HYPERQUADRATIC CONTINUED FRACTIONS AND AUTOMATIC SEQUENCES

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Abstract. We show that three different families of hyperquadratic elements, studied in the literature, have the following property: For these elements, the leading coefficients of the partial quotients in their continued fraction expansion form 2-automatic sequences. We also show that this is not true for algebraic elements in $\mathbb{F}(q)$ in general. Indeed, we use an element studied by Mills and Robbins as counterexample. This element is algebraic in $\mathbb{F}(3)$ but the principal coefficients of the partial quotients does not form an automatic sequence.

1. Introduction

Let $\mathbb{F}_q$ be the finite field containing q elements, with $q = p^s$ where p is a prime number and $s \geqslant 1$ is an integer. We consider the field of power series in $1/T$, with coefficients in $\mathbb{F}_q$, where T is a formal indeterminate. We will denote this field by $\mathbb{F}(q)$. Hence a non-zero element of $\mathbb{F}(q)$ is written as $\alpha = \sum_{k \leqslant k_0} c_k T^k$ with $k_0 \in \mathbb{Z}$, $c_k \in \mathbb{F}_q$, and $c_{k_0} \neq 0$. Noting the analogy of this expansion with a decimal expansion for a real number, it is natural to regard the elements of $\mathbb{F}(q)$ as (formal) numbers and indeed they are analogue to real numbers in many ways.

It is well known that the sequence of coefficients (or digits) $(c_k)_{k \leqslant k_0}$ for α is ultimately periodic if and only if α is rational, that is α belongs to $\mathbb{F}_q(T)$. However, and this is a singularity of the formal case, this sequence of digits can also be characterized for all elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$. The origin of the following theorem can be found in the work of Christol [8] (see also the article of Christol, Kamae, Mendès France, and Rauzy [9]).

Theorem 1 (Christol)**.** *Let α in $\mathbb{F}(q)$ with $q = p^s$. Let $(c_k)_{k \leqslant k_0}$ be the sequence of digits of α and $u(n) = c_{-n}$ for all integers $n \geqslant 0$. Then α is algebraic over $\mathbb{F}_q(T)$ if and only if the following set of subsequences of $(u(n))_{n \geqslant 0}$*

$$K(u) = \left\{ (u(p^i n + j))_{n \geqslant 0} \mid i \geqslant 0,\, 0 \leqslant j < p^i \right\}$$

is finite.

The sequences having the finiteness property stated in this theorem were first introduced in the 1960's by computer scientists. Considered in a larger setting (see the beginning of Section 3), they are now called automatic sequences, and form a class of deterministic sequences which can be defined in several different ways. A full account on this topic and a very complete list of references are to be found in

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the book of Allouche and Shallit [2]. In this note we want to show a different type of connection between automatic sequences and some particular algebraic power series in $\mathbb{F}(q)$.

Firstly, let us describe these particular algebraic elements. Let α be irrational in $\mathbb{F}(q)$. We say that α is hyperquadratic, if there exists $r = p^t$ with $t \geqslant 0$ an integer such that the elements α^{r+1}, α^r, α, and 1 are linked over $\mathbb{F}_q(T)$. Thus an hyperquadratic element is algebraic over $\mathbb{F}_q(T)$ of degree $\leqslant r+1$, and the reader may consult [6] where a precise definition was introduced. The subset of hyperquadratic elements in $\mathbb{F}(q)$ is denoted $\mathcal{H}(q)$. Note that this subset contains the quadratic power series (take $r = 1$) and also the cubic power series (take $r = p$). Originally, these algebraic elements were introduced in the 1970's by Baum and Sweet (see [4]), in the particular case $q = 2$, and later considered in the 1980's by Mills and Robbins [21] and Voloch [25], in all characteristic. It appears that $\mathcal{H}(q)$ contains elements having an arbitrary large algebraic degree. But hyperquadratic power series are rare: an algebraic power series of high algebraic degree has a small probability to be hyperquadratic. For different reasons, this subset $\mathcal{H}(q)$ could be regarded as the analogue of the subset of quadratic real numbers.

Besides, it is well known that any irrational element α in $\mathbb{F}(q)$ can be expanded as an infinite continued fraction where the partial quotients a_n are polynomials in $\mathbb{F}_q[T]$, all of positive degree, except perhaps for the first one, and we will write it as $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$. The explicit description of continued fractions for algebraic power series over a finite field goes back to the works [4, 5] of Baum and Sweet, again when the base field is $\mathbb{F}_2$. It was carried on ten years later by Mills and Robbins in [21]. In the real case, no explicit continued fraction expansion, algebraic of degree $n > 2$, is known. On the other hand this expansion for quadratic real numbers is well known to be ultimately periodic. In the formal case, the situation is more complex. Quadratic power series have also an ultimately periodic continued fraction expansion, but many other hyperquadratic continued fractions can also be explicitly described. Most of the elements in $\mathcal{H}(q)$ have a sequence of partial quotients with unbounded degrees. Indeed Theorem 4 in [21, p. 402] implies the following: If α is hyperquadratic satisfying $u\alpha^{r+1} + v\alpha^r + w\alpha + z = 0$ and we have $r > 1 + \deg(uz - vw)$, then the sequence of the degrees of the partial quotients is unbounded. But there are also expansions with all partial quotients of degree 1. This last phenomenon was discovered firstly by Mills and Robbins in [21], and later studied more deeply by Lasjaunias and Yao in [19]. Even though the pattern of hyperquadratic expansions can sometimes be very sophisticated (see for instance the work [14] of Firicel, where a generalization of the cubic introduced by Baum and Sweet is presented), it is yet doubtful whether this description, even partial, is possible for all power series in $\mathcal{H}(q)$.

Power series in $\mathcal{H}(q)$ have particular properties concerning Diophantine approximation and this is also why they were first considered. The work [20] of Mahler in this area, is fundamental. There, a first historical example of hyperquadratic power series, on which we come back below in this note, was introduced. Note that the irrationality measure (also called approximation exponent, see for instance [17, p. 214]) of a power series can be computed if the explicit continued fraction for this element is known. In this way, for many elements in $\mathcal{H}(q)$, the irrationality measure, often greater than 2 for non-quadratic elements, has been given. Hence, contrary

to the real case, many algebraic power series of degree > 2, most of them hyperquadratic, are known to have an irrationality measure greater than 2. Actually, for algebraic power series which are not hyperquadratic, concerning their continued fraction expansions and their irrationality measure, not so much is known. The reader may consult Schmidt [22], Thakur [24], and Lasjaunias [17], for instance, for more informations and references on this matter.

With each infinite continued fraction in $\mathbb{F}(q)$, we can associate a sequence in $\mathbb{F}_q^*$ in the following way: if $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, then for $n \geq 1$, we define $u(n)$ as the leading coefficient of the polynomial a_n. For several examples in $\mathcal{H}(q)$, we have observed that the sequence $(u(n))_{n\geq 1}$ is automatic. Indeed, a first observation in this area is due to Allouche [1]. In the article [21] of Mills and Robbins, for all $p \geqslant 5$, a particular family of continued fractions in $\mathcal{H}(p)$, having $a_n = \lambda_n T$ for all integers $n \geqslant 1$ with λ_n in $\mathbb{F}_p^*$, was introduced. Shortly after the publication of [21], Allouche could prove in [1] that the sequence $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{F}_p^*$ is automatic (see also the last section of [19], where this question is discussed in a larger context). In the present note (Section 2), we shall describe several families of hyperquadratic continued fractions and we show in Section 3 that the sequence associated with them, as indicated above, is also automatic.

Yet it is an open question to know whether this is true for all elements in $\mathcal{H}(q)$. If the answer were negative, it would be interesting to be able to characterize the elements in $\mathcal{H}(q)$ which have this property. As mentioned above, very little is known, concerning continued fractions, for algebraic power series which are not hyperquadratic. However an element in $\mathbb{F}(3)$, algebraic of degree 4, was introduced by Robbins and Mills in [21]. This element is not hyperquadratic. In this note (Section 4), we show that the sequence in $\mathbb{F}_3^*$, associated as above with its continued fraction expansion, is not automatic.

2. Three families of hyperquadratic continued fractions

In this section we shall use the notation and results found in [18].

Let α be an irrational element in $\mathbb{F}(q)$ with $\alpha = [a_1, \ldots, a_n, \ldots]$ as its continued fraction expansion. We denote by $\mathbb{F}(q)^+$ the subset of $\mathbb{F}(q)$ containing the elements having an integral part of positive degree (i.e. with $\deg(a_1) > 0$). For all integers $n \geqslant 1$, we put $\alpha_n = [a_n, a_{n+1}, \ldots]$ ($\alpha_1 = \alpha$), and we introduce the continuants $x_n, y_n \in \mathbb{F}_q[T]$ such that $x_n/y_n = [a_1, a_2, \ldots, a_n]$. As usual we extend the latter notation to $n = 0$ with $x_0 = 1$ and $y_0 = 0$. Observe that the notation used here for the continuants x_n and y_n is different from the one used in [18], and hopefully simplified.

As above we set $r = p^t$, where $t \geqslant 0$ is an integer. Let $P, Q \in \mathbb{F}_q[T]$ such that $\deg(Q) < \deg(P) < r$. Let $\ell \geqslant 1$ be an integer and $A_\ell = (a_1, a_2, \ldots, a_\ell)$ a vector in $(\mathbb{F}_q[T])^\ell$ such that $\deg(a_i) > 0$ for $1 \leqslant i \leqslant \ell$. Then by Theorem 1 in [18], there exists an infinite continued fraction in $\mathbb{F}(q)$ defined by $\alpha = [a_1, a_2, \ldots, a_\ell, \alpha_{\ell+1}]$ such that $\alpha^r = P\alpha_{\ell+1} + Q$. This element α is hyperquadratic and it is the unique root in $\mathbb{F}(q)^+$ of the following algebraic equation:

$$y_\ell X^{r+1} - x_\ell X^r + (y_{\ell-1}P - y_\ell Q)X + x_\ell Q - x_{\ell-1}P = 0. \tag{1}$$

Note that if $r = 1$, then α is quadratic. In this case P is a nonzero constant polynomial, i.e. $P = \varepsilon \in \mathbb{F}_q^*$ and $Q = 0$. Given ℓ and A_ℓ, we have $\alpha = \varepsilon\alpha_{\ell+1}$, and this implies $a_{\ell+m} = \varepsilon^{(-1)^m} a_m$, for all integers $m \geqslant 1$. Hence the continued

fraction expansion is purely periodic, and a simple computation shows that 2ℓ (resp. $(q-1)\ell$) is a period (maybe not the minimum one) if ℓ is odd (resp. even).

We shall describe three families of continued fractions generated as above.

First family: $\mathcal{F}1$. The simplest and first case that we consider is $(P,Q)=(\varepsilon,0)$ where $\varepsilon\in\mathbb{F}_q^*$ and consequently $\alpha^r=\varepsilon\alpha_{l+1}$. Due to the Frobenius isomorphism, we obtain, in the same way as above for $r=1$, the relation $a_{\ell+m}=\varepsilon^{(-1)^m}a_m^r$ for all integers $m\geqslant 1$. Hence the continued fraction, depending on the arbitrary given ℓ first partial quotients, is fully explicit. These hyperquadratic continued fractions were studied independently by Schmidt [22] and Thakur [23] (particularly for $\varepsilon=1$). Let us recall that the elements in $\mathcal{H}(q)$, called here hyperquadratic, were first named in [16] as algebraic of class I, and then the elements studied by Schmidt and Thakur were called algebraic of class IA. The possibility of choosing arbitrarily the vector A_ℓ has an important consequence. Even though this is not truly the matter of this note, we have already mentioned the irrationality measure $\nu(\alpha)$ of α in $\mathbb{F}(q)$. In his fundamental work [20], Mahler established (following an old result of Liouville in the real case) that if $\alpha\in\mathbb{F}(q)$ is algebraic of degree $d\geqslant 2$ over $\mathbb{F}_q(T)$, then we have $\nu(\alpha)\in[2,d]$. Besides $\nu(\alpha)$ is directly depending on the sequence of the degrees of the partial quotients for α (see for instance [17, p. 214]). For an element of $\mathcal{F}1$, this sequence of degrees $(d_n)_{n\geqslant 1}$, satisfies $d_{\ell+m}=rd_m$ for all integers $m\geqslant 1$, and consequently d_n depends directly on the first ℓ degrees. Hence, Schmidt and Thakur, independently, could obtain (by a sophisticated computation) the irrationality measure for such an element, depending on r and the first ℓ degrees. In this way they could establish the following result: for each rational number μ in the range $[2,+\infty[$, there exists α in $\mathcal{F}1$ such that $\nu(\alpha)=\mu$.

Let us make a last observation on the simplest element in $\mathcal{F}1$. We take $\ell=1$ and $a_1=T$, with $(P,Q)=(1,0)$, then the corresponding continued fraction is $\Theta_1=[T,T^r,T^{r^2},\ldots,T^{r^n},\ldots]$, and Θ_1 satisfies $X=T+1/X^r$. Here the irrationality measure is easy to compute, and indeed we have $\nu(\Theta_1)=r+1$. Since the degree d of Θ_1 satisfies $d\leqslant r+1$, using Mahler's argument, we see that Θ_1 has algebraic degree equal to $r+1$. Note that for a general element α in $\mathcal{F}1$, its exact algebraic degree is an undecided question.

Second and third families: $\mathcal{F}2$ **and** $\mathcal{F}3$**.** For an element α in $\mathcal{F}2$, we assume $(P,Q)=(\varepsilon_1T,\varepsilon_2)$; and for an element α in $\mathcal{F}3$, we assume $(P,Q)=(\varepsilon_1T^2,\varepsilon_2T)$, where $(\varepsilon_1,\varepsilon_2)$ is a pair in $(\mathbb{F}_q^*)^2$. Note that here we have $r>1$ for elements in $\mathcal{F}2$, but $r>2$ for elements in $\mathcal{F}3$. Our aim is to give the explicit continued fraction expansion for α. The integer $\ell\geqslant 1$, as above, is chosen arbitrarily. However we need to impose a restriction on the choice of the vector A_ℓ, in both cases: we assume that T divides a_i for all integers i with $1\leqslant i\leqslant\ell$. Then we can describe the sequence of partial quotients in the continued fraction expansion in both cases. Given A_ℓ, chosen as indicated, for all integers $n\geqslant 0$, we have:

$$a_{\ell+4n+2}=-\frac{\varepsilon_1}{\varepsilon_2}T,\ a_{\ell+4n+3}=-\frac{\varepsilon_2^2a_{2n+2}^r}{\varepsilon_1T},\ a_{\ell+4n+4}=\frac{\varepsilon_1}{\varepsilon_2}T,$$

while $a_{\ell+4n+1}=\frac{a_{2n+1}^r}{\varepsilon_1T}$ for $\alpha\in\mathcal{F}2$, and $a_{\ell+4n+1}=\frac{a_{2n+1}^r}{\varepsilon_1T^2}$ for $\alpha\in\mathcal{F}3$. The method to obtain these formulas is the same in both cases and it has been explained in [18]. Actually the formulas for α in $\mathcal{F}2$ were published in [18, top of p. 334]. However, note that there is a mistake in the final statement given there and the pair $(k,k+1)$

to the right hand side of the first and third formulas must be replaced by the pair $(2k-1, 2k)$. Concerning the formulas for α in $\mathcal{F}3$, they were given by Firicel in [14] (case $\varepsilon_1 = \varepsilon_2 = -1$). The reader is invited to consult this last work where the method is clearly explained. Note that the particular choice of the vector A_ℓ, has been made in order to have an integer (polynomial) when dividing by T or T^2 (however a larger choice could have been possible for elements in $\mathcal{F}3$, in a particular case: for example, if ℓ is odd, then it would be enough to assume that T divides a_i only for odd indices i). Note that if we do not assume the particular choice we made for the vector A_ℓ, the continued fraction for α does exist, but an explicit description is not given.

We make a comment concerning the last family $\mathcal{F}3$. It has been introduced to cover a particular case of historical importance. In his fundamental paper [20], Mahler presented the following power series $\Theta_2 = 1/T + 1/T^r + \cdots + 1/T^{r^n} + \cdots$. Note that Θ_2 can be regarded as a dual of the element Θ_1 in $\mathcal{F}1$, presented above. However we clearly have $\Theta_2 = 1/T + \Theta_2^r$, hence Θ_2 is hyperquadratic and algebraic of degree $\leqslant r$. Mahler observed that we have $\nu(\Theta_2) = r$, and therefore the algebraic degree of Θ_2 is equal to r. Now let us consider the element of $\mathcal{F}3$ defined as above by the pair $(P, Q) = (-T^2, -T)$ with $\ell = 1$ and $a_1 = T$. Then α is the unique root in $\mathbb{F}(p)^+$ of the following algebraic equation:

$$X^{r+1} - TX^r + TX = 0.$$

Set $\beta = 1/\alpha$, and we get $\beta = 1/T + \beta^r$. Since $1/\beta$ is in $\mathbb{F}(p)^+$, we obtain

$$\beta = \Theta_2 = 1/T + 1/T^r + 1/T^{r^2} + \cdots$$

Let us recall that the sequence of partial quotients for Θ_2 has been long known (see for example [14, p. 633] with the references therein. See also [16, p. 224] for a different approach).

As we wrote in the introduction, we are interested in the leading coefficients of partial quotients. If $(a_n)_{n\geqslant 1}$ is the sequence of partial quotients for α, we denote by $u(n)$ the leading coefficient of a_n. If α is in $\mathcal{F}1$, then $u(\ell + m) = \varepsilon^{(-1)^m}(u(m))^r$ for all integers $m \geqslant 1$, and thus $u(s\ell + m) = \varepsilon_1^{(-1)^m} u(m)$ with

$$\varepsilon_1 = \varepsilon^{\sum_{j=0}^{s-1}(-1)^{(s-j-1)\ell} r^j} \in \mathbb{F}_q^*,$$

for we have $r^s = p^{st} = q^t$. As above, one can deduce as once that the sequence $(u(n))_{n\geqslant 1}$ is purely periodic, and $2s\ell$ (resp. $(q-1)s\ell$) is a period (maybe not the minimum one) if $s\ell$ is odd (resp. even).

Now we turn to the case where α belongs to $\mathcal{F}2$ or $\mathcal{F}3$. Both recursive definitions for the sequence of partial quotients, whether α is in $\mathcal{F}2$ or $\mathcal{F}3$, imply the same recursive definition for the corresponding sequence $(u(n))_{n\geqslant 1}$. More precisely, for all integers $\ell \geqslant 1$, given $u(1), u(2), \ldots, u(\ell)$ in $\mathbb{F}_q^*$, we have, for all integers $n \geqslant 0$,

$$\begin{aligned} u(\ell+4n+1) &= \varepsilon_1^{-1}(u(2n+1))^r, & u(\ell+4n+2) &= -\varepsilon_1\varepsilon_2^{-1}, \\ u(\ell+4n+3) &= -\varepsilon_1^{-1}\varepsilon_2^2(u(2n+2))^r, & u(\ell+4n+4) &= \varepsilon_1\varepsilon_2^{-1}. \end{aligned}$$

In the next section we shall see in Theorem 3 that sequences of this type, for all integers $\ell \geqslant 1$, are 2-automatic sequences.

To conclude this section, we go back to the special element $1/\Theta_2$ in $\mathcal{F}3$, and the associated sequence $(u(n))_{n\geqslant 1}$ where $\ell = 1$ and $u(1) = 1$. The latter is remarkable and has been studied extensively (see for example [2, Section 6.5]). We define recursively the sequence of finite words $(W_n)_{n\geqslant 1}$ by $W_1 = 1$, and $W_{n+1} = W_n, -1, W_n^R$,

where commas indicate here concatenation of words, and W_n^R is the reverse of the finite word W_n. Let W be the infinite word beginning with W_n for all integers $n \geqslant 1$. Then one can check that the sequence $(u(n))_{n\geqslant 1}$ coincides with W, and for $p \neq 2$, it is a special paperfolding sequence which is known to be 2-automatic (see for example [2, Theorem 6.5.4]).

3. A family of automatic sequences

In this section, we begin with the definition of automatic sequences. For more details about this subject, see the book [2] of Allouche and Shallit.

Let A be a finite nonempty set, called an alphabet, of which every element is called a letter. Fix $\emptyset$ an element not in A and call it an empty letter over A. Let $n \geqslant 0$ be an integer. If $n = 0$, define $A^0 = \{\emptyset\}$. For $n \geqslant 1$, denote by A^n the set of all finite sequences in A of length n. Put $A^* = \bigcup_{n=0}^{+\infty} A^n$. Every element w of A^* is called a word over A and its length is noted $|w|$, i.e. $|w| = n$ if $w \in A^n$.

Take $w, v \in A^*$. The concatenation between w and v (denoted by $w * v$ or more simply by wv) is the word over A which begins with w and is continued by v.

Now we give below a definition of finite automaton (see for example [13]):

A finite automaton $\mathcal{A} = (S, S_0, \Sigma, \tau)$ consists of

- an alphabet S of states; one state S_0 is distinguished and called initial state.
- a mapping $\tau : S \times \Sigma \to S$, called transition function, where Σ is an alphabet containing at least two elements.

$\forall Q \in S$, put $\tau(Q, \emptyset) = Q$. Extend τ over $S \times \Sigma^*$ (noted again τ) such that

$$\forall Q \in S \text{ and } \forall l, m \in \Sigma^*, \text{ we have } \tau(Q, lm) = \tau(\tau(Q, l), m).$$

Let $k \geqslant 2$ be an integer and $\Sigma_k = \{0, 1, \ldots, k-1\}$. We call $v = (v(n))_{n\geqslant 0}$ a k-automatic sequence if there exist a finite automaton $\mathcal{A} = (S, S_0, \Sigma_k, \tau)$ and a mapping o defined on S such that $v(0) = o(S_0)$, and for all integers $n \geqslant 1$ with standard k-adic expansion $n = \sum_{j=0}^{l} n_j k^j$, we have $v(n) = o(\tau(S_0, n_l \cdots n_0))$.

We recall that all ultimately periodic sequences are k-automatic for all $k \geqslant 2$, adding a prefix to a sequence does not change its automaticity, and that a sequence is k-automatic if and only if it is k^m-automatic for all integers $m \geqslant 1$. In this work, we consider sequences of the form $v = (v(n))_{n\geqslant 1}$, and we say that v is k-automatic if the sequence $(v(n))_{n\geqslant 0}$ is k-automatic, with $v(0)$ fixed arbitrarily. We have the following important characterization: a sequence $v = (v(n))_{n\geqslant 1}$ is k-automatic if and only if its k-kernel

$$K_k(v) = \left\{ (v(k^i n + j))_{n\geqslant 1} \mid i \geqslant 0,\, 0 \leqslant j < k^i \right\}$$

is a finite set. The origin of this characterization for automatic sequences is due to Cobham [11, p. 170, Theorem 1] (see also Eilenberg [13, p. 107, Proposition 3.3], who was one of the first to publish a general treatise on this matter).

Let $v = (v(n))_{n\geqslant 1}$ be a sequence. For all integers $n \geqslant 1$, we define

$$(T_0 v)(n) = v(2n) \text{ and } (T_1 v)(n) = v(2n+1).$$

Then for all integers $n, a \geqslant 1$, and $0 \leqslant b < 2^a$ with binary expansion

$$b = \sum_{j=0}^{a-1} b_j 2^j \qquad (0 \leqslant b_j < 2),$$

with the help of the operators T_0 and T_1, we obtain

$$v(2^a n + b) = (T_{b_{a-1}} \circ T_{b_{a-2}} \circ \cdots \circ T_{b_0} v)(n).$$

In particular, we obtain that v is 2-automatic if and only if both $T_0 v, T_1 v$ are 2-automatic, for we have $K_2(v) = \{v\} \cup K_2(T_0 v) \cup K_2(T_1 v)$.

With these definitions, we have the following theorem, which can be compared with a result of Allouche and Shallit (see [3, Theorem 2.2]).

Theorem 2. *Let $m \geqslant 0$ be an integer, $v = (v(n))_{n \geqslant 1}$ a sequence in an alphabet A, and σ a bijection on A.*

- (0_m) *If $T_0 v$ is 2-automatic, and $(T_1 v)(n+m) = \sigma(v(n))$ for all integers $n \geqslant 1$, then v is 2-automatic;*
- (1_m) *If $T_1 v$ is 2-automatic, and $(T_0 v)(n+m) = \sigma(v(n))$ for all integers $n \geqslant 1$, then v is 2-automatic.*

Proof. Firstly we note that A is finite, thus there exists an integer $l \geqslant 1$ such that $\sigma^l = \mathrm{id}_A$, the identity mapping on A.

Secondly we show that (0_m) implies (1_m). For this, put $u(n) = v(n+1)$, for all integers $n \geqslant 1$. Then $T_0 u = T_1 v$, and for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_1 u)(n+m) &= u(2n+2m+1) = v(2n+2m+2)\\ &= (T_0 v)(n+1+m)\\ &= \sigma(v(n+1)) = \sigma(u(n)).\end{aligned}$$

Hence u is 2-automatic by virtue of (0_m), and so it v, for the latter is obtained from u by adding a letter before.

In the following we shall show (0_m) by induction on m.

If $m = 0$, then under the conditions of (0_0), we have $T_1 v = \sigma(v)$ and

$$K_2(v) = \{\sigma^j(v) \mid 0 \leqslant j < l\} \cup \bigcup_{j=0}^{l-1} \sigma^j(K_2(T_0 v)).$$

Thus $K_2(v)$ is finite since $T_0 v$ is 2-automatic.

If $m = 1$, then $T_0 v$ is 2-automatic, and $(T_1 v)(n+1) = \sigma(v(n))$ for all integers $n \geqslant 1$. Hence $T_1 T_1 v = \sigma(T_0 v)$, and for all integers $n \geqslant 1$, we have

$$(T_0 T_1 v)(n+1) = (T_1 v)(2n+2) = \sigma(v(2n+1)) = \sigma((T_1 v)(n)).$$

Thus $T_1 T_0 T_1 v = \sigma(T_0 T_1 v)$, and for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_0 T_0 T_1 v)(n+1) &= (T_0 T_1 v)(2n+2) = \sigma((T_1 v)(2n+1))\\ &= \sigma^2(v(2n)) = \sigma^2((T_0 v)(n)).\end{aligned}$$

So $T_0 T_0 T_1 v$ is 2-automatic since it is obtained from $\sigma^2(T_0 v)$ by adding a letter before, and the latter is 2-automatic, for $T_0 v$ is. Set $w = T_0 T_1 v$. Then $T_0 w$ is 2-automatic, and $T_1 w = \sigma(w)$. Thus by (0_0), we obtain that $T_0 T_1 v$ is 2-automatic. But $T_1 T_1 v = \sigma(T_0 v)$ is also 2-automatic, consequently $T_1 v$ is 2-automatic, and then v is 2-automatic, for both $T_0 v$ and $T_1 v$ are 2-automatic.

Now let $m \geqslant 1$ be an integer, and assume that (0_j) hold for $0 \leqslant j \leqslant m$. Then (1_j) also hold for $0 \leqslant j \leqslant m$, and we shall show that (0_{m+1}) holds. For this, we distinguish two cases below:

Case 1: m is odd. Then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_0T_1v)(n+[\frac{m}{2}]+1) &= (T_1v)(2n+m+1)=\sigma(v(2n))=\sigma((T_0v)(n)),\\ (T_1T_1v)(n+[\frac{m}{2}]+1) &= (T_1v)(2n+m+2)=\sigma(v(2n+1))=\sigma((T_1v)(n)).\end{aligned}$$

Hence T_0T_1v is 2-automatic for it is obtained from $\sigma(T_0v)$ by adding a prefix of length $[\frac{m}{2}]+1$. Put $w=T_1v$. Then T_0w is 2-automatic, and for all integers $n \geqslant 1$,

$$(T_1w)(n+[\frac{m}{2}]+1)=\sigma(w(n)).$$

Note that $[\frac{m}{2}]+1 \leqslant m$. Thus we can apply $(0_{[\frac{m}{2}]+1})$ with w, and we obtain that w is 2-automatic. Hence v is 2-automatic since both T_0v and T_1v are 2-automatic.

Case 2: m is even. Then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_0T_1v)(n+[\frac{m}{2}]+1) &= (T_1v)(2n+m+2)=\sigma(v(2n+1))=\sigma((T_1v)(n)),\\ (T_1T_1v)(n+[\frac{m}{2}]) &= (T_1v)(2n+m+1)=\sigma(v(2n))=\sigma((T_0v)(n)).\end{aligned}$$

So T_1T_1v is 2-automatic for it is obtained from $\sigma(T_0v)$ by adding a prefix of length $[\frac{m}{2}]$. Put $w=T_1v$. Then T_1w is 2-automatic, and $(T_0w)(n+[\frac{m}{2}]+1)=\sigma(w(n))$, for all integers $n \geqslant 1$. By applying $(1_{[\frac{m}{2}]+1})$ with w, we obtain that w is 2-automatic, and then v is 2-automatic since both T_0v and T_1v are 2-automatic.

Finally we conclude that both (0_m) and (1_m) hold for all integers $m \geqslant 0$. □

We can now prove that the sequences, associated with the elements of $\mathcal{F}2$ and $\mathcal{F}3$ and described at the end of the previous section are 2-automatic. This follows from the more general result stated below, which is a direct application of Theorem 2.

Theorem 3. *Let p be a prime number, $s \geqslant 1$ an integer, and $q=p^s$. Denote by $\mathbb{F}_q$ the finite field in q elements. Set $r=p^t$, with $t \geqslant 0$ an integer. Fix $\alpha,\beta,\gamma,\delta \in \mathbb{F}_q^*$, and $\ell \geqslant 1$ an integer. Let $u=(u(n))_{n\geqslant 1}$ be a sequence in $\mathbb{F}_q^*$ such that for all integers $n \geqslant 0$, we have*

$$\left\{\begin{array}{l} u(\ell+4n+1)=\alpha(u(2n+1))^r,\ u(\ell+4n+2)=\beta,\\ u(\ell+4n+3)=\gamma(u(2n+2))^r,\ u(\ell+4n+4)=\delta.\end{array}\right. \tag{2}$$

Then the sequence u is 2-automatic.

Proof. For all $x,y \in \mathbb{F}_q^*$, we put $\sigma_y(x)=yx^r$. Then σ_y is a bijection on $\mathbb{F}_q^*$. For all integers $n \geqslant 1$, set $u_0(n)=u(2n)$ and $u_1(n)=u(2n+1)$, and we need only show that both u_0 and u_1 are 2-automatic. For this, we distinguish below four cases.

Case I: $\ell=4m+1$, with $m \geqslant 0$ an integer. Then for all integers $n \geqslant 0$, from Formula (2), we deduce

$$\begin{aligned}(T_0u_1)(n+m+1) &= u(4(n+m+1)+1)=u(\ell+4n+4)=\delta,\\ (T_1u_1)(n+m) &= u(4n+4m+3)=u(\ell+4n+2)=\beta.\end{aligned}$$

Since both T_0u_1 and T_1u_1 are ultimately constant, then u_1 is ultimately periodic, and thus 2-automatic.

Similarly, for all integers $n \geqslant 0$, we also have

$$\begin{aligned}(T_0u_0)(n+m+1) &= u(4(n+m+1)) = u(\ell+4n+3)\\ &= \gamma(u(2n+2))^r = \gamma(u_0(n+1))^r,\\ (T_1u_0)(n+m) &= u(4n+4m+2) = u(\ell+4n+1)\\ &= \alpha(u(2n+1))^r = \alpha(u_1(n))^r.\end{aligned}$$

So T_1u_0 is ultimately periodic as u_1, and $(T_0u_0)(n+m) = \sigma_\gamma(u_0(n))$ for all integers $n \geqslant 1$. Then by Theorem 2, we obtain that u_0 is 2-automatic.

Case II: $\ell = 4m+2$, with $m \geqslant 0$ an integer. Then for all integers $n \geqslant 0$, from Formula (2), we deduce

$$\begin{aligned}(T_0u_0)(n+m+1) &= u(4(n+m+1)) = u(\ell+4n+2) = \beta,\\ (T_1u_0)(n+m+1) &= u(4n+4m+6) = u(\ell+4n+4) = \delta.\end{aligned}$$

So u_0 is ultimately periodic, and thus 2-automatic.

Similarly, for all integers $n \geqslant 0$, we also have

$$\begin{aligned}(T_0u_1)(n+m+1) &= u(\ell+4n+3) = \gamma(u(2n+2))^r = \gamma(u_0(n+1))^r,\\ (T_1u_1)(n+m) &= u(\ell+4n+1) = \alpha(u(2n+1))^r = \alpha(u_1(n))^r.\end{aligned}$$

Hence T_0u_1 is ultimately periodic as u_0, and $(T_1u_1)(n+m) = \sigma_\alpha(u_1(n))$ for all integers $n \geqslant 1$. Then by Theorem 2, we obtain that u_1 is 2-automatic.

Case III: $\ell = 4m+3$, with $m \geqslant 0$ an integer. Then for all integers $n \geqslant 0$, from Formula (2), we deduce

$$\begin{aligned}(T_0u_1)(n+m+1) &= u(4(n+m+1)+1) = u(\ell+4n+2) = \beta,\\ (T_1u_1)(n+m+1) &= u(4n+4m+7) = u(\ell+4n+4) = \delta.\end{aligned}$$

So u_1 is ultimately periodic, and thus 2-automatic.

Similarly, for all integers $n \geqslant 0$, we also have

$$\begin{aligned}(T_0u_0)(n+m+1) &= u(4(n+m+1)) = u(\ell+4n+1)\\ &= \alpha(u(2n+1))^r = \alpha(u_1(n))^r,\\ (T_1u_0)(n+m+1) &= u(4n+4m+6) = u(\ell+4n+3)\\ &= \gamma(u(2n+2))^r = \gamma(u_0(n+1))^r.\end{aligned}$$

Thus T_0u_0 is ultimately periodic as u_1, and $(T_1u_0)(n+m) = \sigma_\gamma(u_0(n))$ for all integers $n \geqslant 1$. Then by Theorem 2, we obtain that u_0 is 2-automatic.

Case IV: $\ell = 4m+4$, with $m \geqslant 0$ an integer. Then for all integers $n \geqslant 0$, from Formula (2), we deduce

$$\begin{aligned}(T_0u_0)(n+m+2) &= u(4(n+m+2)) = u(\ell+4n+4) = \delta,\\ (T_1u_0)(n+m+1) &= u(4n+4m+6) = u(\ell+4n+2) = \beta.\end{aligned}$$

So u_0 is ultimately periodic, and thus 2-automatic.

Similarly, for all integers $n \geqslant 0$, we also have

$$\begin{aligned}(T_0u_1)(n+m+1) &= u(\ell+4n+1) = \alpha(u(2n+1))^r = \alpha(u_1(n))^r,\\ (T_1u_1)(n+m+1) &= u(\ell+4n+3) = \gamma(u(2n+2))^r = \gamma(u_0(n+1))^r.\end{aligned}$$

Hence T_1u_1 is ultimately periodic as u_0, and $(T_0u_1)(n+m+1) = \sigma_\alpha(u_1(n))$ for all integers $n \geqslant 1$. Then by Theorem 2, we obtain that u_1 is 2-automatic. $\square$

Remark 1. *As it is pointed out at the end of Section 2, for $p \neq 2$, the sequence associated with the special element $1/\Theta_2$ in $\mathcal{F}3$, is a paperfolding sequence, thus not ultimately periodic (see for example* [2, Theorem 6.5.3]*). Inspired by this example, one can then ask whether the sequence u discussed in Theorem 3 can be ultimately periodic or not for all $q \geqslant 3$ (in the case that $q = 2$, the sequence is constant). Unfortunately for the moment, we do not know the answer of this problem.*

4. A substitutive but not automatic sequence

In this section we are concerned with the following question: is it also true for an algebraic power series, not hyperquadratic, that the sequence of the leading coefficients of the partial quotients form an automatic sequence ? To such a wide question, we will only give a very partial answer, by considering a last example. As we remarked in the introduction, the possibility of describing explicitly the continued fraction expansion for an algebraic power series, which is not hyperquadratic, appears to be remote. However, a particular example, which was introduced by chance in [21], does exist. This example is the object of the theorem below.

First, we recall notions on substitutive sequences (see for example [2]).

Let A be an alphabet with $A = \{A_1, A_2, ..., A_N\}$. A substitution on A is a morphism $\sigma : A^* \to A^*$. With the morphism σ, there is associated a matrix $M_\sigma = (m_{i,j})_{1 \leqslant i,j \leqslant N}$, where $m_{i,j}$ is the number of occurrences of A_i in the word $\sigma(A_j)$. Since M_σ is a non-negative integer square matrix, by the famous Frobenius-Perron theorem (see for example [15]), M_σ has a real eigenvalue α, called the dominating eigenvalue of M_σ, which is an algebraic integer and greater than or equal to the modulus of any other eigenvalue, thus a Perron number. If there exists an integer i ($1 \leqslant i \leqslant N$) such that $\sigma(A_i) = A_i x$ for some $x \in A^* \setminus \{\emptyset\}$, and $\lim_{n\to\infty} |\sigma^n(A_i)| = +\infty$, then σ is said to be prolongable on A_i. Since for all integers $n \geqslant 0$, $\sigma^n(A_i)$ is a prefix of $\sigma^{n+1}(A_i)$, and $|\sigma^n(A_i)|$ tends to infinity with $n \to \infty$, thus the sequence $(\sigma^n(A_i))_{n\geqslant 0}$ converges, and we denote its limits by $\sigma^\infty(A_i)$. The latter is the unique infinite fixed point of σ starting with A_i. Let o be a mapping defined on A, extended pointwisely over $A^* \cup A^{\mathbb{N}}$. We put $v = o(\sigma^\infty(A_i))$, and call it an α-substitutive sequence.

We have the following important characterization for automatic sequences in terms of substitutive sequences, due to Cobham [11]: a sequence $v = (v(n))_{n\geqslant 1}$ is k-automatic if and only if v is a substitutive sequence with σ such that $|\sigma(A_j)| = k$, for all integers $1 \leqslant j \leqslant N$. Note that in this case v is k-substitutive.

Now let α, β be two multiplicatively independent Perron numbers. By generalizing another classical theorem of Cobham [10], Durand has finally shown in [12, Theorem 1, p.1801] the remarkable result that a sequence is both α-substitutive and β-substitutive if and only if it is ultimately periodic.

We can now state and prove the following theorem.

Theorem 4. *The algebraic equation $X^4 + X^2 - TX + 1 = 0$ has a unique root α in $\mathbb{F}(3)$. Let $\alpha = [0, a_1, a_2, \ldots, a_n, \ldots]$ be its continued fraction expansion and $u(n)$ be the leading coefficient of a_n for all integers $n \geqslant 1$. The sequence $W = (u(n))_{n\geqslant 1}$ is the limit of the sequence $(W_n)_{n\geqslant 0}$ of finite words over the alphabet $\{1, 2\}$, defined recursively as follows:*

(3) $W_0 = \emptyset,\ W_1 = 1,$ *and* $W_n = W_{n-1}, 2, W_{n-2}, 2, W_{n-1},$ *for all integers* $n \geqslant 2,$

where commas indicate here concatenation of words. Then α is not hyperquadratic, and the sequence $W = (u(n))_{n\geqslant 1}$ is $(1+\sqrt{2})$-substitutive but not automatic.

Proof. The existence in $\mathbb{F}(p)$ of the root of the quartic equation stated in this theorem was observed firstly by Mills and Robbins in [21], for all prime numbers p. For $p = 3$, in the same work, a conjecture on its continued fraction expansion, based on computer observation, was given. Buck and Robbins established this conjecture in [7]. Shortly after another proof of this conjecture was given in [16]. We have $\alpha = [0, a_1, a_2, \ldots, a_n, \ldots]$ and the sequence of polynomials $(a_n)_{n\geqslant 1}$ is obtained as the limit of a sequence of finite words $(\Omega_n)_{n\geqslant 0}$ with letters in $\mathbb{F}_3[T]$, defined by:

$$\Omega_0 = \emptyset,\ \Omega_1 = T,\ \text{and}\ \Omega_n = \Omega_{n-1}, 2T, \Omega_{n-2}^{(3)}, 2T, \Omega_{n-1}, \tag{4}$$

where commas indicate concatenation of words, and $\Omega_{n-2}^{(3)}$ denote the word obtained by cubing each letter of Ω_{n-2}. Since $x^3 = x$ for all x in $\mathbb{F}_3^*$, we obtain immediately for W the desired formulas (3) from (4).

The fact that α is not hyperquadratic was proved in [16] (see the remark after Theorem A, p. 209). Indeed the knowledge of the continued fraction allows to show that the irrationality measure is equal to 2. However the sequence of partial quotients is clearly unbounded, and it was proved by Voloch [25] that if α were hyperquadratic with an unbounded sequence of partial quotients, then the irrationality measure would be strictly greater than 2 (the reader may consult [17, pp. 215-216], for a presentation of these general statements).

Now we show that W is $(1+\sqrt{2})$-substitutive, but not automatic.

Put $A = \{a, b, c\}$, and define

$$\sigma(a) = abca,\ \sigma(b) = ca,\ \sigma(c) = c,\ o(a) = 1,\ o(b) = o(c) = 2.$$

For all integers $n \geqslant 0$, set $V_n = \sigma^n(a)$. Then for all integers $n \geqslant 2$, we have

$$V_n = \sigma^n(a) = \sigma^{n-1}(abca) = V_{n-1}\sigma^{n-2}(ca)cV_{n-1} = V_{n-1}cV_{n-2}cV_{n-1}.$$

But we also have $W_1 = o(a)$, and $W_2 = 1221 = o(abca) = o(\sigma(a)) = o(V_1)$, thus $o(\sigma^n(a))$ satisfies the same relations as W_{n+1}, consequently they coincide. Set $v = \lim_{n\to\infty} \sigma^n(a)$. Then $\sigma(v) = v$, and $W = o(v)$. So W is substitutive. Finally

$$M_\sigma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

whose characteristic polynomial is equal to $(\lambda - 1)(\lambda^2 - 2\lambda - 1)$, and $1+\sqrt{2}$ is the dominating eigenvalue. Hence W is $(1+\sqrt{2})$-substitutive. Since $1+\sqrt{2}$ is multiplicatively independent with all integers $k \geqslant 2$, according to Cobham's characterization and Durand's theorem, we see that W cannot be k-automatic unless it is ultimately periodic. To conclude the proof, we need only prove that W is not ultimately periodic. To do so, we compute the frequency of 2 in W. For all integers $n \geqslant 0$, put $l_n = |W_n|$. Then we have

$$l_0 = 0,\ l_1 = 1,\ \text{and}\ l_n = 2l_{n-1} + l_{n-2} + 2,\ \text{for all integers}\ n \geqslant 2,$$

from which we deduce $l_n = -1 + \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^n + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^n$, for all integers $n \geqslant 0$. For all integers $n \geqslant 0$, let m_n be the number of occurences of 2 in W_n. Then

$$m_0 = m_1 = 0,\ \text{and}\ m_n = 2m_{n-1} + m_{n-2} + 2,\ \text{for all integers}\ n \geqslant 2,$$

from which we obtain $m_n = -1 + \frac{1}{2}\big((1+\sqrt{2})^n + ((1-\sqrt{2})^n\big)$, for all integers $n \geqslant 0$. If W were ultimately periodic, then the frequency of 2 in W would exist, and it would be a rational number, in contradiction to $\lim_{n\to\infty} m_n/l_n = 2-\sqrt{2}$. $\square$

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