TORSION POINTS OF ABELIAN VARIETIES OVER FUNCTION FIELDS

MARC HINDRY & AMÍLCAR PACHECO

Sketch, notations and ideas for the content of the lectures:

Let $\mathcal{C}$ be a smooth projective geometrically connected curve defined over a finite field $k = \mathbb{F}_q$ of characteristic p, $K = k(\mathcal{C})$ its function field of genus g and A/K a non-constant abelian variety of dimension d. Let $\phi : \mathcal{A} \to \mathcal{C}$ be a Néron model of A over $\mathcal{C}$. The ultimate goal is to show that if the Kodaira-Spencer map of ϕ is non-zero and $p > 2d+1$, then there exists a bound for the order of the torsion subgroup of $A(K)$ depending only on d and g (which alas is not explicit). The proof requires several steps.

1. The differential height and the group of connected components

Denote by $e_{\mathcal{A}/\mathcal{C}}$ the neutral section of ϕ and $\omega_{\mathcal{A}/\mathcal{C}} = e^*_{\mathcal{A}/\mathcal{C}}(\Omega^1_{\mathcal{A}/\mathcal{C}})$. Given a vector bundle $\mathcal{E}$ in $\mathcal{C}$, let $\wedge^{\max}\mathcal{E}$ be its maximal exterior power and $\deg(\mathcal{E}) = \deg(\wedge^{\max}\mathcal{E})$. The differential height of A/K is defined by $h_{A/K} = \deg(\omega_{\mathcal{A}/\mathcal{C}})$. For each place v of K denote by $c_v(A/K)$ the cardinality of the group of connected components of the special fiber of the Néron model of A/K at v which are defined over the residue field κ_v of v. After having taken a finite extension of K we will assume that A/K has everywhere semi-abelian reduction. Let S be the finite set of places of K where A has bad reduction and $s = \#S$. The goal of this section is to present the scheme of proof of the following inequality

$$\sum_{v \in S} c_v(A/K)^{1/d} \deg(v) \ll h_{A/K}.$$

We observe that if $p > 2d+1$ the hypothesis of everywhere semi-abelian reduction is no longer necessary.

2. An *abc* theorem for semi-abelian schemes

This result is an extension of Szpiro's discriminant theorem from elliptic curves with semi-stable reduction to semi-abelian schemes in positive characteristic. Let (τ, B) be the K/k-trace of A and $d_0 = \dim(B)$. Denote by $\mathfrak{F}_{A/K} \in \mathrm{Div}(\mathcal{C})$ the conductor of A/K and let $f_{A/K} = \deg(\mathfrak{F}_{A/K})$. We will sketch the proof of the following inequality : if $\mathrm{Kod}(\phi) \neq 0$ and $p > 2d+1$, then

$$h_{A/K} \leq \frac{1}{2}(d - d_0)(2g - 2 + s).$$

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Here Kod(ϕ) stands for the Kodaira-Spencer map. In the case where Kod(ϕ) = 0, if p^e denotes the inseparable degree of the j-map induced by ϕ on the moduli of abelian varieties, then we have to multiply the right hand side of the inequality by p^e. The method of the proof involves on the one hand the compactification of the moduli space of principally polarized abelian varieties with a level structure and on the other hand the notion of the nilpotence of the Gauss-Manin connection on the de Rham cohomology (through the use of p-curvature of a vector bundle).

3. Rigid geometric uniformization and the order of points

We suppose that A/K has everywhere semi-abelian reduction and that it is principally polarized *via* a symmetric ample line bundle $\mathcal{L}$, then using Raynaud's uniformization of $A(K_v)$ for a place v where A has bad reduction, there exists a Fourier-Jacobi expansion of theta functions (this is contained in Chai's lecture notes on moduli spaces of abelian varieties, essentially these are linear combinations of non-archimedian versions of the usual complex theta functions with integral coefficients).

Denote by $h_{\Theta,A/K}$ the height of A with respect to the theta embedding, $\mathfrak{F}_{A/K,r}$ the reduced conductor of A, i.e., where all multiplicities are equal to 1, and $f_{A/K,r} = \deg(\mathfrak{F}_{A/K,r})$. Denote $\rho_{A/K} = h_{\Theta,A/K}/f_{A/K,r}$, $\sigma_{A/K} = h_{A/K}/f_{A/K}$ and observe (*confer* the previous section) that there exists a constant c depending only on d such that $\rho_{A/K} \leq c\sigma_{A/K}$.

We propose a strategy to prove an analogue of Lang's conjecture for function fields over finite fields, namely, if P is a point of $A(K)$ of infinite order modulo every sub-abelian variety of A, then its Néron-Tate height is bounded below by some constant depending on d, g and the ratio $\rho_{A/K}$ multiplied by $\max\{1, h_{\Theta,A/K}\}$.

Actually, we aim at more : achieving to prove that if the Néron-Tate height of P is bounded from above by such an expression, then P is a point of finite order and its order is bounded from above by a constant depending on d and on $\rho_{A/K}$, a fortiori just on d and $\sigma_{A/K}$. Whence, from the last section, in fact this upper bound will depend just on d and g.

The approach used is an extension of the strategy implemented in David's paper [Minorations de hauteurs de variétés abéliennes, BSMF 1993] in the complex context. In this lecture we will present a sketch of this strategy of the proof of this result.

Would it be possible to obtain a bound depending on the k-gonality of $\mathcal{C}$ (as Poonen does for elliptic curves)?

4. Methods from transcendence theory

Since A is principally polarized every point of A can be identified with a group extension of A by $\mathbb{G}_m$. Similarly, any extension G of A by a multplicative torus $\mathbb{G}_m^m$ is associated to a point of A^m. One also has natural compactification of G (à la Serre) related to $\mathcal{L}$ and a natural ample line bundle $\mathcal{M}$.

On such a multiplicative torus, associated to a k-tuple related to P, and assuming that the Néron-Tate height of P is small enough and constructs a global section of a suitable power $\mathcal{M}^{\otimes n}$ of $\mathcal{M}$ vanishing at the origin with a suitable multiplicity.

One then uses the local uniformizations at each place of bad reduction and analytic tools to prove that such a section must be very small v-adically at a higher multiplicity at the origin at each and every place v of bad reduction of A. A simple one variable Schwarz lemma on an annulus is enough for this purpose.

Putting all the places together, one deduces by the product formula, using the assumption that the Néron-Tate height of P is small that our original section must actually vanish at a higher multiplicity at the origin. For these steps, one notices that the height of the group G itself encodes the height of P.

Classical Philippon zero estimates then ensure that P must be of finite order modulo some sub-abelian variety of A, and even conveniently provide for a bound for that order.