# A NOTE ON HYPERQUADRATIC CONTINUED FRACTIONS IN CHARACTERISTIC 2 WITH PARTIAL QUOTIENTS OF DEGREE 1

by
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**Abstract.** In this note, we describe a family of particular algebraic, and nonquadratic, power series over an arbitrary finite field of characteristic 2, having a continued fraction expansion with all partial quotients of degree one. The main purpose is to point out a common origin with another analogous family in odd characteristic, previously studied by the author.

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## 1. Introduction

Let $p$ be a prime number, $q = p^s$ with $s \geq 1$, and let $\mathbb{F}_q$ be the finite field with $q$ elements. We let $\mathbb{F}_q[T]$, $\mathbb{F}_q(T)$ and $\mathbb{F}(q)$ respectively denote, the ring of polynomials, the field of rational functions and the field of power series in $1/T$ with coefficients in $\mathbb{F}_q$, where $T$ is a formal indeterminate. These fields are valuated by the ultrametric absolute value (and its extension) introduced on $\mathbb{F}_q(T)$ by $|P/Q| = |T|^{\deg(P)-\deg(Q)}$, where $|T| > 1$ is a fixed real number. Hence a non-zero element of $\mathbb{F}(q)$ is written as $\alpha = \sum_{k \leq k_0} a_k T^k$ with $k_0 \in \mathbb{Z}$, $a_k \in \mathbb{F}_q$, and $a_{k_0} \neq 0$ and we have $|\alpha| = |T|^{k_0}$. This field $\mathbb{F}(q)$ is the completion of the field $\mathbb{F}_q(T)$ for this absolute value.

We recall that each irrational (rational) element $\alpha$ of $\mathbb{F}(q)$ can be expanded as an infinite (finite) continued fraction. This will be denoted $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ where the $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) > 0$ for $i > 1$, are the partial quotients. In this note we are concerned with infinite continued fractions in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$. Indeed, we consider algebraic elements of a particular type. Letting $r = p^t$ with $t \geq 0$, we say that $\alpha$ belonging to $\mathbb{F}(q)$ is hyperquadratic (of order $t$) if $\alpha$ is irrational and satisfies an algebraic equation of the particular form $A\alpha^{r+1} + B\alpha^r + C\alpha + D = 0$, where $A, B, C$ and $D$ belong to $\mathbb{F}_q[T]$. For more details on these particular power series, the reader may see the introduction of [BL]. Note that quadratic elements are hyperquadratic. We recall that an element in $\mathbb{F}(q)$ is quadratic if and only if its sequence of partial quotients is ultimately periodic. Many explicit continued fractions are known for nonquadratic but hyperquadratic elements; see the references below. However, these particular algebraic elements play an important role in diophantine approximation and this is why they were first considered. For a general account of continued fractions in power series fields and diophantine approximation, also for more references on this matter, the reader can consult [S, L2].

The origin of the study of continued fractions for hyperquadratic power series is due to Baum and Sweet ([BS1, BS2]). It was been developed in the 1980's by Mills and Robbins [MR]. Mills and Robbins pointed out the existence of hyperquadratic continued fractions with all partial quotients of degree one, in odd characteristic with a prime base field. Different approaches were developed to explain this phenomenon (see [L1, LR1, LR2]), first with the base field $\mathbb{F}_3$, then in all characteristics with an arbitrary base field. However, these methods left many unanswered questions and the study could not be achieved. In [L3] a new approach was introduced and this led to the presentation of different families of explicit hyperquadratic continued fractions, also with unbounded partial quotients. In a recent paper with Yao [LY1], with this new approach, in the case of an arbitrary base field of odd characteristic, we could give the description of a large family, including the historical examples due to Mills and Robbins, of hyperquadratic continued fractions with all partial quotients of degree one.

As often, the case of even characteristic appears to be singular. The simplest case, where the base field is $\mathbb{F}_2$, has been fully treated by Baum and Sweet in [BS2]. They could describe the set of all the continued fractions with partial quotients of degree one (i.e., equal to $T$ or $T+1$). Among them, some are algebraic and for every even integer $d \geq 2$ there is, in this set, an algebraic element of degree $d$. However, it was proved later that none of these algebraic elements (for $d > 2$) were hyperquadratic (see [L2, p. 225]). Later, in a joint paper with Ruch we could exhibit hyperquadratic continued fractions in $\mathbb{F}(2^s)$, with partial quotients all of the form $\lambda T$ where $\lambda \in \mathbb{F}_{2^s}$ (see [LR1, p. 280], with $r = 2$ and also [LR2, p. 556]). Other examples were introduced by Thakur [T, p. 290]. Finally, we shall not omit the impressive note left by Robbins on arXiv [R], in which several conjectures on continued fractions for particular cubic power series in $\mathbb{F}(4)$ and $\mathbb{F}(8)$, with partial quotients of bounded degree, are presented.

Here we present a family of hyperquadratic (for each order $t \geq 1$) continued fractions in $\mathbb{F}(2^s)$ of the form $\alpha = [\lambda_1 T, \lambda_2 T, ...., \lambda_n T, ...]$. In the next section, we state a theorem showing how this sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_{2^s}^*$ is defined recursively from an arbitrary number of initial values. Moreover, in a final remark, we make a comment on the poperties of this sequence. However, the main goal of this note is to show that this family has a common origin with another family of continued fractions in the case of odd characteristic, introduced earlier and fully studied in [LY1]. Indeed, eventhough the case of characteristic 2 is simpler, it appears that the existence of both families is directly connected to a particular universal quadratic power series $\omega = [T, T, ..., T, ....]$ belonging to $\mathbb{F}(p)$ for all prime numbers $p$. Note that this element $\omega$ is actually the analogue, in the formal case, of the celebrated quadratic real number $[1, 1, \dots, 1, \dots] = (1 + \sqrt{5})/2$.

## 2. Results

Our method to obtain the explicit continued fraction for certain hyperquadratic power series was introduced in [L3]. We shall also use the notation, concerning continued fractions and continuants, presented in [LY1, Section 2], to which the reader is invited to refer.

Let $\alpha$ be an irrational element in $\mathbb{F}(q)$ with $\alpha = [a_1, \dots, a_n, \dots]$ as its continued fraction expansion. We let $\mathbb{F}(q)^+$ denote the subset of $\mathbb{F}(q)$ containing the elements having an integral part of positive degree (i.e., with $\deg(a_1) > 0$). For all integers $n \geq 1$, we put $\alpha_n = [a_n, a_{n+1}, \dots]$ ($\alpha_1 = \alpha$), called the $n$-th complete quotient. We introduce the usual continuants $x_n, y_n \in \mathbb{F}_q[T]$ such that $x_n/y_n = [a_1, a_2, \dots, a_n]$ for $n \geq 1$. As usual we extend the latter notation to $n = 0$ with $x_0 = 1$ and $y_0 = 0$. So, in the sequel, with our notation [LY1, p. 266], given a vector $A = (a_1, a_2, \dots, a_n)$, the polynomials $x_n$ and $y_n$ are the continuants $\langle a_1, a_2, \dots, a_n\rangle$ and $\langle a_2, \dots, a_n\rangle$.

As above we set $r = p^t$, where $t \geq 0$ is an integer. Let $P, Q \in \mathbb{F}_q[T]$ such that $\deg(Q) < \deg(P) < r$. Let $\ell \geq 1$ be an integer and $A_\ell = (a_1, a_2, \dots, a_\ell)$ a vector in $(\mathbb{F}_q[T])^\ell$ such that $\deg(a_i) > 0$ for $1 \leq i \leq \ell$. Then by Theorem 1 in [L3, p. 333], there exists an infinite continued fraction in $\mathbb{F}(q)$ defined by $\alpha = [a_1, a_2, \dots, a_\ell, \alpha_{\ell+1}]$ such that $\alpha^r = P\alpha_{\ell+1} + Q$. We consider the four continuants $x_\ell$, $y_\ell$, $x_{\ell-1}$ and $y_{\ell-1}$, in $\mathbb{F}_q[T]$, built from the vector $A_\ell$. Then, this element $\alpha$ is hyperquadratic and it is the unique root in $\mathbb{F}(q)^+$ of the following algebraic equation:

$$y_\ell X^{r+1} - x_\ell X^r + (y_{\ell-1}P - y_\ell Q)X + x_\ell Q - x_{\ell-1}P = 0. \tag{1}$$

A continued fraction defined as above will be called of type $(r, \ell, P, Q)$. Note that such a continued fraction depends on the choice of the vector $A_\ell = (a_1, a_2, \ldots, a_\ell)$ in $(\mathbb{F}_q[T])^\ell$.

Now, let us introduce an important sequence of polynomials in $\mathbb{F}_p[T]$, for all prime numbers $p$ (see [L3, p. 331]). This sequence $(F_n)_{n\geq 0}$ is defined by induction as follows:

$$F_0 = 1, \quad F_1 = T \quad \text{and} \quad F_{n+1} = TF_n + F_{n-1} \quad \text{for} \quad n \geq 1.$$

Note the similarity with the celebrated sequence of Fibonacci numbers. With our notation, for $n \geq 1$, we clearly have

$$F_n/F_{n-1} = [T, T, \ldots, T] \quad \text{and} \quad F_n = \langle T, T, \ldots, T\rangle,$$

where the finite sequence of $T$ has length $n$. Then we define in $\mathbb{F}(p)$ the following infinite continued fraction

$$\omega = [T, T, \ldots, T, \ldots] = \lim_{n\to\infty} F_n/F_{n-1}.$$

Note that $\omega$ is quadratic and satisfies $\omega = T + 1/\omega$. We have the following lemma.

**Lemma 1.** *Let $p$ be a prime. Let the sequence $(F_n)_{n\geq 0}$ in $\mathbb{F}_p[T]$ and $\omega \in \mathbb{F}(p)$ be defined as above. Let $r = p^t$ where $t \geq 1$ is an integer. Then we have*

$$F_{n-1} = (\omega^n - (-\omega^{-1})^n)/(\omega + \omega^{-1}) \quad \textit{for} \quad n \geq 1.$$

$$\textit{If} \quad p = 2 \quad \textit{then} \quad F_{r-1} = T^{r-1} \quad \textit{and} \quad \textit{if} \quad p > 2 \quad \textit{then} \quad F_{r-1} = (T^2 + 4)^{(r-1)/2}.$$

Proof: For $n \geq 0$, we set $\Omega_n = \omega^n - (-\omega^{-1})^n$. Then we have

$$T\Omega_n + \Omega_{n-1} = \omega^{n-1}(T\omega + 1) - (-\omega^{-1})^{n-1}(-T\omega^{-1} + 1). \tag{2}$$

Since $\omega = T + \omega^{-1}$, we get $\omega^2 = T\omega + 1$ and $\omega^{-2} = -T\omega^{-1} + 1$. Hence (2) becomes

$$T\Omega_n + \Omega_{n-1} = \omega^{n+1} - (-\omega^{-1})^{n+1} = \Omega_{n+1}. \tag{3}$$

Hence, by (3), the sequence $(\Omega_n)_{n\geq 0}$ satisfies the same recurrence as the sequence $(F_n)_{n\geq 0}$. We observe that $\Omega_1/(\omega + \omega^{-1}) = 1 = F_0$ and $\Omega_2/(\omega + \omega^{-1}) = \omega - \omega^{-1} = T = F_1$. Since the sequences $(F_n)_{n\geq 0}$ and $(\Omega_{n+1}/(\omega + \omega^{-1}))_{n\geq 0}$ satisfy the same linear recurrence relation and are equal on the first two values, they coincide and we have $F_{n-1} = \Omega_n/(\omega + \omega^{-1})$, for $n \geq 1$, as stated in the lemma. If $p = 2$, using the Frobenius isomorphism, we have $\Omega_r = (\omega + \omega^{-1})^r$ and also $\omega + \omega^{-1} = T$. Therefore, we can write $F_{r-1} = (\omega + \omega^{-1})^r/(\omega + \omega^{-1}) = (\omega + \omega^{-1})^{r-1} = T^{r-1}$. If $p > 2$, again using the Frobenius isomorphism, we also have $\Omega_r = (\omega + \omega^{-1})^r$. But here $\omega - \omega^{-1} = T$ and $(\omega - \omega^{-1})^2 + 4 = (\omega + \omega^{-1})^2$. Consequently, we get $F_{r-1} = (\omega + \omega^{-1})^{r-1} = ((\omega - \omega^{-1})^2 + 4)^{(r-1)/2} = (T^2 + 4)^{(r-1)/2}$. So the proof is complete.

It is easy to check that, for $n \geq 2$, we have $\omega^n = F_{n-1}\omega + F_{n-2}$. Since $\omega = \omega_{l+1}$ for $l \geq 1$, with our terminology, we observe that $\omega$ is a continued fraction of type $(r, \ell, F_{r-1}, F_{r-2})$, for all $r = p^t$ and $\ell \geq 1$. We are now interested in the family of continued fractions of type $(r, \ell, \epsilon_1 F_{r-1}, \epsilon_2 F_{r-2})$, for a pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$. Besides $\omega$, this family contains continued fractions with all partial quotients of degree one, hence the $\ell$ first partial quotients must be chosen as polynomial of degree 1 in $\mathbb{F}_q[T]$. Note that, in the first papers on hyperquadratic continued fractions with bounded partial quotients, the role played by the pair $(F_{r-1}, F_{r-2})$ was not put in evidence. There are two distinct cases according to the characteristic $p = 2$ or $p > 2$. The case of characteristic $p > 2$ was the first to be considered. In [MR], Mills and Robbins described, in the case $q = r = p > 3$ , with $\ell = 2$ and $(a_1, a_2) = (aT, bT)$, a family of such examples (see also the introduction of [L3, p. 332]). However in that case, in general, the solution of (1) will not have all partial quotients of degree 1. This will happen if a mysterious condition, between the coefficients of the polynomials in $A_\ell$ and the pair $(\epsilon_1, \epsilon_2)$, is satisfied. It has been long and complicated to obtain this condition (see [LY1] and particularly the first section for the background information on the matter). Here, we shall now only consider the case of characteristic 2. Due to the simpler form of $F_{r-1}$, given in the previous lemma, we will avoid the sophistication appearing in odd characteristic.

The following general lemma, on the continued fraction algorithm, is the basic tool of our method. It can be found in several articles (see [L3, Lemma 3.1, p. 336] or [LY1, p. 267]).

**Lemma 2.** *For $n \geq 2$, given $n+1$ variables $x_1, x_2, \dots, x_n$ and $x$, we have formally*

$$[[x_1, x_2, \dots, x_n], x] = [x_1, x_2, \dots, x_n, y],$$

*where*

$$y = (-1)^{n-1}\langle x_2, \dots, x_n\rangle^{-2}x - \langle x_2, \dots, x_{n-1}\rangle\langle x_2, \dots, x_n\rangle^{-1}.$$

As a consequence of Lemma 1 and Lemma 2, we have now the following lemma.

**Lemma 3.** *Let $q = 2^s$ and $r = 2^t$ where $s, t \geq 1$ are integers. Let $\alpha = [a_1, a_2, \dots, a_n, \dots]$ be an infinite continued fraction in $\mathbb{F}(q)$. We assume that there is a pair $(\epsilon_1, \epsilon_2) \in (\mathbb{F}_q^*)^2$ and a pair of integers $(i, j)$, where $1 \leq i < j$, such that we have $\alpha_i^r = \epsilon_1 F_{r-1}\alpha_j + \epsilon_2 F_{r-2}$. If $a_i = \lambda_i T$ where $\lambda_i \in \mathbb{F}_q^*$, then we have*

$$a_j = \epsilon_1^{-1}\lambda_i^r T \quad \text{and} \quad a_{j+k} = (\epsilon_2/\epsilon_1)^{(-1)^k} T \quad \text{for} \quad 1 \leq k \leq r-1$$

*and also*

$$\alpha_{i+1}^r = \epsilon_1\epsilon_2^{-2}F_{r-1}\alpha_{j+r} + \epsilon_2^{-1}F_{r-2}.$$

Proof: From the equality $\alpha_i^r = \epsilon_1 F_{r-1}\alpha_j + \epsilon_2 F_{r-2}$, and since $\alpha_i^r = [a_i, \alpha_{i+1}]^r = [a_i^r, \alpha_{i+1}^r]$, we obtain

$$\left[\frac{a_i^r + \epsilon_2 F_{r-2}}{\epsilon_1 F_{r-1}}, \epsilon_1 F_{r-1}\alpha_{i+1}^r\right] = \alpha_j. \tag{4}$$

We denote by $W_i$ the word $T, T, ...T$, where $T$ is repeated $i$ times. We have $(\epsilon_1/\epsilon_2)F_{r-1}/F_{r-2} = (\epsilon_1/\epsilon_2)[W_{r-1}]$. By Lemma 1, we have $F_{r-1} = T^{r-1}$, therefore (4) becomes

$$\left[[\epsilon_1^{-1}\lambda_i^r T, (\epsilon_1/\epsilon_2)[W_{r-1}]], \epsilon_1 F_{r-1}\alpha_{i+1}^r\right] = [[\epsilon_1^{-1}\lambda_i^r T, x_2, x_3, \dots, x_r], \epsilon_1 F_{r-1}\alpha_{i+1}^r] = \alpha_j, \tag{5}$$

where $x_i = (\epsilon_1/\epsilon_2)^{(-1)^i}T$ for $2 \leq i \leq r$. We shall now apply Lemma 2. Using classical properties of continuants (see [LY1, p. 266]), we can write

$$\langle x_2, \dots, x_r\rangle = \langle(\epsilon_1/\epsilon_2)T, (\epsilon_2/\epsilon_1)T, \dots, (\epsilon_1/\epsilon_2)T\rangle = (\epsilon_1/\epsilon_2)\langle T, T, \dots, T\rangle = (\epsilon_1/\epsilon_2)F_{r-1}$$

and also

$$\langle x_2,\dots,x_{r-1}\rangle = \langle (\epsilon_1/\epsilon_2)T,(\epsilon_2/\epsilon_1)T,\dots,(\epsilon_2/\epsilon_1)T\rangle = \langle T,T,\dots,T\rangle = F_{r-2}.$$

We set $x_1=\epsilon_1^{-1}\lambda_i^rT$ and $x=\epsilon_1F_{r-1}\alpha_{i+1}^r$. Thus, according to Lemma 2, we get

$$[[x_1,x_2,x_3,\dots,x_r],x] = [x_1,x_2,\dots,x_r,((\epsilon_1/\epsilon_2)F_{r-1})^{-2}x+((\epsilon_1/\epsilon_2)F_{r-1})^{-1}F_{r-2}]. \tag{6}$$

Combining (5) and (6), since $x=\epsilon_1F_{r-1}\alpha_{i+1}^r$, we obtain

$$\alpha_j = [x_1,x_2,\dots,x_r,(\epsilon_2^2/\epsilon_1)F_{r-1}^{-1}\alpha_{i+1}^r+(\epsilon_2/\epsilon_1)F_{r-1}^{-1}F_{r-2}] = [x_1,x_2,\dots,x_r,y]. \tag{7}$$

Since $|\alpha_{i+1}|>1$, we observe that $|y|>1$. Comparing (7) to $\alpha_j=[a_j,a_{j+1},\dots,a_{j+r-1},\alpha_{j+r}]$, we get $a_j=x_1$, $a_{j+1}=x_2,\dots a_{j+r-1}=x_r$ and $\alpha_{j+r}=y$. With the values given to $x_i$, for $1\le i\le r-1$, we obtain the values for the partial quotients from $a_j$ to $a_{j+r-1}$. Finally, with the value of $y$ in (7), we obtain the last formula of the lemma. So the proof is complete.

We can now state and prove the following theorem.

**Theorem 1.** *Let $q=2^s$ and $r=2^t$ where $s,t\ge 1$ are integers. Let $\ell\ge 1$ be an integer and let $(F_{r-1},F_{r-2})$ in $(\mathbb{F}_2[T])^2$ be the pair defined in Lemma 1. Let $\Lambda_{\ell+2}=(\lambda_1,\lambda_2,\dots,\lambda_\ell,\epsilon_1,\epsilon_2)$ be given in $(\mathbb{F}_q^*)^{\ell+2}$. Let $x_\ell$, $y_\ell$, $x_{\ell-1}$ and $y_{\ell-1}$ be the four continuants, in $\mathbb{F}_q[T]$, built from the vector $A_\ell=(\lambda_1T,\lambda_2T,\dots,\lambda_\ell T)$. We consider the following algebraic equation:*

$$y_\ell X^{r+1}+x_\ell X^r+(\epsilon_1y_{\ell-1}F_{r-1}+\epsilon_2y_\ell F_{r-2})X+\epsilon_2x_\ell F_{r-2}+\epsilon_1x_{\ell-1}F_{r-1}=0. \tag{E}$$

*Then $(E)$ has a unique root $\alpha$ in $\mathbb{F}(q)^+$ and we have $\alpha=[\lambda_1T,\lambda_2T,\dots,\lambda_\ell T,\dots,\lambda_nT,\dots]$. The sequence $(\lambda_n)_{n\ge1}$ in $\mathbb{F}_q^*$ is defined recursively, from the vector $\Lambda_{\ell+2}$, as follows:*

$$\lambda_{\ell+rm+1}=(\epsilon_2/\epsilon_1)\epsilon_2^{(-1)^{m+1}}\lambda_{m+1}^r \quad \textit{and} \quad \lambda_{\ell+rm+i}=(\epsilon_1/\epsilon_2)^{(-1)^i} \quad \textit{for} \quad m\ge 0 \quad \textit{and} \quad 2\le i\le r.$$

Proof: We observe that $(E)$ is simply equation (1), stated above, built from the vector $A_\ell=(\lambda_1T,\lambda_2T,\dots,\lambda_\ell T)$ and the pair $(P,Q)=(\epsilon_1F_{r-1},\epsilon_2F_{r-2})$. According to the theorem from [L3], mentioned before, we know that $(E)$ has a unique root $\alpha$ in $\mathbb{F}(q)^+$. This element is the infinite continued fraction defined by $\alpha=[\lambda_1T,\lambda_2T,\dots,\lambda_\ell T,\alpha_{\ell+1}]$ and $\alpha^r=\epsilon_1F_{r-1}\alpha_{\ell+1}+\epsilon_2F_{r-2}$. Since $a_1=\lambda_1T$, we can apply Lemma 3, with $i=1$ and $j=\ell+1$. Hence, we get

$$a_{\ell+1}=\epsilon_1^{-1}\lambda_1^rT \quad \text{and} \quad a_{\ell+i}=(\epsilon_1/\epsilon_2)^{(-1)^i}T \quad \text{for} \quad 2\le i\le r. \tag{8}$$

We also have

$$\alpha_2^r=\epsilon_1\epsilon_2^{-2}F_{r-1}\alpha_{\ell+r+1}+\epsilon_2^{-1}F_{r-2}.$$

Let us consider the function $f$ from $(\mathbb{F}_q^*)^2$ into $(\mathbb{F}_q^*)^2$, defined by $f(x,y)=(xy^{-2},y^{-1})$. We observe that $f$ is an involution. Hence, we can apply Lemma 3 again (note that $a_2=\lambda_2T$ even if $\ell=1$), replacing $(i,j)$ by $(2,l+r+1)$ and the pair $(\epsilon_1,\epsilon_2)$ by the pair $f(\epsilon_1,\epsilon_2)$. We obtain

$$a_{\ell+r+1}=\epsilon_2^2\epsilon_1^{-1}\lambda_2^rT \quad \text{and} \quad a_{\ell+r+i}=(\epsilon_1/\epsilon_2)^{(-1)^i}T \quad \text{for} \quad 2\le i\le r \tag{9}$$

and

$$\alpha_3^r=\epsilon_1F_{r-1}\alpha_{\ell+r+1}+\epsilon_2F_{r-2}.$$

Hence, by repeated application of this lemma, we see that $a_i = \lambda_i T$ for $i \geq 1$. Note that $(\epsilon_2/\epsilon_1)\epsilon_2^{(-1)^{m+1}}$ is either $\epsilon_1^{-1}$ or $\epsilon_2^2\epsilon_1^{-1}$, according to the parity of the integer $m \geq 0$. From (8) and (9), we get the formulas defining the sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_q^*$, stated in the theorem. So the proof is complete.

**Remark.** In a recent joint paper with J-Y. Yao [LY2], we have observed that the sequence of the leading coefficients of the partial quotients for an hyperquadratic continued fraction could be an automatic sequence. Naturally, the question arises for the sequence described in this theorem. In the particular case $r = 2$, with the terminology introduced above, we note that the element $\alpha$ is of type $(2, \ell, \epsilon_1 T, \epsilon_2)$. Indeed, we have proved in [LY2, Theorem 3] that the corresponding sequence is 2-automatic. Let us indicate another criterion for a sequence in $\mathbb{F}_q$ to be automatic (see [AS, p. 356]): $(\lambda_n)_{n\geq 1}$ in $\mathbb{F}_q$ is p-automatic if the power series $\sum_{i\geq 1}\lambda_i T^{-i} \in \mathbb{F}(q)$ is algebraic over $\mathbb{F}_q(T)$. Concerning the sequence $(\lambda_n)_{n\geq 1}$ described in our theorem, we conjecture the following : Set $\theta = \sum_{i\geq 1}\lambda_i T^{-i} \in \mathbb{F}(q)$, then there exist two elements $A$ and $B$ in $\mathbb{F}_q(T)$, depending on $r$ and on the vector $\Lambda_{\ell+2}$, such that we have $\theta^r + A\theta + B = 0$. Accordingly, by the criterion cited, the conjecture would imply the 2-automaticity of the sequence $(\lambda_n)_{n\geq 1}$ for all $r = 2^t$ and $t \geq 1$. This conjecture is based on a partial study of the sequence $(\lambda_n)_{n\geq 1}$. Indeed, in the simplest case $r = 2$, the conjecture is true and this study was complete enough to allow us to obtain the following:

$$\theta = \sum_{i=1}^{\ell}\lambda_i T^{-i} + (\epsilon_1/\epsilon_2)T^{-\ell}(T+1)^{-2}(\epsilon_2^{(-1)^\ell}T + 1) + (\epsilon_1/\epsilon_2)\epsilon_2^{(-1)^\ell}T^{\ell-1}\rho$$

and

$$\rho^2 + \rho + T^{-2\ell}\sum_{i=1}^{\ell}(1 + \lambda_i^2(\epsilon_2/\epsilon_1)^2\epsilon_2^{(-1)^i-(-1)^\ell})T^{-2i+2} = 0.$$

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