FLAT POWER SERIES OVER A FINITE FIELD

By
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Abstract. We define and describe a class of algebraic continued fractions for power series over a finite field. These continued fraction expansions, for which all the partial quotients are polynomials of degree one, have a regular pattern induced by the Frobenius homomorphism. This is an extension, in the case of positive characteristic, of purely periodic expansions corresponding to quadratic power series.

§1. Introduction.

Let p be a prime number and $q = p^s$ where s is a positive integer. Let $\mathbb{F}_q$ be the finite field with q elements. We consider the ring of polynomials $\mathbb{F}_q[T]$, and the field of rational functions $\mathbb{F}_q(T)$, in an indeterminate T with coefficients in $\mathbb{F}_q$. There is an ultrametric absolute value defined on $\mathbb{F}_q(T)$ by $|0| = 0$ and $|P/Q| = |T|^{\deg P - \deg Q}$ where $|T|$ is a fixed real number greater than one. The field obtained by completion from $\mathbb{F}_q(T)$ for this absolute value will be denoted by $\mathbb{F}(q)$. We call the elements of this field formal numbers over $\mathbb{F}_q$. A non-zero element of this field is represented by a power series in the following way

$$\Theta = \sum_{k \le k_0} \theta_k T^k \quad \text{where} \quad k_0 \in \mathbb{Z},\ \theta_k \in \mathbb{F}_q \text{ and } \theta_{k_0} \ne 0.$$

The absolute value extended to this field is then defined by $|\Theta| = |T|^{k_0}$.

We are concerned with the continued fraction algorithm in this field $\mathbb{F}(q)$. A good survey on the main properties of this algorithm in power series fields can be found in Schmidt's article [S]. We recall that each element $\Theta \in \mathbb{F}(q)$ can be expanded as a continued fraction which we denote by $\Theta = [a_0, a_1, a_2, \ldots, a_n, \ldots]$ where the a_i are polynomials in $\mathbb{F}_q[T]$. These polynomials are called the partial quotients of the expansion. We have $\deg a_i \ge 1$ for $i > 0$. The expansion is finite if and only if Θ is rational. The analogue of Lagrange's theorem holds in $\mathbb{F}(q)$, that is to say the sequence $(a_i)_{i \ge 0}$ is ultimately periodic if and only if Θ is quadratic over the field $\mathbb{F}_q(T)$ (see [S]).

Baum and Sweet ([BS1] and [BS2]) were among the first to consider the field $\mathbb{F}(2)$. Guided by the analogy with the field of real numbers, they studied rational approximation to algebraic elements in $\mathbb{F}(2)$. They could give an example of an algebraic and non-quadratic element having a continued fraction expansion with partial quotients of bounded degree. Later, Mills and Robbins [MR] have taken up this study in a more general context. They were able to give such an example in $\mathbb{F}(p)$ for each prime p greater than 3. In a first approach ($q = 3$, see [L1]) and in a more general context(see [LR]), we have shown the existence of elements in $\mathbb{F}(q)$ algebraic over $\mathbb{F}_q(T)$ which have a continued fraction expansion with partial quotients all of degree one. In this paper, we extend the study of this general pattern of algebraic continued fractions. These elements are obtained as fixed points of the composition of a linear fractional transformation with the Frobenius homomorphism. A special and classical case of this pattern is obtained by replacing the Frobenius homomorphism by the identity, which leads to quadratic numbers having a purely periodic continued fraction expansion.

§2. Definition and characterization of flat formal numbers.

We consider the subset $\mathbb{F}^*(q) = \{\Theta \in \mathbb{F}(q),\ |\Theta| \leq |T|^{-1}\}$. Clearly, if Θ is an irrational element in $\mathbb{F}^*(q)$ then we can expand it as the following continued fraction

$$\Theta = [0, a_1, a_2, \ldots, a_n, \ldots] \quad \text{with } a_i \in \mathbb{F}_q[T] \quad \text{for} \quad i \geq 1.$$

From the sequence $(a_i)_{i\geq 1}$ we define both sequences $(x_i)_{i\geq -1}$ and $(y_i)_{i\geq -1}$ of elements of $\mathbb{F}_q[T]$ by the following recursion

$$\begin{cases} x_{-1} = 1, \quad x_0 = 0 \quad \text{and} \quad x_n = a_n x_{n-1} + x_{n-2} \quad n \geq 1 \\ y_{-1} = 0, \quad y_0 = 1 \quad \text{and} \quad y_n = a_n y_{n-1} + y_{n-2} \quad n \geq 1 \end{cases} \tag{1}$$

We know that $x_i/y_i = [0, a_1, a_2, \ldots, a_i]$ for $i \geq 1$. The sequence $(x_i/y_i)_{i\geq 0}$ is called the sequence of the convergents to Θ. For $n \geq 1$, we obtain from (1), $|y_n| = |a_n||y_{n-1}|$ and $|x_n| = |a_n||x_{n-1}|$. Consequently, for $n \geq 2$, by induction we have

$$|x_n| = \prod_{2\leq i\leq n} |a_i| \qquad \text{and} \qquad |y_n| = \prod_{1\leq i\leq n} |a_i| \tag{2}$$

We recall an important property on the approximation of Θ by its convergents. If $\Theta = [a_0, a_1, \ldots, a_n, \ldots] \in \mathbb{F}(q)$ then we have

$$|\Theta - x_n/y_n| = |a_{n+1}|^{-1}|y_n|^{-2} \qquad \text{for} \quad n \geq 0. \tag{3}$$

If $a_0 = 0$ then, by (3) for $n = 0$, we have $|\Theta| = |a_1|^{-1}$. We consider the sets $E(q) = \{\lambda T + \mu \mid \lambda \in \mathbb{F}_q^*, \mu \in \mathbb{F}_q\}$ and $E_0(q) = \{\lambda T \mid \lambda \in \mathbb{F}_q^*\}$. Finally we define two subsets of irrational numbers in $\mathbb{F}^*(q)$ by

$$\mathcal{E}(q) = \{\Theta = [0, a_1, a_2, \ldots, a_n, \ldots] \mid a_i \in E(q) \quad \text{for} \quad i \geq 1\}$$

and

$$\mathcal{E}_0(q) = \{\Theta = [0, a_1, a_2, \ldots, a_n, \ldots] \mid a_i \in E_0(q) \quad \text{for} \quad i \geq 1\}.$$

We observe that if $\Theta \in \mathcal{E}(q)$, from (2) and for $n \geq 1$, we have $|x_n| = |T|^{n-1}$ and $|y_n| = |T|^n$.

Proposition 1. *Let l and r be two positive integers. Assume that $r = p^t$ with $t \geq 0$ and $l \geq r$. Let $\mathcal{B} = (a_1, a_2, \ldots, a_l) \in E(q)^l$ and $\epsilon \in \mathbb{F}_q^*$. Let x_l, y_l, x_{l-r} and y_{l-r} be the polynomials obtained from $\mathcal{B}$ by (1). We consider the following equation*

$$x = \frac{\epsilon x_l + x_{l-r} x^r}{\epsilon y_l + y_{l-r} x^r} \qquad E(\epsilon, \mathcal{B})$$

Then we have:
1) $E(\epsilon, \mathcal{B})$ has a unique solution in $\mathbb{F}^*(q)$, denoted by $\Theta(\epsilon, \mathcal{B})$.
2) $\Theta(\epsilon, \mathcal{B})$ is an irrational number and $\Theta(\epsilon, \mathcal{B}) = [0, \mathcal{B}, \ldots]$.
3) If $\mathcal{B} \in E_0(q)^l$ and $\Theta(\epsilon, \mathcal{B}) \in \mathcal{E}(q)$ then $\Theta(\epsilon, \mathcal{B}) \in \mathcal{E}_0(q)$.

PROOF: We denote by f the map defined on $\mathbb{F}^*(q)$ by

$$f(x) = \frac{\epsilon x_l + x_{l-r} x^r}{\epsilon y_l + y_{l-r} x^r}.$$

If $x \in \mathbb{F}^*(q)$ we see that $|y_{l-r} x^r| < |y_l|$ and consequently $|\epsilon y_l + y_{l-r} x^r| = |y_l|$. Similarly we have $|\epsilon x_l + x_{l-r} x^r| = |x_l|$. By (2) we also have $|y_l / x_l| = |a_1|$ and thus $|f(x)| = |a_1|^{-1} = |T|^{-1}$. Hence f is a map from $\mathbb{F}^*(q)$ into $\mathbb{F}^*(q)$. For a, $b \in \mathbb{F}^*(q)$, by straightforward calculation and using the Frobenius homomorphism if $r > 1$, we obtain

$$f(a) - f(b) = \frac{\epsilon (y_l x_{l-r} - x_l y_{l-r})(a-b)^r}{(y_{l-r} a^r + \epsilon y_l)(y_{l-r} b^r + \epsilon y_l)} \tag{4}$$

and by taking the absolute value

$$|f(a) - f(b)| = |y_l x_{l-r} - x_l y_{l-r}| |y_l|^{-2} |a - b|^r. \tag{5}$$

By (3) we have $|x_{l-r}/y_{l-r} - x_l/y_l| = |y_{l-r}|^{-2} |a_{l-r+1}|$. This implies that $|y_l x_{l-r} - x_l y_{l-r}| = |y_l y_{l-r}| |y_{l-r}|^{-2} |a_{l-r+1}|$. Since $|a_i| = |T|$ for $1 \leq i \leq l$, (2) implies $|y_i| = |T|^i$ for $1 \leq i \leq l$. Finally we obtain

$$|y_l x_{l-r} - x_l y_{l-r}| = |T|^{r-1}. \tag{6}$$

Therefore (5) becomes

$$|f(a) - f(b)| = |T|^{-2l+r-1} |a - b|^r. \tag{7}$$

For $a, b \in \mathbb{F}^*(q)$ we have $|a - b|^r \leq |a - b|$, then (7) implies

$$|f(a) - f(b)| \leq |T|^{-2} |a - b|. \tag{8}$$

This shows that f is a contraction mapping from $\mathbb{F}^*(q)$ into $\mathbb{F}^*(q)$. Thus, as $\mathbb{F}^*(q)$ is a complete metric subspace of $\mathbb{F}(q)$, the equation $x = f(x)$ has a unique solution in $\mathbb{F}^*(q)$, depending upon $\mathcal{B}$ and ϵ.

Assume now that this solution is rational, say $\Theta = a/b$ with $a, b \in \mathbb{F}_q[T]$ and $\gcd(a,b) = 1$. Then $a/b = f(a/b)$ implies

$$\frac{a}{b} = \frac{\epsilon x_l b^r + x_{l-r} a^r}{\epsilon y_l b^r + y_{l-r} a^r}. \tag{9}$$

We put $u = \epsilon x_l b^r + x_{l-r} a^r$, $v = \epsilon y_l b^r + y_{l-r} a^r$ and $w = x_l y_{l-r} - y_l x_{l-r}$. Then we obtain easily

$$u y_{l-r} - v x_{l-r} = \epsilon b^r w \quad \text{ and } \quad v x_l - u y_l = a^r w. \tag{10}$$

If we set $\delta = \gcd(u,v)$, since $\gcd(a,b) = 1$, both equations (10) imply that δ divides w. Thus $|\delta| \leq |w|$ and, by (6), $|\delta| \leq |T|^{r-1}$. Since $|v| = |y_l||b|^r$, we obtain $|v/\delta| \geq |T|^{l-r+1}|b|^r > |b|$. Hence we have $v/\delta \neq b$ and (9) cannot hold. This brings a contradiction. Further if $\Theta \in \mathbb{F}(q)^*$ and $\Theta = f(\Theta)$, we have seen that $|\Theta| = |T|^{-1}$. By (7) we can write

$$|f(\Theta) - f(0)| = |T|^{-2l+r-1}|\Theta|^r$$

and this is clearly the same as

$$|\Theta - x_l/y_l| = |T|^{-1}|y_l|^{-2}.$$

This last equality proves that x_l/y_l is a convergent to Θ and moreover $|a_{l+1}| = |T|$. Since $x_l/y_l = [0, a_1, a_2, \ldots, a_l]$, we have $\Theta = [0, \mathcal{B}, \ldots]$. The last property to be shown is linked to the fact that, under the given hypothesis, Θ is an odd function of T. An even element in $\mathbb{F}(q)$ is a function of T^2, and an odd element is the product of T by an even element. If $p \neq 2$, and $\Theta \in \mathbb{F}(q)$ then Θ is an odd function of T if $\Theta(-T) = -\Theta(T)$. Clearly Θ is odd if and only if all the coefficients of even degree in the power series expansion of Θ are zero. Equivalently all the partial quotients in the continued fraction expansion are odd polynomials of T. So if Θ is odd and $\Theta \in \mathcal{E}(q)$ then $\Theta \in \mathcal{E}_0(q)$. Now assume that $\mathcal{B} \in E_0(q)^l$. From (1) we observe that x_i is alternatively an odd (if i is even) or even (if i is odd) polynomial of T, for $1 \leq i \leq l$. The same is true for y_i with the opposite parity to x_i. To prove that Θ is odd, we will have to distinguish two cases. First suppose that r is odd, hence x_l and x_{l-r} have opposite parity and $(-1)^r = -1$. Consequently it is easy to check that

$$\Theta(T) = f(\Theta(T)) \quad \text{implies} \quad -\Theta(-T) = f(-\Theta(-T)).$$

If we put $\Theta^*(T) = -\Theta(-T)$ then $|\Theta^*| = |\Theta|$. Since $x = f(x)$ has a unique root in $\mathbb{F}^*(q)$, we must have $\Theta^* = \Theta$, i.e Θ is an odd function of T. Now suppose that r is even, hence x_l and x_{l-r} have the same parity. Clearly Θ^r is even. If x_l is even then $A = \epsilon x_l + x_{l-r}\Theta^r$ is even and $B = \epsilon y_l + y_{l-r}\Theta^r$ is odd. If x_l is odd then A is odd and B is even. In both cases, since $\Theta = A/B$ and observing that the quotient of two numbers of opposite parity is odd, it follows that Θ is odd. So the proof of Proposition 1 is complete.

Remark. Irrational solutions of equations of the type $x = g(x^r)$ where g is a linear fractional transformation with integer coefficients are called algebraic numbers of class I. Various rational approximation properties of these elements have been studied by different authors, see [L2] for references. Our aim is to show, with the above notations, how ϵ and $\mathcal{B}$ can be chosen such that the corresponding number Θ belongs to $\mathcal{E}(q)$. In this case Θ is said to be badly approximable by rational numbers: indeed we have $|\Theta - P/Q||Q|^2 \geq |T|^{-1}$ for all $P, Q \in \mathbb{F}_q[T]$ with $Q \neq 0$.

Definition. *Let p, q, r be integers as above. If there is $\epsilon \in \mathbb{F}_q^*$, an integer $l \geq r$ and $\mathcal{B} \in E(q)^l$ such that $\Theta(\epsilon, \mathcal{B}) \in \mathcal{E}(q)$ then we say that $\Theta(\epsilon, \mathcal{B})$ is a flat formal number of order r. The subset of elements $\Theta \in \mathcal{E}(q)$ for which there is $\epsilon \in \mathbb{F}_q^*$, an integer $l \geq r$ and $\mathcal{B} \in E(q)^l$ such that Θ satisfies $E(\epsilon, \mathcal{B})$ will be denoted by $\mathcal{F}(\epsilon, l, r, q)$. Further we use the notations*

$$\mathcal{F}(l, r, q) = \bigcup_{\epsilon \in \mathbb{F}_q^*} \mathcal{F}(\epsilon, l, r, q) \quad \text{and} \quad \mathcal{F}(r, q) = \bigcup_{l \geq r} \mathcal{F}(l, r, q).$$

At last $\mathcal{F}(q) = \bigcup_{r=p^t} \mathcal{F}(r, q)$ is called the set of flat formal numbers in $\mathbb{F}(q)$.

Different examples of flat numbers have already been given (see [MR], [L1] and [LR]). Moreover we recall that there is a special quadratic number contained in $\mathcal{F}(l, r, q)$ for all possible triple (l, r, q). Indeed, let $q = p^s$ and let $\mathbf{e} \in \mathcal{E}(q)$ be defined by

$$\mathbf{e} = [0, T, T, \ldots, T, \ldots].$$

If $k \geq 1$ and $l \geq k$ are integers, we have shown [LR] that the following equation holds

$$\mathbf{e} = \frac{(-1)^{k-1} x_l + x_{l-k} \mathbf{e}^k}{(-1)^{k-1} y_l + y_{l-k} \mathbf{e}^k}.$$

Hence if $r = p^t$ with $t \geq 0$, replacing k by r in this equation and since $(-1)^{r-1} = 1$ in $\mathbb{F}_q$, we see, with our notations, that $\mathbf{e}$ belongs to $\mathcal{F}(1, l, r, q)$. This formal number $\mathbf{e}$ should be viewed as the analogue of the famous real number $(\sqrt{5} - 1)/2 = [0, 1, \ldots, 1, \ldots]$.

In order to give a characterization of the sequence of the partial quotients in the continued fraction expansion of a flat number, we need to investigate the polynomials x_n and y_n defined inductively from (1). These are functions of the partial quotients a_i, for $i \leq n$. A study of those functions is to be found in Perron's classical treatise on continued fractions. We use the results exposed there and similar notations (see [P] p.3-18). We introduce a sequence of functions $(K_n)_{n \geq -1}$. We put $K_{-1} = 0$ and $K_0 = 1$. For $n \geq 1$, K_n is a function of n variables. We have $K_1(u_1) = u_1$, $K_2(u_1, u_2) = u_1 u_2 + 1$ and the recursive relation, for $n \geq 1$,

$$K_n(u_1, u_2, ..., u_n) = u_n K_{n-1}(u_1, \ldots, u_{n-1}) + K_{n-2}(u_1, \ldots, u_{n-2}).$$

It is interesting to remark that K_n can be expressed by the following determinant

$$K_n(u_1,u_2,\dots,u_n)=\det\begin{vmatrix} u_1 & -1 & 0 & & & \dots & 0\\ 1 & u_2 & -1 & 0 & & \dots & 0\\ 0 & 1 & u_3 & -1 & 0 & \dots & 0\\ \vdots & & & & \ddots & & \vdots\\ 0 & & \dots & & 1 & u_{n-1} & -1\\ 0 & & \dots & & 0 & 1 & u_n \end{vmatrix} \quad (11)$$

From (1) and with these notations, it is clear that we have for $n \geq 0$

$$y_n = K_n(a_1, a_2, \dots, a_n) \qquad \text{and} \qquad x_n = K_{n-1}(a_2, a_3, \dots, a_n).$$

From $\Theta = [0, a_1, \dots] \in \mathcal{E}(q)$ and for $k \geq 0$, we define $\Theta_k = [0, a_{k+1}, a_{k+2}, \dots]$. Hence $\Theta_0 = \Theta$. Moreover, for $k \geq 0$ and $n \geq 0$, we set

$$y_{n,k} = K_n(a_{k+1}, \dots, a_{k+n}) \qquad \text{and} \qquad x_{n,k} = K_{n-1}(a_{k+2}, \dots, a_{k+n}).$$

Clearly $(x_{n,k}/y_{n,k})_{n\geq 0}$ is the sequence of the convergents to Θ_k and we have $x_{n,k}/y_{n,k} = [0, a_{k+1}, \dots, a_{k+n}]$, for $n \geq 1$ and for $k \geq 0$. From these definitions, it is easy to check that we have

$$x_{n,k} = y_{n-1,k+1} \qquad \text{for } k \geq 0 \quad \text{and} \quad n \geq 0. \quad (12)$$

It can also be established by induction that we have for $n \geq m \geq 0$

$$x_n y_m - y_n x_m = (-1)^m y_{n-m-1,m+1} = (-1)^m x_{n-m,m}. \quad (13)$$

From (13), by shifting the sequence of the partial quotients, we obtain for $n \geq m \geq 0$ and $k \geq 0$

$$x_{n,k} y_{m,k} - y_{n,k} x_{m,k} = (-1)^m y_{n-m-1,m+1+k} = (-1)^m x_{n-m,m+k}. \quad (14)$$

From (13), we can also deduce, for $n \geq m \geq 0$,

$$y_n = y_m y_{n-m,m} + y_{m-1} x_{n-m,m}. \quad (15)$$

At last, shifting the sequence of the partial quotients, (15) implies for $n \geq m \geq 0$ and $k \geq 0$

$$y_{n,k} = y_{m,k} y_{n-m,m+k} + y_{m-1,k} x_{n-m,m+k}. \quad (16)$$

With these notations we can characterize the sequence of the partial quotients of a flat formal number.

Proposition 2. *Let p, q, r be integers as above. Let $\Theta = [0, a_1, a_2, \dots] \in \mathcal{E}(q)$. Then there exist $\epsilon \in \mathbb{F}_q^*$ and an integer $l \geq r$ such that $\Theta \in \mathcal{F}(\epsilon, l, r, q)$ if and only if there exists a sequence $(\epsilon_n)_{n\geq 0}$ of elements in $\mathbb{F}_q^*$, with $\epsilon_0 = 1$ and $\epsilon_1 = \epsilon$, such that we have one of the four equivalent properties*

$$(S_1) \qquad \begin{cases} \epsilon_{n+1}x_{nr+l} = \epsilon y_n^r x_l + x_n^r x_{l-r} \\ \epsilon_{n+1}y_{nr+l} = \epsilon y_n^r y_l + x_n^r y_{l-r} \end{cases} \quad \textit{for} \quad n \geq -1.$$

$$(S_2) \qquad \begin{cases} \epsilon_{n+1}x_{nr+l} = \epsilon_{m+1}y_{n-m,m}^r x_{mr+l} + \epsilon_m x_{n-m,m}^r x_{(m-1)r+l} \\ \epsilon_{n+1}y_{nr+l} = \epsilon_{m+1}y_{n-m,m}^r y_{mr+l} + \epsilon_m x_{n-m,m}^r y_{(m-1)r+l} \end{cases}$$

for $n \geq m \geq 0$ or $m = 0$ and $n \geq -1$.

$$(S_3) \qquad \epsilon_{n+1}\epsilon_{n'+1}x_{(n-n')r,n'r+l} = \epsilon_{m+1}\epsilon_m x_{n-n',n'}^r x_{r,(m-1)r+l}$$

for $n \geq n' \geq m \geq 0$ or $m = 0$ and $n \geq n' \geq -1$.

$$(S_4) \qquad \begin{cases} \epsilon_{n+1}\epsilon_{n-1}x_{2r,(n-2)r+l} = \epsilon a_n^r x_{r,l-r} \\ \epsilon_{n+1}\epsilon_n x_{r,(n-1)r+l} = \epsilon x_{r,l-r} \end{cases} \quad \textit{for} \quad n \geq 1.$$

Remark. The case $r = 1$, i.e. when the Frobenius homomorphism is replaced by the identity, is the most simple and also already known since then Θ is quadratic (see [S] p.141-142) . In that case this proposition implies

$$\Theta = [0, a_1, a_2, \dots] \in \mathcal{F}(1, q) \Leftrightarrow (a_i)_{i\geq 1} \text{ is purely periodic.}$$

Assume that $\Theta = [0, a_1, a_2, \dots] \in \mathcal{F}(1, q)$. By (S_4), we have for $n \geq 1$

$$\begin{cases} \epsilon_{n+1}\epsilon_{n-1}a_{n+l} = \epsilon a_n \\ \epsilon_{n+1}\epsilon_n = \epsilon. \end{cases}$$

From the second equation we get $\epsilon_{2n+1} = \epsilon$ and $\epsilon_{2n} = 1$ for $n \geq 0$. Hence the first equation becomes $a_{n+l} = \epsilon^{(-1)^{n+1}} a_n$. By iteration we obtain $a_{n+2l} = \epsilon^{u_{n,2}} a_n$ with $u_{n,2} = (-1)^{n+1} + (-1)^{l+n+1}$. If l is odd $u_{n,2} = 0$ and thus $a_{n+2l} = a_n$ for $n \geq 1$. Otherwise, by further iteration we obtain $a_{n+kl} = \epsilon^{u_{n,k}} a_n$ with $u_{n,k} = (-1)^{n+1} + (-1)^{l+n+1} + \dots + (-1)^{(k-1)l+n+1}$ and since l is even $u_{n,k} = k(-1)^{n+1}$. Hence $\epsilon^{u_{n,k}} = 1$ at least for $k = q-1$ and consequently $a_{n+(q-1)l} = a_n$ for $n \geq 1$. Reciprocally if $\Theta = [0, a_1, \dots, a_L, a_1, \dots, a_L, a_1, \dots]$ then it is well known that $\Theta = (x_L + x_{L-1}\Theta)/(y_L + y_{L-1}\Theta)$ and therefore $\Theta \in \mathcal{F}(1, q)$.

Proof: First we shall prove that $\Theta \in \mathcal{F}(\epsilon, l, r, q)$ if and only if (S_1) holds. Assume that $\Theta = [0, a_1, \dots, a_n, \dots] \in \mathcal{E}(q)$ and satisfies $\Theta = f(\Theta)$ where $f(x) = (\epsilon x_l + x_{l-r}x^r)/(\epsilon y_l + y_{l-r}x^r)$. For $n \geq 0$, we set

$$\begin{cases} u_n = \epsilon x_l y_n^r + x_{l-r}x_n^r \\ v_n = \epsilon y_l y_n^r + y_{l-r}x_n^r \end{cases} \tag{17}$$

Thus we have for $n \geq 0$

$$\frac{u_n}{v_n} = f\left(\frac{x_n}{y_n}\right). \tag{18}$$

According to (7) we can write for $n \geq 0$

$$|\Theta - u_n/v_n| = |f(\Theta) - f(x_n/y_n)| = |T|^{-2l+r-1}|\Theta - x_n/y_n|^r.$$

Since $\Theta \in \mathcal{E}(q)$ implies $|\Theta - x_n/y_n| = |T|^{-1}|y_n|^{-2}$ and clearly $|v_n| = |y_l||y_n|^r$, this becomes

$$|\Theta - u_n/v_n| = |T|^{-1}|v_n|^{-2}. \tag{19}$$

It is known that if $|\Theta - P/Q| < |Q|^{-2}$ then P/Q is a convergent to Θ. Thus there exists an integer m such that $u_n/v_n = x_m/y_m$. Since we also have $|\Theta - x_m/y_m| = |T|^{-1}|y_m|^{-2}$, we deduce $|v_n| = |y_m|$. Further $|y_m| = |T|^m$ and $|v_n| = |T|^{rn+l}$ shows that $m = rn + l$. Consequently $u_n/v_n = x_{rn+l}/y_{rn+l}$, with $\gcd(u_n, v_n) = 1$. This implies that there exists, for $n \geq 0$, $\epsilon_{n+1} \in \mathbb{F}_q^*$ such that

$$\begin{cases} \epsilon_{n+1}x_{nr+l} = \epsilon x_l y_n^r + x_{l-r}x_n^r \\ \epsilon_{n+1}y_{nr+l} = \epsilon y_l y_n^r + y_{l-r}x_n^r \end{cases} \tag{20}$$

Putting $n = 0$ in (20), we see that $\epsilon_1 = \epsilon$. If we define $\epsilon_0 = 1$ we have proved that (S_1) holds for $n \geq -1$. Reciprocally (S_1) implies immediately by division that we have for $n \geq 0$

$$\frac{x_{nr+l}}{y_{nr+l}} = f\left(\frac{x_n}{y_n}\right).$$

Thus, by letting n go to infinity, we obtain the desired equation $\Theta = f(\Theta)$, with $\epsilon = \epsilon_1$. To complete the proof of the proposition we need to show that the four systems (S_1), (S_2), (S_3) and (S_4) are equivalent. We first prove that (S_1) implies (S_2). Let $n \geq m \geq 0$ or $m = 0$ and $n \geq -1$ be integers. By (12), (15) and (16), we have

$$\begin{cases} y_n = y_m y_{n-m,m} + y_{m-1}x_{n-m,m} \\ x_n = x_m y_{n-m,m} + x_{m-1}x_{n-m,m} \end{cases} \tag{21}$$

Using (21), the first equation of (20) can be written

$$\epsilon_{n+1}x_{nr+l} = y_{n-m,m}^r(\epsilon x_l y_m^r + x_{l-r}x_m^r) + x_{n-m,m}^r(\epsilon x_l y_{m-1}^r + x_{l-r}x_{m-1}^r)$$

and finally, applying (20) again,

$$\epsilon_{n+1}x_{nr+l} = \epsilon_{m+1}y_{n-m,m}^r x_{mr+l} + \epsilon_m x_{n-m,m}^r x_{(m-1)r+l}.$$

It is clear that the second equation in (S_2) can be obtained in the same way, hence (S_2) holds. We next prove that (S_3) is implied by (S_2). Let $n \geq n' \geq m \geq 0$ be integers. For brevity sake, (S_2) will be written as

$$\epsilon_{n+1}x_{nr+l} = A_n + B_n \quad \text{and} \quad \epsilon_{n+1}y_{nr+l} = C_n + D_n, \tag{22}$$

and similarly we can write

$$\epsilon_{n'+1}x_{n'r+l} = A_{n'} + B_{n'} \quad \text{and} \quad \epsilon_{n'+1}y_{n'r+l} = C_{n'} + D_{n'}. \tag{23}$$

We put $X = \epsilon_{n+1}x_{nr+l}\epsilon_{n'+1}y_{n'r+l} - \epsilon_{n'+1}x_{n'r+l}\epsilon_{n+1}y_{nr+l}$. By (13) we can write

$$X = \epsilon_{n+1}\epsilon_{n'+1}(-1)^{n'r+l}x_{(n-n')r,n'r+l}. \tag{24}$$

On the other hand, by (22) and (23), we also have

$$\begin{aligned} X &= (A_nC_{n'} - A_{n'}C_n) + (A_nD_{n'} - A_{n'}D_n) \\ &\quad + (B_nC_{n'} - B_{n'}C_n) + (B_nD_{n'} - B_{n'}D_n). \end{aligned} \tag{25}$$

It is easy to check that $A_nC_{n'} - A_{n'}C_n = B_nD_{n'} - B_{n'}D_n = 0$. By (14), we have

$$A_nD_{n'} - A_{n'}D_n = \epsilon_{m+1}\epsilon_m(-1)^{(n'-m)r+1}x_{mr+l}y_{(m-1)r+l}x^r_{n-n',n'} \tag{26}$$

and

$$B_nC_{n'} - B_{n'}C_n = \epsilon_{m+1}\epsilon_m(-1)^{(n'-m)r}x_{(m-1)r+l}y_{mr+l}x^r_{n-n',n'}. \tag{27}$$

Finally we obtain, from (25), (26), (27) and using (14),

$$X = \epsilon_{m+1}\epsilon_m(-1)^{(n'-1)r+l-1}x^r_{n-n',n'}x_{r,(m-1)r+l} \tag{28}$$

We observe that $(-1)^{r+1} = 1$ in $\mathbb{F}_q$ for all p. Hence comparing (24) to (28) we have proved (S_3). Now the first equation of (S_4) is obtained from (S_3) by taking $n' = n-2$ and $m = 0$. So this equation holds for $n \geq 1$. The second equation of (S_4) is obtained from (S_3) by taking $n' = n-1$ and $m = 0$ and hence also holds for $n \geq 1$. Thus clearly (S_3) implies (S_4). It remains to prove that (S_4) implies (S_1). By (12) and (16), we can write for $n \geq 1$

$$x_{2r,(n-2)r+l} = x_{r,(n-2)r+l}y_{r,(n-1)r+l} + x_{r-1,(n-2)r+l}x_{r,(n-1)r+l} \tag{29}$$

From (S_4) we have for $n \geq 1$ and $m \geq 0$

$$\begin{cases} x_{2r,(n-2)r+l} = \epsilon\epsilon^{-1}_{n-1}\epsilon^{-1}_{n+1}a^r_n x_{r,l-r} \\ x_{r,mr+l} = \epsilon\epsilon^{-1}_{m+2}\epsilon^{-1}_{m+1}x_{r,l-r} \end{cases} \tag{30}$$

Combining (29) and (30) we obtain

$$\epsilon_{n+1}y_{r,(n-1)r+l} + \epsilon_{n-1}x_{r-1,(n-2)r+l} = \epsilon_n a^r_n. \tag{31}$$

Again by (12) and (16), we can also write for $n \geq 0$

$$x_{nr+l} = x_{(n-1)r+l}y_{r,(n-1)r+l} + x_{(n-1)r+l-1}x_{r,(n-1)r+l}. \tag{32}$$

Multiplying (32) by ϵ_{n+1} and combining with (31), we obtain

$$\epsilon_{n+1}x_{nr+l} = \epsilon_n a_n^r x_{(n-1)r+l} + X \tag{33}$$

where $X = \epsilon_{n+1}x_{(n-1)r+l-1}x_{r,(n-1)r+l} - \epsilon_{n-1}x_{r-1,(n-2)r+l}x_{(n-1)r+l}$. The second equation of (S_4), implies $\epsilon_{n+1}x_{r,(n-1)r+l} = \epsilon_{n-1}x_{r,(n-2)r+l}$. Therefore we get

$$X = \epsilon_{n-1}\big(x_{r,(n-2)r+l}x_{(n-1)r+l-1} - x_{r-1,(n-2)r+l}x_{(n-1)r+l}\big). \tag{34}$$

Using (13), we can write

$$x_{r-1,(n-2)r+l} = \big(x_{(n-1)r+l-1}y_{(n-2)r+l} - y_{(n-1)r+l-1}x_{(n-2)r+l}\big)\omega$$

and

$$x_{r,(n-2)r+l} = \big(x_{(n-1)r+l}y_{(n-2)r+l} - y_{(n-1)r+l}x_{(n-2)r+l}\big)\omega,$$

with $\omega = (-1)^{(n-2)r+l}$. From these two equalities and (34) it follows that X is equal to

$$\epsilon_{n-1}x_{(n-2)r+l}(x_{(n-1)r+l}y_{(n-1)r+l-1} - y_{(n-1)r+l}x_{(n-1)r+l-1})(-1)^{(n-2)r+l}$$

Finally, by (13), this becomes

$$X = \epsilon_{n-1}x_{(n-2)r+l}. \tag{35}$$

Hence, by (33) and (35), we have proved for $n \geq 1$

$$\epsilon_{n+1}x_{nr+l} = \epsilon_n a_n^r x_{(n-1)r+l} + \epsilon_{n-1}x_{(n-2)r+l}. \tag{36}$$

In a similar way, we could prove the same identity with y instead of x. Now let us prove by induction that the first equation of (S_1), i.e.

$$\epsilon_{n+1}x_{nr+l} = \epsilon y_n^r x_l + x_n^r x_{l-r}, \tag{37}$$

holds for $n \geq -1$. Clearly (37) is true for $n = -1$ and $n = 0$. Assume it is true for $k \leq n$ and $n \geq 0$. From (36) and (37), we can write

$$\epsilon_{n+2}x_{(n+2)r+l} = a_{n+1}^r(\epsilon x_l y_n^r + x_{l-r}x_n^r) + \epsilon x_l y_{n-1}^r + x_{l-r}x_{n-1}^r.$$

Using the Frobenius homomorphism and the recursive definition of the sequences $(x_n)_{n\geq -1}$ and $(y_n)_{n\geq -1}$, we obtain

$$\epsilon_{n+2}x_{(n+2)r+l} = \epsilon x_l y_{n+1}^r + x_{l-r}x_{n+1}^r.$$

Hence (37) holds for $n+1$ and by induction for all $n \geq -1$. Using the corresponding identity to (36) with y instead of x, with the same arguments we also have for $n \geq -1$

$$\epsilon_{n+1}y_{nr+l} = \epsilon y_n^r y_l + x_n^r y_{l-r}.$$

This shows that (S_1) holds for $n \geq -1$. Thus the proof of Proposition 2 is complete.

§3. The case $r = 2$ and some general properties.

In this section we state some consequences of Proposition 2. Clearly the complexity of the system (S_4) is growing with r. Beside the trivial case $r = 1$ considered in the remark following Proposition 2, it is possible to investigate throughly the case $r = 2$ and we do so in Proposition 3 and 4. This case illustrates the fact that in general the existence of a sequence $(a_n)_{n\geq 1}$ of polynomials of degree one in $\mathbb{F}_q[T]$ solution of (S_4) will depend upon the choice of ϵ and also of the first l partial quotients. In Proposition 5 we obtain some general properties on the sequences of the two coefficients of the partial quotients by studying the system (S_4). At last, in Proposition 6, we give some properties of stability for flat numbers.

Proposition 3. *Let $q = 2^s$ with $s \geq 1$ and $l \geq 2$ be integers. Let $\lambda_1, \lambda_2, \ldots, \lambda_l$ and ϵ be given in $\mathbb{F}_q^*$. We consider the sequence $(\lambda_i)_{i\geq 1}$ in $\mathbb{F}_q^*$ defined recursively for $n \geq 1$ by*

$$\begin{cases} \lambda_{l+2n-1} = \lambda_n^2 \lambda_l^{-1} \epsilon^{(-1)^{n+1}} \\ \lambda_{l+2n} = \lambda_l. \end{cases}$$

Let x_{l-2}, y_{l-2}, x_l and y_l be the polynomials built from $(\lambda_1 T, \lambda_2 T, \ldots, \lambda_l T)$ by (1). Let Θ be the irrational element in $\mathbb{F}(q)$ defined by the continued fraction expansion $\Theta = [0, \lambda_1 T, \lambda_2 T, \ldots, \lambda_n T, \ldots]$. Then Θ satisfies the algebraic equation

$$y_{l-2}\Theta^3 + x_{l-2}\Theta^2 + \epsilon y_l \Theta + \epsilon x_l = 0.$$

Proof: Let q and l be as stated above. According to Proposition 1 with $r = 2$, we know that the following equation

$$x = \frac{\epsilon x_l + x_{l-2}x^2}{\epsilon y_l + y_{l-2}x^2} \tag{38}$$

has an irrational solution Θ in $\mathbb{F}(q)$ such that $\Theta = [0, \lambda_1 T, \ldots, \lambda_l T, a_{l+1}, \ldots]$. Moreover since the first l partial quotients are linear, they all are linear and hence we can put $a_i = \lambda_i T$ for $i \geq 1$. By Proposition 2 with $r = 2$, we have $\Theta \in \mathcal{E}(q)$ if and only if there is a sequence $(\epsilon_n)_{n\geq 0}$ such that

$$(S_4) \qquad \begin{cases} \epsilon_{n+1}\epsilon_{n-1}x_{4,2n-4+l} = \epsilon a_n^2 x_{2,l-2} \\ \epsilon_{n+1}\epsilon_n x_{2,2n-2+l} = \epsilon x_{2,l-2} \end{cases} \quad for \quad n \geq 1.$$

It is easy to check that we have

$$\begin{cases} x_{4,2n-4+l} = a_{2n-2+l}a_{2n-1+l}a_{2n+l} + a_{2n-2+l} + a_{2n+l} \\ x_{2,2n-2+l} = a_{2n+l} \\ x_{2,l-2} = a_l. \end{cases} \tag{39}$$

Therefore (S_4) and (39) imply

$$\lambda_{2n-2+l}\lambda_{2n-1+l}\lambda_{2n+l}T^3 + (\lambda_{2n-2+l} + \lambda_{2n+l})T = \epsilon\epsilon_{n+1}^{-1}\epsilon_{n-1}^{-1}\lambda_n^2\lambda_l T^3 \quad (40)$$

and

$$\epsilon_{n+1}\epsilon_n\lambda_{2n+l}T = \epsilon\lambda_l T \quad (41)$$

for $n \geq 1$. From (40) we obtain $\lambda_{2n-2+l} = \lambda_{2n+l}$ and thus $\lambda_{2n+l} = \lambda_l$ for $n \geq 1$. Combining this with (41), it follows that $\epsilon_{n+1}\epsilon_n = \epsilon$ for $n \geq 1$. With the initial conditions, this leads to $\epsilon\epsilon_{n+1}^{-1}\epsilon_{n-1}^{-1} = \epsilon^{(-1)^{n+1}}$. Comparing the coefficients of T^3 in both sides of (40) and since $\lambda_{2n-2+l} = \lambda_{2n+l} = \lambda_l$, we obtain $\lambda_{l+2n-1} = \lambda_n^2\lambda_l^{-1}\epsilon^{(-1)^{n+1}}$ for $n \geq 1$. In conclusion $\Theta = [0, \lambda_1 T, \ldots, \lambda_n T \ldots]$, with the sequence $(\lambda_i)_{i\geq 1}$ defined as in the proposition, satisfies (38). Thus Θ satisfies the desired equation and the proof is complete.

With the notations introduced above, this last proposition shows that, in case of characteristic 2, if $l \geq 2$, $\mathcal{B} \in E_0(q)^l$ and $\epsilon \in \mathbb{F}_q^*$ are arbitrary then $\Theta(\mathcal{B}, \epsilon) \in \mathcal{F}(2, q)$. In the next proposition, also in the case of characteristic 2, we see that if $\mathcal{B}$ is chosen arbitrarily in $E(q)^l$ then in general $\Theta(\mathcal{B}, \epsilon)$ is no longer in $\mathcal{F}(2, q)$. In this proposition we use, whenever this has a sense, the symbol for continued fractions $[u_1, u_2, \ldots, u_m]$ where the u_i are in $\mathbb{F}_q$.

Proposition 4. *Let $q = 2^s$ with $s \geq 1$ and $l \geq 2$ be integers. Let $\mathcal{B} = (a_1, \ldots, a_l) \in E(q)^l$ and $\epsilon \in \mathbb{F}_q^*$ be given. For $1 \leq i \leq l$ we set $a_i = \lambda_i T + \mu_i$ with $\lambda_i \in \mathbb{F}_q^*$ and $\mu_i \in \mathbb{F}_q$. For $1 \leq i \leq l-1$ we set $\alpha_i = \mu_l\lambda_l^{-1}\lambda_i + \mu_i$. We define a subset $I \subset \{1, \ldots, l-1\}$ such that*

$$I = \{i : [\alpha_i][\alpha_{i-1}, \alpha_i] \ldots [\alpha_1, \ldots, \alpha_i] \neq 0\}.$$

For $i \in I$ we set $\omega_i = [0, \alpha_1^2, \ldots, \alpha_i^2] \in \mathbb{F}_q^*$. Finally we define the subset of $\mathbb{F}_q^*$, $G(\mathcal{B}) = \{\omega_i : i \in I\}$. Then we have

1) $\Theta(\epsilon, \mathcal{B}) \in \mathcal{F}(l, 2, q)$ if and only if $\epsilon \notin G(\mathcal{B})$.

2) If $l < q$ then for all $\mathcal{B}$ there exists $\epsilon \in \mathbb{F}_q^*$ such that $\Theta(\epsilon, \mathcal{B}) \in \mathcal{F}(l, 2, q)$.

3) If $l \geq q$ then there exists $\mathcal{B}$ such that for all $\epsilon \in \mathbb{F}_q^*$ we have $\Theta(\epsilon, \mathcal{B}) \notin \mathcal{F}(l, 2, q)$.

Proof: By Proposition 2, with $q = 2^s$ and $l \geq r = 2$, we know that $\Theta(\epsilon, \mathcal{B}) \in \mathcal{F}(l, 2, q)$ if and only if there is a sequence $(\epsilon_n)_{n\geq 0}$ such that (S_4) holds for $n \geq 1$. We have $\Theta(\epsilon, \mathcal{B}) = [0, a_1, \ldots, a_l, a_{l+1}, \ldots]$. Extending the notations of this proposition, we put $a_i = \lambda_i T + \mu_i$ and $\alpha_i = \mu_l\lambda_l^{-1}\lambda_i + \mu_i$, for $i \geq 1$. By (S_4) and (39) as in the previous proposition, an elementary calculation shows that $\Theta(\epsilon, \mathcal{B}) \in \mathcal{F}(l, 2, q)$ if and only if there is a sequence

$(\epsilon_n)_{n\geq 0}$ such that we have for $n \geq 1$

$$\begin{cases} \lambda_{l+2n-1} = \epsilon^{-1}\epsilon_n^2\lambda_n^2\lambda_l^{-1} \\ \lambda_{l+2n} = \epsilon\epsilon_{n+1}^{-1}\epsilon_n^{-1}\lambda_l \\ \mu_{l+n} = \lambda_{l+n}\lambda_l^{-1}\mu_l \\ \epsilon_{n+1} = \epsilon_n\alpha_n^2 + \epsilon_{n-1} \end{cases} \tag{42}$$

The third equation shows that $\alpha_n = 0$, for $n \geq l$. Hence, the last equation implies $\epsilon_{n+1} = \epsilon_{n-1}$, for $n \geq l$. Consequently (42) will have a unique solution $(a_i)_{i\geq 1}$ if and only if $\mathcal{B}$ and ϵ are such that $\epsilon_i \neq 0$ for $2 \leq i \leq l$. So we only have to study the last equation of (42), for $1 \leq n \leq l-1$. This equation can be written as

$$\eta_{n+1} = \alpha_n^2 + \frac{1}{\eta_n} \quad \text{with} \quad \eta_n = \frac{\epsilon_n}{\epsilon_{n-1}}.$$

This can be written formally, using the usual symbol for continued fraction, as

$$\eta_n = [\alpha_{n-1}^2, \alpha_{n-2}^2, \ldots, \alpha_1^2, \epsilon] \quad \text{for} \quad 2 \leq n \leq l.$$

It is clear that $\epsilon_i \neq 0$ for $2 \leq i \leq l$ is equivalent to $\eta_i \neq 0$ for $2 \leq i \leq l$. If $i \in I$ then ω_i exists in $\mathbb{F}_q^*$ and we have $\eta_{i+1} \neq 0$ if and only if $\epsilon \neq w_i$. Therefore (42) has a solution if and only if $\epsilon \notin G(\mathcal{B})$, which ends the proof of the first part of the proposition.

Moreover since $|G(\mathcal{B})| \leq l-1$, if $l < q$ then we have $|G(\mathcal{B})| < |\mathbb{F}_q^*|$. Therefore we can always find $\epsilon \in \mathbb{F}_q^*$ and $\epsilon \notin G(\mathcal{B})$ and this proves the second part.

At last assume that $l \geq q$ and thus $|\mathbb{F}_q^*| \leq l-1$. In that case it is possible to choose $\mathcal{B}$ and particularly the $q-1$ first α_i such that the ω_i for $1 \leq i \leq q-1$ take all the values in $\mathbb{F}_q^*$. So $\epsilon \notin G(\mathcal{B})$ is impossible and the proof is complete.

Now we return to the general case. So we have $q = p^s$, $r = p^t$ where p is an arbitrary prime number and $s \geq 1$, $t \geq 0$ are integers. For the next proposition we introduce the following notations. If $\Theta \in \mathcal{E}(q)$ we put $\Theta = [0, a_1, a_2, \ldots]$ where $a_i = \lambda_i T + \mu_i$ with $\lambda_i \in \mathbb{F}_q^*$ and $\mu_i \in \mathbb{F}_q$ for $i \geq 1$. We put $\gamma_i = \mu_i\lambda_i^{-1}$ for $i \geq 1$. Moreover, to simplify the writing, we set for $j \geq 1$ and $i \geq 1$

$$\pi_{i,j} = \prod_{k=j}^{j+i-1} \lambda_k \quad \text{and} \quad \sigma_{i,j} = \sum_{k=j}^{j+i-1} \gamma_k$$

with $\pi_{0,j} = 1$ and $\sigma_{0,j} = 0$. Now we can state the following proposition.

Proposition 5. *Let $\epsilon \in \mathbb{F}_q^*$ and $\Theta \in \mathcal{F}(\epsilon, l, r, q)$. Let $(\epsilon_n)_{n\geq 0}$ be the sequence of elements in $\mathbb{F}_q^*$ corresponding to Θ by Proposition 2. Then we have :*

1) $\epsilon_n = \pi_{nr,l-r+2}(\pi_{n,1})^{-r}$ *for* $n \geq 1$.

2) $(\pi_{r-1,(n-2)r+l+2})^2 \; \lambda_{(n-2)r+l+1}\lambda_{(n-1)r+l+1} = (\lambda_n\lambda_{n-1})^r$ *for* $n \geq 2$.

3) if $r \geq 2$, $\sigma_{r,nr+l+1} = \sigma_{r,(n-1)r+l+2} = 0$ *and* $\gamma_{nr+l+1} = \gamma_{l+1}$ *for* $n \geq 0$.

Proof: Let us consider $x_{i,k}$, for $k \geq 0$ and $i \geq 1$, as a polynomial in T. With the above notations, we easily see by induction that

$$x_{i,k} = \pi_{i-1,k+2}T^{i-1} + \pi_{i-1,k+2}\sigma_{i-1,k+2}T^{i-2} + \dots \tag{43}$$

From (S_4), combining the two equations, we have for $n \geq 1$

$$\epsilon_{n+1}\epsilon_{n-1}x_{2r,(n-2)r+l} = a_n^r\epsilon_{n+1}\epsilon_n x_{r,(n-1)r+l}$$

By comparing the coefficients of highest degree in T on both sides of this equation, we obtain directly with (43)

$$\epsilon_{n-1}\pi_{2r-1,(n-2)r+l+2} = \lambda_n^r\epsilon_n\pi_{r-1,(n-1)r+l+2} \quad \text{for} \quad n \geq 1.$$

This becomes

$$\frac{\epsilon_n}{\epsilon_{n-1}} = \frac{\pi_{r,(n-2)r+l+2}}{\lambda_n^r} \quad \text{for} \quad n \geq 1. \tag{44}$$

Since $\epsilon_n = \prod_{k=1}^n \epsilon_k/\epsilon_{k-1}$, we obtain the first point of this proposition.

From (S_4), combining the two equations, we have for $n \geq 2$

$$\epsilon_n\epsilon_{n-2}x_{2r,(n-3)r+l} = a_{n-1}^r\epsilon_{n-1}\epsilon_{n-2}x_{r,(n-3)r+l}$$

By comparing the coefficients of highest degree in T on both sides of this equation and using (43) again, we get

$$\epsilon_n\pi_{2r-1,(n-3)r+l+2} = \lambda_{n-1}^r\epsilon_{n-1}\pi_{r-1,(n-3)r+l+2}$$

and this becomes

$$\frac{\epsilon_n}{\epsilon_{n-1}} = \frac{\lambda_{n-1}^r}{\pi_{r,(n-2)r+l+1}} \quad \text{for} \quad n \geq 2. \tag{45}$$

Finally, comparing (44) and (45), we obtain for $n \geq 2$ the second point of the proposition.

For the last point we will use (S_3). By comparing the coefficients of highest degree in T in both sides of (S_3), we get the following equality

$$X = \epsilon_{n+1}\epsilon_{n'+1}\pi_{(n-n')r-1,n'r+l+2} = \epsilon_{m+1}\epsilon_m\pi_{n-n'-1,n'+2}^r\pi_{r-1,(m-1)r+l+2}.$$

Now comparing the coefficients of degree in T just below the highest, using (43) and assuming that $r \geq 2$, we obtain

$$X\sigma_{(n-n')r-1,n'r+l+2} = X\sigma_{r-1,(m-1)r+l+2}.$$

Thus we have $\sigma_{(n-n')r-1,n'r+l+2} = \sigma_{r-1,(m-1)r+l+2}$ for $n \geq n' \geq m \geq 0$ or $n \geq n' \geq -1$ and $m = 0$. Taking now $n' = n-1$, we get

$$\sigma_{r-1,(n-1)r+l+2} = \sigma_{r-1,l-r+2} \quad \text{for} \quad n \geq 0. \tag{46}$$

While taking $n' = n-2$, implies

$$\sigma_{2r-1,(n-2)r+l+2} = \sigma_{r-1,l-r+2} \quad \text{for} \quad n \geq 1. \tag{47}$$

By (46) and (47) we obtain $\sigma_{2r-1,(n-2)r+l+2} = \sigma_{r-1,(n-2)r+l+2}$ for $n \geq 1$. Hence we have $\sigma_{r,(n-2)r+l+2} = 0$ for $n \geq 1$ or equivalently

$$\sigma_{r,(n-1)r+l+2} = 0 \quad \text{for} \quad n \geq 0. \tag{48}$$

Therefore, for $n \geq 0$, we can write

$$\sigma_{r-1,(n-1)r+l+2} + \gamma_{nr+l+1} = \sigma_{r-1,l-r+2} + \gamma_{l+1}$$

and , by (46), this implies

$$\gamma_{nr+l+1} = \gamma_{l+1} \quad \text{for} \quad n \geq 0. \tag{49}$$

Finally, from (48) and (49), it follows that

$$\sigma_{r,nr+l+1} = 0 \quad \text{for} \quad n \geq 0.$$

So the proof of this proposition is complete.

Remark. We recall here another property of the sequence $(\epsilon_n)_{n\geq 0}$ which has been proved in [LR]. With the above notations, if $\mu_i = 0$ for $i \geq 1$ then $\epsilon_{2n} = 1$ and $\epsilon_{2n+1} = \epsilon$ for $n \geq 0$ (as in Proposition 3 above). Further, by taking the value at zero in the polynomials in T of both equations of (S_4), we observe that the sequence $(\mu_i)_{i\geq 1}$ satisfies the same system (S_4) as the sequence of the partial quotients $(a_i)_{i\geq 1}$ does.

We state a last proposition.

Proposition 6. *Let $\epsilon \in \mathbb{F}_q^*$ and $\Theta \in \mathcal{F}(\epsilon, l, r, q)$. Then we have :*

1) If $\lambda \in \mathbb{F}_q^*$ and $\mu \in \mathbb{F}_q$ then $\Theta'(T) = \Theta(\lambda T + \mu) \in \mathcal{F}(\epsilon, l, r, q)$.

2) If $\lambda \in \mathbb{F}_q^*$ then $\Theta' = \lambda\Theta \in \mathcal{F}(\epsilon', l, r, q)$ with

$$\begin{cases} \epsilon' = \epsilon\lambda^r & \text{if} \quad r \text{ is even} \\ \epsilon' = \epsilon\lambda^{(-1)^{l+1}+r} & \text{if} \quad r \text{ is odd} \end{cases}$$

3) There exists $\epsilon(k) \in \mathbb{F}_q^*$ such that $\Theta \in \mathcal{F}(\epsilon(k), l_k, r^k, q)$ for every integer $k \geq 1$, with $l_k = (1 + r + \cdots + r^{k-1})l$.

4) There exists $\epsilon(k) \in \mathbb{F}_q^*$ such that $\Theta_k \in \mathcal{F}(\epsilon(k), l_k, r, q)$ for every integer $k \geq 0$, with $l_k = l + k(r-1)$.

PROOF: The first point of this proposition is obvious. Indeed if $\Theta(T) = [0, a_1(T), a_2(T), \ldots]$ then $\Theta'(T) = [0, a_1(\lambda T + \mu), a_2(\lambda T + \mu), \ldots]$. Consequently the four polynomials x_l', y_l', x_{l-r}' and y_{l-r}', corresponding to Θ' are obtained from x_l, y_l, x_{l-r} and y_{l-r} changing T into $\lambda T + \mu$. Thus if Θ satisfies $E(\epsilon, \mathcal{B})$ then Θ' satisfies $E(\epsilon, \mathcal{B}')$ where $\mathcal{B}'$ is obtained from $\mathcal{B}$ changing T into $\lambda T + \mu$. Observe that $\Theta \in \mathcal{E}(q)$ if and only if $\Theta' \in \mathcal{E}(q)$.

If $\Theta = [0, a_1, a_2, \ldots]$ and $\lambda \in \mathbb{F}_q^*$ it is clear that $\Theta' = \lambda\Theta = [0, \lambda^{-1}a_1, \lambda a_2, \lambda^{-1}a_3, \lambda a_4, \ldots]$. Again denoting by x_l', y_l', x_{l-r}' and y_{l-r}' the four polynomials corresponding to Θ', it is easy to check that we have for $n \geq 1$

$$\begin{cases} x_n' = x_n & y_n' = \lambda^{-1}y_n \quad \text{if } n \text{ is odd} \\ x_n' = \lambda x_n & y_n' = y_n \quad \text{if } n \text{ is even} \end{cases} \tag{50}$$

Observe that $\Theta \in \mathcal{E}(q)$ if and only if $\Theta' \in \mathcal{E}(q)$. We suppose now that Θ satisfies $E(\epsilon, \mathcal{B})$. We will have to consider four cases according to the different parities of r and l. Assume first that r is odd and l is even. Then , by (50), we obtain

$$\Theta' = \lambda\Theta = \lambda\left(\frac{\epsilon x_l + x_{l-r}\Theta^r}{\epsilon y_l + y_{l-r}\Theta^r}\right) = \frac{\epsilon\lambda^{r-1}x_l' + x_{l-r}'(\Theta')^r}{\epsilon\lambda^{r-1}y_l' + y_{l-r}'(\Theta')^r}.$$

This proves that Θ' belongs to $\mathcal{F}(\epsilon', l, r, q)$ with $\epsilon' = \epsilon\lambda^{r-1}$. The proof in the three cases left is obtained in the same way and so we omit it.

Let us prove the third point of the proposition. We use induction on k. Since $\Theta \in \mathcal{F}(\epsilon, l, r, q)$, the result is true for $k = 1$ with $\epsilon(1) = \epsilon$. We write

$$\Theta = \frac{\epsilon x_l + x_{l-r}\Theta^r}{\epsilon y_l + y_{l-r}\Theta^r} = f_{\epsilon,l}(\Theta^r).$$

Assume that there is $\epsilon(k) \in \mathbb{F}_q^*$ such that we have

$$\Theta = f_{\epsilon(k),l_k}(\Theta^{r^k}) \quad \text{with} \quad l_k = (1 + r + \cdots + r^{k-1})l.$$

Then we can write

$$\Theta = f_{\epsilon,l}\left(\left(f_{\epsilon(k),l_k}(\Theta^{r^k})\right)^r\right)$$

Using the Frobenius homomorphism, a simple calculation leads to

$$\Theta = \frac{A + B\Theta^{r^{k+1}}}{C + D\Theta^{r^{k+1}}} \tag{51}$$

where

$$\begin{cases} A = \epsilon(k)^r(\epsilon x_l y_{l_k}^r + x_{l-r}x_{l_k}^r), & B = \epsilon x_l y_{l_k - r^k}^r + x_{l-r}x_{l_k - r^k}^r \\ C = \epsilon(k)^r(\epsilon y_l y_{l_k}^r + y_{l-r}x_{l_k}^r), & D = \epsilon y_l y_{l_k - r^k}^r + y_{l-r}x_{l_k - r^k}^r. \end{cases}$$

Now we know, by Proposition 2, that there is a sequence $(\epsilon_n)_{n\geq 0}$ of elements in $\mathbb{F}_q^*$, with $\epsilon_0 = 1$ and $\epsilon_1 = \epsilon$, such that we have

$$(S_1) \qquad \begin{cases} \epsilon_{n+1}x_{nr+l} = \epsilon y_n^r x_l + x_n^r x_{l-r} \\ \epsilon_{n+1}y_{nr+l} = \epsilon y_n^r y_l + x_n^r y_{l-r} \qquad for \qquad n \geq -1, \end{cases}$$

Replacing n by l_k or $l_k - r^k$ in both equations of (S_1), we obtain directly

$$\begin{cases} A = \epsilon(k)^r \epsilon_{l_k+1} x_{rl_k+l}, \quad B = \epsilon_{l_k-r^k+1} x_{rl_k+l-r^{k+1}} \\ C = \epsilon(k)^r \epsilon_{l_k+1} y_{rl_k+l}, \quad D = \epsilon_{l_k-r^k+1} y_{rl_k+l-r^{k+1}}. \end{cases} \tag{52}$$

Since $rl_k + l = l_{k+1}$, putting $\epsilon(k+1) = \epsilon(k)^r \epsilon_{l_k+1} \epsilon_{l_k-r^k+1}^{-1}$, we see that (51) and (52) imply $\Theta \in \mathcal{F}(\epsilon(k+1), l_{k+1}, r^{k+1}, q)$. Thus the proof of this point is complete.

Let us prove the last point. Since $\Theta = [0, a_1, a_2, \dots] \in \mathcal{F}(\epsilon, l, r, q)$, we know that there is a sequence $(\epsilon_n)_{n\geq 0}$ of elements in $\mathbb{F}_q^*$, with $\epsilon_0 = 1$ and $\epsilon_1 = \epsilon$, such that we have

$$(S_4) \qquad \begin{cases} \epsilon_{n+1}\epsilon_{n-1}x_{2r,(n-2)r+l} = a_n^r \epsilon x_{r,l-r} \\ \epsilon_{n+1}\epsilon_n x_{r,(n-1)r+l} = \epsilon x_{r,l-r} \qquad for \qquad n \geq 1. \end{cases}$$

It is easy to check that this is equivalent to

$$\begin{cases} \epsilon_{n-1}x_{2r,(n-2)r+l} = \epsilon_n a_n^r x_{r,(n-1)r+l} \\ \epsilon_{n+1}x_{r,(n-1)r+l} = \epsilon_{n-1}x_{r,(n-2)r+l} \qquad for \qquad n \geq 1. \end{cases} \tag{53}$$

Given $k \geq 0$, wet set $\tilde{a}_n = a_{n+k}$ for $n \geq 1$ and $\tilde{\epsilon}_n = \epsilon_{n+k}\epsilon_k^{-1}$ for $n \geq 0$. Further for $i \geq 1$ and $j \geq 0$ we set $\tilde{x}_{i,j} = x_{i,j+k}$. Now we write (53) replacing n by $n+k$. With these notations and with $l_k = l + k(r-1)$, we have

$$\begin{cases} \tilde{\epsilon}_{n-1}\tilde{x}_{2r,(n-2)r+l_k} = \tilde{\epsilon}_n \tilde{a}_n^r \tilde{x}_{r,(n-1)r+l_k} \\ \tilde{\epsilon}_{n+1}\tilde{x}_{r,(n-1)r+l_k} = \tilde{\epsilon}_{n-1}\tilde{x}_{r,(n-2)r+l_k} \qquad for \qquad n \geq 1. \end{cases} \tag{54}$$

We observe that $\tilde{\epsilon}_0 = 1$ and that $\tilde{x}_{i,j}$ is obtained from the sequence $(\tilde{a}_n)_{n\geq 1}$ as $x_{i,j}$ is obtained from the sequence $(a_n)_{n\geq 1}$. By (54), we see that the sequences $(\tilde{a}_n)_{n\geq 1}$ and $(\tilde{\epsilon}_n)_{n\geq 1}$ satisfy (S_4). Consequently $\Theta_k = [0, \tilde{a}_1, \tilde{a}_2, \dots]$ is an element of $\mathcal{F}(\epsilon(k), l_k, r, q)$ with $\epsilon(k) = \epsilon_{k+1}\epsilon_k^{-1}$. So the proof of Proposition 6 is complete.

Before concluding we make a last observation. Since flat numbers are algebraic over $\mathbb{F}_q(T)$, the question of their exact degree is open. If $\Theta \in \mathcal{F}(r, q)$ then clearly we have $2 \leq [\mathbb{F}_q(\Theta, T) : \mathbb{F}_q(T)] \leq r + 1$. Although we could not prove it, it is reasonable to believe that, except in the trivial case $r = 1$, we have $[\mathbb{F}_q(\Theta, T) : \mathbb{F}_q(T)] = 2$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ and $\lambda_3 \in \mathbb{F}_q$ such that $\Theta(T) = \lambda_1 \mathbf{e}(\lambda_2 T + \lambda_3)$. In connection with this remark, it is easy to check that if $\Theta \in \mathcal{F}(l, r, q) \cap \mathcal{F}(l', r, q)$ for $l \neq l'$ then Θ^r is quadratic over $\mathbb{F}_q(T)$ and thus Θ also.

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References

[**BS1**] L. Baum and M. Sweet, *Continued fractions of algebraic power series in characteristic 2*, Annals of Mathematics **103** (1976), 593–610.

[**BS2**] L. Baum and M. Sweet, *Badly approximable power series in characteristic 2*, Annals of Mathematics **105** (1977), 573–580.

[**L1**] A. Lasjaunias, *Quartic power series in* $\mathbb{F}_3((T^{-1}))$ *with bounded partial quotients*, Acta Arithmetica **XCV.1** (2000), 49-59.

[**L2**] A. Lasjaunias, *A Survey on Diophantine Approximation in Fields of power Series*, Monatshefte fur Mathematik **130** (2000), 211-229.

[**LR**] A. Lasjaunias and J-J. Ruch, *Algebraic and badly approximable power series over a finite field*, (To appear in Finite Fields and Their Applications).

[**MR**] W. Mills and D. Robbins, *Continued fractions for certain algebraic power series*, Journal of Number Theory **23** (1986), 388–404.

[**P**] O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Chelsea Publishing Company, New York, 1950.

[**S**] W. Schmidt, *On continued fractions and diophantine approximation in power series fields*, Acta Arithmetica **XCV.2** (2000), 139-165.

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