Aerodynamic Constraints for Vortex Trapping Airfoils

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Riassunto. Si mostra come è possibile ricavare la geometria del profilo capace di catturare un vortice che realizza una data distribuzione di velocità, essendo tale distribuzione di velocità assegnata in funzione dell’ascissa curvilinea fissata sul contorno del profilo. Vengono altresì espressi i vincoli che detta distribuzione deve soddisfare affinché il problema risulti ben posto.

Abstract. The inverse problem for a vortex capturing airfoil is dealt with. A design velocity distribution is given as a function of the curvilinear abscissa of the airfoil contour. The velocity distribution constraints that lead to a well posed problem are found.

Aerodynamic bodies have certain architectures and shapes that are the result of the progress that has been made in understanding the flow physics. For example, the discovery of the mechanism of lift has led to sophisticated wing designs of minimum induced drag. By the end of the sixties, improvements in the understanding of compressible flows led to the design of shockless airfoils at supercritical Mach numbers [1]. Today, the advances in aircraft performance concern the scrutiny of transition and turbulence phenomena, but the mechanism that will grant a break-through in delaying transition or reducing turbulent drag will probably be active rather than passive. By active mechanism we mean an artificial modification of the flow which is induced by the actual flow status, therefore based on sensors and actuators which are able of an automatic reaction. In this context we have studied, using the inverse boundary value problem within potential flow theory (see for example [2]), the design of new airfoil shapes that might become of some interest thanks to active control. An airfoil is considered with a cavity of such a shape that a region of vortical flow is trapped inside the cavity. Ideally, it represents a virtual flap that provides high lift with no mechanical part being involved. In particular, there are equilibrium configurations of a point vortex that satisfy the Kutta condition at the trailing edge both for a flat plate, see[3], and for the Joukowski airfoil [4]. Alternately, a trapped vortex can be used to let the flow overcome
the regions of unfavorable pressure gradient. In [5] it is shown how such airfoils should be designed to allow the development of a cyclic boundary layer inside the cavity. It is also shown that the correct high Reynolds number limit of the flow in the cavity is the Batchelor model, i.e., a region of constant vorticity.

These ideas are not so far as they might seem from application. The EKIP aircraft, see [5], is a bluff body that hosts a turbofan which allows an attached flow over its upper body surface by a series of vortex trapping cavities. The vortices are stabilized using passive suction. In practice, the captured vortices result to be highly unstable due to upstream receptivity and secondary separations inside the cavity. We have already dedicated work to the active stabilization of such flows [6] [7]. In this paper attention is concentrated on a related problem, namely, the solution of an inverse aerodynamic problem for vortex trapping airfoils.

The inverse problem in aerodynamics has its roots in the works of Mangler [8] and Lighthill [9] who solved the problem of determining an airfoil, given the velocity distribution over its surface. Lighthill discovered that the velocity distribution had to satisfy certain integral constraints in order to obtain closed airfoils and to verify the given velocity at infinity. Such conditions go under the name of solvability conditions. The inverse problem formulation for a vortex trapping airfoil involves additional constraints on the desired velocity distribution. Cavities that allow the formation of a stable vortical region are in particular sought. To this end, the vortical region inside the cavity is modeled by a single point vortex. Such a model is approximated as it concentrates all the vorticity of the cavity in one point. Experimental results as well as theoretical results [5] show that the vorticity is mostly constant in the cavity except in the boundary layer, according to the Batchelor model. Nevertheless, as we show in the following sections, the point vortex model allows the expression, in closed form, of the additional constraints on the prescribed velocity distribution. Thus we obtain an airfoil which generates the given velocity on its boundary and sustains the presence of a vortex in stable equilibrium.

We heavily rely on the techniques presented in [2] and [10]. The well known inverse boundary value problem for a single isolated airfoil, is briefly presented in the next paragraph to make the ideas and the terminology clear. Then we give the main result of this paper, namely the constraints that have to be satisfied by the prescribed velocity on the airfoil to obtain a vortex capturing airfoil. Additional details are found in [11].

**Simple airfoil**

Let us represent the airfoil on the complex plane \( z = x + iy \) (fig. 1(a)). The airfoil contour \( L_z \) is closed. The trailing edge \( T \) is sharp. It is located on the axis origin and the value of its external angle is given as \( \varepsilon \pi \), with \( \varepsilon \in [1, 2] \) with \( \varepsilon \neq 1 \). The airfoil chord forms an unknown angle \( \alpha \) with the \( x \)-axis, while the asymptotic flow velocity \( v_\infty \) has the same direction of the positive \( x \)-axis. A curvilinear abscissa \( s \) runs clockwise along \( L_z \), with the origin \( s = 0 \) located
on the trailing edge $T$. The contour length $L$ is given. The unit vector tangent to the contour is denoted by $\hat{\tau}$.

The inverse problem consists in finding the airfoil shape for a given free stream flow velocity $v_\infty$ and a given velocity distribution $v_p = v_p(s)$ on the airfoil contour $L_z$ (fig. 1(b)). The solution is obtained by finding the conformal mapping $z = z(\zeta)$ that maps the unit circle of the $\zeta$-plane (fig. 2(a)) onto the airfoil of $z$-plane once the complex potential pertinent to the given velocity distribution has been determined under the constraints $\lim_{\zeta \to \infty} z(\zeta) = \infty$ and $z(1) = 0$.

Let $w = w(z) = \varphi(x, y) + i\psi(x, y)$ be the complex potential. Since $\psi =$ cost on an impermeable solid wall, we assume $\psi = 0$ on the airfoil boundary $L_z$. Moreover, we are free to set $\varphi(s^*) = 0$, with $s^*$ denoting the curvilinear abscissa of the stagnation point (fig. 1(a)), so that

$$\varphi(s) = \int_{s^*}^s v(s) \, ds. \quad (1)$$

By setting $\varphi_1 = \varphi(L)$, $\varphi_0 = \varphi(0)$ it follows that the airfoil circulation is $\Gamma = \varphi_1 - \varphi_0$, with $\Gamma > 0$ when clockwise.

According to the invariance of complex potential for conformal mapping $\tilde{w}(\zeta) = \tilde{\varphi} + i\tilde{\psi} = w(z(\zeta))$. The expression of $\tilde{w}(\zeta)$ is known:

$$\tilde{w}(\zeta) = u_0 e^{-i\beta} \zeta + \frac{u_0 e^{i\beta}}{\zeta} - \frac{\Gamma}{2\pi i} \ln \zeta + c_0 \quad (2)$$
where \(u_0 e^{-i\beta}\) is the complex velocity \(d\bar{w}/d\zeta\) at infinity and \(c_0\) is a real constant.

Let \(\zeta = r \exp(i\gamma)\), therefore

\[
v_c(\gamma) = \left( \frac{\partial \tilde{\varphi}}{\partial \gamma} \right)_{r=1} = -2u_0 \sin(\gamma - \beta) - \frac{\Gamma}{2\pi}
\]

is the velocity distribution on the unit circle contour relevant to the given velocity distribution \(v_p(s)\) on the airfoil contour.

The values of \(u_0\) and \(\beta\) are determined by enforcing the conditions: \(\tilde{\varphi}(\gamma^*) = \varphi(s^*) = 0\), \(\tilde{\varphi}(0) = \varphi(L) = \varphi_1\), \(v_c(\gamma^*) = 0\) and \(v_c(0) = v_c(2\pi) = 0\), that is:

\[
\begin{align*}
2u_0 \cos(\gamma^* - \beta) - \frac{\Gamma}{2\pi}\gamma^* + c_0 &= 0 \\
2u_0 \cos \beta + c_0 &= \varphi_1 \\
-2u_0 \sin(\gamma^* - \beta) - \frac{\Gamma}{2\pi} &= 0 \\
2u_0 \sin \beta - \frac{\Gamma}{2\pi} &= 0
\end{align*}
\]

which are solved with respect to \(u_0\), \(\beta\), \(c_0\) and the stagnation angle \(\gamma^*\).

The function \(s = s(\gamma)\) relating the unit circle angle \(\gamma\) to the airfoil arc length \(s\) is implicitly expressed by the equation \(\varphi(s) = \tilde{\varphi}(\gamma)\) and its derivative \(s'(\gamma)\) can be deduced by \(v_p(s) = \varphi'(s)\), \(v_c(\gamma) = \tilde{\varphi}'(\gamma)\) and eq. (3), that is

\[
s'(\gamma) = \frac{\tilde{\varphi}'(\gamma)}{\varphi'(s)} = \frac{-2u_0 \sin(\gamma - \beta) - \frac{\Gamma}{2\pi}}{v_p[s(\gamma)]} = \frac{-4u_0 \sin \frac{\gamma}{2} \cos \left( \frac{\gamma}{2} - \beta \right)}{v_p[s(\gamma)]}
\]

which is a continuous negative function for \(0 < \gamma < 2\pi\) and which has a zero of order \(\varepsilon - 1\) for \(\gamma = 0\) and \(\gamma = 2\pi\).

The determination of the logarithm of the mapping derivative

\[
\log \frac{dz}{d\zeta} = \log \left| \frac{dz}{d\zeta} \right| + i \arg \left( \frac{dz}{d\zeta} \right)
\]

is obtained according to the classical solution of the problem of finding an analytic function whose real part

\[
\log \left| \frac{dz}{d\zeta} \right|_{|\zeta|=1} = \log |s'(\gamma)|.
\]

is known on the unit circle contour.

The function \(\log(dz/d\zeta)\) has a logarithmic singularity in \(\zeta = 1\) which can be avoided by considering the function

\[
\omega(\zeta) = P(\zeta) + i Q(\zeta) = \log \frac{dz}{d\zeta} - (\varepsilon - 1) \log \left( 1 - \frac{1}{\zeta} \right)
\]
whose real part is known as well
\[ P(\gamma) = \ln |s'(\gamma)| - (\varepsilon - 1) \log \left( 2 \sin \frac{\gamma}{2} \right). \]  

(9)

According to the Schwarz formula, for the region exterior to the unit circle
\[ \omega(\zeta) = -\frac{1}{2\pi} \int_0^{2\pi} P(\tau) e^{i\tau} + \zeta e^{i\tau} - \zeta \, d\tau + i Q(\infty) \]  

(10)

with \( Q(\infty) = \arg(dz/d\zeta)_{\zeta=\infty} = -\beta \). By integrating eq. (8) we obtain the inverse problem solution
\[ z(\zeta) = \int_1^\zeta e^{\omega(\zeta')} \left( 1 - \frac{1}{\zeta} \right)^{\varepsilon-1} d\zeta'. \]  

(11)

The previously described problem is in general ill-posed for the velocity at infinity \( v_\infty \) and the airfoil velocity distribution \( v_p(s) \) cannot be arbitrary chosen but have to satisfy three constraints, as found by Lighthill [9].

The first constraint expresses the compatibility between \( v_p(s) \) and \( v_\infty \), and is found by considering that the mapping derivative at infinity has to satisfy
\[ \frac{dz}{d\zeta}_{\zeta=\infty} = u_0 \exp(-i\beta)/v_\infty, \quad \text{that is, according to eqs.} \ (8), (10) \]
\[ \frac{1}{2\pi} \int_0^{2\pi} P(\tau) \, d\tau = \log \frac{u_0}{v_\infty}. \]  

(12)

Other two constraints are expressed by the airfoil closure conditions. For the airfoil to be closed, the integral \( \oint (dz/d\zeta) \, d\zeta \) taken along the unit circle contour must be zero. Since \( dz/d\zeta \) is regular in the domain exterior to the unit circle, its residual at infinity is null. According to eqs. (8),(10) this condition yields
\[ \int_0^{2\pi} P(\tau) \cos \tau \, d\tau = \pi(\varepsilon - 1), \quad \int_0^{2\pi} P(\tau) \sin \tau \, d\tau = 0. \]  

(13)

In the following section it will be shown that a quasi-solution can be obtained using a variational approach, where by quasi-solution we intended the solution for a modified velocity distribution that satisfies the constraints and is the closest to the original one according to a given criterion.

**Vortex Trapping Airfoil**

Examples of airfoils that are capable of trapping a vortex in stable equilibrium are described in [5] [7]. As shown in fig. 3(b), these airfoils are characterized by a sharp edge \( A \) in addition to the trailing edge \( T \). The flow separates at \( A \) and a vortex is trapped in the subsequent cavity.

The solution method of the inverse problem is the same as that shown in the previous section. For a given velocity distribution pertinent to a captured vortex (fig. 3(a)), the solution is found by determining the function \( z = z(\zeta) \) that transforms the unit circle of the \( \zeta \)-plane (fig. 3(c)) onto the airfoil of the
The complex potential $\tilde{w}(\zeta) = w(z(\zeta))$ is:

$$
\tilde{w}(\zeta) = u_0 e^{-i\beta} + \frac{u_0 e^{i\beta}}{\zeta} - \frac{K}{2\pi i} \ln \frac{\zeta - \zeta_0}{\zeta - \zeta_0} - \frac{\Gamma + K}{2\pi i} \ln \zeta + c
$$

where $\zeta_0 = 1/\zeta^*$, $\Gamma$ is the total circulation, $K$, $\zeta_0$ are the circulation and the location of the trapped vortex, respectively, and $c$ is a complex constant. The complex velocity is therefore

$$
\frac{d\tilde{w}}{d\zeta} = u_0 e^{-i\beta} - \frac{u_0 e^{i\beta}}{\zeta^2} - \frac{K}{2\pi i} \frac{\zeta_0 - \zeta}{\zeta_0 - \zeta} - \frac{\Gamma + K}{2\pi i} \zeta
$$

The flow past the circle has four stagnation points, two are located on the edge images $A, T$, the other two are denoted by $B$ and $D$, as shown in fig. 3. The value $\Gamma$ is provided by $\Gamma = \varphi(L) - \varphi(0)$, with $\varphi(s) = \int_{s_0}^{s} v_p(s) ds$. The values of $\beta, u_0, K, \zeta_0, c, \gamma_A, \gamma_B, \gamma_D$ are determined by the system of ten equations:

$$
\tilde{\varphi}(\gamma_B) - \tilde{\varphi}(2\pi) = \Delta \varphi_{TB}, \quad \tilde{\varphi}(\gamma_A) - \tilde{\varphi}(\gamma_B) = \Delta \varphi_{BA}
$$

$$
\tilde{\varphi}(\gamma_D) - \tilde{\varphi}(\gamma_A) = \Delta \varphi_{AD}, \quad \tilde{\varphi}(0) - \tilde{\varphi}(\gamma_D) = \Delta \varphi_{DT}
$$

$$
\tilde{\varphi}'(0) = \tilde{\varphi}'(\gamma_D) = \tilde{\varphi}'(\gamma_A) = \tilde{\varphi}'(\gamma_B) = \tilde{\psi}(\gamma) = \tilde{\varphi}(\gamma_A) = 0
$$
As a consequence, the function \( \tilde{\varphi}(\gamma) \) is known. Since \( \tilde{\varphi}(\gamma) = \varphi(s) \), the function \( s = s(\gamma) \) is known in implicit form. Hence the derivative \( s'(\gamma) = -|d\gamma/d\zeta|_{|\zeta|=1} \) is also known, and therefore we obtain the same problem of the previous section, namely that of determining \( \log(d\zeta/d\gamma) \) once its real part is known on the contour. Now \( \log(d\zeta/d\gamma) \) has two logarithmic singularities due to the two edges \( A \) and \( T \). These are avoided by considering the function

\[
\omega(\zeta) = P(\zeta) + iQ(\zeta) = \ln \frac{dz}{d\zeta} - (\varepsilon_\pi - 1) \ln \left(1 - \frac{1}{\zeta}\right) - (\varepsilon_A - 1) \ln \left(1 - \frac{\zeta}{\zeta_A}\right)
\]

(17)

where \( \varepsilon_\pi \) and \( \varepsilon_A \) are the external angle of the \( T \) and \( A \) edges, respectively.

\( \omega(\gamma) \) is known and the Schwarz formula (10) determines \( \omega(\zeta) \) that yields the inverse problem solution.

Following the same remarks of the simple airfoil case, three constraints on \( v_p(s) \) enforcing the closure conditions and the compatibility at infinity:

\[
\int_0^{2\pi} P(\tau) \, d\tau = B_1, \quad \int_0^{2\pi} P(\tau) \cos \tau \, d\tau = B_2, \quad \int_0^{2\pi} P(\tau) \sin \tau \, d\tau = B_3.
\]

(18)

with

\[
B_1 = 2\pi \ln \frac{u_0}{v_\infty}, \quad B_2 = \pi \left[\varepsilon_\pi - 1 + (\varepsilon_A - 1) \cos \gamma_A\right], \quad B_3 = \pi (\varepsilon_A - 1) \sin \gamma_A.
\]

So far the discussion of vortex equilibrium has been neglected. It is quite remarkable that vortex equilibrium implies another two constraints on \( v_p(s) \) that can be expressed in a form analogous to eq. (18). According to the Routh rule [12], a vortex is standing in equilibrium in the physical \( z \)-plane if the velocity \( \zeta_0 \) of its free image on the transformed \( \zeta \)-plane is

\[
\dot{\zeta}_0 = -\frac{K}{4\pi i} \left[ \frac{d}{d\zeta} \ln \left(\frac{dz}{d\zeta}\right) \right]_{\zeta_0}
\]

(19)

with

\[
\dot{\zeta}_0 = u_0 e^{-i\beta} - \frac{u_0 e^{-i\beta}}{\zeta_0} + \frac{K}{2\pi i} \frac{1}{\zeta_0 - \zeta_0} + \frac{1}{2\pi i \zeta_0}.
\]

(20)

According to the Schwarz formula (10)

\[
\left[ \frac{d}{d\zeta} \ln \left(\frac{dz}{d\zeta}\right) \right]_{\zeta_0} = -\frac{1}{2\pi} \int_0^{2\pi} P(\tau) \frac{2 e^{i\tau}}{(e^{i\tau} - \zeta_0)^2} \, d\tau + \frac{1}{\zeta_0 (\zeta_0 - 1)} + (\varepsilon_\pi - 1) + (\varepsilon_A - 1) + \frac{1}{\zeta_0 (\zeta_0 - 1)}
\]

(21)

and eq. (19) can be rewritten as

\[
\left\{ \begin{array}{c}
\int_0^{2\pi} P(\tau) \cos \tau + \rho_0^2 \cos(\tau - 2\gamma_0) - 2\rho_0 \cos \gamma_0 \, d\tau = B_4 \\
\int_0^{2\pi} P(\tau) \sin \tau + \rho_0^2 \sin(\tau - 2\gamma_0) - 2\rho_0 \sin \gamma_0 \, d\tau = B_5
\end{array} \right.
\]

(22)
with $\rho_0 = |\zeta_0|$, $\gamma_0 = \arg(\zeta_0)$ and

$$B_4 = \pi \left\{ \left( \varepsilon_T - 1 \right) \frac{\rho_0 \cos 2\gamma_0 - \cos \gamma_0}{\rho_0 (\rho_0^2 + 1 - 2\rho_0 \cos \gamma_0)} + \left( \varepsilon_A - 1 \right) \frac{\rho_0 \cos (2\gamma_0 - 2\gamma_A) - \cos \gamma_0}{\rho_0 (\rho_0^2 + 1 - 2\rho_0 \cos (\gamma_0 - \gamma_A))} \right\}$$

$$-4\pi \frac{\varepsilon_0}{K} \left[ \sin \beta + \frac{1}{\rho_0^2} \sin (\beta - 2\gamma_0) \right] - \frac{2\rho_0}{\rho_0^2 - 1} \cos \gamma_0 - \frac{2}{\rho_0} \left( 1 + \frac{\Gamma}{K} \right) \cos \gamma_0$$

$$B_5 = \pi \left\{ \left( \varepsilon_T - 1 \right) \frac{\rho_0 \sin 2\gamma_0 - \sin \gamma_0}{\rho_0 (\rho_0^2 + 1 - 2\rho_0 \cos \gamma_0)} + \left( \varepsilon_A - 1 \right) \frac{\rho_0 \sin (2\gamma_0 - 2\gamma_A) - \sin \gamma_0}{\rho_0 (\rho_0^2 + 1 - 2\rho_0 \cos (\gamma_0 - \gamma_A))} \right\}$$

$$-4\pi \frac{\varepsilon_0}{K} \left[ \cos \beta - \frac{1}{\rho_0^2} \cos (\beta - 2\gamma_0) \right] + \frac{2\rho_0}{\rho_0^2 - 1} \sin \gamma_0 - \frac{2}{\rho_0} \left( 1 + \frac{\Gamma}{K} \right) \sin \gamma_0$$

The five constraints (18), (22) that have to be satisfied in order to obtain a well posed problem have the form

$$\int_0^{2\pi} P(\tau) f_k(\tau) d\tau = B_k \quad \text{con} \ k = 1, 2, 3, 4, 5 \quad (23)$$

where the functions $f_k(\tau)$ can be inferred from eqs. (18), (22)

The constrains are, in general, not satisfied by the $P(\gamma)$ pertinent to an arbitrarily given $v_p(s)$. Therefore, we look for a quasi solution for a $P_*(\gamma)$ that satisfies the constraints and is the closest, in the $L^2$ sense, to the original $P(\gamma)$, that is, we look for the minimum of the functional

$$J(T) = \int_0^{2\pi} T^2(\tau) f_0(\tau) d\tau$$

where $f_0(\tau)$ is a weight function and $T(\tau) = P_*(\tau) - P(\tau)$. The minimum is found under the five constraints

$$\int_0^{2\pi} T(\tau) f_k(\tau) d\tau = C_k \quad (24)$$

with

$$C_k = B_k - \int_0^{2\pi} P(\tau) f_k(\tau) d\tau, \quad k = 1, 2, 3, 4, 5$$

According to the Lagrangian multipliers technique, $T(\tau)$ is obtained by finding the minimum of the unconstrained functional

$$L(T) = J(T) - \sum_{k=1}^5 \mu_k \left[ \int_0^{2\pi} T(\tau) f_k(\tau) d\tau - C_k \right]$$

where $\mu_k$ are the Lagrangian multipliers.

Letting the first variation of $L(T)$, with respect to $T(\tau)$, be zero, one obtains

$$T(\tau) = \frac{1}{2f_0(\tau)} \sum_{k=1}^5 \mu_k f_k(\tau) \quad (25)$$
which, enforcing the constraints (24), yields

$$\sum_{k=1}^{5} \mu_k \int_{0}^{2\pi} \frac{f_j(\tau)f_k(\tau)}{2f_0(\tau)} d\tau = C_j, \quad j = 1, 2, 3, 4, 5$$

that allows the computation of the multipliers $\mu_k$ and hence $T(\tau)$.

The case of a vortex trapping airfoil obtained by solving the inverse problem and satisfying all of the five constraints including the equilibrium conditions for the vortex is shown in figs. 4, 5 as an example.

![Figure 4: Vortex trapping airfoil.](image1.png)

![Figure 5: Original velocity distribution and that one that minimizes the Lagrangian.](image2.png)