)(\zeta_0)$. But then, by lemma 3.2 (1), there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$ if and only if there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $\omega \circ \pi' = \rho_\zeta$. To prove the existence of such an ω observe that:

$$(*) \ \ \{pr_1 \circ \pi'' \circ \rho_\zeta, \dots, pr_n \circ \pi'' \circ \rho_\zeta\} = \{u_1, \dots, u_n\}.$$

Indeed, the inclusion $\subset$ is straightforward and to prove the converse inclusion $\supset$, it is enough to prove that the $pr_i \circ \pi'' \circ \rho_\zeta$, $1 \leq i \leq n$ are all distincts. But since $pr_i \circ \pi'' \circ \pi' = u_i \neq u_j = pr_j \circ \pi'' \circ \pi'$, $1 \leq i \neq j \leq n$ and $\pi' : X_0 \to \hat{X}$ is a strict epimorphism, $pr_i \circ \pi'' \neq pr_j \circ \pi''$, $1 \leq i \neq j \leq n$ as well. And, as X_0 is connected, $\rho_\zeta : X_0 \to \hat{X}$ is automatically a strict epimorphism hence $pr_i \circ \pi'' \circ \rho_\zeta \neq pr_j \circ \pi'' \circ \rho_\zeta$, $1 \leq i \neq j \leq n$. From $(*)$, there exists a permutation $\sigma \in \mathcal{S}_n$ such that $pr_{\sigma(i)} \circ \pi'' \circ \rho_\zeta = pr_i \circ \pi'' \circ \pi'$, $i = 1, \dots, n$ and from the universal property of product there exist a unique isomorphism $\sigma : X^n \tilde{\rightarrow} X^n$ such that $pr_i \circ \sigma = pr_{\sigma(i)}$, $i = 1, \dots, n$. Hence $pr_i \circ \pi'' \circ \pi' = pr_i \circ \sigma \circ \pi'' \circ \rho_\zeta$, $i = 1, \dots, n$, which forces $\pi'' \circ \pi' = \sigma \circ \pi'' \circ \rho_\zeta$. But, then, from the unicity of the decomposition in axiom (3), there exists an automorphism $\omega : \hat{X} \tilde{\rightarrow} \hat{X}$ satisfying $\sigma \circ \pi'' = \pi'' \circ \omega$ and $\omega \circ \pi' = \rho_\zeta$.

It remains to prove the minimality of $\hat{X}$. Let $Y \in \mathcal{C}$ Galois and $q : Y \to X$ a morphism in $\mathcal{C}$. Fix $\eta_i \in F(Y)$ such that $F(q)(\eta_i) = \zeta_i$, $i = 1, \dots, n$. Since $Y \in \mathcal{C}$ is Galois, there exists $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\eta_1) = \eta_i$, $i = 1, \dots, n$. This defines a unique morphism $\kappa := (q \circ \omega_1, \dots, q \circ \omega_n) : Y \to X^n$ such that $pr_i \circ \kappa = q \circ \omega_i$,

$i = 1, \dots, n$. By axiom (3), $\kappa : Y \to X^n$ factors as $Y \xrightarrow{\kappa'} Z' \xrightarrow{\pi''} X^n = Z' \sqcup Z''$ with π' a strict epimorphism in $\mathcal{C}$ and π'' a monomorphism inducing an isomorphism onto Z'. it follows from lemma 3.2 (3) (ii) that Z' is connected and $F(\kappa)(\eta_1) = (\zeta_1, \dots, \zeta_n) = \hat{\zeta}_0$ hence Z' is the connected component of $\hat{\zeta}_0$ in X^n that is $\hat{X}$. $\square$

In particular, $\hat{X}$ is unique up to isomorphism; it is called the *Galois closure of* X.

The following lemma will allow us to restrict to connected objects.

Let $X_0,\ X_1, \dots, X_r \in \mathcal{C}$ connected, set:

$$X := \bigsqcup_{i=1}^{r} X_i$$

and let $i_{X_i} : X_i \hookrightarrow X$ denote the canonical monomorphism, $i = 1, \dots, r$. One has a well-defined injective map:

$$\sqcup_{i=1}^{r} i_{X_i} \circ : \bigsqcup_{i=1}^{r} \mathrm{Hom}_{\mathcal{C}}(X_0, X_i) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X_0, X).$$

And, actually:

Lemma 3.4. *The map:*

$$\sqcup_{i=1}^{r} i_{X_i} \circ : \bigsqcup_{i=1}^{r} \mathrm{Hom}_{\mathcal{C}}(X_0, X_i) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(X_0, X)$$

is bijective

Proof. From axiom (3), any $u : X_0 \to X$ factors as $X_0 \xrightarrow{u'} X' \xrightarrow{u''} X = X' \sqcup X''$ with u' a strict epimorphism and u'' a monomorphism inducing an isomorphism onto X'. As X_0 is connected, it follows from lemma 3.2 (3) (ii) that X' is also connected, so X' is one of the connected component X_i, $i = 1, \dots, r$ of X. This shows that the above injective map is surjective hence bijective as claimed. $\square$

For any $X_0 \in \mathcal{C}$ Galois let $\mathcal{C}^{X_0} \subset \mathcal{C}$ denote the full subcategory whose objects are the $X \in \mathcal{C}$ such that X_0 dominates any connected component of X in $\mathcal{C}$. Write $F^{X_0} := F|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to FSets$ for the restriction of $F : \mathcal{C} \to FSets$ to $\mathcal{C}^{X_0}$. The next proposition is the "finite level" version of theorem 2.8 and can be regarded as the core of its proof.

Proposition 3.5. (Galois correspondence)

(1) Any $\zeta_0 \in F(X_0)$ induces a functor isomorphism:

$$ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} \tilde{\to} F^{X_0}.$$

In particular, this induces an isomorphism of groups:

$$u^{\zeta_0} : \mathrm{Aut}_{Fct}(F^{X_0}) \tilde{\to} \mathrm{Aut}_{Fct}(\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}}) = \mathrm{Aut}_{\mathcal{C}}(X_0)^{op}$$

(where the second equality is just Yoneda lemma) and which can be explicitly described:

$$u^{\zeta_0}(\theta) = ev_{\zeta_0}^{-1}(\theta(X_0)(\zeta_0)).$$

(2) The functor $F^{X_0} : \mathcal{C}^{X_0} \to FSets$ factors through an equivalence of categories:

$$\begin{tikzcd} \mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & FSets \\ \mathcal{C}(\mathrm{Aut}_{\mathcal{C}}(X_0)^{op}) \arrow[ur, "For"'] & \end{tikzcd}$$

Proof.

(1) For any morphism $u : Y \to X$ in $\mathcal{C}^{X_0}$, it follows from the fact that $F : \mathcal{C} \to FSets$ is a functor that the following diagram commutes:

$$\begin{tikzcd}
F(Y) \arrow[r, "F(u)"] & F(X) \\
\mathrm{Hom}_{\mathcal{C}}(X_0, Y) \arrow[u, "ev_{\zeta_0(Y)}"] \arrow[r, "u\circ"'] & \mathrm{Hom}_{\mathcal{C}}(X_0, X) \arrow[u, "ev_{\zeta_0(X)}"']
\end{tikzcd},$$

that is, $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} \tilde{\to} F^{X_0}$ is a functor morphism.

Also, since X_0 is connected, $ev_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \hookrightarrow F(X)$ is injective, $X \in \mathcal{C}^{X_0}$.

- If X is connected it follows from lemma 3.2 (3) (i) that any morphism $u : X_0 \to X$ in $\mathcal{C}$ is automatically a strict epimorphism. Write $F(X) = \{\zeta_1, \dots, \zeta_n\}$ and let $\zeta_{0i} \in F(X_0)$ such that $F(u)(\zeta_{0i}) = \zeta_i$, $i = 1, \dots, n$. Since $X_0 \in \mathcal{C}$ is Galois, there exists $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\zeta_0) = \zeta_{0i}$, $i = 1, \dots, n$, which proves that $ev_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \twoheadrightarrow F(X)$ is surjective hence bijective.

- If X is not connected, the conclusion follows from proposition 3.1, lemma 3.4 and axiom (5).

(2) For simplicity set $G := \mathrm{Aut}_{\mathcal{C}}(X_0)$. From (1), we can identify $F^{X_0} : \mathcal{C}^{X_0} \to FSets$ with:

$$\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to FSets,$$

over which G^{op} acts naturally *via* composition on the right, whence a factorization:

$$\begin{tikzcd}
\mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & FSets \\
\mathcal{C}(G^{op}) \arrow[ur, "For"'] &
\end{tikzcd}$$

We will write "$\circ$" for the composition law in G and "$\vee$" for the composition law in G^{op}. It remains to prove that $F^{X_0} : \mathcal{C}^{X_0} \to \mathcal{C}(G^{op})$ is an equivalence of categories.

- <u>F^{X_0} is essentially surjective</u>: Let $E \in \mathcal{C}(G^{op})$. By the same argument as in (1), one may assume that E is connected in $\mathcal{C}(G^{op})$ that is a transitive left G^{op}-set. Thus we get an epimorphism in G^{op}-Sets:

$$\begin{array}{cccc} p_e^0 : & G^{op} & \twoheadrightarrow & E \\ & \omega & \mapsto & \omega \cdot e \end{array}$$

Set $f_e := p_e^0 \circ ev_{\zeta_0}^{-1} : F(X_0) \twoheadrightarrow E$. Then, for any $s \in S_e := \mathrm{Stab}_{G^{op}}(e)$, and $\omega \in G$, one has:

$$\begin{aligned} f_e \circ F(s)(ev_{\zeta_0}(\omega)) &= p_e^0 \circ ev_{\zeta_0}^{-1} \circ ev_{\zeta_0}(s \circ \omega) \\ &= (s \circ \omega) \cdot e \\ &= (\omega \vee s) \cdot e \\ &= \omega \cdot (s \cdot e) \\ &= \omega \cdot e \\ &= f_e(ev_{\zeta_0}(\omega)). \end{aligned}$$

So, by the universal property of quotient, $f_e : F(X_0) \twoheadrightarrow E$ factors through:

$$\begin{tikzcd}
G^{op} \arrow[r, two heads, "ev_{\zeta_0}"] \arrow[dr, "p_e^0"'] & F(X_0) \arrow[r] \arrow[d, "f_e"] & F(X_0)/S_e \arrow[dl, "\overline{f}_e"] \\
& E &
\end{tikzcd}$$

But if $p_e : X_0 \to X_0/S_e$ denotes the categorical quotient of X_0 by $S_e \subset G$ assumed to exist by axiom (2), it follows from axiom (5) that $F(X_0) \twoheadrightarrow F(X_0)/S_e$ is $F(p_e) : F(X_0) \twoheadrightarrow F(X_0/S_e)$. Since X_0 is connected, G acts simply on $F(X_0)$ hence:

$$|F(X_0)/S_e| = |F(X_0)|/|S_e| = [G : S_e] = |E|.$$

So $\overline{f}_e : F(X_0)/S_e = F(X_0/S_e) \twoheadrightarrow E$ is actually an isomorphism in G^{op}-Sets.

- $\underline{F^{X_0} \text{ is fully faithfull}}$: Let $X,\ Y \in \mathcal{C}^{X_0}$. Again, by the same argument as in (1), one may assume that $X,\ Y$ are connected in $\mathcal{C}$. The faithfulness of F^{X_0} directly follows from proposition 3.2 (1). As for the fullness, for any morphism $u : F(X) \to F(Y)$ in $\mathcal{C}(G^{op})$, fix $e \in F(X)$. Since $u : F(X) \to F(Y)$ in a morphism in $\mathcal{C}(G^{op})$ one has $S_e \subset S_{u(e)}$ hence $p_{u(e)} : X_0 \to X_0/S_{u(e)}$ factors through:

$$\begin{tikzcd} X_0 \arrow[r, "p_e"] \arrow[d, "p_{u(e)}"'] & X_0/S_e \arrow[dl, "p_{e,u(e)}"] \\ X_0/S_{u(e)} & \end{tikzcd}$$

whence, from the proof of essential surjectivity, one gets the commutative diagram:

$$\begin{tikzcd} & F(X_0) \arrow[dl, "F(p_e)"'] \arrow[dr, "F(p_{u(e)})"] & \\ F(X_0/S_e) \arrow[rr, "F(p_{e,u(e)})"] \arrow[d, "\overline{f}_e"', "\simeq"] & & F(X_0/S_{u(e)}) \arrow[d, "\simeq"', "\overline{f}_{u(e)}"] \\ F(X) \arrow[rr, "u"'] & & F(Y). \end{tikzcd}$$

□

Exercise 3.6. Let $X_0 \in \mathcal{C}$ Galois and $X \in \mathcal{C}^{X_0}$ which, from proposition 3.5 can be identified with the quotient of X_0 by a subgroup $S_X \subset \mathrm{Aut}_{\mathcal{C}}(X_0)$. Show that X is Galois in $\mathcal{C}$ if and only if S_X is normal in $\mathrm{Aut}_{\mathcal{C}}(X_0)$ and that then, one has a short exact sequence of finite groups:

$$1 \to S_X \to \mathrm{Aut}_{\mathcal{C}}(X_0) \to \mathrm{Aut}_{\mathcal{C}}(X) \to 1.$$

3.3. **Strict pro-representability of $F : \mathcal{C} \to FSets$.** The category $Pro(\mathcal{C})$ associated with $\mathcal{C}$ is the category whose objects are projective systems $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j \in I,\ i \geq j}$ in $\mathcal{C}$ indexed by partially ordered filtrant sets $(I, \leq)$ and whose morphisms from $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j \in I,\ i \geq j}$ to $\underline{X}' = (\phi'_{i,j} : X'_i \to X'_j)_{i,j \in I',\ i \geq j}$ are:

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, \underline{X}') := \varprojlim_{i' \in I'} \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, X'_{i'}).$$

Note that $\mathcal{C}$ can be regarded canonically as a full subcategory of $Pro(\mathcal{C})$ and that $F : \mathcal{C} \to FSets$ extends canonically to a functor $Pro(F) : Pro(\mathcal{C}) \to Pro(FSets)$.

The functor $F : \mathcal{C} \to FSets$ is said to be *pro-representable in* $\mathcal{C}$ if there exists $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j \in I,\ i \geq j} \in Pro(\mathcal{C})$ and a functor isomorphism:

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, -)|_{\mathcal{C}} \tilde{\to} F$$

and it is said to be *strictly pro-representable in* $\mathcal{C}$ if it is pro-representable in $\mathcal{C}$ by an object $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j \in I,\ i \geq j} \in Pro(\mathcal{C})$ whose transition morphisms $\phi_{i,j} : X_i \twoheadrightarrow X_j$ are epimorphisms, $i, j \in I,\ i \geq j$.

3.3.1. *Projective structures on Galois objects.* Let $\mathcal{G}$ denote the set of all Galois objects (or more precisely, a system of representatives of the isomorphism classes of Galois objects) in $\mathcal{C}$. From proposition 3.2 (2) and proposition 3.3, $(\mathcal{G}, \leq)$ is directed. Fix $\underline{\zeta} = (\zeta_X)_{X \in \mathcal{G}} \in \prod_{X \in \mathcal{G}} F(X)$. Then, from proposition 3.2 (1), for any $X,\ Y \in \mathcal{G}$ with $X \leq Y$, there exists a unique $\phi^{\zeta}_{X,Y} : Y \to X$ in $\mathcal{C}$ such that $\phi^{\zeta}_{X,Y}(\zeta_Y) = \zeta_X$. The unicity of $\phi^{\zeta}_{X,Y} : Y \to X$ implies that for any $X,\ Y,\ Z \in \mathcal{G}$ with $X \leq Y \leq Z$ one has:

$$\phi^{\zeta}_{X,Y} \circ \phi^{\zeta}_{Y,Z} = \phi^{\zeta}_{X,Z}.$$

This endows $\mathcal{G}$ with a structure of projective system $\mathcal{G}^{\underline{\zeta}} := (\phi^{\zeta}_{X,Y} : Y \twoheadrightarrow X)_{X,\ Y \in \mathcal{G},\ X \leq Y}$ and one has:

Proposition 3.7. *The fibre functor $F : \mathcal{C} \to FSets$ is strictly pro-representable in $\mathcal{C}$ by $\mathcal{G}^{\underline{\zeta}}$. More precisely, the evaluation morphisms $ev_{\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}^X} \to F|_{\mathcal{C}^X}$, $X \in \mathcal{G}$ induce a functor isomorphism:*

$$ev_{\underline{\zeta}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}} \tilde{\to} F.$$

Proof. From propostion 3.2 (3) (i), the transition morphisms are automatically strict epimorphisms. The remaining part of the assertion follows directly from the construction and proposition 3.5. □

The projective structure $\mathcal{G}^{\underline{\zeta}}$ also induces a projective structure on the $\mathrm{Aut}_{\mathcal{C}}(X)$, $X \in \mathcal{G}$. More precisely, we have:

Lemma 3.8. *For any $X,\ Y \in \mathcal{G}$ with $X \leq Y$, for any morphisms $\phi,\ \psi : Y \to X$ in $\mathcal{C}$ and for any $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ there is a unique automorphisms $\omega_X := r_{\phi,\psi}(\omega_Y) : X \tilde{\to} X$ in $\mathcal{C}$ such that the following diagram commutes:*

$$\begin{tikzcd} Y \arrow[r, "\omega_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "\omega_X"'] & X. \end{tikzcd}$$

Proof. Since X is connected, $\psi : Y \to X$ is automatically a strict epimorphism and, in particular, the map:

$$\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

is injective. But it follows from proposition 3.5 that $|\mathrm{Hom}_{\mathcal{C}}(Y,X)| = |F(X)|$ and from the fact that X is Galois that $|F(X)| = |\mathrm{Aut}_{\mathcal{C}}(X)|$. As a result the map:

$$\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

is actually bijective and, in particular, there existe a unique automorphism $\omega_X : X \tilde{\to} X$ in $\mathcal{C}$ such that $\phi \circ \omega_Y = \omega_X \circ \psi$. □

So one gets a well-defined sujective map:

$$r_{\phi,\psi} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X),$$

which is automatically a group epimorphism when $\phi = \psi$. In particular, one gets a projective system of finite groups:

$$(r^{\underline{\zeta}}_{X,Y} := r_{\phi^{\underline{\zeta}}_{X,Y},\phi^{\underline{\zeta}}_{X,Y}} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X))_{X,Y \in \mathcal{G},\ X \leq Y}.$$

Set:

$$\Pi := \varprojlim \mathrm{Aut}_{\mathcal{C}}(X).$$

Then Π^{op} acts naturally on:

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}}$$

by composition on the right, which induces a group monomorphism:

$$\Pi^{op} \hookrightarrow \mathrm{Aut}_{Fct}(\varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}})$$

and the functor isomorphism

$$ev_{\underline{\zeta}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}} \tilde{\to} F$$

thus induces a group monomorphism:

$$\begin{array}{rccl} u^{\underline{\zeta}} : & \pi_1(\mathcal{C};F) & \hookrightarrow & \Pi^{op} \\ & \theta & \mapsto & (ev^{-1}_{\zeta_X}(\theta(X)(\zeta_X)))_{X \in \mathcal{G}} \end{array}$$

and, actually:

Proposition 3.9. *The group monomorphism $u^{\underline{\zeta}} : \pi_1(\mathcal{C};F) \to \Pi^{op}$ is an isomorphism of profinite groups.*

Proof. We first show that $u_{\underline{\zeta}}:\pi_1(\mathcal{C};F)\to\Pi^{op}$ is a group isomorphism by constructing an inverse. Let $\underline{\omega}:=(\omega_X)_{X\in\mathcal{G}}\in\Pi$. For any $Z\in\mathcal{C}$ connected, let $\hat{Z}$ denote the Galois closure of Z in $\mathcal{C}$ and consider the bijective map:

$$\theta_{\underline{\omega}}(Z):F(Z)\overset{ev_{\zeta_{\hat{Z}}}^{-1}}{\tilde{\to}}\mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z)\overset{\circ\omega_{\hat{Z}}}{\tilde{\to}}\mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z)\overset{ev_{\zeta_{\hat{Z}}}}{\tilde{\to}}F(Z).$$

One checks that this defines a functor automorphism and that $u^{\underline{\zeta}}(\theta_{\underline{\omega}})=\underline{\omega}$.

Next, we show that $u_{\underline{\zeta}}:\pi_1(\mathcal{C};F)\to\Pi^{op}$ is continuous. For this, it is enough to check that the:

$$\pi_1(\mathcal{C};F)\overset{u^{\underline{\zeta}}}{\to}\Pi^{op}\to\mathrm{Aut}_{\mathcal{C}}(X)^{op},\ X\in\mathcal{G}$$

are, which is straightforward by the definition of the topology on $\pi_1(\mathcal{C};F)$. Finally, since $\pi_1(\mathcal{C};F)$ is compact, $u_{\underline{\zeta}}^{-1}$ is continuous as well. □

3.3.2. *Conclusion.* We can now carry out the proof of theorem 2.8

(1) From proposition 3.7 and proposition 3.9, this amount to showing that:

$$F^{\underline{\zeta}}:\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{G}^{\underline{\zeta}},-)|_{\mathcal{C}}:\mathcal{C}\to FSets$$

factors through an equivalence of category $F^{\underline{\zeta}}:\mathcal{C}\to\mathcal{C}(\Pi^{op})$. But this follows almost straightforwardly from proposition 3.5. Indeed,

- $F^{\underline{\zeta}}$ is essentially surjective: For any $E\in\mathcal{C}(\Pi^{op})$ since E is equipped with the discrete topology, the action of Π^{op} on E factors through a finite quotient $\mathrm{Aut}_{\mathcal{C}}(X)$ with $X\in\mathcal{G}$ and we can apply proposition 3.5 in $\mathcal{C}^X$.

- $F^{\underline{\zeta}}$ is fully faithful: For any $Z,\ Z'\in\mathcal{C}$, there exists $X\in\mathcal{G}$ such that $X\geq Z$, $X\geq Z'$ and, again, this allows us to apply proposition 3.5 in $\mathcal{C}^X$.

(2) This immediately follows from proposition 3.7. Indeed, let $F_i:\mathcal{C}\to FSets$, $i=1,2$ be fibre functors. Then any $\underline{\zeta}^i\in\prod_{X\in\mathcal{G}}F^i(X)$ induces a functor isomorphism:

$$ev^{F_i}_{\underline{\zeta}^i}:\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{G}^{\underline{\zeta}^i},-)|_{\mathcal{C}}\tilde{\to}F_i.$$

So it is enough to prove that $\underline{\mathcal{G}}^{\underline{\zeta}^1}$ and $\underline{\mathcal{G}}^{\underline{\zeta}^2}$ are isomorphic in $Pro(\mathcal{C})$. But one has:

$$\varprojlim_X\varinjlim_Y\mathrm{Hom}_{\mathcal{C}}(Y,X)=\varprojlim_X\varinjlim_Y\mathrm{Aut}_{\mathcal{C}}(X)=\varprojlim_X\mathrm{Aut}_{\mathcal{C}}(X)\ ,$$

where the first equality follows from proposition 3.5 (1). So it is actually enough to prove that

$$\varprojlim\mathrm{Aut}_{\mathcal{C}}(X)\neq\emptyset,$$

where the structure of projective system on the $\mathrm{Aut}_{\mathcal{C}}(X)$, $X\in\mathcal{G}$ is given by the surjective maps defined in lemma 3.8:

$$r_{\phi^1_{X,Y},\phi^2_{X,Y}}:\mathrm{Aut}_{\mathcal{C}}(Y)\twoheadrightarrow\mathrm{Aut}_{\mathcal{C}}(X),\ X,\ Y\in\mathcal{G},\ X\leq Y.$$

And this follows from the fact that a projective system of non-empty finite sets is non-empty. □

4. Fundamental functors and functoriality

4.1. **Fundamental functors.** Let $\mathcal{C}$, $\mathcal{C}'$ be two Galois categories. Then a covariant functor $H:\mathcal{C}\to\mathcal{C}'$ is a *fundamental* (or *exact*, according to the terminology of [SGA1]) functor from $\mathcal{C}$ to $\mathcal{C}'$ if there exists a fibre functor $F':\mathcal{C}'\to FSets$ for $\mathcal{C}'$ such that $F'\circ H:\mathcal{C}\to FSets$ is again a fibre functor for $\mathcal{C}$ or, equivalently (since, from theorem 2.8 (2), two fibre functors are always isomorphic), if for any fibre functor $F':\mathcal{C}'\to FSets$ for $\mathcal{C}'$ the functor $F'\circ H:\mathcal{C}\to FSets$ is again a fibre functor for $\mathcal{C}$.

Let $u:\Pi'\to\Pi$ be a morphism of profinite groups. Then any $E\in\mathcal{C}(\Pi)$ can be endowed with a continuous action of Π' *via* $u:\Pi'\to\Pi$, which defines a canonical fundamental functor:

$$H_u:\mathcal{C}(\Pi)\to\mathcal{C}(\Pi').$$

Conversely, let $H : \mathcal{C} \to \mathcal{C}'$ be a fundamental functor. Let $F' : \mathcal{C}' \to FSets$ be a fibre functor for $\mathcal{C}'$ and set $F := F' \circ H : \mathcal{C} \to FSets$, $\Pi := \pi_1(\mathcal{C}; F)$, $\Pi' := \pi_1(\mathcal{C}'; F')$. Then for any $\Theta' \in \Pi'$, one has $\Theta' \circ H \in \Pi$, which defines a canonical morphism of profinite groups:

$$u_H : \Pi' \to \Pi.$$

One checks that $u_{H_u} = u$ and that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(\Pi) \arrow[r, "H_{u_H}"] & \mathcal{C}(\Pi') \\ \mathcal{C} \arrow[u, "F"] \arrow[r, "H"'] & \mathcal{C}'. \arrow[u, "F'"'] \end{tikzcd}$$

Furthermore, given a fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$ and two fundamental functors $H_1, H_2 : \mathcal{C} \to \mathcal{C}'$ such that $F' \circ H_1 = F' \circ H_2 =: F$, any morphism of functors $\alpha : H_1 \to H_2$ induces an endomorphism of functor $u_\alpha : F \to F$ such that:

$$u_\alpha \circ u_{H_1}(\theta') = u_{H_2}(\theta') \circ u_\alpha,\ \theta' \in \Pi'.$$

Thus, one the one hand, let *Gal* denote the 2-category whose objects are Galois categories pointed with fibre functors and where 1-morphisms from $(\mathcal{C}; F)$ to $(\mathcal{C}'; F')$ are fundamental functors $H : \mathcal{C} \to \mathcal{C}'$ such that $F' \circ H = F$ and 2-morphisms are isomorphisms between fundamental functors. And, on the other hand, let *Pro* denote the 2-category whose objects are profinite groups and where 1-morphisms are morphisms of profinite groups and 2-morphisms from $u_1 : \Pi' \to \Pi$ to $u_2 : \Pi' \to \Pi$ are elements $\theta \in \Pi$ such that $\theta \circ u_1(-) \circ \theta^{-1} = u_2$. Then, the functor $(\mathcal{C}, F) \to \pi_1(\mathcal{C}; F)$ from *Gal* to *Pro* is an equivalence of 2-categories with pseudo-inverse $\Pi \to (\mathcal{C}(\Pi), For)$. In the next subsection, we compare the properties of the fundamental functor $H : \mathcal{C} \to \mathcal{C}'$ and of the corresponding morphism of profinite groups $u : \Pi' \to \Pi$.

Example 4.1. Any continuous map $\phi : B' \to B$ of connected, locally arcwise connected and locally simply connected topological spaces defines a canonical functor:

$$\begin{array}{rlcl} H: & \mathcal{C}_B^{top} & \to & \mathcal{C}_{B'}^{top} \\ & f : X \to B & \mapsto & p_2 : X \times_{f,B,\phi} B' \to B'. \end{array}$$

and for any $b' \in B'$, one has:

$$\begin{aligned} F_{b'} \circ H(f) &= p_2^{-1}(b') \\ &= \{(x, b') \mid x \in X \text{ such that } f(x) = \phi(b')\} \\ &= f^{-1}(\phi(b')). \end{aligned}$$

Hence $H : \mathcal{C}_B \to \mathcal{C}_{B'}$ is a fundamental functor. In that case, the corresponding morphism of profinite groups is just the canonical morphism:

$$\hat{\phi} : \widehat{\pi_1^{top}(B'; b')} \to \widehat{\pi_1^{top}(B; \phi(b'))}$$

induced from $\phi : \pi_1^{top}(B'; b') \to \pi_1^{top}(B; \phi(b'))$.

4.2. **Functoriality.** From subsection 4.1, one may assume that $\mathcal{C} = \mathcal{C}(\Pi)$, $\mathcal{C}' = \mathcal{C}(\Pi')$ and $H = H_u$ for some morphism of profinite groups $u : \Pi' \to \Pi$.

Given $(X, \zeta) \in \mathcal{C}^{pt}$, we will write $(X, \zeta)_0 := (X_0, \zeta)$, where X_0 denotes the connected component of ζ in X.

We will say that an object $X \in \mathcal{C}$ *has a section in* $\mathcal{C}$ if $e_{\mathcal{C}} \geq X$ and that an object $X \in \mathcal{C}$ is *totally split in* $\mathcal{C}$ if it is isomorphic to a finite coproduct of final objects.

Lemma 4.2. *With the above notation:*

(1) *For any open subgroup* $U \subset \Pi$,
   - $\mathrm{im}(u) \subset U$ *if and only if* $(e_{\mathcal{C}'}, *) \geq (H(\Pi/U), 1))$ *in* $\mathcal{C}'^{pt}$*;*
   - *Let:*

$$\mathrm{K}_\Pi(\mathrm{im}(u)) \lhd \Pi$$

   denote the smallest normal subgroup in Π *containing* $\mathrm{im}(u)$*. Then* $\mathrm{K}_\Pi(\mathrm{im}(u)) \subset U$ *if and only if* $H(\Pi/U)$ *is totally split in* $\mathcal{C}'$.

In particular, $u:\Pi'\to\Pi$ *is trivial if and only if for any object* X *in* $\mathcal{C}$, $H(X)$ *is totally split in* $\mathcal{C}'$.

(2) *For any open subgroup* $U'\subset\Pi'$,
 - $\ker(u)\subset U'$ *if and only if there exists an open subgroup* $U\subset\Pi$ *such that:* $(H(\Pi/U),1)_0\geq(\Pi'/U',1)$ *in* $\mathcal{C}'^{pt}$.
 - *if, furthermore,* $u:\Pi'\twoheadrightarrow\Pi$ *is an epimorphism, then* $\mathrm{Ker}(u)\subset U'$ *if and only if there exists an open subgroup* $U\subset\Pi$ *and an isomorphism* $(H(\Pi/U),1)_0\tilde{\to}(\Pi'/U',1)$ *in* $\mathcal{C}'^{pt}$.

 In particular,
 - $u:\Pi'\hookrightarrow\Pi$ *is a monomorphism if and only if for any connected object* $X'\in\mathcal{C}'$ *there exists a connected object* $X\in\mathcal{C}$ *and a connected component* $H(X)_0$ *of* $H(X)$ *in* $\mathcal{C}$ *such that* $H(X)_0\geq X'$ *in* $\mathcal{C}'$.
 - *if, furthermore,* $u:\Pi'\twoheadrightarrow\Pi$ *is an epimorphism, then* $u:\Pi'\twoheadrightarrow\Pi$ *is an isomorphism if and only if* $H:\mathcal{C}\to\mathcal{C}'$ *is essentially surjective.*

Proof. Recall that, given a profinite group Π, a closed subgroup $S\subset\Pi$ is the intersection of all the open subgroups $U\subset\Pi$ containing S thus, in particular, $\{1\}$ is the intersection of all open subgroups of Π. This yields the characterization of trivial morphisms and monomorphisms from the preceding assertions in (1) and (2).

(1) For the first assertion of (1), note that $e_{\mathcal{C}'}=*$ and that $(e_{\mathcal{C}'},*),\geq(H(\Pi/U),1))$ in $\mathcal{C}'^{pt}$ if and only if the unique map $\phi:*\to H(\Pi/U)$ sending $*$ to U is a morphism in $\mathcal{C}'$ that is, if and only if for any $\theta'\in\Pi'$,

$$U=\phi(*)=\phi(\theta'\cdot *)=\theta'\cdot\phi(*)=u(\theta')U.$$

For the second assertion of (1), note that $\mathrm{K}_\Pi(\mathrm{Im}(u))\subset U$ if and only if for any $g\in\Pi/U$, the map $\phi_g:*\to H(\Pi/U)$ sending $*$ to gU is a morphism in $\mathcal{C}'$. This yields a surjective morphism $\sqcup_{g\in\Pi/U}\phi_g:\sqcup_{g\in\Pi/U}*\to H(\Pi/U)$ in $\mathcal{C}'$, which is automatically injective by cardinality. Conversely, for any isomorphism $\sqcup_{i\in I}\phi_i:\sqcup_{i\in I}*\to H(\Pi/U)$ in $\mathcal{C}'$, set $i_i:*\to H(\Pi/U)$ for the morphism $*\hookrightarrow\sqcup_{i\in I}*\to H(\Pi/U)$ in $\mathcal{C}'$; by construction $i_i=\phi_{i_i(*)}$.

(2) Since U' is closed of finite index in Π' and both Π and Π' are compact, $u(U')$ is closed of finite index in $\mathrm{im}(u)$ hence open. So there exists an open subgroup $U\subset\Pi$ such that $U\cap\mathrm{im}(u)\subset u(U')$. By definition, the connected component of 1 in $H(\Pi/U)$ in $\mathcal{C}'$ is:

$$\mathrm{im}(u)U/U\simeq\mathrm{im}(u)/(U\cap\mathrm{im}(u))\simeq\Pi'/u^{-1}(U).$$

But $u^{-1}(U)=u^{-1}(U\cap\mathrm{Im}(u))\subset U'$, whence a canonical epimorphism $(\mathrm{Im}(u)U/U,1)\to(\Pi'/U',1)$ in $\mathcal{C}'^{pt}$. If, furthermore, $\mathrm{im}(u)=\Pi$, then one can take $U=u(U')$ and ϕ is nothing but the inverse of the canonical isomorphism $\Pi'/U'\tilde{\to}\Pi/U$. Conversely, assume that there exists an open subgroup $U\subset\Pi$ and a morphism $\phi:(\mathrm{Im}(u)U/U,1)\to(\Pi'/U',1)$ in $\mathcal{C}'^{pt}$. Then, for any $g'\in\Pi'$, one has: $\phi(u(g')U)=g'\cdot\phi(1)=g'U'$. In particular, if $u(g')\in U$ then $g'U=\phi(u(g')U)=\phi(U)=U'$ whence $\ker(u)\subset u^{-1}(U)\subset U'$. Eventually, note that since $\ker(u)$ is normal in Π', the condition $\ker(u)\subset U'$ does not depend on the choice of $\zeta\in F(X)$ defining the isomorphism $X'\tilde{\to}\Pi'/U'$. □

Proposition 4.3.

(1) *The following three assertions are equivalent:*
 (i) $u:\Pi'\twoheadrightarrow\Pi$ *is an epimorphism;*
 (ii) $H:\mathcal{C}\to\mathcal{C}'$ *sends connected objects to connected objects;*
 (iii) $H:\mathcal{C}\to\mathcal{C}'$ *is fully faithful.*

(2) $u:\Pi'\hookrightarrow\Pi$ *is a monomorphism if and only if for any object* X' *in* $\mathcal{C}'$ *there exists an object* X *in* $\mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *which dominates* X' *in* $\mathcal{C}'$.

(3) $u:\Pi'\tilde{\to}\Pi$ *is an isomorphism if and only if* $H:\mathcal{C}\to\mathcal{C}'$ *is an equivalence of categories.*

(4) *If* $\mathcal{C}\overset{H}{\to}\mathcal{C}'\overset{H'}{\to}\mathcal{C}''$ *is a sequence of fundamental functors of Galois categories with corresponding sequence of profinite groups* $\Pi\overset{u}{\leftarrow}\Pi'\overset{u'}{\leftarrow}\Pi''$. *Then,*
 - $\ker(u)\supset\mathrm{im}(u')$ *if and only if* $H'(H(X))$ *is totally split in* $\mathcal{C}''$, $X\in\mathcal{C}$;

- $\ker(u) \subset \mathrm{im}(u')$ *if and only if for any connected object* $X' \in \mathcal{C}'$ *such that* $H'(X')$ *has a section in* $\mathcal{C}''$*, there exists* $X \in \mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *which dominates* X' *in* $\mathcal{C}'$*.*

Proof. Assertion (2) and (4) follow from lemma 4.2 (2). Assertions (3) follows from lemma 4.2 and (1). So we are only to prove assertion (1). We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). For (i) $\Rightarrow$ (ii), assume that $u : \Pi' \twoheadrightarrow \Pi$ is an epimorphism. Then, for any connected object X in $\mathcal{C}(\Pi)$, the group Π acts transitively on X. But $H(X)$ is just X equipped with the Π'-action $g' \cdot x = u(g') \cdot x$, $g' \in \Pi'$. Hence Π' acts transitively on $H(X)$ as well or, equivalently, $H(X)$ is connected. For (ii) $\Rightarrow$ (i), assume that if $X \in \mathcal{C}$ is connected then $H(X)$ is also connected in $\mathcal{C}'$. This holds, in particular, for any finite quotient Π/N of Π with N a normal open subgroup of Π that is, the canonical morphism $u_N : \Pi' \xrightarrow{u} \Pi \overset{pr_N}{\twoheadrightarrow} \Pi/N$ is a continuous epimorphism. Hence so is $u = \underleftarrow{\lim} u_N$. The implication $\Rightarrow$ (iii) is straightforward. Finally, for (iii) $\Rightarrow$ (i), observe that given an open subgroup $U \subset \Pi$, $U \neq \Pi$ there is no morphism from $*$ to Π/U in $\mathcal{C}$. Hence, if $H : \mathcal{C} \to \mathcal{C}'$ is fully (faithful), there is no morphism as well from $*$ to $H(\Pi/U)$ in $\mathcal{C}'$. But, from lemma 4.2, this is equivalent to $\mathrm{im}(u) \not\subset U$. □

Exercise 4.4. Given a Galois category $\mathcal{C}$ with fibre functor $F : \mathcal{C} \to FSets$ and $X_0 \in \mathcal{C}$ connected, let $\mathcal{C}_{X_0}$ denote the *category of* X_0*-objects* that is the category whose objects are morphism $\phi : X \to X_0$ in $\mathcal{C}$ and whose morphisms from $\phi' : X' \to X_0$ to $\phi : X \to X_0$ are the morphisms $\psi : X' \to X$ in $\mathcal{C}$ such that $\phi \circ \psi = \phi'$. For any $\zeta \in F(X_0)$, set

$$F_{(X_0,\zeta)} : \begin{array}{lcl} \mathcal{C}_{X_0} & \to & FSets \\ \phi : X \to X_0 & \mapsto & F(\phi)^{-1}(\zeta). \end{array}$$

Then,

(1) show that $\mathcal{C}_{X_0}$ is Galois with fibre functors $F_{(X_0,\zeta)} : \mathcal{C}_{X_0} \to FSets$, $\zeta \in F(X_0)$ and that, furthermore, the canonical functor

$$H : \begin{array}{lcl} \mathcal{C} & \to & \mathcal{C}_{X_0} \\ X & \mapsto & p_2 : X \times X_0 \to X_0 \end{array}$$

has the property that $F_{(X_0,\zeta)} \circ H = F$, $\zeta \in F(X_0)$ and induces a profinite group monomorphism: $\pi_1(\mathcal{C}_{X_0}; F_{(X_0,\zeta)}) \hookrightarrow \pi_1(\mathcal{C}; F)$ with image $\mathrm{Stab}_{\pi_1(\mathcal{C};F)}(\zeta)$;

(2) show that $H(\hat{X}_0)$ is totally split in $\mathcal{C}_{X_0}$ and that if X_0 is the Galois closure $\hat{X}$ of some connected object $X \in \mathcal{C}$ then $H(X)$ is totally split in $\mathcal{C}_{\hat{X}}$.

5. Etale Covers

The aim of this section is to prove that the category of finite étale covers of a connected scheme is Galois (see theorem 5.10). The proof of this result is carried out in subsection 5.3. In subsections 5.1 and 5.2, we introduce the notion of étale covers and give some of their elementary properties.

Convention: All the schemes are *locally noetherian.* We make this hypothesis for simplicity and will not repeat it later. For instance, it will sometimes be used explicitly in the proofs but not mentioned in the corresponding statement. Be aware that some results stated in the following sections remain valid without the noetherianity assumptions but *some do not.*

5.1. **Etale algebras.** Given a ring R, let Alg/R denote the category of R-algebras. Also, given a ring R, we write $R^\times$ for the group of invertible elements in R.

Lemma 5.1. *Let A be a finite dimensional algebra over a field k. Then the following properties are equivalent:*

(1) A is isomorphic (as k-algebra) to a finite product of finite separable field extensions of k;
(2) $A \otimes_k \overline{k}$ is isomorphic (as $\overline{k}$-algebra) to a finite product of copies of $\overline{k}$;
(3) $A \otimes_k \overline{k}$ is reduced;
(4) $\Omega_{A|k} = 0$.

Proof. We first prove that a finite dimensional algebra A over a field k is reduced if and only if it is isomorphic (as k-algebra) to a finite product of finite field extensions of k. The 'if' part is straightforward. As for the 'only if' part, write $A = \prod_{i=1}^r A_i$ as the finite product of its connected components. Since it is enough to prove that A_i is (as k-algebra) a finite field extension of k *i.e.* that $A_i \setminus \{0\} = A_i^\times$, $i = 1, \dots, r$, we may assume that A is a finite dimensional connected algebra over k. Let $a \in A \setminus \{0\}$. Since A is finite dimensional over k, it is artinian hence $Aa^n = Aa^{n+1}$ for $n \gg 0$. In particular, there exists $b \in A$ such that $a^n = ba^{n+1} = ba^n a = b^2 a^{n+2} = \cdots = b^n a^{2n}$ hence $a^n b^n = (a^n b^n)^2$, which forces $a^n b^n = 0$ or 1 since A has no non-trivial idempotent. But $a^n b^n = 0$ would imply $a^n = (a^n b^n) a^n = 0$, which is impossible since $a \neq 0$ and A is reduced. Hence $a(a^{n-1} b^n) = a^n b^n = 1$ so $a \in A^\times$. This proves that A is a field and, as it is also finite dimensional over k, it is a finite field extension of k. This already proves (2) $\Leftrightarrow$ (3). We are going to prove (2) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (1): Set $\overline{A} := A/\sqrt{0}$. Then $\overline{A}$ is reduced hence, from the above, is isomorphic (as k-algebra) to $\prod_{i=1}^r K_i$ with K_i a finite field extension of k, $i = 1, \dots, r$. Now, any morphism $A \to \overline{k}$ of k-algebras factors through one of the K_i hence

$$N := |\mathrm{Hom}_{Alg/k}(A, \overline{k})| = \sum_{i=1}^r |\mathrm{Hom}_{Alg/k}(K_i, \overline{k})|.$$

Since:

$$|\mathrm{Hom}_{Alg/k}(K_i, \overline{k})| \leq [K_i : k]$$

with equality if and only if K_i is a finite separable field extension of k and

$$\dim_k(\overline{A}) = \sum_{i=1}^r [K_i : k] \leq \dim_k(A),$$

one has $N \leq \dim_k(A)$ and $N = \dim_k(A)$ if and only if $A = \overline{A}$ and:

$$|\mathrm{Hom}_{Alg/k}(K_i, \overline{k})| = [K_i : k],\ i = 1, \dots r$$

that is, if and only if $A = \overline{A}$ and K_i is a finite separable field extension of k, $i = 1, \dots, r$. But the universal property of tensor product implies that:

$$\mathrm{Hom}_{Alg/k}(A, \overline{k}) = \mathrm{Hom}_{Alg/\overline{k}}(A \otimes_k \overline{k}, \overline{k})$$

hence:

$$N = |\mathrm{Hom}_{Alg/\overline{k}}(A \otimes_k \overline{k}, \overline{k})| = \dim_{\overline{k}}(A \otimes_k \overline{k}) = \dim_k(A).$$

(1) $\Rightarrow$ (4): Write:

$$A = \prod_{i=1}^r K_i$$

as a finite product of finite separable field extensions of k. Then the maximal ideals of A are the kernel of the projection maps $\mathfrak{m}_i := \ker(A \twoheadrightarrow K_i)$, $i = 1, \dots, r$ and $\Omega^1_{A|k} = 0$ if and only if $(\Omega^1_{A|k})_{\mathfrak{m}_i} = \Omega_{K_i|k} = 0$, $i = 1, \dots, r$. Hence, one can assume that $A = K$ is a finite separable field extension of k. But, then, by the primitive element theorem, $K = k[X]/P$ for some irreducible separable polynomial $P \in k[X]$ hence $\Omega^1_{K|k} = KdT/P'(t)dT$ (where t denotes the image of X in k) with $P'(t) \neq 0$ since P is separable.

(4) $\Rightarrow$ (3): $\Omega_{A|k} = 0$ implies that $\Omega_{A \otimes_k \overline{k}|\overline{k}} = \Omega_{A|k} \otimes_k \overline{k} = 0$. So, one may assume that $k = \overline{k}$ is algebraically closed. Since A is Artinian any prime ideal is maximal and $|\mathrm{spec}(A)| < +\infty$. Write $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ for the finitely many prime (=maximal) ideals of A. Then, by the Chinese remainder theorem, one has the short exact sequence of A-modules:

$$0 \to \sqrt{0} \to A \xrightarrow{\phi} \prod_{i=1}^r A/\mathfrak{m}_i \to 1.$$

As $[A/\mathfrak{m}_i : k] < +\infty$ and k is algebraically closed, one actually has $A/\mathfrak{m}_i = k$, $i = 1, \dots, r$. Let $e_i \in A$, $i = 1, \dots r$ such that (i) $\phi(e_i) = (\delta_{i,j})_{1 \leq j \leq r}$, $i = 1, \dots, r$, (ii) $e_i e_j \in (\sqrt{0})^2$, $1 \leq i \neq j \leq r$ and (iii) $e_i - e_i^2 \in (\sqrt{0})^2$, $i = 1, \dots, r$. Such a r-tuple can always be constructed. Indeed, start from $e_i \in A$, $i = 1, \dots, r$ satisfying (i); then the e_i^2, $i = 1, \dots, r$ satisfy (i) and (ii). Also, as A is artinian and thus, for all $i = 1, \dots, r$ the chain of ideals:

$$\langle e_i \rangle \supset \langle e_i^2 \rangle \supset \cdots$$

stabilizes, we can find $n \geq 1$ and $a_i \in A$ such that for all $i = 1, \dots, r$ one has:

$$a_i e_i^{2n} = e_i^n.$$

We set $\epsilon_i := (a_i e_i^n)^2 (= a_i e_i^n)$. Then $\phi(\epsilon_i) = \delta_{ij}$, $\epsilon_i\epsilon_j \in (\sqrt{0})^2$ for $1 \leq i \neq j \leq r$ and:

$$\epsilon_i^2 = (a_i e_i^n)^2 = a_i(a_i e_i^{2n}) = a_i e_i^n = \epsilon_i.$$

Hence the ϵ_i, $i = 1, \dots, r$ satisfy (i), (ii), (iii). Let $\lambda_i : A \to A/\mathfrak{m}_i$ denote the ith component of ϕ and, for every $a \in A$, define $\lambda(a) := \sum_{i=1}^r \lambda_i(a)e_i$. Then, by definition, $a - \underline{\lambda}(a) \in \sqrt{0}$, $a \in A$ and one can check that the following map:

$$\begin{array}{rccl} d: & A & \to & \sqrt{0}/(\sqrt{0})^2 \\ & a & \to & (a - \underline{\lambda}(a)) mod (\sqrt{0})^2 \end{array}$$

defines a k-derivation hence is 0 by assumption, which forces $\sqrt{0} = (\sqrt{0})^2$. But, as A is a artinian, $\sqrt{0}$ is nilpotent hence $\sqrt{0} = (\sqrt{0})^2$ implies $\sqrt{0} = 0$ that is $A = \overline{A}$. $\square$

A finite dimensional algebra A over a field k satisfying the equivalent properties of lemma 5.1 is said to be *étale over* k. We will write $FEAlg/k \subset Alg/k$ for the full subcategory of finite étale algebras over k.

5.2. **Etale covers.** Let Sch denote the category of schemes and, given a scheme S, let Sch/S denote the category of S-schemes.

Given a scheme S, we will write $\mathcal{O}_S$ for its structural sheaf and, given a point $s \in S$, we will write $\mathcal{O}_{S,s}$, $\mathfrak{m}_s$ and $k(s)$ for the local ring, maximal ideal and residue field at s respectively. Also, we will write $\overline{s}$ for any geometric point associated with s, that is any morphism $\overline{s} : \mathrm{spec}(\Omega) \to S$ with image s and such that Ω is an algebraically closed field.

A morphism $\phi : X \to S$ that is locally of finite type is *unramified at* $x \in X$ if $\mathfrak{m}_{\phi(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)$ is a finite separable extension of $k(\phi(x))$ (or, equivalently, if $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(\phi(x))$ is a finite separable field extension of $k(s)$) and it is *unramified* if it is unramified at all $x \in X$. A morphism $\phi : X \to S$ that is locally of finite type is *étale at* $x \in X$ if $\phi : X \to S$ is both flat and unramified at $x \in X$ and it is *étale* if it is étale at all $x \in X$. A morphism $\phi : X \to S$ is an *étale cover of* S if it is finite, surjective and étale.

We will often use the following characterization of finite flat morphisms and finite unramified morphisms respectively. Recall that, given a finite morphism $\phi : X \to S$, the $\mathcal{O}_S$-module $\phi_*\mathcal{O}_X$ is coherent.

Lemma 5.2. *Let $\phi : X \to S$ be a finite morphism. Then,*

1. *$\phi : X \to S$ is flat if and only if $\phi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module;*
2. *The following properties are equivalent:*
   - (a) *$\phi : X \to S$ is unramified;*
   - (b) *$\Omega^1_{X|S} = 0$;*
   - (c) *$\Delta_{X|S} : X \to X \times_S X$ is an open immersion (hence induces an isomorphism onto an open and closed subscheme of $X \times_S X$).*
   - (d) *$(\phi_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) = \mathcal{O}_{X_s}(X_s)$ is a finite étale algebra over $\kappa(s)$, $s \in S$;*

Proof.

1. As the question is local on X we may assume that $\phi : X \to S$ is induced by a finite, flat A-algebra $\phi^\# : A \to B$ with A noetherian. Then B is a flat A-module if and only if $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, $\mathfrak{p} \in S$. But as $A_\mathfrak{p}$ is a local noetherian ring and $B_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module if and only if $B_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. To conclude, for each $\mathfrak{p} \in S$, write:

$$B_\mathfrak{p} = \bigoplus_{i=1}^r A_\mathfrak{p} \frac{b_i}{s},$$

where $s \in A \setminus \mathfrak{p}$. This defines an exact sequence of A_s-modules:

$$0 \to K \to A_s^r \stackrel{(\frac{b_1}{s}, \dots \frac{b_r}{s})}{\to} B_s \to Q \to 0.$$

As A_s is noetherian, K is a finitely generated A_s-module hence its support $\mathrm{supp}(K)$ is the closed subset $V(\mathrm{Ann}(K)) \subset \mathrm{spec}(A_s)$. Similarly, as B_s is a finitely generated A_s-module, Q is a finitely generated

A_s-module as well hence with closed support $\mathrm{supp}(Q) = V(\mathrm{Ann}(Q)) \subset \mathrm{spec}(A_s)$. But, by definition of the support, $U_{\mathfrak{p}} := \mathrm{supp}(K) \cap \mathrm{supp}(Q)$ is an open neighbourhood of $\mathfrak{p}$ in S such that:

$$\phi_* \mathcal{O}_X|_{U_{\mathfrak{p}}} \simeq \mathcal{O}_{U_{\mathfrak{p}}}.$$

This shows that if $\phi : X \to S$ is flat then $\phi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module. The converse implication is straightforward.

(2) We prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Since $\Omega^1_{X|S} = 0$ if and only if $\Omega_{X|S,x} = 0$, $x \in X$, one may again assume that $\phi : X \to S$ is induced by a finite A-algebra $\phi^\# : A \to B$ with A noetherian. Also, as $\Omega^1_{B|A}$ is a finitely generated B-module, by Nakayama lemma, it is enough to show that:

$$\Omega^1_{B|A} \otimes_B k(\mathfrak{q}) = 0,\ \mathfrak{q} \in X.$$

But it follows from the fact that $f : X \to S$ is unramified that for any $\mathfrak{q} \in X$ above $\mathfrak{p} \in S$ one has:

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = k(\mathfrak{q}).$$

Whence:

$$\begin{aligned} \Omega^1_{B|A} \otimes_B k(\mathfrak{q}) &= \Omega^1_{B|A} \otimes_A k(\mathfrak{p}) \\ &= \Omega_{B \otimes_A k(\mathfrak{p})|k(\mathfrak{p})} \\ &= \Omega_{k(\mathfrak{q})|k(\mathfrak{p})} \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{q})$ is a finite separable field extension.

(b) $\Rightarrow$ (c): As $\phi : X \to S$ is separated, the diagonal morphism $\Delta_{X|S} : X \to X \times_S X$ is a closed immersion and, in particular:

$$\Delta_{X|S}(X) = \mathrm{supp}(\Delta_{X|S*}\mathcal{O}_X).$$

Let:

$$\mathcal{I} := \mathrm{Ker}(\Delta^\#_{X|S} : \mathcal{O}_{X\times_S X} \to (\Delta_{X|S})_*\mathcal{O}_X) \subset \mathcal{O}_{X\times_S X}$$

denote the corresponding sheaf of ideals. By assumption $\Omega^1_{X|S} = 0 = \Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2)$. In particular,

$$\mathcal{I}_{\Delta_{X|S}(x)}/\mathcal{I}^2_{\Delta_{X|S}(x)} = (\Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2))_x = 0,\ x \in X$$

or, equivalently, $\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)}$, $x \in X$. But, as S is noetherian and $\phi : X \to S$ is finite, X is noetherian hence $\mathcal{I}$ is coherent. So, by Nakayama,

$$\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)},\ x \in X$$

forces

$$\mathcal{I}_{\Delta_{X|S}(x)} = 0,\ x \in X.$$

Thus $\Delta_{X|S}(X)$ is contained in the open subset $U := X \times_S X \setminus \mathrm{supp}(\mathcal{I})$. On the other hand, for all $u \in U$, the morphism induced on stalks:

$$\Delta^\#_{X|S,u} : \mathcal{O}_{X\times_S X,u} \tilde{\to} (\Delta_{X|S*}\mathcal{O}_X)_u$$

is an isomorphism. So U is contained in $\mathrm{supp}(\Delta_{X|S*}\mathcal{O}_X) = \Delta_{X|S}(X)$ hence $\Delta_{X|S}(X) = U$ and $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is an open immersion.

(c) $\Rightarrow$ (d): For any geometric points $\overline{s} : \mathrm{spec}(\Omega) \to S$ and $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{s}}$, consider the cartesian diagram:

$$\begin{tikzcd}
X \arrow[d, "\Delta_{X|S}"'] \arrow[dr, phantom, "\square"] & X_{\overline{s}} \arrow[l] \arrow[d, "\Delta_{X_{\overline{s}}|\Omega}"'] \arrow[dr, phantom, "\square"] & \mathrm{spec}(\Omega) \arrow[l, "\overline{x}"'] \arrow[d, "(Id\times\overline{x})"'] \\
X \times_S X & X_{\overline{s}} \times_\Omega X_{\overline{s}} \arrow[l] & \mathrm{spec}(\Omega) \times_\Omega X_{\overline{s}}. \arrow[l, "(\overline{x}\times Id)"]
\end{tikzcd}$$

Since open immersions are stable under base changes, $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ is again an open immersion hence induces an isomorphism onto a closed and open subscheme of $X_{\overline{s}}$ that is, since $\mathrm{spec}(\Omega)$ is connected and $X_{\overline{s}}$ is finite, a connected component of $X_{\overline{s}}$. As a result,

$$X_{\overline{s}} = \bigsqcup_{\overline{x}:\mathrm{spec}(\Omega)\to X_{\overline{s}}} \mathrm{spec}(\Omega)$$

is a coproduct of $|X_{\overline{s}}|$ copies of $\mathrm{spec}(\Omega)$.

(d) $\Rightarrow$ (a): As the question is local on X, we may assume, one more time, that $\phi : X \to S$ is induced by a finite A-algebra $\phi^{\#} : A \to B$ with A noetherian. By assumption,

$$B \otimes_A k(\mathfrak{p}) = \prod_{1\leq i\leq n} k_i$$

is, as a $k(\mathfrak{p})$-algebra, the product of finitely many finite separable field extensions of $k(\mathfrak{p})$. In particular, any ideal in $\mathrm{spec}(B \otimes_A k(\mathfrak{p}))$ is maximal and equal to one of the:

$$\mathfrak{m}_j := \ker(\prod_{1\leq i\leq n} k_i \twoheadrightarrow k_j),\ j = 1, \dots, n.$$

But, then, for any $\mathfrak{q} \in X$ above $\mathfrak{p} \in S$ whose image in $\mathrm{spec}(B \otimes_A k(\mathfrak{p}))$ is $\mathfrak{m}_j$ for some $1 \leq j \leq n$, one has:

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = (B \otimes_A k(\mathfrak{p}))_{\mathfrak{m}_j} = k_j,$$

which, by assumption, is a finite separable field extension of $k(\mathfrak{p})$. $\square$

Remark 5.3. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) also hold for morphisms which are locally of finite type.

Example 5.4. Assume that $S = \mathrm{spec}(A)$ is affine and let $P \in A[T]$ be a monic polynomial such that $P' \neq 0$. Set $B := A[T]/PA[T]$ and $C := B_b$ where $b \in B$ is such that $P'(t)$ becomes invertible in B_b (here t denotes the image of T in B). Then $\mathrm{spec}(C) \to S$ is an étale morphism. Such morphisms are called *standard étale morphisms.*

Actually, any étale morphism is locally of this type.

Theorem 5.5. (Local structure of étale morphisms) *Let A be a noetherian local ring and set $S = \mathrm{spec}(A)$. Let $\phi : X \to S$ an unramified (resp. étale) morphism. Then, for any $x \in X$, there exists an open neighbourhood U of x such that one has a factorization:*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{spec}(C), \arrow[dl] \\ S & \end{tikzcd}$$

where $\mathrm{spec}(C) \to S$ is a standard étale morphism and $U \hookrightarrow \mathrm{spec}(C)$ is an immersion (resp. an open immersion).

Proof. See [Mi80, Thm. 3.14 and Rem. 3.15]. $\square$

For any étale cover $\phi : X \to S$, the rank function:

$$\begin{array}{rccl} r_-(\phi) \colon & S & \to & \mathbb{Z}_{\geq 0} \\ & s & \mapsto & r_s(\phi) := \mathrm{rank}_{\mathcal{O}_{S,s}}((\phi_*\mathcal{O}_X)_s) = \mathrm{rank}_{k(s)}(\mathcal{O}_{X_s}(X_s)) = \dim_{\overline{k(s)}}(\mathcal{O}_{X_s}(X_s) \otimes_{k(s)} \overline{k(s)}) = |X_{\overline{s}}| \end{array}$$

is locally constant hence constant, since S is connected; we say that $r(\phi)$ is the *rank of $\phi : X \to S$.*

Eventually, let us recall the following two standard lemmas.

Lemma 5.6. (Stability) *If P is a property of morphisms of schemes which is (i) stable under composition and (ii) stable under arbitrary base-change then (iv) P is stable by fibre products. If furthermore (iii) closed immersions have P then, (v) for any $X \xrightarrow{f} Y \xrightarrow{g} Z$, if g is separated and $g \circ f$ has P then f has P.*

The properties $P=$ surjective, flat, unramified, étale satisfy (i) and (ii) hence (iv). The properties $P=$ separated, proper, finite satisfy (i), (ii), (iii) hence (iv) and (v).

Lemma 5.7. (Topological properties of finite morphisms)

(1) A finite morphism is closed;
(2) A finite flat morphism is open.

Remark 5.8.

(1) Since being finite is stable under base-change, lemma 5.7 (1) shows that a finite morphism is universally closed. Since finite morphisms are affine hence separated, this shows that finite morphisms are proper.
(2) Lemma 5.7 (2) also hold for flat morphisms which are locally of finite type.

Corollary 5.9. *Let S be a connected scheme. Then any finite étale morphism $\phi : X \to S$ is automatically an étale cover. Furthermore, $\phi : X \tilde{\to} S$ is an isomorphism if and only if $r(\phi) = 1$.*

Proof. From lemma 5.7, the set $\phi(X)$ is both open and closed in S, which is connected. Hence $\phi(X) = S$. As for the second part of the assertion, the "if" implication is straightforward so we are only to prove the "only if" part. The condition $r(\phi) = 1$ already implies that $\phi : X \to S$ is bijective. But as $\phi : X \to S$ is continuous and, by lemma 5.7 (2), open, it is automatically an homeomorphism. So $\phi : X \to S$ is an isomorphism if and only if $\phi^{\#}_s : \mathcal{O}_{S,s} \tilde{\to} (\phi_*\mathcal{O}_X)_s$ is an isomorphism, $s \in S$. This amounts to showing that any finite, faithfully flat A-algebra $A \hookrightarrow B$ such that $B = Ab$ as A-module is surjective that is $b \in A$. By assumtion, there exists $a \in A$ such that $ab = 1$ and, as B is finite over A, there exists a monic polynomial $P_b = T^d + \sum_{i=0}^{d-1} r_i T^i \in A[T]$ such that $P_b(b) = 0$ hence, multiplying this equality by a^{d-1}, one gets $b = -\sum_{i=0}^{d-1} r_i a^{d-1-i} \in A$. $\square$

5.3. **The category of étale covers of a connected scheme.**

5.3.1. *Statement of the main theorem.* Let S be a connected scheme and denote by $\mathcal{C}_S \subset Sch/S$ the full subcategory whose objects are étale covers of S.

Given a geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the underlying set associated to the scheme $X_{\overline{s}} := X \times_{\phi,S,\overline{s}} \mathrm{spec}(\Omega)$ will be denoted by $X^{set}_{\overline{s}}$. One thus obtains a functor:

$$\begin{array}{llll} F_{\overline{s}} : & \mathcal{C}_S & \to & FSets \\ & \phi : X \to S & \to & X^{set}_{\overline{s}}. \end{array}$$

Theorem 5.10. *The category of étale covers of S is Galois. And for any geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the functor $F_{\overline{s}} : \mathcal{C}_S \to FSets$ is a fibre functor for $\mathcal{C}_S$.*

Remark 5.11. For any geometric point $s : \mathrm{spec}(\Omega) \to S$, the functor $F_{\overline{s}} : \mathcal{C}_S \to FSets$ is a fibre functor for $\mathcal{C}_S$ but all fibre functors are not necessarily of this form. For instance, given an algebraically closed field Ω and a morphism $f : \mathbb{P}^1_\Omega \to S$ then the functor:

$$\begin{array}{llll} F_f : & \mathcal{C}_S & \to & FSets \\ & \phi : X \to S & \to & \pi_0(X \times_{\phi,S,f} \mathbb{P}^1_\Omega) \end{array}$$

is also a fibre functor for $\mathcal{C}_S$.

By analogy with topology, for any geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the profinite group:

$$\pi_1(S; \overline{s}) := \pi_1(\mathcal{C}_S; F_{\overline{s}})$$

is called the *étale fundamental group of S with base point $\overline{s}$*. Similarly, for any two geometric points $\overline{s}_i : \mathrm{spec}(\Omega_i) \to S$, $i = 1, 2$, the set:

$$\pi_1(S; \overline{s}_1, \overline{s}_2) := \pi_1(\mathcal{C}_S; F_{\overline{s}_1}, F_{\overline{s}_2})$$

is called the set of *étale paths from $\overline{s}_1$ to $\overline{s}_2$*. (Note that Ω_1 and Ω_2 may have different characteristics).

From theorem 2.8, the set of étale paths $\pi_1(S;\overline{s}_1,\overline{s}_2)$ from $\overline{s}_1$ to $\overline{s}_2$ is non-empty and the profinite group $\pi_1(S;\overline{s}_1)$ is noncanonicaly isomorphic to $\pi_1(S;\overline{s}_2)$ with an isomorphism that is canonical up to inner automorphisms.

Eventually, given a morphism $f: S' \to S$ of connected schemes and a geometric point $\overline{s}' : \mathrm{spec}(\Omega) \to S'$, the universal property of fibre product implies that the base change functor $f^* : \mathcal{C}_S \to \mathcal{C}_{S'}$ satisfies $F_{\overline{s}'} \circ f^* = F_{f(\overline{s}')}$. Hence $f^* : \mathcal{C}_S \to \mathcal{C}_{S'}$ is a fundamental functor and one gets, correspondingly, a morphism of profinite groups:

$$\pi_1(f) : \pi_1(S';\overline{s}') \to \pi_1(S;\overline{s}),$$

whose properties can be read out of those of $f: S' \to S$ using the results of subsection 4.2.

5.3.2. *Proof.* We check axioms (1) to (6) of the definition of a Galois category.

Axiom (1): The category of étale covers of S has a final object: $Id_S : S \to S$ and, from lemma 5.6, the fibre product (in the category of S-schemes) of any two étale covers of S over a third one is again an étale cover of S.

Axiom (2): The category of étale covers of S has an initial object: $\emptyset$ and the coproduct (in the category of S-schemes) of two étale covers of S is again an étale cover of S. A more delicate point is:

Lemma 5.12. *Categorical quotients by finite groups of automorphisms exist in $\mathcal{C}_S$.*

Proof of the lemma. Let $\phi : X \to S$ be an étale cover and let $G \subset \mathrm{Aut}_{Sch/S}(\phi)$ be a finite subgroup.

Step 1: Assume first that $S = \mathrm{spec}(A)$ is an affine scheme. Since étale cover are, in particular, finite hence affine morphisms, $\phi : X \to S$ is induced by a finite A-algebra $\phi^\# : A \to B$. But, then, it follows from the equivalence of category between the category of affine S-schemes and $(Alg/A)^{op}$ that the factorization

$$\begin{tikzcd} X \arrow[r, "p_G"] \arrow[d, "\phi"'] & \mathrm{spec}(B^{G^{op}}) =: G \setminus X \arrow[dl, "\phi_G"] \\ S & \end{tikzcd}$$

is the categorical quotient of $\phi : X \to S$ by G in the category of affine S-schemes. So, as $\mathcal{C}_S$ is a full subcategory of the category of affine S-schemes, to prove that $\phi_G : G \setminus X \to S$ is the categorical quotient of $\phi : X \to S$ by G in $\mathcal{C}_S$ it only remains to prove that $\phi_G : G \setminus X \to S$ is in $\mathcal{C}_S$.

Step 1-1 (trivialization): *An affine, surjective morphism $\phi : X \to S$ is an étale cover of S if and only if there exists a finite faithfully flat morphism $f : S' \to S$ such that the first projection $\phi' : X' := S' \times_{f,S,\phi} X \to S'$ is a totally split étale cover of S'.*

In other words, an affine surjective morphism $\phi : X \to S$ is an étale cover if and only if it is locally trivial for the Grothendieck topology whose covering families are finite, faithfully flat morphisms.

Proof. We first prove the "only if" implication. As $f : S' \to S$ is finite and faithfully flat, it follows from lemma 5.2 (1) that for any $s \in S$ there exists an open affine neighbourhood $U = \mathrm{spec}(A)$ of s such that $f|^U_{f^{-1}(U)} : f^{-1}(U) \to U$ is induced by a finite A-algebra $f^\# : A \hookrightarrow A'$ with $A' = A^r$. Also, as $\phi : X \to S$ is affine and surjective, $\phi|^U_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ corresponds to a A-algebra $\phi^\# : A \hookrightarrow B$. By assumption $B \otimes_A A' = A'^s$ as A'-algebras hence $B \otimes_A A' = A^{rs}$ as A-modules. But, on the other hand, $B \otimes_A A' = B \otimes_A A^r = B^r$ as B-modules hence as A-modules. In particular, B is a direct factor of A^{rs} as A-module hence is flat over A. This shows that $\phi : X \to S$ is flat. Also, as B is a submodule of the finitely generated A-module A^{rs} and A is noetherian, B is also a finitely generated A-module. This shows that $\phi : X \to S$ is finite. With the notation:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\phi'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\phi"] \\ S' \arrow[r, "f"'] & S, \end{tikzcd}$$

it follows from lemma 5.2 (2) (c) that $f'^*\Omega_{X|S} = \Omega_{X'|S'} = 0$ that is, $(f'^*\Omega_{X|S})_{x'} = \Omega_{X|S,f'(x')} = 0$, $x' \in X'$. But $f' : X' \to X$ is the base change of the surjective morphism $f : S' \to S$ hence it is surjective as well, which implies $\Omega_{X|S} = 0$. This shows that $\phi : X \to S$ is finite étale.

We now prove the "if" implication by induction on $r(\phi) \geq 1$. If $r(\phi) = 1$ it follows from corollary 5.9 that $\phi : X \tilde{\to} S$ is an isomorphism and the statement is straightforward with $f = Id_S$. If $r(\phi) > 1$, from lemma 5.2 (2) (d), the diagonal morphism $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is both a closed and open immersion hence $X \times_S X$ can be written as a coproduct $X \sqcup X'$, where $\Delta_{X|S}(X)$ is identified with X and $X' := X \times_S X \setminus \Delta_{X|S}(X)$. In particular, $i_{X'} : X' \hookrightarrow X \times_S X$ is both a closed and open immersion as well hence a finite étale morphism. Also, as $\phi : X \to S$ is finite étale, its base change $p_1 : X \times_{\phi,S,\phi} X \to X$ is finite étale as well so the composite $\phi' : X' \stackrel{i_{X'}}{\to} X \times_S X \stackrel{p_1}{\to} X$ is finite étale. But as $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is a section of $p_1 : X \times_S X \to X$, one has: $r(\phi') = r(p_1) - 1 = r(\phi) - 1$. So, by induction hypothesis, there exists a finite faithfully flat morphism $f : S' \to X$ such that $S' \times_{f,X,\phi'} X' \to S'$ is a totally split étale cover of S'. But, then, the composite $\phi \circ f : S' \to S$ is also finite and faithfully flat. Hence the conclusion follows from the formal computation based on elementary properties of fibre product of schemes:

$$S' \times_{\phi\circ f,S,\phi} X = S' \times_{f,X,p_1} (X \times_S X) = S' \times_{f,X,p_1} (X \sqcup X') = (S' \times_{f,X,p_1} X) \sqcup (S' \times_{f,X,p_1} X'). \ \square$$

Step 1-2: We want to apply step 1-1 to the quotient morphism $\phi_G : G \setminus X \to S$. For this, apply first step 1-1 to the étale cover $\phi : X \to S$ to obtain a faithfully flat A-algebra $A \to A'$ such that $B \otimes_A A' = A'^n$ as A'-algebras. Tensoring the exact sequence of A-algebras:

$$0 \to B^{G^{op}} \to B \stackrel{\sum_{g\in G^{op}}(Id_B - g\cdot)}{\longrightarrow} \bigoplus_{g\in G^{op}} B$$

by the flat A-algebra A', one gets the exact sequence of B'-algebras:

$$0 \to B^{G^{op}} \otimes_A A' \to B \otimes_A A' \stackrel{\sum_{g\in G^{op}}(Id_B - g\cdot)\otimes_A Id_{A'}}{\longrightarrow} \bigoplus_{g\in G} B \otimes_A A',$$

whence:

$$(*) \ \ B^{G^{op}} \otimes_A A' = (B \otimes_A A')^{G^{op}} = (A'^n)^{G^{op}}.$$

But G^{op} is a subgroup of $\mathrm{Aut}_{Alg/A'}(A'^n)$, which is nothing but the symmetric group $\mathcal{S}_n$ acting on the canonical coordinates $E := \{1, \cdots, n\}$ in A'^n. Hence:

$$(A'^E)^{G^{op}} = \bigoplus_{G\setminus E} A'.$$

In terms of schemes, if $f : S' \to S$ denotes the faithfully flat morphism corresponding to $A \hookrightarrow A'$ then $S' \times_{f,S,\phi} X$ is just the coproduct of n copies of S' over which G acts by permutation and $(*)$ becomes:

$$S' \times_{f,S,\phi_G} (G \setminus X) = G \setminus (\bigsqcup_E S') = \bigsqcup_{G\setminus E} S'.$$

Step 2: Reduce to step 1 by covering S with affine open subschemes (local existence) and using the unicity of categorical quotient up to canonical isomorphism (glueing). $\square$

Remark 5.13. One can actually show that, in the affine case, $G \setminus X = \mathrm{spec}(B^{G^{op}})$ is actually the categorical quotient of $\phi : X \to S$ by G is the category of all S-schemes (*Cf.* [MumF82, Prop. 0.1]).

Exercise 5.14. Show that categorical quotients of étale covers by finite groups of automorphisms commute with arbitrary base-changes.

Axiom (3): Before dealing with axiom (3), let us recall that, in the category of S-schemes, open immersions are monomorphisms and that:

Theorem 5.15. (Grothendieck - see [Mi80, Thm. 2.17]) *In the category of S-schemes, faithfully flat morphisms of finite type are strict epimorphisms.*

Lemma 5.16. *Given a commutative diagram of schemes:*

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "\psi"'] & X \arrow[dl, "\phi"] \\ S, & \end{tikzcd}$$

if $\phi : X \to S$, $\psi : Y \to S$ are finite étale morphisms then $u : Y \to X$ is a finite étale morphism as well.

Proof of the lemma. Write $u = p_2 \circ \Gamma_u$, where $\Gamma_u : Y \to Y \times_S X$ is the graph of u, identified with the base-change:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\Gamma_u"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_{X|S}"] \\ Y \times_S X \arrow[r, "u \times_S Id_X"'] & X \times_S X \end{tikzcd}$$

and $p_2 : Y \times_S X \to X$ is the base-change defined by:

$$\begin{tikzcd} Y \times_S X \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\phi"'] & S. \end{tikzcd}$$

From lemma 5.2 (2) (d), the diagonal morphism $\Delta_{X|S} : X \to X \times_S X$ is finite étale hence it follows from the first part of lemma 5.6 that $\Gamma_u : Y \to Y \times_S X$ is finite, étale as well. Similarly, as $\psi : Y \to S$ is finite étale, $p_2 : Y \times_S X \to X$ is finite étale as well. Hence, the conclusion follows from the second part of lemma 5.6. □

For any two étale covers $\phi : X \to S$, $\psi : Y \to S$ and for any morphism $u : X \to Y$ over S, it follows from lemma 5.16 that $u : Y \to X$ is a finite, étale morphism hence is both open (flatness) and closed (finite). In particular, one can write X as a coproduct $X = X' \sqcup X''$, where $X' := u(Y)$, $X'' := X \setminus X'$ are both open and closed in X and u factors as $u : Y \overset{u|^{X'} = u'}{\to} X' \overset{i'_{X'} = u''}{\to} X = X' \sqcup X''$ with u' a faithfully flat morphism hence a strict epimorphism in $R_S^{\text{ét}}$ and u'' an open immersion hence a monomorphism in $\mathcal{C}_S$.□

Axiom (4): For any étale cover $\phi : X \to S$ one has $F_{\overline{s}}(\phi) = *$ if and only if $r(\phi) = 1$, which, in turn, is equivalent to $\phi : X \tilde{\to} S$. Also, it follows straightforwardly from the universal property of fibre product and the definition of $F_{\overline{s}}$ that $F_{\overline{s}}$ commutes with fibre products.

Axiom (5): The fact that $F_{\overline{s}}$ commutes with finite coproducts and transforms strict epimorphisms into strict epimorphisms is straightforward. So it only remains to prove that $F_{\overline{s}}$ commutes with categorical quotients by finite groups of automorphisms. Let $\phi : X \to S$ be an étale cover and $G \subset \mathrm{Aut}_{Sch/S}(\phi)$ a finite subgroup. Since the assertion is local on S, it follows from step 1-1 in axiom (2) that we may assume that $\phi : X \to S$ is totally split and that G acts on X by permuting the copies of S. But, then, the assertion is immediate since $G \setminus X = \bigsqcup_{G \setminus F_{\overline{s}}(\phi)} S$.

Axiom (6): For any two étale covers $\phi : X \to S$, $\psi : Y \to S$ let $u : X \to Y$ be a morphism over S such that $F_{\overline{s}}(u) : F_{\overline{s}}(\psi) \tilde{\to} F_{\overline{s}}(\phi)$ is bijective. It follows from lemma 5.16 that $u : Y \to X$ is finite étale but, by assumption, it is also surjective hence $u : Y \to X$ is an étale cover. Moreover, still by assumption, it has rank 1 hence it is an isomorphism by corollary 5.9. □

6. Examples

Given a scheme X over an affine scheme $\mathrm{spec}(A)$, we will write $X \to A$ instead of $X \to \mathrm{spec}(A)$ for the structural morphism and given a A-algebra $A \to B$, we will write X_B for $X \times_A \mathrm{spec}(B)$. Similarly, given a morphism $f : X \to Y$ of schemes over $\mathrm{spec}(A)$, we will write $f_B : X_B \to Y_B$ for its base-change by $\mathrm{spec}(B) \to \mathrm{spec}(A)$. Also, given a morphism $f : Y \to X$ and a morphism $X \to X'$ we will often say that $f' : Y' \to X'$ is a model of $f : Y \to X$ over X' if there is a cartesian square:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "f'"] \\ X \arrow[r] & X'. \end{tikzcd}$$

6.1. **Spectrum of a field.** Let k be a field, $k \hookrightarrow \overline{k}$ a fixed algebraic closure of k and $k^s \subset \overline{k}$ the separable closure of k in $\overline{k}$; write $\Gamma_k := \mathrm{Aut}_{Alg/k}(k^s)$ for the absolute Galois group of k. Set $S := \mathrm{spec}(k)$. Then the datum of $k \hookrightarrow \overline{k}$ defines a geometric point $\overline{s} : \mathrm{spec}(\overline{k}) \to S$ and:

Proposition 6.1. *There is a canonical isomorphism of profinite groups:*

$$c_{\overline{s}} : \pi_1(S; \overline{s}) \tilde{\to} \Gamma_k.$$

Proof. The Galois objects in $\mathcal{C}_S$ are the $\mathrm{spec}(K) \to S$ induced by finite Galois field extensions $k \hookrightarrow K$; write $\mathcal{G}_S \subset \mathcal{C}_S$ for the full subcategory of Galois objects. The datum of $k \hookrightarrow \overline{k}$ allows us to identify k with a subfield of $\overline{k}$ and define a canonical section of the forgetful functor: $For : \mathcal{G}_S^{pt} \to \mathcal{G}_S$ by associating to each Galois object $\mathrm{spec}(K) \to S$ its isomorphic copy $\mathrm{spec}(K_\Omega) \to S$, where K_Ω is the unique subfield of $\overline{k}$ containing k and isomorphic to K as k-algebra. Then, on the one hand, the restriction morphisms $|_{K_\Omega} : \Gamma_k \to \mathrm{Aut}_{Alg/k}(K_\Omega)$ induce an isomorphism of profinite groups:

$$\Gamma_k \tilde{\to} \varprojlim_{K_\Omega} \mathrm{Aut}_{Alg/k}(K_\Omega).$$

And, on the other hand, by the equivalence of categories:

$$\begin{array}{ccc} \mathcal{C}_S & \to & (FEAlg/k)^{op} \\ \phi : X \to S & \mapsto & \phi^{\#}(X) : k \hookrightarrow \mathcal{O}_X(X) \end{array}$$

one can identify:

$$\mathrm{Aut}_{Alg/k}(K_\Omega) = \mathrm{Aut}_{\mathcal{C}_S}(\mathrm{spec}(K_\Omega))^{op}.$$

But then, from proposition 3.9, one also has the canonical evaluation isomorphism of profinite groups:

$$\pi_1(S; \overline{s}) \tilde{\to} \varprojlim_{K_\Omega} \mathrm{Aut}_{\mathcal{C}_S}(\mathrm{spec}(K_\Omega))^{op},$$

which concludes the proof. $\square$

6.2. **The first homotopy sequence and applications.**

6.2.1. *Stein factorization.* A scheme X over a field k is *separable over* k if, for any field extension K of k the sheme $X \times_k K$ is reduced. This is equivalent to requiring that X be reduced and that, for any generic point η of X, the extension $k \hookrightarrow k(\eta)$ be separable (recall that an arbitrary field extension $k \hookrightarrow K$ is separable if any finitely generated subextension admits a separating transcendence basis and that any field extension of a perfect field is separable.) In particular, if k is perfect, this is equivalent to requiring that X be reduced. More generally, a scheme X over a scheme S is *separable over* S if it is flat over S and for any $s \in S$ the scheme X_s is separable over $k(s)$. Separable morphisms satisfy the following elementary properties:
- Any base change of a separable morphism is separable.
- If $X \to S$ is separable and $X' \to X$ is étale then $X' \to S$ is separable.

Theorem 6.2. (Stein factorization of a proper morphism) *Let $f : X \to S$ be a morphism such that $f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-algebra. Then $f_*\mathcal{O}_X$ defines an S-scheme:*

$$p : S' = \mathrm{spec}(f_*\mathcal{O}_X) \to S$$

and $f : X \to S$ factors canonically as:

$$\begin{tikzcd} S & S' \arrow[l, "p"'] \\ X. \arrow[u, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Furthermore,

(1) If $f : X \to S$ is proper then
  (a) $p : S' \to S$ is finite and $f' : X \to S'$ is proper and with geometrically connected fibres;
  (b) - The set of connected components of X_s is one-to-one with $S_s'^{set}$, $s \in S$;
  - The set of connected components of $X_{\overline{s}}$ is one-to-one with $S_{\overline{s}}'^{set}$, $s \in S$.
  In particular, if $f_*\mathcal{O}_X = \mathcal{O}_S$ then $f : X \to S$ has geometrically connected fibres.

(2) If $f : X \to S$ is proper and separable then $p : S' \to S$ is an étale cover. In particular, $f_*\mathcal{O}_X = \mathcal{O}_S$ if and only if $f : X \to S$ has geometrically connected fibres.

Corollary 6.3. *Let $f : X \to S$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Then, if S is connected, X is connected as well.*

Proof. It follows from (1) (b) of theorem 6.2 that, if $f_*\mathcal{O}_X = \mathcal{O}_S$ then $f : X \to S$ is geometrically connected and, in particular, has connected fibres. But, as $f : X \to S$ is proper, it is closed and $f_*\mathcal{O}_X$ is coherent hence:

$$f(X) = \mathrm{supp}(f_*\mathcal{O}_X).$$

So $f_*\mathcal{O}_X = \mathcal{O}_S$ also implies that $f : X \to S$ is surjective. As a result, if $f_*\mathcal{O}_X = \mathcal{O}_S$ the morphism $f : X \to S$ is closed, surjective, with connected fibres so, if S is connected, this forces X to be connected as well. □

6.2.2. *The first homotopy sequence.* Let S be a connected scheme, $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $s \in S$. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ with image again denoted by x_Ω in X and s_Ω in S.

Theorem 6.4. (First homotopy sequence) *Consider the canonical sequence of profinite groups induced by $(X_{\overline{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$:*

$$\pi_1(X_{\overline{s}}; x_\Omega) \xrightarrow{i} \pi_1(X; x_\Omega) \xrightarrow{p} \pi_1(S; s_\Omega).$$

Then $p : \pi_1(X; x_\Omega) \twoheadrightarrow \pi_1(S; s_\Omega)$ is an epimorphism and $\mathrm{im}(i) \subset \ker(p)$*. If, furthermore, $f : X \to S$ is separable then* $\mathrm{im}(i) = \ker(p)$.

A first consequence of theorem 6.4 is that the étale fundamental group of a connected, proper scheme over k is invariant by algebraically closed field extension. More precisely, let k be an algebraically closed field, X a scheme connected and proper over k and $k \hookrightarrow \Omega$ an algebraically closed field extension of k. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_\Omega$ with image again denoted by x_Ω in X.

Corollary 6.5. *The canonical morphism of profinite groups:*

$$\pi_1(X_\Omega; x_\Omega) \tilde{\to} \pi_1(X; x_\Omega)$$

induced by $(X_\Omega; x_\Omega) \to (X; x_\Omega)$ is an isomorphism.

Proof. We first prove:

Lemma 6.6. (Product) *Let k be an algebraically closed field, X a connected, proper scheme over k and Y a connected scheme over k. For any $x : \mathrm{spec}(k) \to X$ and $y : \mathrm{spec}(k) \to Y$, the canonical morphism of profinite groups:*

$$\pi_1(X \times_k Y; (x, y)) \to \pi_1(X; x) \times \pi_1(Y; y)$$

induced by the projections $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ is an isomorphism.

Proof of the lemma. From theorem A.2, one may assume that X is reduced hence, as k is algebraically closed, that X is separable over k. As X is proper, separable, geometrically connected and surjective over k, so is its base change $p_Y : X \times_k Y \to Y$. So, it follows from theorem 6.2 (2) that $p_{Y*}\mathcal{O}_{X\times_k Y} = \mathcal{O}_Y$. Thus, one can apply theorem 6.4 to $p_Y : X \times_k Y \to Y$ to get an exact sequence:

$$\pi_1((X \times_k Y)_y; x) \to \pi_1(X \times_k Y; (x, y)) \to \pi_1(Y; y) \to 1.$$

Furthermore, $X = (X \times_k Y)_y \to X \times_k Y \stackrel{p_X}{\to} X$ is the identity so $p_X : X \times_k Y \to X$ yields a section of $\pi_1(X; x) \to \pi_1(X \times_k Y; (x, y))$. $\square$

Note that if $y : \mathrm{spec}(\Omega) \to Y$ is any geometric point then the above only shows that $\pi_1(X\times_k Y; (x, y))\tilde{\to}\pi_1(X_\Omega; x)\times \pi_1(Y; y)$.

Proof of corollary 6.5. We apply the criterion of proposition 4.3.

<u>Surjectivity:</u> Let $\phi : Y \to X$ be a connected étale cover. We are to prove that Y_Ω is again connected. But, as k is algebraically closed, if Y is connected then it is automatically geometrically connected over k and, in particular, Y_Ω is connected.

<u>Injectivity:</u> One has to prove that for any connected étale cover $\phi : Y \to X_\Omega$, there exists an étale cover $\tilde{\phi} : \tilde{Y} \to X$ which is a model of ϕ over X. We begin with a general lemma.

Lemma 6.7. *Let X be a connected scheme of finite type over a field k and let $k \hookrightarrow \Omega$ be a field extension of k. Then, for any étale cover $\phi : Y \to X_\Omega$, there exists a finitely generated k-algebra R contained in Ω and an affine morphism of finite type $\tilde{\phi} : \tilde{Y} \to X_R$ which is a model of $\phi : Y \to X_\Omega$ over X_R. Furthermore, if η denotes the generic point of* $\mathrm{spec}(R)$*, then $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$ is an étale cover.*

Proof of the lemma. Since X is quasi-compact, there exists a finite covering of X by Zariski-open subschemes $X_i := \mathrm{spec}(A_i) \hookrightarrow X$, $i = 1, \dots, n$, where the A_i are finitely generated k-algebra. As $\phi : Y \to X_\Omega$ is affine, we can write $U_i := \phi^{-1}(X_{i\Omega}) = \mathrm{spec}(B_i)$, where B_i is of the form:

$$B_i = A_i \otimes_k \Omega[\underline{T}]/\langle P_{i,1}, \dots, P_{i,r_i}\rangle.$$

For each $1 \leq j \leq r_i$, the $\underline{\alpha}$th coefficient of $P_{i,j}$ is of the form:

$$\sum_k r_{i,j,\underline{\alpha},k} \otimes_k \lambda_{i,j,\underline{\alpha},k}$$

with $r_{i,j,\underline{\alpha},k} \in A_i$, $\lambda_{i,j,\underline{\alpha},k} \in \Omega$. So, let R_i denote the sub k-algebra of Ω generated by the $\lambda_{i,j,\underline{\alpha},k}$ then B_i can also be written as:

$$B_i = A_i \otimes_k R_i[\underline{T}]/\langle P_{i,1}, \dots, P_{i,r_i}\rangle \otimes_{R_i} \Omega.$$

Let R denote the sub-k-algebra of Ω generated by the R_i, $i = 1, \dots, n$. Then $k \hookrightarrow R$ is a finitely generated k-algebra and up to enlarging R, one may assume that the glueing data on the $U_i \cap U_j$ descend to R then one can construct $\tilde{\phi}$ by glueing the $\mathrm{spec}(A_i \otimes_k R[\underline{T}]/\langle P_{i,1}, \dots, P_{i,r_i}\rangle)$ along these descended gluing data. By construction $\tilde{\phi}$ is affine.

To conclude, since $k(\eta) \hookrightarrow \Omega$ is faithfully flat and $\phi : Y \to X_\Omega$ is finite and faithfully flat, the same is automatically true for $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$, which is then étale since $\phi : Y \to X_\Omega$ is. $\square$

So, applying lemma 6.7 to $\phi : Y \to X_\Omega$ and up to replacing R by R_r for some $r \in R \setminus \{0\}$, one may assume that $\phi : Y \to X_\Omega$ is the base-change of some étale cover $\phi^0 : Y^0 \to X_R$. Note that, since $Y^0_\Omega = Y$ is connected, both Y^0_η and Y^0 are connected as well. Fix $s : \mathrm{spec}(k) \to S$. Since the fundamental group does not depend on the fibre functor, one can assume that $k(x) = k$. Then, from lemma 6.6, one gets the canonical isomorphism of profinite groups:

$$\pi_1(X \times_k S; (x, s))\tilde{\to}\pi_1(X; x) \times \pi_1(S; s).$$

Let $U \subset \pi_1(X \times_k S; (x, s))$ be the open subgroup corresponding to the étale cover $\phi^0 : Y^0 \to X \times_k S$ and let $U_X \subset \pi_1(X; x)$ and $U_S \subset \pi_1(S; s)$ be open subgroups such that $U_X \times U_S \subset U$. Then U_X and U_S correspond to connected étale covers $\psi_X : \tilde{X} \to X$ and $\psi_S : \tilde{S} \to S$ such that $\phi^0 : Y^0 \to X \times_k S$ is a quotient of $\psi_X \times_k \psi_S : \tilde{X} \times_k \tilde{S} \to X \times_k S$. Consider the following cartesian diagram:

$$\begin{tikzcd}
& & \tilde{X} \times_k \tilde{S} \arrow[dll] \arrow[dl, dotted] \arrow[ddl] \\
Y^0 \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{Y}^0 \arrow[l] \arrow[d] & \\
X \times_k S & X \times_k \tilde{S} \arrow[l] &
\end{tikzcd}$$

Since $k(\eta) \subset \Omega$ and Ω is algebraically closed, one may assume that any point $\tilde{s} \in \tilde{S}$ above $s \in S$ has residue field contained in Ω and, in particular, one can consider the associated Ω-point $\tilde{s}_\Omega : \mathrm{spec}(\Omega) \to \tilde{S}$. Then, one has the cartesian diagram:

$$\begin{tikzcd}
\tilde{Y}^0_S \arrow[d] \arrow[dr, phantom, "\square"] & Y_\Omega \arrow[l] \arrow[d] \\
X \times_k \tilde{S} & X_\Omega . \arrow[l, "Id_X \times_k \tilde{s}_\Omega"']
\end{tikzcd}$$

Again, since Y_Ω is connected, $\tilde{Y}^0$ is connected as well, from which it follows that $\tilde{Y}^0 \to X \times_k \tilde{S}$ corresponds to an open subgroup $V \subset \pi_1(X \times_k \tilde{S}) = \pi_1(X) \times U_S$ containing $\pi_1(\tilde{X} \times_k \tilde{S}) = U_X \times U_S$. Hence $V = U \times U_S$ for some open subgroup $U_X \subset U \subset \pi_1(X)$ hence $\tilde{Y}^0 \to X \times_k \tilde{S}$ is of the form $\tilde{Y} \times_k \tilde{S} \to X \times_k \tilde{S}$ for some étale cover $\tilde{\phi} : \tilde{Y} \to X$. $\square$

Remark 6.8. An argument due to F. Pop [Sz09, p. 190-191] shows that corollary 6.5 remains true for connected schemes of finite type over k as soon as $\pi_1(X; x_\Omega)$ (or $\pi_1(X_\Omega; x_\Omega)$) is finitely generated. However, in general, corollary 6.5 is no longer true for non-proper schemes. Indeed, let k be an algebraically closed field of characteristic $p > 0$. From the long cohomology exact sequence associated with Artin-Schreier short exact sequence:

$$0 \to (\mathbb{Z}/p)_{\mathbb{A}^1_k} \to \mathbb{G}_{a,\mathbb{A}^1_k} \xrightarrow{\wp} \mathbb{G}_{a,\mathbb{A}^1_k} \to 0$$

(and taking into account that, as $\mathbb{A}^1_k$ is affine, $\mathrm{H}^1(\mathbb{A}^1_k, \mathbb{G}_a) = 0$) one gets:

$$k[T]/\wp k[T] = \mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})/\wp \mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \tilde{\to} \mathrm{H}^1_{et}(\mathbb{A}^1_k, \mathbb{Z}/p) = \mathrm{Hom}(\pi_1(\mathbb{A}^1_k, 0), \mathbb{Z}/p).$$

An additive section of the canonical epimorphism $k[T] \twoheadrightarrow k[T]/\wp k[T]$ is given by the representatives:

$$\sum_{n>0,(n,p)=1} a_n T^n, \ a_n \in k,$$

which shows that $\pi_1(\mathbb{A}^1_k, 0)$ is not of finite type and depends on the base field k.
More generally, one can show [Bo00], [G00] that if S is a smooth connected curve over an algebraically closed field of characteristic $p > 0$ then the pro-p completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$ is a free pro-p group of rank r, where:

- if S is proper over k then r is the p-rank of the jacobian variety $\mathrm{J}_{S|k}$;
- if S is affine over k then r is the cardinality of k.

This determines completely the pro-p completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$. In sections 8, 9 and 10, we will see that the prime-to-p completion $\pi_1(S)^{(p)'}$ of $\pi_1(S)$ is also completely determined. However, except when $\pi_1(S)$ is abelian, this does not determine $\pi_1(S)$ entirely (see remark 11.5).

6.2.3. *Proof of theorem 6.4.* We apply, again, the criterion of proposition 4.3. We begin with an elementary lemma, stating that the inclusion $\mathrm{im}(i) \subset \ker(p)$ is true under less restrictive hypotheses.

Lemma 6.9. *Let X, S be connected schemes, $f : X \to S$ a geometrically connected morphism and $s \in S$. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ with image again denoted by x_Ω in X and s_Ω in S and consider the canonical sequence of profinite groups induced by $(X_{\overline{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$:*

$$\pi_1(X_{\overline{s}}; x_\Omega) \xrightarrow{i} \pi_1(X; x_\Omega) \xrightarrow{p} \pi_1(S; s_\Omega).$$

Then, one always has $\mathrm{im}(i) \subset \ker(p)$.

Proof. Let $\phi : S' \to S$ be an étale cover and consider the following notation:

$$\begin{tikzcd}
S'_{\overline{s}} \arrow[rr] \arrow[d] & \arrow[d, phantom, "\square"] & S' \arrow[d, "\phi"] \\
X_{\overline{s}} \arrow[r] \arrow[dr] & X \arrow[r, "f"'] \arrow[d, phantom, "\square"] & S \\
& k(\overline{s}) \arrow[ur, "\overline{s}"'] &
\end{tikzcd}.$$

We are to prove that $\overline{S}' \to X_{\overline{s}}$ is totally split. But, this is just formal computation based on elementary properties of fibre product of schemes:

$$\begin{aligned}
S'_{\overline{s}} \quad = X_{\overline{s}} \times_{S,\phi} S' \quad &= (X \times_{f,S,\overline{s}} \mathrm{spec}(k(\overline{s}))) \times_{S,\phi} S' \\
&= X \times_{f,S} (\mathrm{spec}(k(\overline{s})) \times_{\overline{s},S,\phi} S') \\
&= X \times_{f,S} \sqcup_{S'_{\overline{s}}} \mathrm{spec}(k(\overline{s})) \\
&= \sqcup_{S'_{\overline{s}}} X_{\overline{s}}.
\end{aligned}$$

□

We return to the proof of theorem 6.4. For simplicity, write $\overline{X} := X_{\overline{s}}$.

<u>Exactness on the right:</u> We are to prove that for any connected étale cover $\phi : S' \to S$ and with the notation for base change:

$$\begin{tikzcd}
X' \arrow[r, "\phi'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \arrow[r, "\phi"'] & S,
\end{tikzcd}$$

the scheme X' is again connected. But, one has:

$$f'_*(\mathcal{O}_{X'}) = f'_*(\phi'^*\mathcal{O}_X) = \phi^* f_*\mathcal{O}_X \overset{(*)}{=} \phi^*\mathcal{O}_S = \mathcal{O}_{S'},$$

where (*) follows from the assumption that $f_*\mathcal{O}_X = \mathcal{O}_S$. Hence, as $f' : X' \to S'$ is proper, it follows from theorem 6.2 (1) (b) that X' is connected.

<u>Exactness in the middle:</u> From lemma 6.9, this amounts to show that $\ker(p) \subset \mathrm{im}(i)$. Let $\phi : X' \to X$ be a connected étale cover and consider the notation:

$$\begin{tikzcd}
X' \arrow[r, "\phi"] & X \arrow[r, "f"] & S \\
\overline{X}' \arrow[r, "\overline{\phi}"'] \arrow[u] \arrow[ur, phantom, "\square"] & \overline{X} \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & k(\overline{s}) \arrow[u, "\overline{s}"']
\end{tikzcd}$$

Assume that $\overline{\phi} : \overline{X}' \to \overline{X}$ admits a section $\sigma : \overline{X} \to \overline{X}'$. We are to prove that $\phi : X' \to X$ comes, by base-change, from a connected étale cover $S' \to S$.

Since $\phi : X' \to X$ is finite étale and $f : X \to S$ is proper and separable, $g := f \circ \phi : X' \to S$ is also proper and separable. Consider its Stein factorization $X' \xrightarrow{g'} S' \xrightarrow{p} S$. From theorem 6.2 (2), the morphism $p : S' \to S$ is étale. Furthermore, as X' is connected and $g' : X' \to S'$ is surjective, S' is connected. Consider the following commutative diagram:

(1)
$$\begin{tikzcd}
& X' \arrow[dl, "\phi"'] \arrow[d, dotted, "\alpha"] \arrow[dd, bend left, "g'"] \\
X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X'' \arrow[l, "p_X"] \arrow[d, "f'"] \\
S & S' \arrow[l, "p"]
\end{tikzcd}.$$

<u>Claim:</u> $\alpha : X \tilde{\to} X''$ *is an isomorphism.*

Proof of the claim. As $p : S' \to S$ is an étale cover, its base-change $p_X : X'' \to X$ is an étale cover as well. Since S' is connected, it follows from the exactness on the right that X'' is connected as well hence, from lemma 5.16 and corollary 5.9 the morphism $\alpha : X' \to X''$ is an étale cover. So, it only remains to prove that $r(\alpha) = 1$. For this, consider the base-change of (1) *via* $\overline{s} : \mathrm{spec}(k(\overline{s})) \to S$.

(2)
$$\begin{tikzcd}
& X'_{\overline{s}} \arrow[dl, "\phi_{\overline{s}}"'] \arrow[d, "\alpha_{\overline{s}}"] \arrow[dd, bend left, "g_{\overline{s}}"] \\
X_{\overline{s}} \arrow[ur, bend left, "\sigma"] \arrow[d, "f_{\overline{s}}"'] \arrow[dr, phantom, "\square"] & X''_{\overline{s}} \arrow[l, "p_{X_{\overline{s}}}"] \arrow[d, "f'_{\overline{s}}"] \\
k(\overline{s}) & S'_{\overline{s}} \arrow[l, "p_{\overline{s}}"]
\end{tikzcd}.$$

Since $\alpha_{\overline{s}} : X'_{\overline{s}} \to X''_{\overline{s}}$ is an étale cover, it induces a surjective map $\pi_0(X'_{\overline{s}}) \twoheadrightarrow \pi_0(X''_{\overline{s}})$, where $\pi_0(-)$ denotes the set of connected components. But, as both $g' : X' \to S'$ and $f' : X'' \to S'$ are geometrically connected, $|\pi_0(X'_{\overline{s}})| = |\pi_0(X''_{\overline{s}})| (= r(p))$ hence, actually, the map $\pi_0(X'_{\overline{s}}) \twoheadrightarrow \pi_0(X''_{\overline{s}})$ is bijective. So it is enough to find $X'_{\overline{s}0} \in \pi_0(X'_{\overline{s}})$ such that $\alpha_{\overline{s}} : X'_{\overline{s}} \to X''_{\overline{s}}$ induces an isomorphism from $X'_{\overline{s}0}$ to $\alpha_{\overline{s}}(X'_{\overline{s}0})$. For this, consider $X'_{\overline{s}0} := \sigma(X\overline{s})$ and set $X''_{\overline{s}0} := \alpha_{\overline{s}}(X'_{\overline{s}0})$. Then σ induces an isomorphism from $X_{\overline{s}}$ to $X'_{\overline{s}}$ and, as $p_{X\overline{s}} : X''_{\overline{s}} \to X_{\overline{s}}$ is totally split, it induces an isomorphism from $X''_{\overline{s}0}$ to $X_{\overline{s}}$. Hence the conclusion follows from

$$\sigma|^{X'_{\overline{s}0}} \circ p_{X\overline{s}}|_{X''_{\overline{s}0}} \circ \alpha\overline{s}|^{X''_{\overline{s}0}}_{X'_{\overline{s}0}} = Id_{X''_{\overline{s}0}}$$

Remark 6.10. The assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ can be omitted and the conclusion of theorem **??** then becomes that the following canonical exact sequence of profinite groups is exact:

$$\pi_1(\overline{X}_1, \overline{x}_1) \xrightarrow{i_1} \pi_1(X, x_{(1)}) \xrightarrow{p_1} \pi_1(S, \overline{s}_1) \to \pi_0(\overline{X}_1) \to \pi_0(X) \to \pi_0(S) \to 1$$

Theorem 6.4 will also play a crucial part in the construction of the specialization morphism in section 9.

6.3. **Abelian varieties.** A main reference for abelian varieties is [Mum70]. See also [Mi86] for a concise introduction.

Let k be an algebraically closed field and A an abelian variety over k. For each $n \geq 1$ let $A[n]$ denote the group of k-points underlying the kernel of the multiplication-by-n morphism:

$$[n_A] : A \to A.$$

For each prime ℓ, the multiplication-by-ℓ morphism induces a projective system structure on the $A[\ell^n]$, $n \geq 0$ and one sets:

$$T_\ell(A) := \varprojlim A[\ell^n].$$

If ℓ is prime to the characteristic of k then $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$ whereas if $\ell = p$ is the characteristic of k then $T_p(A) \simeq \mathbb{Z}_p^r$, where g and $r(\leq g)$ denote the dimension and p-rank of A respectively [Mum70, Chap. IV, §18].

Theorem 6.11. *There is a canonical isomorphism:*

$$\pi_1(A;0_A)\tilde{\to}\prod_{\ell:\mathrm{prime}}T_\ell(A).$$

Proof. The proof below was suggested to me by the referee. For another proof based on rigidity, see [Mum70, Chap. IV, §18].

Given a profinite group Π and a prime ℓ, let $\Pi^{(\ell)}$ denote its pro-ℓ completion that is its maximal pro-ℓ quotient, which can also be described as:

$$\Pi^{(\ell)}=\varprojlim \Pi/N,$$

where the projective limit is over all normal open subgroups of index a power of ℓ in Π.

Claim 1: $\pi_1(A;0_A)$ *is abelian. In particular,*

$$\pi_1(A,0)=\prod_{\ell:\mathrm{prime}}\pi_1(A,0)^{(\ell)}.$$

Proof of claim 1. From lemma 6.6, the multiplication map $\mu:A\times_k A\to A$ on A induces a morphism of profinite groups:

$$\pi_1(\mu):\pi_1(A;0_A)\times\pi_1(A;0_A)\to\pi_1(A;0_A).$$

The canonical section $\sigma_1=A\to A\times_k A$ of the first projection $p_1:A\times_k A\to A$ induces the morphism of profinite groups:

$$\begin{array}{rccl}\pi_1(\sigma_1): & \pi_1(A;0_A) & \to & \pi_1(A;0_A)\times\pi_1(A;0_A)\\ & \gamma & \mapsto & (\gamma,1)\end{array}$$

and, by functoriality, $\pi_1(\mu)\circ\pi_1(\sigma_1)=Id$. The same holds for the second projection and since σ_1 and σ_2 commute, one gets:

$$\begin{array}{rl}\pi_1(\mu)(\gamma_1,\gamma_2) & =\pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1)\pi_1(\sigma_2)(\gamma_2))\\ & =\pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1))\pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2))=\gamma_1\gamma_2\\ & =\pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2)\pi_1(\sigma_1)(\gamma_1))\\ & =\pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2))\pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1))=\gamma_2\gamma_1.\end{array}$$

Claim 2 (Serre-Lang): *Let $\phi:X\to A$ be a connected étale cover. Then X carries a unique structure of abelian variety such that $\phi:X\to A$ becomes a separable isogeny.*

Proof of claim 2. The idea is to construct first the group structure on one fibre and, then, extend it automatically by the formalism of Galois categories. Let $x:\mathrm{spec}(k)\to X$ such that $\phi(x)=0_A$. Then the pointed connected étale cover $\phi:(X;x)\to(A;0_A)$ corresponds to a transitive $\pi_1(A;0_A)$-set M together with a distinguished point $m\in M$. Since $\pi_1(A;0_A)$ is abelian, the map:

$$\begin{array}{rccl}\mu_M: & M\times M & \to & M\\ & (\gamma_1 m,\gamma_2 m) & \mapsto & \gamma_1\gamma_2 m\end{array}$$

is well defined, maps (m,m) to m and is $\pi_1(A;0_A)\times\pi_1(A;0_A)$-equivariant if we endow M with the structure of $\pi_1(A;0_A)\times\pi_1(A;0_A)$-set induced by $\pi_1(\mu)$ (which corresponds to the étale cover $X\times_{\phi,A,\mu}(A\times_k A)\to A\times_k A$). Hence it corresponds to a morphism $\mu_X^0:X\times_k X\to X\times_{\phi,A,\mu}(A\times_k A)$ above $A\times_k A$ or, equivalently, to a morphism $\mu_X:X\times_k X\to X$ fitting in:

$$\begin{tikzcd}
X\times_k X \arrow[r, "\mu_X^0"] \arrow[rr, bend left, "\mu_X"] \arrow[d, "\phi\times_k\phi"'] & X\times_{\phi,A,\mu}(A\times_k A) \arrow[r] \arrow[dl] \arrow[dr, phantom, "\square"] & X \arrow[d, "\phi"] \\
A\times_k A \arrow[rr, "\mu"'] & & A.
\end{tikzcd}$$

and mapping (x,x) to x. By the same arguments, one constructs $i_X:X\to X$ above $[-1_A]:A\to A$ mapping x to x, checks that this endows X with the structure of an algebraic group with unity x (hence, of an abelian variety since X is connected and $\phi : X \to A$ is proper) and such that $\phi : X \to A$ becomes a morphism of algebraic groups (hence a separable isogeny since $\phi : X \to S$ is an étale cover).

Now let $\phi : X \to A$ be a degree n isogeny. Then $\ker(\phi) \subset \ker([n_X])$ hence one has a canonical commutative diagram:

$$\begin{tikzcd} & X/\ker(\phi) \arrow[dl, "\psi"'] \arrow[r, equal] & A \\ X & X. \arrow[l, "{[n_X]}"] \arrow[u] \arrow[ur, "\phi"'] & \end{tikzcd}$$

From the surjectivity of ϕ, one also has $\phi \circ \psi = [n_A]$. When ℓ is a prime different from the characteristic p of k, combining this remark and claim 2, one gets that $([\ell^n] : A \to A)_{n\geq 0}$ is cofinal among the finite étale covers of A with degree a power of ℓ that is

$$\pi_1(A; 0_A)^{(\ell)} = \varprojlim A[\ell^n] = T_\ell(A).$$

When $\ell = p$, one has to be more careful since, when p divides n, the isogeny $[n_A] : A \to A$ is no longer étale. However, it factors as:

$$\begin{tikzcd} A \arrow[r, "\psi_n"] \arrow[d, "{[n_A]}"'] & B_n \arrow[dl, "\phi_n"] \\ A, & \end{tikzcd}$$

where $\phi_n : B_n \to A$ is an étale isogeny and $\psi_n : A \to B_n$ is a purely inseparable isogeny. In particular, one has:

$$\begin{aligned} \mathrm{Aut}(B_n/A) &= \mathrm{Aut}(k(B_n)/k(A)) \\ &= \mathrm{Aut}(k(A) \overset{[n_A]^\#}{\hookrightarrow} k(A)) \\ &= A[n](k) \end{aligned}$$

and, if $\phi : X \to A$ is a degree n étale isogeny, one gets a factorization $\phi_n = \phi \circ \psi$. Thus, in that case, $(\phi_{p^n} : B_{p^n} \to A)_{n\geq 0}$ is cofinal among the finite étale covers of A with degree a power of p hence, as $\mathrm{Aut}(B_{p^n}/A) = A[p^n](k)$, one has, again:

$$\pi_1(A; 0_A)^{(p)} = \varprojlim A[p^n](k) = T_p(A). \ \square$$

Now, assume that $k = \mathbb{C}$ and that $A = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice. Then, on the one hand, the universal covering of A is just the quotient map $\mathbb{C}^g \to A$ and has group $\pi_1^{top}(A(\mathbb{C}); 0_A) \simeq \Lambda$ whereas, on the other hand, for any prime ℓ:

$$\begin{aligned} T_\ell(A) &= \varprojlim A[\ell^n] \\ &= \varprojlim \frac{1}{\ell^n}\Lambda/\Lambda \\ &= \varprojlim \Lambda/\ell^n\Lambda \\ &= \Lambda^{(\ell)}, \end{aligned}$$

whence

$$\pi_1(A; 0_A) = \prod_{\ell:prime} T_\ell(A) = \prod_{\ell:prime} \pi_1^{top}(A(\mathbb{C}); 0_A)^{(\ell)} = \pi_1^{top}(\widehat{A(\mathbb{C})}; 0_A).$$

This is a special case of the much more general Grauert-Remmert theorem 8.1 but, basically, the only one where one has a purely algebraically proof of it.

6.4. **Normal schemes.** Let S be a normal connected (hence integral) scheme.

Lemma 6.12. *Let $k(S) \hookrightarrow L$ be a finite separable field extension. Then the normalization of S in $k(S) \hookrightarrow L$ is finite.*

Proof. Without loss of generality, we may assume that $S = \mathrm{spec}(A)$ is affine that is, we are to prove that given an integrally closed, noetherian ring A with fraction field K and a finite separable field extension $K \hookrightarrow L$, the integral closure B of A in $K \hookrightarrow L$ is a finitely generated A-module. Since $K \hookrightarrow L$ is separable, the trace form:

$$\begin{array}{rccc} \langle -,-\rangle : & L \times L & \to & K \\ & (x,y) & \mapsto & Tr_{L|K}(xy) \end{array}$$

is non-degenerate. Set $n := [L : K]$ and let $b_1, \dots, b_n \in B$ be a basis of L over K. Let $b_1^*, \dots, b_n^* \in L$ denote its dual with respect to $\langle -,-\rangle : L \times L \to K$. Then, since $Tr_{L|K}(B) \subset A$, one has $B \subset \oplus_{i=1}^n Ab_i^*$ hence B is a finitely generated A-module as well since A is noetherian. $\square$

When S is normal, we can improve theorem 5.5 as follows.

Lemma 6.13. *Let A be a noetherian integrally closed local ring with fraction field K and set $S = \mathrm{spec}(A)$. Let $\phi : X \to S$ an unramified (resp. étale) morphism. Then, for any $x \in X$, there exists an open affine neighborhood U of x such that one has a factorization:*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{spec}(C), \arrow[dl] \\ S & \end{tikzcd}$$

where $\mathrm{spec}(C) \to S$ is a standard étale morphism where $B = A[T]/PA[T]$ can be chosen in such a way that the monic polynomial $P \in A[T]$ becomes irreducible in $K[T]$ and $U \hookrightarrow \mathrm{spec}(C)$ is an immersion (resp. an open immersion).

Proof Let $\mathfrak{m}$ denote the maximal ideal of A and, correspondingly, let s denote the closed point of S. From theorem 5.5, one may assume that $\phi : X \to S$ is induced by an A-algebra of the form $A \to B_b$ with $B = A[T]/PA[T]$ and $b \in B$ such that $P'(t)$ is invertible in B_b. Since A is integrally closed, any monic factor of P in $K[T]$ is in $A[T]$. Let $x \in X_s$ and fix an irreducible monic factor Q of P mapping to 0 in $k(x)$. Write $P = QR$ in $A[T]$. As $\overline{P} \in k(s)[T]$ is separable, $\overline{Q}$ and $\overline{R}$ are coprime in $k(s)[T]$ or, equivalently:

$$\langle \overline{Q}, \overline{R} \rangle = k(s)[T].$$

But, then, as Q is monic $M := A[T]/\langle Q, R\rangle$ is a finitely generated A-module so, from Nakayama, $A[T] = \langle Q, R\rangle$. This, by the Chinese remainder theorem:

$$A[T]/PA[T] = A[T]/QA[T] \times A[T]/RA[T].$$

Set $B_1 := A[T]/QA[T]$ and let b_1 denote the image of b in B_1. Then the open subscheme $U_1 := \mathrm{spec}(B_{1b_1}) \hookrightarrow X$ contains x and:

$$U_1 := \mathrm{spec}(B_{1b_1}) \hookrightarrow X \to S$$

is a standard morphism of the required form. $\square$

Lemma 6.14. *Let $\phi : X \to S$ be an étale cover. Then X is also normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component X_0 of X, the induced étale cover $X_0 \to S$ is the normalization of S in $k(S) \hookrightarrow k(X_0)$.*

Proof. We first prove the assertion when $S = \mathrm{spec}(A)$ with A a noetherian integrally closed local ring and $\phi : X \to S$ is a standard morphism as in lemma 6.13. Let $K(= k(S))$ denote the fraction field of A. By assumption, $L := C \otimes_A K = K[T]/PK[T]$ is a finite separable field extension of K. Let A^c denote the integral closure of A in $K \hookrightarrow L$. Since B is integral over A, one has $A \subset B \subset A^c \subset L$ hence $B_b \subset (A^c)_b = ((A^c)_b)^c \subset L$. So, to show that C is integrally closed in $K \hookrightarrow L$, it is enough to show that $A^c \subset B_b$. So let $\alpha \in A^c$ and write:

$$\alpha = \sum_{i=0}^{n-1} a_i t^i,$$

with $a_i \in K$, $i = 1, \dots n$ and $n = \deg(P)$. As $K \hookrightarrow L$ is separable of degree n, there are exactlly n distinct morphisms of K-algebras:

$$\phi_i : L \hookrightarrow \overline{K}$$

Let $V_n(t) := V(\phi_1(t), \dots, \phi_n(t))$ denote the Vandermonde matrix associated with $\phi_1(t), \dots, \phi_n(t)$. Then one has:

$$|V_n(t)|(a_i)_{0\leq i\leq n-1} = {}^tCom(V_n(t))(\phi_i(\alpha))_{1\leq i\leq n}$$

(where ${}^tCom(-)$ denotes the transpose of the comatrix and $|-|$ the determinant). Hence, as the $\phi_i(t)$ and the $\phi_i(\alpha)$ are all integral over A, the $|V_n(t)|a_i$ are also all integral over A. By assumption, the a_i are in K and $|V_n(t)|$ is in K since it is symmetric in the $\phi_i(t)$. So, as A is integrally closed, the $|V_n(t)|a_i$ are in A, from which the conclusion follows since $|V_n(t)|$ is a unit in C (recall that $P'(t)$ is invertible in C).

We now turn to the general case. From lemma 6.13, the above already shows that X is normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. So, without loss of generality we may assume that X is a normal connected hence integral scheme. But then, for any open subscheme $U \subset S$, the ring $\mathcal{O}_X(\phi^{-1}(U))$ is integral ring and its local rings are all integrally closed so $\mathcal{O}_X(\phi^{-1}(U))$ is integrally closed as well and, since it is also integral over $\mathcal{O}_S(U)$, it is the integral closure of $\mathcal{O}_S(U)$ in $k(S) \hookrightarrow k(X)$. □

The following provides a converse to lemma 6.14:

Lemma 6.15. *Let $k(S) \hookrightarrow L$ be a finite separable field extension which is unramified over S. Then the normalization $\phi : X \to S$ of S in $k(S) \hookrightarrow L$ is an étale cover.*

Proof. Since S is locally noetherian, $\phi : X \to S$ is finite by lemma 6.12; it is also surjective [AM69, Thm. 5.10] and, by construction it is unramified. So we are only to prove that $\phi : X \to S$ is flat, namely that $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$ is a flat algebra, $x \in X$. One has a commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{X,x} & C \arrow[l, two heads] \\ \mathcal{O}_{S,\phi(x)} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

where $\mathcal{O}_{S,\phi(x)} \to C$ is a standard algebra as in lemma 6.13, $C \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective and, as $\phi : X \to S$ is surjective, $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$. In particular,

$$\mathcal{O}_{S,\phi(x)} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \hookrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$$

is injective as well hence:

$$C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$$

is non-zero. But, as $C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$ is a field, the above morphism is actually injective and, as $\mathcal{O}_{S,\phi(x)} \to k(S)$ is faithfully flat, this implies that $C \twoheadrightarrow \mathcal{O}_{X,x}$ is injective hence bijective. □

Lemma 6.14 shows that there is a well-defined functor:

$$\begin{array}{rlcl} R\colon & \mathcal{C}_S & \to & (FEAlg/k(S))^{op} \\ & X \to S & \mapsto & k(S) \hookrightarrow R(X) := \prod_{X_0 \in \pi_0(X)} k(X_0). \end{array}$$

Let $FEAlg/k(S)/S \subset FEAlg/k(S)$ denote the full subcategory of finite étale algebras $k(S) \hookrightarrow R$ which are unramified over S. Lemmas 6.14 and 6.15 show:

Theorem 6.16. *The functor $R : \mathcal{C}_S \to FEAlg/k(S)$ is fully faithfull and induces an équivalence of categories $R : \mathcal{C}_S \to FEAlg/k(S)/S$ with pseudo-inverse the normalization functor.*

Let $\eta \in S$ denote the generic point of S hence $k(\eta) = k(S)$. Let $k(\eta) \hookrightarrow \Omega$ be an algebraically closed field extension defining geometric points $\overline{s}_\eta : \mathrm{spec}(\Omega) \to \mathrm{spec}(k(\eta))$ and $\overline{\eta} : \mathrm{spec}(\Omega) \to S$. From theorem 6.16, the base-change functor

$$\eta^* : \mathcal{C}_S \to \mathcal{C}_{\mathrm{spec}(k(\eta))}$$

is fully faithfull hence, from proposition 4.3 (1), induces an epimorphism of profinite groups:

$$\pi_1(\eta) : \pi_1(\mathrm{spec}(k(\eta); \overline{s}_\eta) \twoheadrightarrow \pi_1(S; \overline{s})$$

whose kernel is the absolute Galois group of the maximal algebraic extension $k(\eta) \hookrightarrow M_{k(S),S}$ of $k(\eta)$ in Ω which is unramified over S.

Example 6.17. Let S be a curve, smooth and geometrically connected over a field k and let $S \hookrightarrow S^{cpt}$ be the smooth compactification of S. Write $S^{cpt} \setminus S = \{P_1, \dots, P_r\}$. Then the extension $k(S) \hookrightarrow M_{k(S),S}$ is just the maximal algebraic extension of $k(S)$ in Ω unramified outside the places $P_1, \dots, P_r$.

7. Geometrically connected schemes of finite type

Let S be a scheme geometrically connected and of finite type over a field k. Fix a geometric point $\overline{s}$: $\mathrm{spec}(k(\overline{s})) \to S_{k^s}$ with image again denoted by $\overline{s}$ in S and $\mathrm{spec}(k)$.

Proposition 7.1. *The morphisms $(S_{k^s}, \overline{s}) \to (S, \overline{s}) \to (\mathrm{spec}(k), \overline{s})$ induce a canonical short exact sequence of profinite groups:*

$$1 \to \pi_1(S_{k^s}; \overline{s}) \xrightarrow{i} \pi_1(S; \overline{s}) \xrightarrow{p} \pi_1(\mathrm{spec}(k); \overline{s}) \to 1.$$

Example 7.2. Assume furthermore that S is normal. Then the assumption that S is geometrically connected over k is equivalent to the assumption that $\overline{k} \cap k(S) = k$ and, with the notation of subsection 6.4, the short exact sequence above is just the one obtained from usual Galois theory:

$$1 \to \mathrm{Gal}(M_{k(S),S}|k^s(S)) \to \mathrm{Gal}(M_{k(S),S}|k(S)) \to \Gamma_k \to 1.$$

Proof. We use, again, the criteria of proposition 4.3.

<u>Exactness on the right:</u> As S is geometrically connected over k, the scheme S_K is also connected for any finite separable field extension $k \hookrightarrow K$.

<u>Exactness on the left:</u> For any étale cover $f : X \to S_{k^s}$ we are to prove that there exists an étale cover $f : \tilde{X} \to S$ such that $f_{k(s)}$ dominates f. From lemma 6.7, there exists a finite separable field extension $k \hookrightarrow K$ and an étale cover $\tilde{f} : \tilde{X} \to S_K$ which is a model of $f : X \to S_{k^s}$ over S_K. But then, the composite $f : \tilde{X} \to S_K \to S$ is again an étale cover whose base-change *via* $S_{k^s} \to S$ is the coproduct of $[K : k]$ copies of f hence, in particular, dominates f.

<u>Exactness in the middle:</u> From lemma 6.9, this amounts to show that $\ker(p) \subset \mathrm{im}(i)$. For any connected étale cover $\phi : X \to S$ such that $\phi_{k^s} : X_{k^s} \to S_{k^s}$ admits a section, say $\sigma : S_{k^s} \hookrightarrow X_{k^s}$, we are to prove that there exists a finite separable field extension $k \hookrightarrow K$ such that the base change of $\mathrm{spec}(K) \to \mathrm{spec}(k)$ *via* $S \to \mathrm{spec}(k)$ dominates $\phi : X \to S$. So, let $k \hookrightarrow K$ be a finite separable field extension over which $\sigma : S_{k^s} \hookrightarrow X_{k^s}$ admits a model $\sigma_K : S_K \hookrightarrow X_K$. This defines a morphism from S_K to X over S by composing $\sigma_K : S_K \hookrightarrow X_K$ with $X_K \to X$. □

Proposition 7.1 shows that the fundamental group $\pi_1(S)$ of a scheme S geometrically connected and of finite type over a field k can be canonically decomposed into a geometric part $\pi_1(S_{k^s})$ and an arithmetic part Γ_k. This raises several problems:

(1) Determine the geometric part $\pi_1(S_{k^s})$;
(2) Describe the sections of $\pi_1(S) \twoheadrightarrow \Gamma_k$;
(3) Describe the outer representation $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s}))$.

In the end of these notes, we are going to explain how problem (1) can be solved (fully in characteristic 0 and partly in positive characteristic). Basically, this is done in three steps (one step in characteristic 0):

(a) G.A.G.A. theorems (see section 8), which show that the étale fundamental group of a connected scheme locally of finite type over $\mathbb{C}$ is the profinite completion of the topological fundamental of its underline topological space. The latter can often be explicitly computed by methods from algebraic topology. From the invariance of fundamental groups under algebraically closed field extensions (see subsection 6.2), this yields the determination of most of the étale fundamental groups of connected schemes locally of finite type over algebraically closed field in characteristic 0.

(b) Specialization theory (see section 9), which says that if $f : X \to S$ is a proper separable morphism with geometrically connected fibres and $s_0,\ s_1 \in S$ are such that s_0 is a specialization of s_1, there is an epimorphism of profinite groups:
$$sp : \pi_1(X_{\overline{s}_1}) \twoheadrightarrow \pi_1(X_{\overline{s}_0}).$$

(c) The Zariski-Nagata purity theorem (see section 10.1), which yields information about the kernel of the above specialization epimorphism when $f : X \to S$ is furthermore assumed to be smooth and, in particular, shows that it induces an isomorphism on the prime-to-p completions, where p denotes the residue characteristic of s_0. Note that, however, to understand the prime-to-p completion of the étale fundamental group in positive characteristic $p > 0$ by this method, one has to face the deep problem of lifting schemes from characteristic p to characteristic 0; we will give an illustration of this in the proof of theorem 11.1. Concerning the pro-p completion and the determination of the full étale fundamental groups of curves in positive characteristic $p > 0$, see remarks 6.8 and 11.5.

Problems (2) and (3) are still widely open.

The section conjecture provides a conjectural answer to problem (2) when k is a finitely generated field of characteristic 0 and S is a smooth, separated, geometrically connected hyperbolic curve over k. More precisely, let $S \hookrightarrow S^{cpt}$ denote the smooth compactification of S. Any $s \in S(k)$ induces a ($\pi_1(S_{k^s})$-conjugacy class of) section(s) $s : \Gamma_k \to \pi_1(S)$. More generally, given a point $\tilde{s} \in S^{cpt}(k)$, if $I(\tilde{s})$ and $D(\tilde{s})$ denote the inertia and decomposition group of $\tilde{s}$ in $\Gamma_{k(S^{cpt})}$ respectively, then the short exact sequence:
$$1 \to I(\tilde{s}) \to D(\tilde{s}) \to \Gamma_k \to 1$$
always splits but this splitting is not unique up to inner conjugation by elements of $\Gamma_{\overline{k}(S^{cpt})}$ hence, any point $\tilde{s} \in S^{cpt}(k) \setminus S(k)$ gives rise to several ($\pi_1(S_{k^s})$-conjugacy class of) sections. A section $s : \Gamma_k \to \pi_1(S)$ is said to be *geometric* if it raises from a point $\tilde{s} \in S^{cpt}(k)$ and is said to be *unbranched* if $s(\Gamma_k)$ is contained in no decomposition group of a point $\tilde{s} \in S^{cpt}(k) \setminus S(k)$ in $\pi_1(S)$. Let $\Sigma(S)$ denote the set of conjugacy classes of sections of $\pi_1(S) \twoheadrightarrow \Gamma_k$. A basic form of the section conjecture can thus be formulated as follows:

Conjecture 7.3. (Section conjecture) *For any smooth, separated and geometrically connected curve S over a finitely generated field k of characteristic 0 the canonical map $S(k) \to \Sigma(S)$ is injective and induces a bijection onto the set of $\pi_1(S_{k^s})$-conjugacy classes of unbranched sections. Furthermore, any section is a geometric section.*

The injectivity part of the section conjecture was already known to A. Grothendieck (basically as a consequence of Lang-Néron theorem with some technical adjustements in the non-proper case); it is the surjectivity part which is difficult. It easily follows from the formalism of Galois categories, Mordell conjecture and Uchida's theorem [U77] that the section conjecture (for all hyperbolic curves over k) is equivalent to:

Conjecture 7.4. (Section conjecture - reformulation) *For any smooth, separated and geometrically connected curve S over a finitely generated field k of characteristic 0 one has $S(k) \neq \emptyset$ if and only if $\Sigma(S) \neq \emptyset$.*

One can formulate a pro-p variant of the section conjecture. Let $K^{(p)}$ denote the kernel of the pro-p completion $\pi_1(S_{k^s}) \twoheadrightarrow \pi_1(S_{k^s})^{(p)}$; by definition $K^{(p)}$ is characteristic in $\pi_1(S_{k^s})$ hence normal in $\pi_1(S)$. So, defining $\pi_1(S)^{[p]} := \pi_1(S)/K^{(p)}$, one gets a short exact sequence of profinite groups:
$$1 \to \pi_1(S_{k^s})^{(p)} \to \pi_1(S)^{[p]} \to \Gamma_k \to 1$$
Let $\Sigma^{(p)}(S)$ denote the set of conjugacy classes of sections of $\pi_1(S)^{[p]} \twoheadrightarrow \Gamma_k$ and consider the composite map:
$$S(k) \to \Sigma(S) \to \Sigma^{(p)}(S).$$
Then, S. Mochizuki showed that this remains injective [Mo99] but Y. Hoshi showed that it is no longer surjective [Ho10b].

One can also formulate a birational variant of the section conjecture, where the short exact sequence of profinite group:

$$1 \to \pi_1(S_{k^s}) \to \pi_1(S) \to \Gamma_k \to 1$$

is replaced by the usual short exact sequence from Galois theory of field extensions:

$$1 \to \Gamma_{k^s(S)} \to \Gamma_{k(S)} \to \Gamma_k \to 1$$

In that case, there are some examples where the answer is known to be positive [St07] and the birational section conjecture itself was proved by J. Koenigsmann when k is replaced by a p-adic field [K05].

As for problem (3), it leads to a whole bunch of questions and conjectures usually gathered under the common denomination of *anabelian geometry*. Among those problems one can mention, for instance:

- Is $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s}))$ injective? The answer is known to be positive for smooth, separated, geometrically connected hyperbolic curves over sub-p-adic fields (*i.e.* subfields of finitely generated extensions of $\mathbb{Q}_p$). The affine case when k is a number field was proved by M. Mastumoto [M96], the general case was completed by Y. Hoshi and S. Mochizuki when k is a sub-p-adic field [HoMo10].
- Given a prime ℓ, up to what extend does the kernel of the outer pro-ℓ representation $\rho^{(\ell)} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})^{(\ell)})$ determine the isomorphism class of S? Under some technical conditions Y. Hoshi [Ho10a] and S. Mochizuki [Mo03] obtained partial results for affine hyperbolic curves of genus ≤ 1.
- Up to what extend does the outer (resp. the outer pro-ℓ) representation $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})$ (resp. $\rho^{(\ell)} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})^{(\ell)}))$ determine S? When S is assumed to be an hyperbolic curve, this rather vague question is often referred to as *Grothendieck's anabelian conjecture*. One motivation for it is Tate conjecture for abelian varieties. Indeed, given two proper hyperbolic curves S_1, S_2 over a finitely generated field k of characteristic 0 then, or any prime ℓ if the outer pro-ℓ abelianized representations:
$$\rho_i^{(\ell),ab} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{i\overline{k}})^{(\ell),ab}) = \mathrm{Aut}(T_\ell(J_{S_i|k}))$$
coincide for $i = 1, 2$ then, $J_{S_1|k}$ and $J_{S_2|k}$ are isogenous. In particular, from the isogeny theorem, there are only finitely many isomorphism classes of proper hyperbolic curves X with the same outer pro-ℓ abelianized representation. It is thus reasonable to expect that taking into account the whole outer pro-ℓ representation or, even more, the whole outer representation, will determine entirely the isomorphism classes of hyperbolic curves. Note that the assumption that S is hyperbolic implies that $\pi_1(S_{\overline{k}})$ has trivial center hence that:
$$\pi_1(S) = \mathrm{Aut}(\pi_1(S_{\overline{k}})) \times_{\mathrm{Out}(\pi_1(S_{\overline{k}})),\rho} \Gamma_k$$
so $\pi_1(S) \twoheadrightarrow \Gamma_k$ can be recovered from $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{\overline{k}}))$. More precisely, one can formulate Grothendieck's anabelian conjecture for hyperbolic curves as follows. Let Pro_k^{open} denote the category of profinite groups G equipped with an epimorphism $p : G \twoheadrightarrow \Gamma_k$ and where morphisms from $p_1 : G_1 \twoheadrightarrow \Gamma_k$ to $p_2 : G_2 \twoheadrightarrow \Gamma_k$ are morphisms from G_1 to G_2 in Pro with representatives $\phi : G_1 \to G_2$ such that:
  (i) $\rho_2 \circ \phi = \rho_1$ modulo inner conjugation by elements of Γ_k;
  (ii) $\mathrm{im}(\phi)$ is open in G_2.

Conjecture 7.5. (Grothendieck's anabelian conjecture for hyperbolic curves) *Let k be a finitely generated field of characteristic 0. Then the functor $\pi_1(-)$ from the category of smooth, separated, geometrically hyperbolic curves over k with dominant morphisms to Pro_k^{open} is fully faithfull.*

After works of K. Uchida [U77], A. Tamagawa proved conjecture 7.5 for affine hyperbolic curves [T97]. Using technics from p-adic Hodge theory, S. Mochizuki then proved the general form of conjecture 7.5 (and, more generally, its pro-ℓ-variant for k a sub-ℓ-adic field) [Mo99]. For an introduction to this subject, see [NMoT01]. For more elaborate surveys, see [Sz00], [H00] and the Bourbaki lecture by G. Faltings [F98].

One can formulate birational, higher dimensional variants, variants over finite fields or function fields of conjecture 7.5. These questions are currently intensively studied. For more recent results, see the works of Y. Hoshi, S. Mochizuki, H. Nakamura, F. Pop, M. Saïdi, J. Stix, A. Tamagawa *etc.*

8. G.A.G.A. theorems

In this section, we review implications of the so-called G.A.G.A. theorems (named after J.-P. Serre's fundamental paper [S56] *Géométrie algébrique et géométrie analytique*) to the description of étale fundamental groups of schemes locally of finite type over $\mathbb{C}$. The main result is theorem 8.1, which states that this is nothing but the profinite completion of the topological fundamental group of the underlying topological space. However, the definition of what is meant by "underlying topological space" is not so clear *a priori* and the definition - as well of the proof - goes through the complex analytic space X^{an} which can canonically be associated to any scheme X locally of finite type over $\mathbb{C}$. In subsection 8.1, we give the definition of complex analytic spaces, sketch the construction of the analytification functor $X \mapsto X^{an}$ and provide a partial dictionnary of properties which it preserves. In subsection 8.2, we state the main G.A.G.A. theorem alluded to above. The proof of this theorem is beyond the scope of these notes. For a clear exposition based on [S56] and [Hi64], we refer to [SGA1, Chap XII, §5].

8.1. **Complex analytic spaces.** As schemes over $\mathbb{C}$ are obtained by glueing affine schemes over $\mathbb{C}$ in the category $LR/\mathbb{C}$ of locally-ringed spaces in $\mathbb{C}$-algebras, complex analytic spaces are obtained by glueing "affine" complex analytic spaces in $LR/\mathbb{C}$.

Affine complex analytic spaces are defined as follows. Let $U \subset \mathbb{C}^n$ denote the polydisc of all $\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that $|z_i| < 1$, $i = 1, \dots, n$ and, given analytic functions $f_1, \dots, f_r : U \to \mathbb{C}$, let $\mathfrak{U}(f_1, \dots, f_r)$ denote the locally ringed space in $\mathbb{C}$-algebra whose underlying topological space the closed subset:

$$\bigcap_{i=1}^{r} f_i^{-1}(0) \subset U$$

endowed with the topology inherited from the transcendent topology on U and whose structural sheaf is:

$$\mathcal{O}_U / < f_1, \dots, f_r >,$$

where $\mathcal{O}_U$ is the sheaf of germs of analytic functions on U.

The category $An_{\mathbb{C}}$ of complex analytic spaces is then the full subcategory of $LR/\mathbb{C}$ whose objects $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ are locally isomorphic to affine complex analytic spaces.

Now, let X be a scheme locally of finite type over $\mathbb{C}$

<u>Claim:</u> *The functor* $\mathrm{Hom}_{LR/\mathbb{C}}(-, X) : An_{\mathbb{C}}^{op} \to$ *Sets is representable that is there exists a complex analytic space* X^{an} *and a morphism* $\phi_X : X^{an} \to X$ *in* $LR/\mathbb{C}$ *inducing a functor isomorphism*

$$\phi_X \circ : \mathrm{Hom}_{An_{\mathbb{C}}}(-, X^{an}) \tilde{\to} \mathrm{Hom}_{LR_{\mathbb{C}-Alg}}(-, X)|_{An_{\mathbb{C}}^{op}}.$$

Furthermore, for any $x \in X^{an}$*, the canonical morphism induced on completions of local rings* $\hat{\mathcal{O}}_{X,\phi_X(x)} \tilde{\to} \hat{\mathcal{O}}_{X^{an},x}$ *is an isomorphism.*

Proof (sketch of)

(1) Assume that X^{an} exists for a given scheme X, locally of finite type over $\mathbb{C}$. Then:
   (a) U^{an} exists for any open subscheme $U \hookrightarrow X$ ($U^{an} = \phi_X^{-1}(U)$ with the structure of complex analytic space induced from the one of X);
   (b) Z^{an} exists for any closed subscheme $Z \hookrightarrow X$ (if $\mathcal{I}_Z$ denotes the coherent sheaf of ideals of $\mathcal{O}_X$ defining Z then $\phi_X^{an}\mathcal{I}_Z =: \mathcal{I}_Z^{an}$ is again a coherent sheaf of ideals of $\mathcal{O}_{X^{an}}$ hence defines a closed analytic subspace $Z^{an} \hookrightarrow X^{an}$).

(2) Assume that X_i^{an} exists for a given scheme X_i, locally of finite type over $\mathbb{C}$, $i = 1, 2$. Then $(X_1 \times_{\mathbb{C}} X_2)^{an}$ exists and is $X_1^{an} \times X_2^{an}$.

(3) $(\mathbb{A}^1_{\mathbb{C}})^{an}$ exists ($= \mathbb{A}^1(\mathbb{C})$) hence it follows from (2) that$(\mathbb{A}^n_{\mathbb{C}})^{an}$ exists for $n \geq 1$. Then, it follows from (1) (b) that X^{an} exists for any affine scheme, locally of finite type over $\mathbb{C}$.

(4) Now, given any scheme X locally of finite type over $\mathbb{C}$, consider a covering of X by open affine subschemes $X_i \hookrightarrow X$, $i \in I$ and set $X_{i,j} := X_i \cap X_j$, $i, j \in I$. From (3) and (1) (a), one knows that X_i^{an} and $X_{i,j}^{an}$

exist, $i,j \in I$. Then the analytic space X^{an} obtained by glueing the X_i^{an} along the $X_{i,j}^{an}$ satisfies the required universal property. □

The morphism $\phi_X : X^{an} \to X$ is unique up to a unique X-isomorphism and is called the *complex analytic space associated with* X or the *analytification of* X. In particular, given a $\mathbb{C}$-morphism $f : X \to Y$ of schemes locally of finite type over $\mathbb{C}$, it follows from the universal property of $\phi_Y : Y^{an} \to Y$ that there exists a unique morphism $f^{an} : X^{an} \to Y^{an}$ in $An_{\mathbb{C}}$ such that $\phi_Y \circ f^{an} = f \circ \phi_X$. One readily checks that this gives rise to a functor:

$$(-)^{an} : Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}},$$

where $Sch^{LFT}/\mathbb{C}$ denotes the category of schemes locally of finite type over $\mathbb{C}$.

There is a nice dictionary between the properties of X (resp. $X \to Y$) and those of X^{an} (resp. $X^{an} \to Y^{an}$). Morally, all those which are encoded in the completion of the local rings are preserved. For instance:

(1) Let P be the property of being connected, irreducible, regular, normal, reduced, of dimension d. Then X has P if and only if X^{an} has P;

(2) Let P be the property of being surjective, dominant, a closed immersion, finite, an isomorphism, a monomorphism, an open immersion, flat, unramified, étale, smooth. Then $X \to Y$ has P if and only if $X^{an} \to Y^{an}$ has P.

Concerning the categories $\mathrm{Mod}(X)$ and $\mathrm{Mod}(X^{an})$ of $\mathcal{O}_X$-modules and $\mathcal{O}_{X^{an}}$ respectively, one can easily show that the functor:

$$\phi_X^* : \mathrm{Mod}(X) \to \mathrm{Mod}(X^{an})$$

is exact, faithful, conservative and sends coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{X^{an}}$-modules.

8.2. **Main G.A.G.A. theorem.** The most important result of [S56] is that, when X is assumed to be projective over $\mathbb{C}$, the functor $\phi_X^* : \mathrm{Mod}(X) \to \mathrm{Mod}(X^{an})$ induces an equivalence of categories from coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{X^{an}}$-modules. By technical arguments such as Chow's lemma, this can be extended to schemes proper over $\mathbb{C}$. From the equivalence of categories between finite morphisms $Y \to X$ (resp. $Y^{an} \to X^{an}$) and coherent $\mathcal{O}_X$-algebras (resp. coherent $\mathcal{O}_{X^{an}}$-algebras), one easily deduces that for a proper schemes X over $\mathbb{C}$ the categories of finite étale covers of X and X^{an} are equivalent. Working more, one gets:

Theorem 8.1. ([SGA1, XII, Thm. 5.1]) *For any scheme X locally of finite type over $\mathbb{C}$, the functor $(-)^{an}$: $Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}}$ induces an equivalence from the category of étale covers of X to the category of étale covers of X^{an}.*

The category of étale covers of X^{an} is equivalent to the category of finite topological covers of the underlying transcendent topological space X^{top} of X^{an}. Indeed, observe that if $f : Y \to X^{top}$ is a finite topological cover then the local trivializations endow Y with a unique structure of analytic space (induced from X^{an}) and such that, with this structure, $f : Y \to X^{top}$ becomes an analytic cover. Conversely, if $f : Y \to X^{an}$ is an étale cover then, from theorem 5.5, for any $y \in Y$ one can find open affine neighborhoods $V = \mathrm{spec}(B)$ of y and $U = \mathrm{spec}(A)$ of $f(y)$ such that $f(V) \subset U$, $B = A[X]/ < f >$ and $(\frac{\partial f}{\partial X})_y \in \mathcal{O}_{Y,y}^{\times}$ hence the local inversion theorem gives local trivializations. So, for any $x \in X$ one has a canonical isomorphism of profinite groups :

$$\widehat{\pi_1^{top}(X^{top}}, x) \simeq \pi_1(X, x).$$

Example 8.2. Let X be a smooth connected curve over $\mathbb{C}$ of type (g,r) (that is the smooth compactification $\tilde{X}$ of X has genus g and $|\tilde{X} \setminus X| = r$). Then, for any $x \in X$ one has a canonical profinite group isomorphism $\widehat{\Gamma}_{g,r} \simeq \pi_1(X,x)$, where $\Gamma_{g,r}$ denotes the group defined by the generators $a_1, \dots, a_g, b_1, \dots, b_g, \gamma_1, \dots, \gamma_r$ with the single relation $[a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_r = 1$. From section 6.4, $\pi_1(X,x)$ can also be described as the Galois group $\mathrm{Gal}(M_{\mathbb{C}(X),X}|\mathbb{C}(X))$ of the maximal algebraic extension $M_{\mathbb{C}(X),X}$ of $\mathbb{C}(X)$ in $\overline{\mathbb{C}(X)}$ étale over X.

In particular, if $g = 0$ then $\pi_1(X,x)$ is the pro-free group on $r-1$ generators, so, any finite group G generated by $\leq r-1$ elements is a quotient of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{t_1, \dots, t_r\}, x)$ or, equivalently, appears as the Galois group of a Galois extension $\mathbb{C}(T) \hookrightarrow K$ unramified everywhere except over $t_1, \dots, t_r$. This solves the inverse Galois problem over $\mathbb{C}(T)$.

Exercise 8.3. Show that the étale fundamental group of an algebraic group over an algebraically closed field of characteristic 0 is commutative.

9. Specialization

9.1. **Statements.** Let S be a connected scheme and $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ (so, in particular, $f : X \to S$ is surjective, geometrically connected and X is connected). Fix $s_0,\ s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$ and geometric points $\overline{x}_i : \mathrm{spec}(\Omega_i) \to X_{\overline{s}_i}$, $i = 0, 1$. Denote again by $\overline{x}_i$ the images of $\overline{x}_i$ in X_{s_i} and X_i and by $\overline{s}_i$ the image of $\overline{x}_i$ in S, $i = 0, 1$.

The theory of specialization of fundamental groups consists, essentially, in comparing $\pi_1(X_{\overline{s}_1};\overline{x}_1)$ and $\pi_1(X_{\overline{s}_0};\overline{x}_0)$. The main result is the following.

Theorem 9.1. (Semi-continuity of fundamental groups) *There exists a morphism of profinite groups*

$$sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \to \pi_1(X_{\overline{s}_0};\overline{x}_0),$$

canonically defined up to inner automorphisms of $\pi_1(\overline{X}_0, \overline{x}_0)$. *If, furthermore,* $f : X \to S$ *is separable, then* $sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0};\overline{x}_0)$ *is an epimorphism.*

The morphism $sp : \pi_1(X_{\overline{s}_1},\overline{x}_1) \to \pi_1(X_{\overline{s}_0},\overline{x}_0)$ is called the *specialization morphism from* s_1 *to* s_0.

The proof of theorem 9.1 relies on the first homotopy sequence, already studied in subsection 6.2 but that we restate below with our notation.

Theorem 9.2. (First homotopy sequence) *Consider the canonical sequence of profinite groups induced by* $(X_{\overline{s}_1},\overline{x}_1) \to (X,\overline{x}_1) \to (S,\overline{s}_1)$*:*

$$\pi_1(X_{\overline{s}_1};\overline{x}_1) \xrightarrow{i_1} \pi_1(X;\overline{x}_1) \xrightarrow{p_1} \pi_1(S;\overline{s}_1). \tag{3}$$

Then $p_1 : \pi_1(X;\overline{x}_1) \twoheadrightarrow \pi_1(S;\overline{s}_1)$ *is an epimorphism and* $\mathrm{im}(i_1) \subset \ker(p_1)$. *If, furthermore,* $f : X \to S$ *is separable then* $\mathrm{im}(i_1) = \ker(p_1)$.

and the second homotopy sequence:

Theorem 9.3. (Second homotopy sequence) *Assume that* $S = \mathrm{Spec}(A)$ *with* A *a local complete noetherian ring and that* s_0 *is the closed point of* S. *Then, the canonical sequence of profinite groups induced by* $(X_{\overline{s}_0},\overline{x}_0) \to (X,\overline{x}_0) \to (S,\overline{s}_0)$*:*

$$1 \to \pi_1(X_{\overline{s}_0};\overline{x}_0) \xrightarrow{i_0} \pi_1(X;\overline{x}_0) \xrightarrow{p_0} \pi_1(S;\overline{s}_0) \to 1 \tag{4}$$

is exact and the canonical morphism $\Gamma_{k(s_0)}\tilde{\to}\pi_1(S;\overline{s}_0)$ *is an isomorphism. In particular, the canonical morphism* $\pi_1(X_{s_0};\overline{x}_0)\tilde{\to}\pi_1(X;\overline{x}_0)$ *is an isomorphism and if* $x_0 \in X(k(s_0))$ *then the above short exact sequence splits.*

9.2. **Construction of the specialization morphism.**
Assume first that $S = \mathrm{Spec}(A)$ with A a local complete noetherian ring and that s_0 is the closed point of S, $s_1 \in S$ is any point of S. Then, one has the following canonical diagram of profinite groups, which commutes up to inner automorphisms:

$$\begin{tikzcd}
(4) & 1 \arrow[r] & \pi_1(X_{\overline{s}_0};\overline{x}_0) \arrow[r, "i_0"] & \pi_1(X;\overline{x}_0) \arrow[r, "p_0"] & \pi_1(S;\overline{s}_0) \arrow[r] & 1 \\
(3) & & \pi_1(X_{\overline{s}_1};\overline{x}_1) \arrow[r, "i_1"] \arrow[u, dotted, "\exists!\, sp"'] & \pi_1(X;x\overline{x}_1) \arrow[r, "p_1"] \arrow[u, "\alpha_X"'] & \pi_1(S;\overline{s}_1) \arrow[r] \arrow[u, "\alpha_S"'] & 1,
\end{tikzcd}$$

where the vertical arrows $\alpha_X : \pi_1(X;\overline{x}_1)\tilde{\to}\pi_1(X;\overline{x}_0)$ and $\alpha_S : \pi_1(S;\overline{s}_1)\tilde{\to}\pi_1(S;\overline{s}_0)$ are the canonical (up to inner automorphisms) isomorphisms of theorem 2.8.

Now, since $p_0 \circ \alpha_X \circ i_1 " = " \alpha_S \circ p_1 \circ i_1 \overset{(*)}{=} 0$ (here $" = "$ means equal up to inner automorphisms and equality $(*)$ comes from theorem **??**), it follows from theorem 9.3 that:

$$\mathrm{im}(\alpha_X \circ i_1) \subset \ker(p_0) = \mathrm{im}(i_0)$$

and, hence, there exists a morphism of profinite groups:

$$sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \to \pi_1(X_{\overline{s}_0};\overline{x}_0),$$

unique up to inner automorphisms and such that $\alpha_X \circ p_1$" = "$i_0 \circ sp$.

If, furthermore, $\mathrm{im}(i_1) = \ker(p_1)$, a straightforward diagram chasing shows that:

$$sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0};\overline{x}_0)$$

is an epimorphism.

We come back to the case where S is any locally noetherian scheme and $s_0,\ s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$. One then has a commutative diagram (where we abbreviate spec(K) by K when K is a field):

$$\begin{tikzcd}
\overline{k(s_1)} \arrow[r] & k(s_1) \arrow[rr, "s_1"] \arrow[dr] & & S & k(s_0) \arrow[l, "s_0"'] \arrow[dl] \arrow[dd, equal] & \overline{k(s_0)} \arrow[l] \arrow[dd, equal] \\
& & \mathrm{spec}(\mathcal{O}_{S,s_1}) \arrow[r] & \mathrm{spec}(\mathcal{O}_{S,s_0}) \arrow[u] & & \\
\overline{k(\hat{s}_1)} \arrow[r] \arrow[uu] & k(\hat{s}_1) \arrow[rr, "\hat{s}_1"'] \arrow[uu] & & \mathrm{spec}(\widehat{\mathcal{O}}_{S,s_0}) \arrow[u] & k(\hat{s}_0) \arrow[l, "\hat{s}_0"] & \overline{k(\hat{s}_0)}, \arrow[l]
\end{tikzcd}$$

where the existence of $\hat{s}_1$ is ensured by the fact that $\mathcal{O}_{S,s_0} \to \widehat{\mathcal{O}}_{S,s_0}$ is faithfully (flat). Choose a geometric point $\overline{\hat{x}}_1$ of $\overline{X}_{\overline{s}_1} := X_{\overline{s}_1} \times_{\overline{k(s_1)}} \overline{k(\hat{s}_1)}$ over $\overline{x}_1$. Since $X_{\widehat{\mathcal{O}}_{S,s_0}} \to \mathrm{spec}(\widehat{\mathcal{O}}_{S,s_0})$ is proper (and separable as soon as $f : X \to S$ is) , it follows from (1) that one has a canonical specialization morphism:

$$(*)\ \ sp : \pi_1(\overline{X}_{\overline{s}_1};\overline{\hat{x}}_1) \to \pi_1(X_{\overline{s}_0};\overline{x}_0)$$

and, from corollary 6.6, the canonical morphism:

$$(**)\ \ \pi_1(\overline{X}_{\overline{s}_1};\overline{\hat{x}}_1)\tilde{\to}\pi_1(X_{\overline{s}_1};\overline{x}_1)$$

is an isomorphism. Thus the specialization isomorphism is obtained by composing the inverse of $(**)$ with $(*)$.

9.3. **Proof of theorem 9.3.** The proof resorts to difficult results from [EGA3]; we will only sketch it but give references for the missing details. See also [I05] for a more detailed treatment.

<u>**Claim 1:**</u> *If A is a local artinian ring, the conclusions of theorem 9.3 hold.*

Proof of claim 1. Recall that, in an Artin ring, any prime ideal is maximal hence the nilradical and the Jacobson radical coincide. In particular, if A is local, the nilpotent elements of A are precisely those of its maximal ideal. From theorem A.2, one may thus assume that $A = k(s_0)$ and, then, the conclusion $\pi_1(S,\overline{s}_0) \simeq \Gamma_{k(s_0)}$ is straightforward. Let $k(s_0)^i$ denote the inseparable closure of $k(s_0)$ in $\overline{k(s_0)}$ and write $X^i_{s_0} := X \times_S k(s_0)^i$. Then the cartesian diagram:

(5)
$$\begin{tikzcd}
X_{\overline{s}_0} \arrow[r] & X \arrow[r] & S \\
X_{\overline{s}_0} \arrow[r] \arrow[u, equal] \arrow[ur, phantom, "\square"] & X^i_{s_0} \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \mathrm{Spec}(k(s_0)^i) \arrow[u]
\end{tikzcd}$$

induces a commutative diagram of morphisms of profinite groups:

(6)
$$\begin{tikzcd}
\pi_1(X_{\overline{s}_0};\overline{x}_0) \arrow[r] & \pi_1(X;x_{(0)}) \arrow[r] & \pi_1(S;\overline{s}_0) \\
\pi_1(X_{\overline{s}_0};\overline{x}_0) \arrow[r] \arrow[u] & \pi_1(X_0^i;x_{(0)}^i) \arrow[r] \arrow[u] & \pi_1(\mathrm{Spec}(k(s_0)^i);\overline{s}_0^i) \arrow[u]
\end{tikzcd}$$

Now, since each of the vertical arrows in (5) is faithfully flat, quasi-compact and radiciel, it follows from corollary A.4 that the vertical arrows in (6) are isomorphisms of profinite groups. Hence it is enough to prove that the bottom line of (6) is exact that is one may assume that $k(s_0)$ is perfect.

But, then, $\overline{k(s_0)}$ can be written as the inductive limit of its finite Galois subextensions $k(s_0) \hookrightarrow k_i \hookrightarrow \overline{k(s_0)}$, $i \in I$ hence, writing again $\overline{x}_0$ for the image of $\overline{x}_0$ in X_{k_i}, it follows from lemma 6.7 that the morphism:

$$X_{\overline{s}_0} \to \varinjlim X_{ki}$$

induces an isomorphism of profinite groups:

$$\pi_1(X_{\overline{s}_0};\overline{x}_0)\tilde{\to}\varprojlim \pi_1(X_{k_i};\overline{x}_0).$$

But, for each $i \in I$, the étale cover $X_{k_i} \to X$ is Galois with group $\mathrm{Aut}_{Alg/k(s_0)}(k_i)$ so, from proposition 4.4 one has a short exact sequence of profinite groups:

$$1 \to \pi_1(X_{k_i};\overline{x}_0) \to \pi_1(X;\overline{x}_0) \to \mathrm{Aut}_{Alg/k(s_0)}(k_i) \to 1.$$

Using that the projective limit functor is exact in the category of profinite groups, we thus get the expected short exact sequence of profinite groups:

$$1 \to \varprojlim \pi_1(X_{k_i};\overline{x}_0) \to \pi_1(X;\overline{x}_0) \to \Gamma_{k(s_0)} \to 1.$$

<u>Claim 2:</u> *The closed immersion $i_{X_{s_0}} : X_{s_0} \hookrightarrow X$ induces an equivalence of categories $\mathcal{C}_X \to \mathcal{C}_{X_{s_0}}$ hence, in particular, an isomorphism of profinite groups:*

$$\pi_1(X_{s_0};\overline{x}_0)\tilde{\to}\pi_1(X;\overline{x}_0).$$

Proof of claim 2. One has to prove:

(1) For any étale covers $p : Y \to X$, $p' : Y' \to X$ the canonical map
$$\mathrm{Hom}_{\mathcal{C}_X}(p,p') \to \mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p \times_X X_{s_0}, p' \times_X X_{s_0})$$
is bijective;

(2) For any étale cover $p_0 : Y_0 \to X_{s_0}$ there exists an étale cover $p : Y \to X$ which is a model of $p_0 : Y_0 \to X_{s_0}$ over X.

The proof of these two assertions is based on Grothendieck's Comparison and Existence theorems in algebraico-formal geometry. We first state simplified versions of these theorems.

Let S be a noetherian scheme and let $p : X \to S$ be a proper morphism. Let $\mathcal{I} \subset \mathcal{O}_S$ be a coherent sheaf of ideals. Then the descending chains $\cdots \subset \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \cdots \subset \mathcal{I}$ corresponds to a chain of closed subschemes $S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow \cdots \hookrightarrow S$. We will use the notation in the diagram below:

$$\begin{tikzcd}
S & S_n \arrow[l, hook'] & \cdots \arrow[l, hook'] & S_1 \arrow[l, hook'] & S_0 \arrow[l, hook'] \\
X \arrow[u, "p"] \arrow[ur, phantom, "\square"] & X_n \arrow[l, hook'] \arrow[u, "p_n"] \arrow[ur, phantom, "\square"] & \cdots \arrow[l, hook'] & X_1 \arrow[l, hook'] \arrow[u, "p_1"] \arrow[ur, phantom, "\square"] & X_0. \arrow[l, hook'] \arrow[u, "p_0"]
\end{tikzcd}$$

and write $i_n : X_n \hookrightarrow X$, $n \geq 0$. For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, set $\mathcal{F}_n := i_n^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$, $n \geq 0$. Then $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module and the canonical morphim of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{F}_n$ induces morphims of $\mathcal{O}_S$-modules $\mathrm{R}^q p_*\mathcal{F} \to \mathrm{R}^q p_*\mathcal{F}_n$, $q \geq 0$ hence morphims of $\mathcal{O}_{S_n}$-modules:

$$(\mathrm{R}^q p_*\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_n} \to \mathrm{R}^q p_*\mathcal{F}_n,\ q \geq 0$$

and, taking projective limit, canonical morphisms:

$$\varprojlim((\mathrm{R}^q p_*\mathcal{F})\otimes_{\mathcal{O}_S}\mathcal{O}_{S_n})\rightarrow\varprojlim \mathrm{R}^q p_*\mathcal{F}_n,\ q\geq 0.$$

When $S=\mathrm{spec}(A)$ is affine and $I\subset A$ is the ideal corresponding to $\mathcal{I}\subset\mathcal{O}_S$, the above isomorphism becomes:

$$\mathrm{H}^q(X,\mathcal{F})\otimes_A\widehat{A}\tilde{\rightarrow}\varprojlim \mathrm{H}^q(X_n,\mathcal{F}_n),\ q\geq 0,$$

where $\widehat{A}$ denotes the completion of A with respect to the I-adic topology.

Theorem 9.4. (Comparison theorem [EGA3, (4.1.5)]) *The canonical morphisms:*

$$\varprojlim((\mathrm{R}^q p_*\mathcal{F})\otimes_{\mathcal{O}_S}\mathcal{O}_{S_n})\tilde{\rightarrow}\varprojlim \mathrm{R}^q p_*\mathcal{F}_n,\ q\geq 0$$

are isomorphisms.

Theorem 9.5. (Existence theorem [EGA3, (5.1.4)]) *Assume, furthermore that $S=\mathrm{spec}(A)$ is affine and that A is complete with respect to the I-adic topology. Let $\mathcal{F}_n$, $n\geq 0$ be coherent $\mathcal{O}_{X_n}$-modules such that $\mathcal{F}_{n+1}\otimes_{\mathcal{O}_{X_{n+1}}}\mathcal{O}_{X_n}\tilde{\rightarrow}\mathcal{F}_n$, $n\geq 0$. Then there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_{X_n}\tilde{\rightarrow}\mathcal{F}_n$, $n\geq 0$.*

Also, for any étale cover $p:Y\rightarrow X$, observe that $\mathcal{A}(p):=p_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-algebra of finite rank and that, denoting by $FLFAlg/\mathcal{O}_X$ the category of locally free $\mathcal{O}_X$-algebra of finite rank the functor:

$$\begin{array}{llll}\mathcal{A}: & \mathcal{C}_X & \rightarrow & FLFAlg/\mathcal{O}_X\\ & p:Y\rightarrow X & \rightarrow & \mathcal{A}(p)\end{array}$$

is fully faithful.

<u>Proof of (1)</u>: One has canonical functorial isomorphisms:

$$\begin{array}{lll}\mathrm{Hom}_{\mathcal{C}_X}(p,p') & \tilde{\rightarrow} & \mathrm{H}^0(X,\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'),\mathcal{A}(p)))\\ & \tilde{\rightarrow} & \varprojlim \mathrm{H}^0(X_n,\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'),\mathcal{A}(p))\otimes_{\mathcal{O}_X}\mathcal{O}_{X_n}),\end{array}$$

where the first isomorphism comes from the fact that $\mathcal{A}$ is fully faithful and the second isomorphism is just the comparison theorem applied to $q=0$, $\mathcal{F}=\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'),\mathcal{A}(p))$ and I the maximal ideal of A, observing that, since A is complete with respect to the I-adic topology, $A=\widehat{A}$.

Furthermore, as $\mathcal{A}(p)$, $\mathcal{A}(p')$ are locally free $\mathcal{O}_X$-module, one has canonical isomorphisms:

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}(X)}(\mathcal{A}(p'),\mathcal{A}(p))\otimes_{\mathcal{O}_X}\mathcal{O}_{X_n}\tilde{\rightarrow}\underline{\mathrm{Hom}}_{\mathrm{Mod}(X_n)}(\mathcal{A}(p'_n),\mathcal{A}(p_n))$$

But these preserve the structure of $\mathcal{O}_X$-algebra morphisms hence one also gets, by restriction:

$$\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'),\mathcal{A}(p))\otimes_{\mathcal{O}_X}\mathcal{O}_{X_n}\tilde{\rightarrow}\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_{X_n}}(\mathcal{A}(p'_n),\mathcal{A}(p_n)).$$

Whence,

$$\begin{array}{lll}\mathrm{Hom}_{\mathcal{C}_X}(p,p') & \tilde{\rightarrow} & \varprojlim \mathrm{H}^0(X_n,\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'),\mathcal{A}(p))\otimes_{\mathcal{O}_X}\mathcal{O}_{X_n})\\ & \tilde{\rightarrow} & \varprojlim \mathrm{H}^0(X_n,\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_{X_n}}(\mathcal{A}(p'_n),\mathcal{A}(p_n)))\\ & \tilde{\rightarrow} & \varprojlim \mathrm{Hom}_{\mathcal{C}_{X_n}}(p_n,p'_n)\\ & \tilde{\rightarrow} & \varprojlim \mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p_0,p'_0),\end{array}$$

where the last isomorphism comes from the fact $\mathrm{Hom}_{\mathcal{C}_{X_n}}(p_n,p'_n)\tilde{\rightarrow}\mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p_0,p'_0)$, $n\geq 0$ by theorem A.2.

<u>Proof of (2)</u>: By theorem A.2, there exists étale covers $p_n:Y_n\rightarrow X_n$, $n\geq 0$ such that $p_n\tilde{\rightarrow}p_{n+1}\times_{X_{n+1}}X_n$, or, equivalently, $\mathcal{A}(p_{n+1})\otimes_{\mathcal{O}_{X_{n+1}}}\mathcal{O}_{X_n}\tilde{\rightarrow}\mathcal{A}(p_n)$, $n\geq 0$. So, by the Existence theorem, there exists a locally free $\mathcal{O}_X$-algebra of finite rank $\mathcal{A}$ such that $\mathcal{A}\otimes_{\mathcal{O}_{X_n}}\mathcal{O}_X\tilde{\rightarrow}\mathcal{A}(p_n)$, $n\geq 0$ hence, setting $p:Y=\mathrm{spec}\ (\mathcal{A})\rightarrow X$ one has $p\times_X X_{s_0}\tilde{\rightarrow}p_0$.

It remains to show that $p:Y=\mathrm{spec}\ (\mathcal{A})\rightarrow X$ is an étale cover. For this, see [Mur67, p. 159-161].

One can now conclude the proof. From claim 1 applied to $A = k(s_0)$, $X = X_{s_0}$, one gets the short exact sequence of profinite groups:

$$1 \to \pi_1(X_{\overline{s}_0}; \overline{x}_0) \to \pi_1(X_{s_0}; \overline{x}_0) \to \Gamma_{k(s_0)} \to 1.$$

Now, from claim 2 one has the canonical profinite group isomorphisms $\pi_1(X; \overline{x}_0)\tilde{\to}\pi_1(X_{s_0}; \overline{x}_0)$ and (for $X = S$) $\pi_1(S; \overline{s}_0)\tilde{\to}\Gamma_{k(s_0)}$, which yields the required short exact sequence.

Eventually, for the last assertion of theorem 9.3, just observe that, as above, one can assume that $A = k(s_0)$ thus, if $x \in X(k(s_0))$, it produces a section $x : S \to X$ of $f : X \to S$ such that $x \circ s_0 = x$ thus a section $\Gamma_{k(s_0)} \to \pi_1(X; \overline{x}_0)$ of (4). □

10. Purity and applications

In this section, we use Zariski-Nagata purity theorem to prove that the étale fundamental group is a birational invariant in the category of proper regular schemes over a field and to determine the kernel of the specialization epimorphism constructed in section 9.

Theorem 10.1. (Zariski-Nagata purity theorem [SGA2, Chap. X, thm. 3.4]) *Let X, Y be integral schemes with X normal and Y regular. Let $f : X \to Y$ be a quasi-finite dominant morphism and let $Z_f \subset X$ denote the closed subset of all $x \in X$ such that $f : X \to Y$ is not étale at x. Then, either $Z_f = X$ or Z_f is pure of codimension* 1 *(that is, for any generic point $\eta \in Z_f$, one has* $\dim(\mathcal{O}_{X,\eta}) = 1$*).*

10.1. Birational invariance of the étale fundamental group.

Corollary 10.2. *Let X be a connected, regular scheme and let $i_U : U \hookrightarrow X$ be an open subscheme such that $X \setminus U$ has codimension ≥ 2 in X. Then $i_U : U \hookrightarrow X$ induces an equivalence of categories:*

$$i_U^* : \mathcal{C}_X \to \mathcal{C}_U$$

hence an isomorphism of profinite groups:

$$\pi_1(i_U) : \pi_1(U)\tilde{\to}\pi_1(X)$$

Proof. As X is connected, locally noetherian and regular (hence with integral local rings), X is irreducible. Since X is normal and $X \setminus U \subset X$ is a closed subset of codimension ≥ 2, the functor $i_U^* : \mathcal{C}_X \to \mathcal{C}_U$ is fully faithfull [L00, Thm. 4.1.14] hence, one only has to prove that it is also essentially surjective that is, for any étale cover $p_U : V \to U$ there exists a (necessarily unique by the above) étale cover $p : Y \to X$ such that $p_U : V \to U$ is the base-change of $p : Y \to X$ *via* $i_U := U \hookrightarrow X$. One may assume that V is connected hence, it follows from lemma 6.14 that V is the normalization of U in $k(X) = k(U) \hookrightarrow k(V)$. Let $p : Y \to X$ be the normalization of X in $k(X) \hookrightarrow k(V)$. Then, on the one hand, it follows from the universal property of normalization that $p_U : V \to U$ is the base-change of $p : Y \to X$ *via* $i_U := U \hookrightarrow X$ as expected. On the other hand, since X is normal and $k(X) \hookrightarrow k(V)$ is a finite separable field extension, $p : Y \to X$ is finite, dominant and, from lemma 6.15, étale on:

$$p^{-1}(U) = V = Y \setminus p^{-1}(X \setminus U).$$

But $X \setminus U$ has codimension ≥ 2 in X hence, since $p : Y \to X$ is finite, $p^{-1}(X \setminus U)$ has codimension ≥ 2 in Y as well. Thus, it follows from theorem 10.1 that $p : Y \to X$ is étale. □

Let X be a connected, regular scheme, Y a connected scheme and $f : X \rightsquigarrow Y$ be a rational map. Write $U_f \subset X$ for the maximal open subset on which $f : X \rightsquigarrow Y$ is defined and assume that $X \setminus U_f$ has codimension ≥ 2 in X. Then, corresponding to the sequence of base-change functors:

$$\mathcal{C}_Y \stackrel{f|_{U_f}^*}{\to} \mathcal{C}_{U_f} \stackrel{i_{U_f}^*}{\leftarrow} \mathcal{C}_X$$

one has, for any geometric point $x \in U_f$, the sequence of morphisms of profinite groups:

$$\pi_1(X; x) \stackrel{\pi_1(i_{U_f})}{\tilde{\leftarrow}} \pi_1(U_f; x) \stackrel{\pi_1(f|_{U_f})}{\to} \pi_1(Y; f(x)).$$

So, if $\mathcal{C}$ denotes the category of all connected, regular schemes pointed by geometric points in codimension 1 together with dominant rational maps defined on an open subscheme whose complement has codimension ≥ 2 one gets a well-defined functor $\pi_1(-)$ from $\mathcal{C}$ to the category of profinite groups. In particular, let k be a field, X, Y two schemes proper over k, connected and regular and $f : X \leftrightsquigarrow Y$ a birational map of schemes over k. Then f is always defined over an open subscheme $i_{U_f} : U_f \hookrightarrow X$ such that $X \setminus U_f$ has codimension ≥ 2 in X and the same holds for f^{-1}. So, from corollary 10.2, one gets a sequence of isomorphisms of profinite groups:

$$\pi_1(X) \overset{\pi_1(i_{U_f})^{-1}}{\tilde{\to}} \pi_1(U_f) \overset{\pi_1(f|_{U_f}^{U_{f^{-1}}})}{\tilde{\to}} \pi_1(U_{f^{-1}}) \overset{\pi_1(i_{U_f^{-1}})}{\tilde{\to}} \pi_1(Y).$$

Example 10.3. Let k be any field and consider the blowing-up $f : B_x \to \mathbb{P}^2_k$ of $\mathbb{P}^2_k$ at any point $x \in \mathbb{P}^2_k$. Then for any geometric point $b \in B_x$:

$$\pi_1(B_x; b) \tilde{\to} \pi_1(\mathbb{P}^2_k; f(b)).$$

However, B_x and $\mathbb{P}^2_k$ are not k-isomorphic (any two curves in $\mathbb{P}^2_k$ intersects whereas the exceptional divisor E in B_x does not intersect the inverse images of the curves in $\mathbb{P}^2_k$ passing away from x). This shows that one has to be careful when formulating higher dimensional variants of conjecture 7.5.

10.2. **Kernel of the specialization morphism.** We retain the notation of §9. Let S be a locally noetherian scheme and $X \to S$ a *smooth*, proper, geometrically connected morphism. The aim of this section is to determine the kernel of the specialization epimorphism:

$$sp : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0}; \overline{x}_0)$$

constructed in section 9 namely, to prove:

Theorem 10.4. *For any finite group G of order prime to the residue characteristic p of S at s_0 and for any profinite group epimorphism $\phi : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \twoheadrightarrow G$ there exists an epimorphism of profinite groups $\phi_0 : \pi_1(X_{\overline{s}_0}; \overline{x}_0) \twoheadrightarrow G$ such that $\phi_0 \circ sp = \phi$. In particular, sp induces an isomorphism of profinite groups:*

$$sp^{(p)'} : \pi_1(X_{\overline{s}_1}; \overline{x}_1)^{(p)'} \tilde{\to} \pi_1(X_{\overline{s}_0}; \overline{x}_0)^{(p)'},$$

where $(-)^{(p)'}$ denotes the prime-to-p profinite completion.

Proof. After reducing to the case where $S = \mathrm{spec}(\mathcal{O})$ with $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field, the proof of theorem 10.4 amounts to showing the following. Given an étale cover $Y \to X_{s_1}$ Galois with group G of prime-to-p order n, there exists a finite field subextension $K \hookrightarrow L \hookrightarrow K^s$ such that the extension $k(X).L \hookrightarrow k(Y).L$ be unramified over $X \times_S S^L$, where $S^L := \mathrm{spec}(\mathcal{O}^L)$. Zariski-Nagata purity theorem actually shows that it is enough to construct $K \hookrightarrow L$ in such a way that $k(X).L \hookrightarrow k(Y).L$ be unramified only over the points above the generic point of the closed fiber of X. Such a L can be constructed by Abhyankar's lemma.

Claim 1: *One may assume that $S = \mathrm{spec}(\mathcal{O})$, with $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field.*

Proof of claim 1. Let $s_0 = t_0, t_1, \ldots, t_r = s_1 \in S$ such that $t_i \in \overline{\{t_{i+1}\}}$ and $\mathcal{O}_{\overline{\{t_{i+1}\}}, t_i}$ has dimension 1, $i = 0, \ldots, r-1$. Then, one has the sequence of specialization epimorphisms:

$$\pi_1(X_{\overline{s}_1}) \twoheadrightarrow \pi_1(X_{\overline{t}_{r-1}}) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(X_{\overline{t}_1}) \twoheadrightarrow \pi_1(X_{\overline{s}_0}).$$

Thus, without loss of generality, we may assume that $\dim(\mathcal{O}_{\overline{\{s_1\}}, s_0}) = 1$. Next, let R denote the strict henselianization of the integral closure of $\mathcal{O}_{\overline{\{s_1\}}, s_0}$ and let $R \hookrightarrow \hat{R}$ denotes its completion. Then $\hat{R}$ is a complete discrete valuation ring with separably closed residue field and the canonical morphism $\mathrm{spec}(\hat{R}) \to S$ maps the generic point of $\mathrm{spec}(\hat{R})$ to s_1 and the closed point of $\mathrm{spec}(\hat{R})$ to s_0.

We will use the following notation for $\mathcal{O}$. Given a finite Galois extension L/K we will write $\mathcal{O}^L$ for the integral closure of $\mathcal{O}$ in L and $e_{L/K}(\mathcal{O})$ for the order of the inertia group of $\mathcal{O}$ in L/K. Now fix an algebraic closure $K \hookrightarrow \overline{K}$ of the fraction field K of $\mathcal{O}$ and let $K \hookrightarrow K^s$ be the separable closure of K in $\overline{K}$. For simpliciy, we remove the reference to the base point in the notation below.

From theorem 9.3, and the construction of the specialization morphism, one has the following situation:

$$\begin{tikzcd} \pi_1(X_{\overline{s}_1}) \arrow[r] \arrow[drr, "sp"'] & \pi_1(X) & \pi_1(X_{s_0}) \arrow[l, "\simeq"'] \\ & & \pi_1(X_{\overline{s}_1}). \arrow[u, hook] \end{tikzcd}$$

which shows that:

$$\ker(sp) = \ker(\pi_1(X_{\overline{s}_1}) \to \pi_1(X)).$$

Consider the following factorization of $\overline{s}_1 : \mathrm{spec}(\overline{K}) \to S$:

$$\begin{tikzcd} \mathrm{spec}(\overline{K}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\overline{s}_1"] & \mathrm{spec}(K) \arrow[r, "s_0"] & S \\ \mathrm{spec}(K^s) \arrow[urr, "s_1^s"'] & & \end{tikzcd}$$

Since $\mathrm{spec}(\overline{K}) \to \mathrm{spec}(K^s)$ is faithfully flat, quasi-compact and radiciel, it follows from corollary A.4 that the morphism of profinite groups:

$$\pi_1(X_{\overline{s}_1}) \tilde{\to} \pi_1(X_{s_1^s})$$

is an isomorphism. Hence:

$$\ker(sp) = \ker(\pi_1(X_{s_1^s}) \to \pi_1(X)).$$

Let $K \hookrightarrow L$ be a finite field extension. Then $\mathcal{O}^L$ is again a complete discrete valuation ring. Set $S^L := \mathrm{spec}(\mathcal{O}^L)$ and write $s_{L,1}$, $s_{L,0}$ for its generic and closed points respectively. Note that $k(s_0) = k(s_{L,0}) = k$ since k is algebraically closed.

<u>Claim 2:</u> *The morphism of profinite groups:*

$$\pi_1(X \times_S S^L) \tilde{\to} \pi_1(X)$$

induced by $X \times_S S^L \to X$ is an isomorphism.

Proof of claim 2. From theorem 9.3, one has the following commutative diagram with exact row:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1((X \times_S S^L)_{s_{L,0}}) \arrow[r] \arrow[d] & \pi_1(X \times_S S^L) \arrow[r] \arrow[d] & \pi_1(S^L) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \pi_1(X_{s_0}) \arrow[r] & \pi_1(X) \arrow[r] & \pi_1(S) \arrow[r] & 1. \end{tikzcd}$$

But since $k(s_0) = k(s_{L,0}) = k$ is algebraically closed one has $\pi_1(S) = \Gamma_{k(s_0)} = 1$, $\pi_1(S^L) = \Gamma_{k(s_{L,0})} = 1$ and $X_{s_0} = (X \times_S S^L)_{s_{L,0}}$, whence the conclusion.

So, one can replace freely K by any finite separable field extension.

From lemma 4.2 (2), the assertion of theorem 10.4 amounts to showing that for any étale cover $Y \to X_{s_1^s}$ Galois with group G of prime-to-p order n, there exists a finite separable field subextension $K \hookrightarrow L \hookrightarrow K^s$ and an étale cover $Y^L \to X \times_S S^L$ Galois with group G which is a model of $Y \to X_{s_1^s}$ over $X \times_S S^L$.

Since K^s is the inductive limit of the finite extensions of K contained in K^s, by the argument of the proof of proposition 6.7, there exists a finite separable extension $K \hookrightarrow L$ and an étale cover $Y^{0L} \to X_L$ Galois with group G which is a model of $Y \to X_{s_1^s}$ over X_L. Thus, from claim 2, we are to prove:

Claim 3: *For any étale cover $Y \to X_{s_1}$ Galois with group G of prime-to-p order n, there exists a finite field subextension $K \hookrightarrow L \hookrightarrow K^s$ and an étale cover $Y^L \to X \times_S S^L$ Galois with group G which is a model of $Y_L \to X_L$ over $X \times_S S^L$.*

Proof of claim 3. Observe first that, for any finite separable subextension $K \hookrightarrow L \hookrightarrow K^s$, as S^L is regular and $X \times_S S^L \to S^L$ is smooth then $X \times_S S^L$ is regular as well (hence, in particular, normal). Also, since $X \times_S S^L \to S^L$ is closed (since proper), surjective and with connected fibres an since S^L is connected, $X \times_S S^L$ is connected as well hence being noetherian and normal, it is irreducible. So, one can consider the normalization $Y^L \to X \times_S S^L$ of $X \times_S S^L$ in

$$k(X \times_S S^L) = k(X_L) \hookrightarrow k(Y_L).$$

From the universal property of normalization, $Y^L \to X \times_S S^L$ is a model of $Y_L \to X_L$ over $X \times_S S^L$). From theorem 6.16, it only remains to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over $X \times_S S^L$. Since $X \times_S S^L$ is regular, from Zariski-Nagata purity theorem 10.1, we are only to to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over the codimension 1 points of $X \times_S S^L$. But as all the codimension 1 points of X are either contained in the generic fibre X_{s_1} or the generic point ζ of the closed fibre X_{s_0}, we are only to to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over the points of $X \times_S S^L$ lying over ζ in $S \times_S S^L \to X$.

For this, let π be a uniformizing parameter of $\mathcal{O}$; it is also a uniformizing parameter of $\mathcal{O}_{X,\zeta}$. Set $L := K[T]/\langle T^n - \pi\rangle$. Then, $k(X_L) = k(X) \cdot L = k(X)[T]/\langle T^n - \pi\rangle$ is a degree n extension of $k(X)$, tamely ramified over $\mathcal{O}_{X,\zeta}$ with inertia group of order n by Kummer theory. Now, apply lemma 10.5 below to the extensions $k(Y)/k(X)$ and $k(X^L)/k(X)$ to obtain that the compositum $k(Y)\dot{k}(X^L)$ is unramified over $\mathcal{O}_{X\times_S S^L,\zeta^L}$ for any point ζ^L in $X \times_S S^L$ above ζ. □

Lemma 10.5. (Abhyankar's lemma) *Let L/K and M/K be two finite Galois extensions tamely ramified over $\mathcal{O}$ and assume that $e_{M|K}(\mathcal{O})$ divides $e_{L|K}(\mathcal{O})$. Then, for any maximal ideal $\mathfrak{m}_L$ of $\mathcal{O}^L$, the compositum $L.M$ is unramified over $\mathcal{O}^L_{\mathfrak{m}_L}$.*

11. Proper schemes over algebraically closed fields

In this last section, we would like to prove the following:

Theorem 11.1. *The étale fundamental group of a proper connected scheme over an algebraically closed field is topologically finitely generated.*

A striking consequence of this theorem is that a proper connected scheme over an algebraically closed field has only finitely many isomorphism classes of étale covers of bounded degree.

Proof. We proceed by induction on the dimension d to reduce to the case of curves. However, to make the induction step work, we need the two intermediary claims 1 and 2 below.

Claim 1: *Fix an integer $d \geq 0$ and assume that theorem 11.1 holds for all projective normal connected and d-dimensional schemes over an algebraically closed field k. Then theorem 11.1 holds for all proper connected and d-dimensional schemes over k.*

Proof of claim 1. Let X be a proper connected and d-dimensional scheme over an agebraically closed field k. The first ingredient is:

Theorem 11.2. (Chow's lemma [EGA2, Cor. 5.6.2]) *Let S be a noetherian scheme. Then, for any $X \to S$ proper there exists $X' \to S$ projective and a surjective birational morphism $X' \to X$ over S.*

Applying Chow's lemma to the structural morphism $X \to \mathrm{spec}(k)$, one obtains a scheme X' projective over k and a surjective birational morphism $X' \to X$ over k, which is automatically proper since both X' and X are proper over k. Then, from theorem A.5 and corollary A.7, the profinite group $\pi_1(X)$ is topologically finitely generated as soon as $\pi_1(X'_0)$ is for each connected component $X'_0 \in \pi_0(X')$.

Assume that X' is connected. The underlying reduced closed subscheme $X'^{red} \hookrightarrow X'$ is projective over k since X' is. Also, as X'^{red} is of finite type over k, its normalization $\tilde{X}'^{red} \to X'^{red}$ is a finite and, in particular, $\tilde{X}'^{red}$ is projective over k as well. And, from theorem A.5 and corollary A.7, $\pi_1(X')$ is topologically finitely generated as soon as $\pi_1(\tilde{X}_0'^{red})$ is for each connected component $\tilde{X}_0'^{red}$ of $\tilde{X}'^{red}$.

Claim 2: *Let X be projective, normal connected and d-dimensonal scheme over an algebraically closed field k. Then there exists a proper, connected and $d-1$-dimensional scheme Y over k and an epimorphism of profinite groups:*

$$\pi_1(Y) \twoheadrightarrow \pi_1(X).$$

Proof of claim 2. Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a closed immersion and let $H \hookrightarrow \mathbb{P}^n_k$ be an hyperplane such that $X \not\subset H$ then the corresponding hyperplane section $X.H$ (regarded as a scheme with the induced reduced scheme structure) has dimension $\leq d-1$. The fact that $Y := X.H$ has the required properties results from the following application of Bertini theorem and the Stein factorization theorem:

Theorem 11.3. ([J83, Thm. 7.1]) *Let X be a proper scheme over k, let $f : X \to \mathbb{P}^n_k$ be a morphism over k and $L \hookrightarrow \mathbb{P}^n_k$ a linear projective subscheme. Assume that:*
(i) X is irreducible;
(ii) $\dim(f(X)) + \dim(L) > n$.
Then $f^{-1}(L)$ is connected and non-empty.

Since X is connected, noetherian with integral local ring, X is irreducible and one can apply theorem 11.3 to the closed immersion $i : X \hookrightarrow \mathbb{P}^n_k$ to obtain that $X \cdot H$ is (projective) and connected over k. It remains to prove that the morphism of profinite groups $\pi_1(X \cdot H) \to \pi_1(X)$ induced by the closed immersion $X \cdot H \hookrightarrow X$ is an epimorphism. But this follows again from theorem 11.3. Indeed, for any connected étale cover $Y \to X$, the scheme Y is again connected, noetherian with integral local ring (Y is normal since X is) hence irreducible and, from theorem 11.3 applied to $Y \to X \overset{i}{\hookrightarrow} \mathbb{P}^n_k$, one gets that $Y \times_X (X \cdot H)$ is connected.

Combining claims 1 and 2, one reduce by induction on the dimension d to the case of 0 and 1-dimensional projective normal connected schemes over k. (First apply claim 1 to show that theorem 11.1 for d-dimensional proper connected schemes over k is equivalent to theorem 11.1 for d-dimensional projective normal connected schemes over k, then apply claim 2 to show that theorem 11.1 for d-dimensional projective normal connected schemes over k is implied by theorem 11.1 for $d-1$-dimensional proper connected schemes over k and so on).

If $d = 0$ then $X = \text{spec}(k)$ and $\pi_1(X) = \Gamma_k = \{1\}$. So, let X be a projective, smooth, connected curve of genus say g.

Write Q for the prime field of k. Since X is of finite type over k, there exists a subextension $Q \hookrightarrow k_0 \hookrightarrow k$ of finite transcendence degree over Q and a model X_0 of X over k_0.

Assume first that Q has characteristic 0. Since k_0 is of finite transcendence degree over Q, one can find a field embedding $k_0 \hookrightarrow \mathbb{C}$ hence, from lemma 6.5, one has the following isomorphism of profinite groups:

$$\pi_1(X) = \pi_1(X_0 \times_{k_0} k) = \pi_1(X_0 \times_{k_0} \overline{k}_0) = \pi_1(X_0 \times_{k_0} \mathbb{C}).$$

So, one can assume that $k = \mathbb{C}$. It then follows from example 8.2 that one has an isomorphism of profinite groups:

$$\pi_1(X) \tilde{\to} \widehat{\Gamma}_{g,0}.$$

Assume now that Q has characteristic $p > 0$. The key ingredients here are the specialization theorem and the following consequence of Grothendieck's existence theorem for lifting smooth projective curves from characteristic > 0 to characteristic 0:

Theorem 11.4. [SGA1, III, Cor. 7.3] *Let $S := \text{spec}(A)$ with A a complete local noetherian ring with residue field k and closed point $s_0 \in S$. For any smooth and projective scheme X_1 over k, if:*

$$\mathrm{H}^2(X_1, (\Omega^1_{X_1|k})^\vee) = \mathrm{H}^2(X_1, \mathcal{O}_{X_1}) = 0$$

then X_1 has a smooth and projective model $X \to S$ over S.

By Grothendieck's vanishing theorem for cohomology [Hart77, Chap. III, thm. 2.7], the hypotheses of theorem 11.4 are always satisfied when X is a smooth projective curve. So, write A for the ring $W(k)$ of Witt vectors over k; it is a complete discrete valuation ring with residue field k and fraction field K of characteristic 0. Set $S := \mathrm{spec}(A)$ and let s_0, s_1 denote the generic and closed point of S respectively. From theorem 11.4, there exists a smooth projective curve $\mathcal{X} \to S$ such that:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ k \arrow[r, "s_1"'] & S. \end{tikzcd}$$

Since $\mathcal{X} \to S$ is proper and smooth (hence separable), it follows from theorem 9.1 that the specialization morphism is an epimorphism:

$$sp : \pi_1(\mathcal{X}_{\overline{s}_1}) \twoheadrightarrow \pi_1(\mathcal{X}_{\overline{s}_0} = X).$$

Hence the conclusion follows from $\pi_1(\mathcal{X}_{\overline{s}_1}) = \widehat{\Gamma}_{g,0}$. $\square$

Remark 11.5. Let S be a smooth, separated and geometrically connected curve over an algebraically closed field k of characteristic $p > 0$, let g denote the genus of its smooth compactification $S \hookrightarrow S^{cpt}$ and r the degree of $S \setminus S^{cpt}$. From remark 6.8, the pro-p-completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$ is known and, from theorem 10.4 and the proof of theorem 11.1, the prime-to-p completion $\pi_1(S)^{(p)'}$ of $\pi_1(S)$ is known as well (and equal to $\widehat{\Gamma_{g,r}}^{(p)'}$). But this does not determine $\pi_1(S)$ entirely (except when $(g, r) = (0, i)$, $i = 0, 1, 2$ or $(g, r) = (1, 0)$). However, in direction of a more precise determination of $\pi_1(S)$ one had the following conjecture:

Conjecture 11.6. (Abhyankar's conjecture) *With the above notation, any finite group G such that $G^{(p)'}$ is quotient of $\pi_1(S)^{(p)'} = \widehat{\Gamma_{g,r}}^{(p)'}$ (or, equivalently, is generated by $\leq 2g + r - 1$ elements) is a quotient of $\pi_1(S)$.*

Abhyankar's conjecture for $S = \mathbb{A}^1_k$ was proved by M. Raynaud [R94] and the general case was proved by D. Harbater, by reducing it to the case of the affine line [Harb94]. Note that, in the affine case, $\pi_1(S)$ is not topologically finitely generated so the knowledge of its finite quotients does not determine its isomorphism class.

Appendix A. Digest of descent theory for étale fundamental groups

A.1. **The formalism of descent.** We recall briefly the formalism of descent. Let S be a scheme and $\mathcal{C}_S$ a subcategory of the category of S-schemes closed under fibre product. A *fibred category over* $\mathcal{C}_S$ is a pseudofunctor $\mathfrak{X} : \mathcal{C}_S \to Cat$ that is the data of:
- for any $U \in \mathcal{C}_S$, a category $\mathfrak{X}_U$ (sometimes called the fibre of $\mathfrak{X}$ over $U \to S$);
- for any morphism $\phi : V \to U$ in $\mathcal{C}_S$, a base change functor $\phi^\star : \mathfrak{X}_U \to \mathfrak{X}_V$;
- for any morphisms $W \overset{\chi}{\to} V \overset{\phi}{\to} U$ in $\mathcal{C}_S$, a functor isomorphism $\alpha_{\chi,\phi} : \chi^\star\phi^\star \tilde{\to} (\phi \circ \chi)^\star$ satisfying the usual cocycle relations that is, for any morphisms $X \overset{\psi}{\to} W \overset{\chi}{\to} V \overset{\phi}{\to} U$ in $\mathcal{C}_S$, the following diagrams are commutative:

$$\begin{tikzcd} \psi^\star\chi^\star\phi^\star \arrow[r, "\psi^\star(\alpha_{\chi,\phi})"] \arrow[d, "\alpha_{\psi,\chi}(\phi^\star)"'] & \psi^\star(\phi \circ \chi)^\star \arrow[d, "\alpha_{\psi,\phi\circ\chi}"] \\ (\chi \circ \psi)^\star\phi^\star \arrow[r, "\alpha_{\chi\circ\psi,\phi}"'] & (\phi \circ \chi \circ \psi)^\star. \end{tikzcd}$$

Given a morphism $\phi : U' \to U$ in $\mathcal{C}_S$, write $U'' := U' \times_U U'$, $U''' := U' \times_U U' \times_U U'$, $p_i : U'' \to U'$, $i = 1, 2$, $p_{i,j} : U''' \to U''$, $1 \leq i < j \leq 3$, $u_i : U''' \to U'$, $i = 1, 2,\ 3$ for the canonical projections.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of descent for* $\mathfrak{X}$ if for any $x, y \in \mathfrak{X}_U$ and any morphism $f' : \phi^\star x \to \phi^\star y$ in $\mathfrak{X}_{U'}$ such that the following diagram commute:

$$\begin{tikzcd}
& p_1^\star\phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi}(x)"'] & p_1^\star y \arrow[dr, "\alpha_{p_1,\phi}(y)"] & \\
{\phi'}^\star(x) \arrow[rrr, dotted, "p_1^\star f'", "p_2^\star f'"'] \arrow[dr, "\alpha_{p_2,\phi}(x)"'] & & & {\phi'}^\star(y) \arrow[dl, "\alpha_{p_2,\phi}(y)"] \\
& p_2^\star\phi^\star(x) \arrow[r, "p_2^\star f'"] & p_1^\star y &
\end{tikzcd}$$

there exists a unique morphism $f : x \to y$ in $\mathfrak{X}_U$ such that $\phi^\star f = f'$.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of effective descent for* $\mathfrak{X}$ if $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ and if for any $x' \in \mathfrak{X}_{U'}$ and any isomorphism $u : p_1^\star(x')\tilde{\to}p_2^\star(x')$ in $\mathfrak{X}_{U''}$ such that the following diagram commute

$$\begin{tikzcd}
& p_{1,3}^\star p_1^\star(x') \arrow[r, "p_{1,3}^\star u"] \arrow[dl, "\alpha_{p_{1,3},p_1}(x')"'] & p_{1,3}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,3},p_2}(x')"] & \\
u_1^\star(x') \arrow[rrr, dotted, "p_{1,3}^\star u"] \arrow[dddr, dotted, "p_{1,2}^\star u"] & & & u_3^\star(x') \\
p_{1,2}^\star p_1^\star(x') \arrow[u, "\alpha_{p_{1,2},p_1}(x')"] \arrow[d, "p_{1,2}^\star u"'] & & & p_{2,3}^\star p_2^\star(x') \arrow[u, "\alpha_{p_{2,3},p_2}(x')"'] \\
p_{1,2}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,2},p_2}(x')"'] & & & p_{2,3}^\star p_1^\star(x') \arrow[u, "p_{2,3}^\star u"'] \arrow[dl, "\alpha_{p_{2,3},p_1}(x')"] \\
& u_2^\star(x') \arrow[r, equal] & u_2^\star(x') \arrow[uuur, dotted, "p_{2,3}^\star u"] &
\end{tikzcd}$$

there is a (necessarily unique since $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$) $x \in \mathfrak{X}_U$ and an isomorphism $f' : \phi^\star(x)\tilde{\to}x'$ in $\mathfrak{X}_{U'}$ such that the following diagram commute

$$\begin{tikzcd}
& p_1^\star\phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi}(x)"'] & p_1^\star(x') \arrow[dd, "u"] \\
{\phi'}^\star(x) \arrow[urr, dotted, "p_1^\star f'"] \arrow[drr, dotted, "p_2^\star f'"] & & \\
& p_2^\star\phi^\star(x) \arrow[r, "p_2^\star f'"] \arrow[ul, "\alpha_{p_2,\phi}(x)"] & p_2^\star(x')
\end{tikzcd}$$

The pair $\{x',\ u : p_1^\star(x')\tilde{\to}p_2^\star(x')\}$ is called a descent datum for $\mathfrak{X}$ relatively to $\phi : U' \to U$. Denoting by $\mathfrak{D}(\phi)$ the category of descent data for $\mathfrak{X}$ relatively to $\phi : U' \to U$, saying that $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is fully faithfull and saying that $\phi : U' \to U$ is a morphism of effective descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is an equivalence of category.

Example A.1. The basic example is that any faithfully flat and quasi-compact morphism $\phi : U' \to U$ is a morphism of effective descent for the fibreed category of quasi-coherent modules. See for instance [V05] for a comprehensive introduction to descent technics.

A.2. **Selected results.** The fibred categories we will now focus our attention on are the categories of finite étale covers. We only mention results that are used in these notes. For the proofs, we refer to [SGA1, Chap. VIII and IX] .

Theorem A.2. *Let X be a scheme and $i : X^{red} \hookrightarrow X$ be the underlying reduced closed subscheme. Then the functor $i^\star : \mathcal{C}_X \to \mathcal{C}_{X^{red}}$ is an equivalence of categories. In particular, if X is connected, it induces an isomorphism of profinite groups:*

$$\pi_1(i) : \pi_1(X^{red}) \tilde{\to} \pi_1(X).$$

Theorem A.3. *Let S be a scheme and let $f : S' \to S$ be a morphism which is either:*
- finite and surjective or
- faithfully flat and quasi-compact.
Then $f : S' \to S$ is a morphism of effective descent for the fibred category of étale, separated schemes of finite type.

Corollary A.4. *Let S be a scheme and let $f : S' \to S$ be a morphism which is either:*
- finite, radiciel and surjective or
- faithfully flat, quasi-compact and radiciel.
Then $f : S' \to S$ induces an equivalence of categories $\mathcal{C}_S \to \mathcal{C}_{S'}$.

Theorem A.5. *Let S be a scheme and let $f : S' \to S$ be a proper and surjective morphism. Then $f : S' \to S$ is a morphism of effective descent for the fibre category of étale covers.*

A.3. **Comparison of fundamental groups for morphism of effective descent.**
Assume that $f : S' \to S$ is a morphism of effective descent for the fibre category of étale covers. Our aim is to interpret this in terms of fundamental groups.

Consider the usual notation S'', S''' and:

$$\begin{array}{l} p_i : S'' \to S',\ i = 1, 2, \\ p_{i,j} : S''' \to S'',\ 1 \leq i < j \leq 3, \\ u_i : S''' \to S',\ = 1, 2, 3. \end{array}$$

Assume that S, S', S'', S''' are disjoint union of connected schemes, then, with $E' := \pi_0(S')$, $E'' := \pi_0(S'')$, $E''' := \pi_0(S''')$, also set:

$$\begin{array}{l} q_i = \pi_0(p_i) : E'' \to E',\ i = 1, 2, \\ q_{i,j} = \pi_0(p_{i,j}) : E''' \to E'',\ 1 \leq i < j \leq 3, \\ v_i = \pi_0(u_i) : E''' \to E',\ i = 1, 2, 3. \end{array}$$

Write $\mathcal{C} := \mathcal{C}_S$, $\mathcal{C}' := \mathcal{C}_{S'}$, $\mathcal{C}'' := \mathcal{C}_{S''}$, $\mathcal{C}''' := \mathcal{C}_{S'''}$. We assume that S is connected.

Fix $s'_0 \in E'$ and for each $s' \in E'$, fix an element $\overline{s'} \in E''$ such that

$$q_1(\overline{s'}) = s'_0 \text{ and } q_2(\overline{s'}) = s'.$$

Also, for any $s' \in E'$ (resp. $s'' \in E''$, $s''' \in E'''$) fix a geometric point $\underline{s}' \in s'$ (resp. $\underline{s}'' \in s''$, $\underline{s}'' \in s''$) and write $\pi_{s'} := \pi_1(s'; \underline{s}')$ (resp. $\pi_{s''} := \pi_1(s''; \underline{s}'')$, $\pi_{s'''} := \pi_1(s'''; \underline{s}'''))$ for the corresponding fundamental group.

Since for any $s'' \in E''$ $p_i(\underline{s}'')$ and $\underline{q_i(s'')}$ lie in the same connected component of S', one gets étale paths $\alpha_i^{s''} : F''_{\underline{s''}} \circ p_i^\star = F'_{p_i(\underline{s}'')} \tilde{\to} F'_{\underline{q_i(s'')}}$, hence profinite group morphisms:

$$q_i^{s''} : \pi_{s''} \to \pi_1(q_i(s''), p_i(\underline{s}'')) \simeq \pi_{q_i(s'')},\ i = 1, 2.$$

Similarly, one gets étale paths $\alpha_{i,j}^{s'''} : F'''_{\underline{s'''}} \circ p_{i,j}^\star = F''_{p_{i,j}(\underline{s}''')} \tilde{\to} F''_{\underline{q_{i,j}(s''')}}$ and profinite group morphisms:

$$q_{i,j}^{s'''} : \pi_{s'''} \to \pi_1(q_{i,j}(s'''), p_i(\underline{s}''')) \simeq \pi_{q_{i,j}(s''')},\ 1 \leq i < j \leq 3.$$

Eventually, from the étale paths

$$F'''_{s'''}\circ p^\star_{1,2}\circ p^\star_1\tilde{\to}F_{\underline{v_1(s''')}}\tilde{\leftarrow}F'''_{s'''}\circ p^\star_{1,3}\circ p^\star_1;$$
$$F'''_{s'''}\circ p^\star_{1,2}\circ p^\star_2\tilde{\to}F_{\underline{v_2(s''')}}\tilde{\leftarrow}F'''_{s'''}\circ p^\star_{2,3}\circ p^\star_1;$$
$$F'''_{s'''}\circ p^\star_{1,3}\circ p^\star_2\tilde{\to}F_{\underline{v_3(s''')}}\tilde{\leftarrow}F'''_{s'''}\circ p^\star_{2,3}\circ p^\star_2;$$

one gets $a_i^{s'''}\in\pi_{v_i(s''')}$, $i=1,2,3$ such that

$$q_1^{q_{1,2}(s''')}\circ q_{1,2}^{s'''}=\mathrm{int}(a_1^{s'''})\circ q_1^{q_{1,3}(s''')}\circ q_{1,3}^{s'''};$$
$$q_2^{q_{1,2}(s''')}\circ q_{1,2}^{s'''}=\mathrm{int}(a_2^{s'''})\circ q_1^{q_{2,3}(s''')}\circ q_{2,3}^{s'''};$$
$$q_2^{q_{1,3}(s''')}\circ q_{1,2}^{s'''}=\mathrm{int}(a_3^{s'''})\circ q_2^{q_{2,3}(s''')}\circ q_{2,3}^{s'''};$$

Since $f:S'\to S$ is a morphism of effective descent, the above data allows us to recover $\mathcal{C}$ from $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$ up to an equivalence of category hence to reconstruct $\pi_1(S,\underline{p(s'_0)})$ from the $\pi_{s'}$, $\pi_{s''}$, $\pi_{s'''}$.

More precisely, the category $\mathcal{C}'$ with descent data for $f:S'\to S$ is equivalent to the category $\mathcal{C}(\{\pi_{s'}\}_{s'\in E'})$ together with a collection of functor automorphisms $g_{s''}:Id\tilde{\to}Id$, $s''\in E''$ satisfying the following relations:

(1) $g_{s''}q_1^{s''}(\gamma'')=q_1^{s''}(\gamma'')g_{s''}$, $s''\in E''$;
(2) $g_{\overline{s'}}=g_{\overline{s'_0}}$, $s'\in E'$;
(3) $a_3^{s'''}g_{q_{1,3}(s''')}a_1^{s'''}=g_{q_{2,3}(s''')}a_2^{s'''}g_{q_{1,2}(s''')}$, $s'''\in E'''$,

So, set

$$\Phi:=\coprod_{s'\in S'}\pi_{s'}\coprod_{s''\in E''}\hat{\mathbb{Z}}g_{s''}/<(1),(2),(3)>,$$

where $\coprod$ stands for the free product in the category of profinite groups and let $\mathcal{N}$ be the class of all normal subgroups $N\lhd\Phi$ such that $[\Phi:N]$ and $[\pi_{s'}:i_{s'}^{-1}(N)]$ are finite (here $i_s:\pi_s\hookrightarrow\coprod_{s'\in S'}\pi_{s'}\coprod_{s''\in E''}\hat{\mathbb{Z}}g_{s''}\twoheadrightarrow\Phi$ denotes the canonical morphism). Then writing

$$\pi:=\varprojlim_{N\in\mathcal{N}}\Phi/N$$

one gets that the category $\mathcal{C}'$ with descent data for $f:S'\to S$ is also equivalent to the category $\mathcal{C}(\pi)$. Whence:

Theorem A.6. *With the above assumptions and notation, one has a canonical profinite group isomorphism*

$$\pi_1(S,p(s'_0))\tilde{\to}\pi.$$

Corollary A.7. *With the above assumptions and notation, if E' and E'' are finite and if the $\pi_{s'}$, $s'\in E'$ are topologically of finite type then so is $\pi_1(S,p(s'_0))$.*

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