A SURVEY OF DIOPHANTINE APPROXIMATION IN FIELDS OF POWER SERIES

A. Lasjaunias

The fields of power series (or perhaps better called formal numbers) are analogues of the field of real numbers. Many questions in number theory which have been studied in the setting of the real numbers can be transposed in the setting of the power series. The study of rational approximation to algebraic real numbers has been intensively developped starting from the middle of the nineteenth century with the work of Liouville up to the celebrated theorem of Roth established in 1955. In the last thirty years, several mathematicians have studied diophantine approximation in fields of power series. We present here a summary of the present knowledge on this subject, emphasizing the analogies and differences with the situation in the real numbers case.

1. The fields of power series .

Let K be a field. If T is an indeterminate, we consider the ring $K[T]$ of polynomials in T with coefficients in K, and the field $K(T)$ of rational functions in T with coefficients in K. An ultrametric absolute value in $K(T)$ is defined by $|0| = 0$ and $|P/Q| = |T|^{\deg P - \deg Q}$, where $|T|$ is a fixed real number greater than 1. We consider the completion field of $K(T)$ for this absolute value, which is denoted $K((T^{-1}))$ and called the field of power series over K. Then if $\alpha \in K((T^{-1}))$, and $\alpha \neq 0$, we can write

$$\alpha = \sum_{k \leq k_0} a_k T^k \quad \text{where } k_0 \in \mathbb{Z} \quad , \quad a_k \in K \quad \text{and} \quad a_{k_0} \neq 0.$$

The degree of $\alpha \neq 0$ is defined by extension as $\deg \alpha = k_0$. So the absolute value is extended in $K((T^{-1}))$, and we have $|\alpha| = |T|^{\deg \alpha}$ if $\alpha \neq 0$.

This construction is clearly similar to the construction of the real numbers. The analogues of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are respectively $K[T]$, $K(T)$ and $K((T^{-1}))$.
Here we study the approximation of the elements of $K((T^{-1}))$ by the elements of $K(T)$. Particularly we consider this approximation for the elements of $K((T^{-1}))$ which are algebraic over $K(T)$.

This analogy between the field of real numbers and the field of formal power series can be considered from another point of view. Indeed the sequence of the coefficients of a power series can be compared to the sequence of the digits in the decimal expansion of a real number. We illustrate here this parallelism with an analogue of a classical result for the real numbers. Let $\alpha \in K((T^{-1}))$. It is easy to prove that the sequence of the coefficients of the power series representing α is ultimately periodic if and only if there exist integers $n \geq 0$ and $m \geq 1$ such that $T^n(T^m - 1)\,\alpha \in$

$K[T]$. Now if K is a finite field, it is a classical result that every $Q \in K[T]$, $Q \neq 0$, divides $T^n(T^m - 1)$ for some integers $n, m \in \mathbb{N}$. Consequently we can state the following theorem.

Theorem 1.1. *Let $\alpha \in \mathbb{F}_q((T^{-1}))$ with $\alpha = \sum_{k \leq k_0} a_k T^k$. Then $\alpha \in \mathbb{F}_q(T)$ if and only if the sequence $(a_{-k})_{k \geq 0}$ is ultimately periodic.*

It is interesting to notice the correspondence with the classical result on the rational numbers and their decimal expansion. Observe that the proof is obtained in the same way as above, replacing T by 10, and using the fact that every positive integer divides $10^n(10^m - 1)$ for some integers $n \geq 0$ and $m \geq 1$.

As we are concerned with algebraic power series, it is important to mention here another result in the same direction as the previous one. Again when the base field K is finite, Christol [C] was able to show that an element in $K((T^{-1}))$ which is algebraic over $K(T)$ can be characterised by a property of the sequence of the coefficients in its power series expansion. We state Christol's theorem.

Theorem 1.2. *Let $\alpha \in \mathbb{F}_q((T^{-1}))$ with $\alpha = \sum_{k \leq k_0} a_k T^k$. We put $u_n = a_{-n}$ for $n \geq 0$. Then α is algebraic over $\mathbb{F}_q(T)$ if and only if the set of subsequences*

$$\{(u_{q^i n + r})_{n \geq 0} \quad \textit{for} \quad i \geq 0 \quad \textit{and} \quad 0 \leq r \leq q^i - 1\}$$

is finite.

We have just seen the importance of the case where K is a finite field, which somehow simplifies the situation. In the following we will observe this importance in other contexts, nevertheless our base field K is arbitrary when no restriction is mentioned.

2. Analogues of Liouville and Roth theorems.

The starting point in diophantine approximation for algebraic real numbers is a famous theorem obtained by Liouville in 1850. We have the following theorem, due to K. Mahler [M], which is an adaptation of Liouville's theorem in the case of power series.

Theorem 2.1. *Let K be a field. Let α be an element of $K((T^{-1}))$ algebraic over $K(T)$ of degree $n > 1$ and $P, Q \in K[T]$ with $Q \neq 0$. Then there exists a positive real constant A, depending only upon α, such that*

$$|\alpha - P/Q| \geq A|Q|^{-n}.$$

In 1955, Roth published his famous theorem about the rational approximation of algebraic real numbers. We also have the following theorem, due to Uchiyama [U], which is an adaptation of Roth's theorem in our context.

Theorem 2.2. *Let K be a field of characteristic 0. If α is an algebraic irrational element of $K((T^{-1}))$ then, for every $\epsilon > 0$, we have*

$$|\alpha - P/Q| \geq |Q|^{-(2+\epsilon)}$$

for all $P, Q \in K[T]$ with $|Q|$ sufficiently large.

It is well known, as we will see, that this theorem does not hold if the characteristic of K is positive. Thus rational approximation of algebraic power series is much more complex when the base field has positive characteristic. But although the equivalent of Roth's theorem holds in the power series case when the base field is of characteristic 0, we will see also that the rational approximation of algebraic elements in $K((T^{-1}))$ is different from the situation we know in the classical case of the real numbers.

3. The continued fraction algorithm.

As in the classical context of the real numbers, we have a continued fraction algorithm in $K((T^{-1}))$. The continued fraction theory in $K((T^{-1}))$ is in some ways simpler than in the real number case because of the ultra-metric absolute value in the first field. For a general reference on this subject, see [S2]. If $\alpha \in K((T^{-1}))$ we can write

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + ... = [a_0, a_1, a_2, ...] \quad \text{where} \quad a_i \in K[T].$$

The a_i are called the partial quotients and we have $\deg a_i > 0$ for $i > 0$. This continued fraction is finite if and only if $\alpha \in K(T)$. As in the classical theory we define recursively the two sequences of polynomials $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ by

$$P_n = a_n P_{n-1} + P_{n-2} \quad \text{and} \quad Q_n = a_n Q_{n-1} + Q_{n-2},$$

with the initial conditions $P_0 = a_0$, $P_1 = a_0 a_1 + 1$, $Q_0 = 1$ and $Q_1 = a_1$. We have $P_{n+1}Q_n - Q_{n+1}P_n = (-1)^n$, whence P_n and Q_n are coprime. The rational P_n/Q_n is called a convergent to α and we have $P_n/Q_n = [a_0, a_1, a_2, ..., a_n]$. Because of the ultrametric absolute value we have

$$|\alpha - P_n/Q_n| = |P_{n+1}/Q_{n+1} - P_n/Q_n| = |Q_n Q_{n+1}|^{-1} = |a_{n+1}|^{-1}|Q_n|^{-2}.$$

It is interesting to notice that there is a simple characterisation of a convergent: if $P, Q \in K[T]$ and $Q \neq 0$ then P/Q is a convergent to α if and only if $|\alpha - P/Q| < |Q|^{-2}$.

At last we mention an important result, whose analogue in the classical case is well known.

Theorem 3.1. *Let K be a finite field and $\alpha \in K((T^{-1}))$ irrational. Then the sequence of partial quotients in the continued fraction expansion of α is ultimately periodic if and only if α is quadratic over $K(T)$.*

4. The approximation exponent.

In order to mesure the quality of the rational approximation, we introduce now the following notation and definitions.
Let α be an irrational element of $K((T^{-1}))$. For all real numbers μ, we define

$$B(\alpha,\mu)=\liminf_{|Q|\to\infty}|Q|^{\mu}|\alpha-P/Q|$$

where P and Q run over polynomials in $K[T]$ with $Q\neq 0$. Now the *approximation exponent* of α is defined by

$$\nu(\alpha)=\sup\{\mu\in\mathbb{R}\ :\ B(\alpha,\mu)<\infty\}.$$

It is clear that we have

$$B(\alpha,\mu)=\infty\ \text{ if } \mu>\nu(\alpha)\ ,\ B(\alpha,\mu)=0\ \text{ if } \mu<\nu(\alpha)\ \text{ and }\ 0\leq B(\alpha,\nu(\alpha))\leq\infty.$$

The approximation exponent can also be defined directly by

$$\nu(\alpha)=\limsup_{|Q|\to\infty}(-\frac{\log|\alpha-P/Q|}{\log|Q|})$$

where P and Q run over polynomials in $K[T]$ with $Q\neq 0$. Observe that the same definition could hold for the approximation exponent of a real number, replacing P and $Q\neq 0$ by rational integers and the absolute value being the usual one.

We recall that if P_n/Q_n is a convergent to α, we have

$$|\alpha-P_n/Q_n|=|Q_n|^{-(1+\deg Q_{n+1}/\deg Q_n)}.$$

Since the best rational approximations to α are its convergents, in the above notation we have

$$\nu(\alpha)=1+\limsup_k(\deg Q_{k+1}/\deg Q_k)=2+\limsup_k(\deg a_{k+1}/\sum_{1\leq i\leq k}\deg a_i).$$

The approximation exponent for a real number x would be

$$\nu(x)=1+\limsup_n(\ln q_{n+1}/\ln q_n),$$

where $(p_n/q_n)_{n\geq 0}$ is the sequence of the convergents to x.

This notation is a slight modification of that introduced by B. de Mathan [dM1]. According to W. Schmidt [S2], it is also possible to define the *approximation spectrum* of α. This is the set of the accumulation points of the sequence $(1+(\deg Q_{k+1}/\deg Q_k))_{k\geq 0}$. This set is denoted $S(\alpha)$. Then $\nu(\alpha)$ is the upper bound of $S(\alpha)$.

Since $|\alpha-P_n/Q_n|\leq|T|^{-1}|Q_n|^{-2}$, for all irrational $\alpha\in K((T^{-1}))$ we have $B(\alpha,2)\leq|T|^{-1}$. Thus $\nu(\alpha)\geq 2$. Moreover, using continued fractions, it is clear that for every $\lambda \in [2, +\infty]$ there exists an irrational $\alpha \in K((T^{-1}))$ such that $\nu(\alpha) = \lambda$. The same is true in the real number case.

Mahler's version of Liouville's theorem says that if $\alpha \in K((T^{-1}))$ is algebraic over $K(T)$ of degree $n > 1$ then $B(\alpha, n) > 0$. Consequently, for all $\alpha \in K((T^{-1}))$ algebraic over $K(T)$ of degree $n > 1$ we have $\nu(\alpha) \in [2, n]$. Moreover, because of Uchiyama's version of Roth's theorem, if K has characteristic 0 then $\nu(\alpha) = 2$.

We will now use the following vocabulary : If $\alpha \in K((T^{-1}))$, we say that

- α is badly approximable if $\nu(\alpha) = 2$ and $B(\alpha, 2) > 0$. This is equivalent to saying that the partial quotients in the continued fraction for α are bounded.
- α is normally approximable if $\nu(\alpha) = 2$ and $B(\alpha, 2) = 0$.
- α is well approximable if $\nu(\alpha) > 2$.

Clearly, by Mahler's theorem, all quadratic power series are badly approximable. This fact is also a consequence of their particular continued fraction expansion. We will see that there are algebraic power series in each of these three classes. By Liouville's theorem all quadratic real numbers are badly approximable and by Roth's theorem no algebraic real number can be well approximable. Nevertheless, it is necessary to underline that there is no example of algebraic real number of degree $n > 2$ that is known to be badly or normally approximable. This is certainly a very important open question in number theory.

Now we want to give two classical examples of algebraic elements when the base field has positive characteristic. Let p be a prime number and let q be a positive power of p. Both examples illustrate the disturbance brought by the Frobenius homomorphism.
Example 1 : First we define $\alpha \in \mathbb{F}_p((T^{-1}))$ by $\alpha = [0, T, T^q, .., T^{q^k}, ...]$. Then, because of the Frobenius homomorphism, we have $\alpha = 1/(T + \alpha^q)$. From the continued fraction expansion we deduce $\nu(\alpha) = q+1$ and $S(\alpha) = \{q + 1\}$. If α is algebraic over $K(T)$ of degree d, we know that $\nu(\alpha) \leq d$ and therefore $d = q + 1$.
Example 2 : Now we define $\alpha \in \mathbb{F}_p((T^{-1}))$ by $\alpha = \sum_{k \geq 0} T^{-q^k}$. Again, because of the Frobenius homomorphism, we have $\alpha = 1/T + \alpha^q$. If we write $U_n/V_n = \sum_{0 \leq k \leq n} T^{-q^k}$, we have $|\alpha - U_n/V_n| = |V_n|^{-q}$. Hence we see that $\nu(\alpha) \geq q$. Consequently α is algebraic over $K(T)$ of degree q and $\nu(\alpha) = q$.

Unlike the real numbers case, where no explicit continued fraction is known for an algebraic number of degree ≥ 3, we can describe the continued fraction for this element (see [L1], p. 224).

Theorem 4.1. *Let p be a prime number and $q > 2$ a power of p. Let $\alpha \in \mathbb{F}_p((T^{-1}))$ be defined by $\alpha = 1/T + \alpha^q$ and $|\alpha| < 1$. Let us define the* *sequence* $(\Omega_n)_{n>0}$ *of finite sequences of elements of* $K[T]$ *recursively by*

$$\Omega_1 = T \quad \text{and} \quad \Omega_n = \Omega_{n-1}, -T^{(q-2)q^{n-2}}, -\Omega'_{n-1} \quad \text{for } n \geq 2.$$

In this formula, if $\Omega = a_1, a_2, ..., a_m$*, then* $\Omega' = a_m, a_{m-1}, ..., a_1$ *and* $-\Omega = -a_1, -a_2, ..., -a_m$ *and commas denote juxtaposition of sequences. Denote by* Ω_∞ *the infinite sequence begining by* Ω_n *for all* $n \geq 1$ *. Then the continued fraction expansion for* α *is* $[0; \Omega_\infty]$.

For $q = 2$, α is a quadratic element and we can see that $\alpha = [0, T+1, (T)]$ (Here the brackets indicate the periodic part of the expansion). From this continued fraction expansion we can determine the approximation spectrum.

Corollary 4.1. *Let* $k \geq 1$ *be an integer and* $k = \sum_{0 \leq i \leq m} k_i 2^i$ *its representation in base 2, then we define* $\omega(k, q) = \sum_{0 \leq i \leq m} k_i q^i$*. The approximation spectrum of the element* α *defined in theorem 4.1 is*

$$S(\alpha) = \{(u_k)_{k \geq 1}\} \quad \text{with} \quad u_k = 2 + \frac{q-2}{(q-1)\omega(k,q) + 2 - q}.$$

Observe that the sequence $(u_k)_{k \geq 1}$ is decreasing and that we have $u_1 = q$ and $\lim_{k \to \infty} u_k = 2$.

This second example was first introduced by K. Mahler [M] to show the specificity of diophantine approximation in positive characteristic.

5. The subset of algebraic elements of class I.

Here we suppose that the base field K has a positive characteristic p. As we have seen in the two preceeding examples the rational approximation of algebraic elements is disturbed by the Frobenius homomorphism. Therefore it is important to consider a special subset of algebraic elements in $K((T^{-1}))$.

For an integer $s \geq 1$, we will denote $\mathcal{H}_s$ the set of irrational algebraic elements in $K((T^{-1}))$ which satisfy an algebraic equation of the following type:

$$\text{(I)} \qquad x = (Ax^{p^s} + B)/(Cx^{p^s} + D)$$

where A, B, C, D are in $K[T]$ with $AD - BC \neq 0$. We put $q = p^s$ and set $\mathcal{H} = \bigcup_{s \geq 1} \mathcal{H}_s$. We say that an element in $\mathcal{H}$ is of class I.

It is clear that if $\alpha \in K((T^{-1}))$ is algebraic of degree less than 4 then it is an element of class I, since $1, \alpha, \alpha^q$ and α^{q+1} are linearly dependent over $K(T)$. Besides, it is easy to show that there are algebraic elements in $K((T^{-1}))$ which are not of class I. The two examples given in section **4** are of class I.

The rational approximation for elements of class I has been studied by Voloch [V1] and more deeply by de Mathan [dM1]. They could show that if the partial quotients in the continued fraction expansion of such an element α are unbounded, then $\nu(\alpha) > 2$. In other words there are no normally approximable elements of class I. By the work of B. de Mathan [dM1], we know moreover that for elements of class I, the approximation exponent $\nu(\alpha)$ is a rational number and $B(\alpha, \nu(\alpha)) \neq 0, \infty$.

Many elements of class I are well approximable, as for instance the two previous examples. Indeed it is possible to show with the above notation that if $q > 1 + \deg(AD - BC)$, then $\nu(\alpha) > 2$ (see [L2], p. 53). In particular, if $AD - BC \in K^*$, then this condition is fullfiled for all q. The algebraic irrational elements satisfying equation (I) with the condition $AD - BC \in K^*$ are called of class IA. Those special elements have been considered and studied by W. Schmidt [S2] and D. Thakur [T]. The element given as *example 1* in section **4** belongs to this class. For those elements the continued fraction expansion can be explicitely described, and thus one can determine their approximation spectrum, which is a finite set [S2].

6. Badly approximable elements of class I.

In 1976, Baum and Sweet [BS1] were the first to study the rational approximation of particular algebraic elements of class I for $K = \mathbb{F}_2$. One famous example is the unique solution in $\mathbb{F}_2((T^{-1}))$ of the equation

$$TX^3 + X + T = 0.$$

This cubic element α has the special property of being badly approximable. More precisely we have

$$|\alpha - P/Q| \geq |T|^{-2}|Q|^{-2}$$

for all $P, Q \in \mathbb{F}_2[T]$ with $Q \neq 0$. It is equivalent to say that the partial quotients of the continued fraction expansion of this element are of degree bounded by 2. Generalizing their methods, we have obtained the following result [L2].

Theorem 6.1. *Let l be a positive integer. Let $D \in \mathbb{F}_2[T]$ be such that $D(0) = 1$. We consider the algebraic equation*

$$(E) \qquad Tx^3 + Dx + T^l = 0.$$

Let α be an irrational solution of (E) in $\mathbb{F}_2((T^{-1}))$. Then

i) if $|\alpha| \geq |T|^{-(l+1)}$, the sequence of the partial quotients of the continued fraction expansion for α is bounded by $|T|^{l+1}$.

ii) if $|\alpha| < |T|^{-(l+1)}$, the sequence of the partial quotients of the continued fraction expansion for α is unbounded.

The existence of an irrational solution of (E) depends on the choice of D and l. We can indicate some cases where this solution exists and is unique in $\mathbb{F}_2((T^{-1}))$. So for $l = 1$ and $D = 1$, the solution of (E) is the cubic example given by L. Baum and M. Sweet. In this case we have $|\alpha| = 1$, and the partial quotients of its continued fraction expansion are bounded by $|T|^2$. Also if $m = \deg D$ and if $1 \leq l \leq m$ with $(l, m) \neq (1, 1)$ then equation (E) has a unique solution α with $|\alpha| = |T|^{l-m}$. In this last situation, if this solution is irrational, the theorem implies that the partial quotients of its continued fraction expansion are bounded by $|T|^{l+1}$ if and only if $\lfloor m/2 \rfloor \leq l \leq m$.

In 1986 Mills and Robbins [MR] studied the continued fraction expansion for the cubic example given by Baum and Sweet. They were the first to consider the subset of algebraic elements of class I. Then they were able to describe an algorithm to compute the partial quotients of the continued fraction expansion for an element of class I. As a consequence of this work, they gave for each prime $p \geq 3$ an example of a non-quadratic algebraic power series with coefficients in $\mathbb{F}_p$ whose partial quotients are all of degree one. We have studied the case where the base field is $\mathbb{F}_3$. We looked for non-quadratic algebraic power series whose partial quotients are all $\pm T$ or $\pm T \pm 1$. We observed that a continued fraction with such partial quotients can satisfy a quartic equation if these polynomials are arranged in a precise pattern. We illustrate this with the following result [L3].

Theorem 6.2. *Let k be a non-negative integer. We define the sequence of integers $(u_n)_{n\geq 0}$ by*

$$u_0 = k \quad \text{and} \quad u_{n+1} = 3u_n + 4 \quad \text{for } n \geq 0.$$

If $a \in \mathbb{F}_3[T]$ and $n \geq 0$ is an integer, $a^{[n]}$ denotes the sequence $a, a, ..., a$ where a is repeated n times and $a^{[0]}$ denotes the empty sequence. Then for $n \geq 0$, we define a finite sequence $H_n(T)$ of elements of $\mathbb{F}_3[T]$ by

$$H_n(T) = T + 1, T^{[u_n]}, T + 1.$$

Let $H_\infty(k)$ be the infinite sequence defined by juxtaposition

$$H_\infty(k) = H_0(T), H_1(-T), H_2(T), ..., H_n((-1)^n T), ...$$

Let $\omega(k)$ be the element of $\mathbb{F}_3((T^{-1}))$ defined by the continued fraction expansion

$$\omega(k) = [0, H_\infty(k)].$$

Let $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ be the usual sequences for the numerators and the denominators of the convergents of $\omega(k)$.
Then $\omega(k)$ is the unique solution in $\mathbb{F}_3((T^{-1}))$ of the irreducible quartic equation

$$q_k x^4 - p_k x^3 + q_{k+3} x - p_{k+3} = 0.$$

For example, if $k = 0$ then

$$\omega(0) = [0, T+1, T+1, -T+1, -T^{[4]}, -T+1, T+1, T^{[16]}, T+1, -T+1, ...]$$

and this element satisfies the equation $x = (T^2+1)/(T^3+T^2-T-x^3)$. To prove this theorem, we first show, with the above notations, that for $n \geq 0$ we have

$$p_{3n+k+3}/q_{3n+k+3} = (p_k p_n^3 + q_n^3 p_{k+3})/(q_k p_n^3 + q_n^3 q_{k+3}).$$

Since $\lim_{m\to\infty} p_m/q_m = \omega(k)$, this equality implies that $\omega(k)$ satisfies the desired equation.

At last, concerning badly approximable elements of class I, we mention here recent work by D. Robbins [R], in which he studied systematically the roots of a cubic equation with polynomial coefficients in $\mathbb{F}_2[T]$. Also when the base field is a finite extension of $\mathbb{F}_2$, D. Thakur [T] has given examples of algebraic elements of class I with bounded partial quotients. The question of determining which elements of class I are badly approximable remains open.

7. Determination of the approximation exponent.

It is clear that the approximation exponent and the approximation spectrum can be determined when the continued fraction of the element is explicitly known, as we have seen for the two previous examples in section **4**. It is important to notice that in this manner W. Schmidt [S2] and D. Thakur [T] have independently obtained the following result.

Theorem 7.1. *Let K be a finite field. For every rational number $\mu > 2$ there exists an algebraic element $\alpha \in K((T^{-1}))$ of class* IA *such that $\nu(\alpha) = \mu$.*

Now we will show how it is possible in some cases to compute the approximation exponent for an algebraic element, without knowing the whole continued fraction. This will be possible if this approximation exponent is large enough, that is to say not close to 2 . We will give applications to algebraic elements which are of class I and also to others which are not.

The basic idea in the following result is due to Voloch [V1]. It has been improved by de Mathan [dM2].

Theorem 7.2. *Let K be a field. Let $\alpha \in K((T^{-1}))$. Assume that there is a sequence $(P_n, Q_n)_{n\geq 0}$, with $P_n, Q_n \in K[T]$, satisfying the following conditions :*

(1) *There are two real constants $\lambda > 0$ and $\mu > 1$, such that*

$$|Q_n| = \lambda |Q_{n-1}|^\mu \quad \text{and} \quad |Q_n| > |Q_{n-1}| \quad \text{for all } n \geq 1.$$

(2) *There are two real constants $\rho > 0$ and $\gamma > 1 + \sqrt{\mu}$, such that*

$$|\alpha - P_n/Q_n| = \rho |Q_n|^{-\gamma} \quad \text{for all } n \geq 0.$$

Then we have $\nu(\alpha) = \gamma$. *Further, if* $\gcd(P_n, Q_n) = 1$ *for* $n \geq 0$, *we have* $B(\alpha, \nu(\alpha)) = \rho$, *and if the sequence* $(\gcd(P_n, Q_n))_{n\geq 0}$ *is bounded then* $B(\alpha, \nu(\alpha)) \neq 0, \infty$.

Using this proposition it is possible to compute the approximation exponent for many elements of class I. We give as an example the following result.

Theorem 7.3. *Let* K *be a field of characteristic* p. *Let* $n > 2$ *be an integer prime with* p. *Let* $P, Q \in K[T]$ *coprime, of same degree and unitary. Assume that* $P/Q \notin K(T)^n$. *Let* q *be the smallest power of* p *such that* n *divides* $q-1$. *Set* $\lambda = \deg(P-Q)/\deg Q$. *Then the equation* $x^n = P/Q$ *has a unique root* α *in* $K((T^{-1}))$ *with* $|\alpha - 1| < 1$. *If* $\lambda = 0$ *or* $\lambda < 1-(1+\sqrt{q})/n$ *then* $\nu(\alpha) = n(1-\lambda)$. *Moreover we have* $\nu(\alpha) = n$ *if and only if there exist* P_0, Q_0 *and* $C \in K[T]$ *such that* $P/Q = (P_0/Q_0)^n(1 + 1/C)$.

It was proved by Osgood (see [O2] p. 109) that if $\gcd(n, p) = 1$ then the n-th root of $1+1/T$ in $\mathbb{F}_p((T^{-1}))$ has an approximation exponent equal to n. We have obtained in Theorem 7.3 the converse of this result. In some cases it is possible to improve the above theorem. For instance, with $p = 2$ and $n = 3$ and with the same conditions and notation as above, we have $\nu(\alpha) = 3(1 - \lambda)$ if $\lambda < (2 - \sqrt{2})/3$.

We give another application to algebraic elements which are not of class I. These algebraic elements are also defined by an equation involving the Frobenius homomorphism. We have just selected an example.

Theorem 7.4. *Let* p *be a prime number with* $p \geq 11$. *We consider the algebraic equation*

$$x = T^{-1} + x^p + T^{-2}x^{p^2}.$$

This equation has p *roots in* $\mathbb{F}_p((T^{-1}))$ *denoted* α_i *for* $i = 0, ..., p-1$ *with* $|\alpha_0| < 1$. *There exist* $\omega \in \mathbb{F}_p((T^{-1}))$ *with* $|\omega| = 1$ *and* $\alpha_i = \alpha_0 + i\omega$ *for* $i = 1, ..., p-1$. *We have*

$$\nu(\alpha_0) = p(p^2 - 1)/(p^2 + 1) \quad \text{and} \quad \nu(\alpha_i) = (p^2 - 1)/(2p) \text{ for } i = 1, .., p-1.$$

Now we consider the image under a rational function of an element α of class I. This image is generally no longer an element of class I. If we know the approximation exponent of α, it is sometimes possible to deduce that of its image. This phenomenon was observed by Voloch (see [V2] p. 322). We have the following result.

Theorem 7.5. *Let* K *be a field of characteristic* p. *Let* $\alpha \in K((T^{-1}))$ *be an algebraic element of class* I. *Let* q *be a power of* p *involved in an equation satisfied by* α. *Let* $R \in K[T](X)$. *Assume that* $R'_X(\alpha) \neq 0$. *If* $R = U/V$ *where* U *and* V *are two coprime polynomials in* $K[T, X]$, *we set* $d = \max(\deg_X U, \deg_X V)$. *Then if* $\nu(\alpha) > d(\sqrt{q} + 1)$, *we have*

$$\nu(R(\alpha)) = \nu(\alpha)/d \quad \text{and} \quad B(R(\alpha), \nu(R(\alpha))) \neq 0, \infty.$$

We give here a special application of this proposition to the two examples mentioned in section **4** :
For *example 1*, we have $\nu(\alpha) = q + 1$. Let k be a positive integer prime with p. Then if $k < (q+1)/(\sqrt{q}+1)$, one has $\nu(\alpha^k) = (q+1)/k$.
For *example 2*, we have $\nu(\alpha) = q$. Let k be a positive integer prime with p. Then if $k < q/(\sqrt{q}+1)$, one has $\nu(\alpha^k) = q/k$.

The proofs of Theorems 7.2, 7.3, 7.4 and 7.5 can be found in my Ph.D. thesis [L0].

8. Thue's method.

In 1908, A. Thue [Th] proved a famous theorem which was the first step on the path leading to Roth's theorem. This result is the following.

Theorem 8.1. *Let α be a real algebraic number of degree $n > 1$, then for all $\epsilon > 0$ we have*

$$|\alpha - p/q| \geq q^{-(n/2+1+\epsilon)}$$

for all $(p, q) \in \mathbb{Z}^2$ with $q > 0$ sufficiently large.

We have tried to adapt his proof in our context in order to obtain a similar result. Of course we had to consider algebraic elements which are not too well approximable by rationals. As we have seen such elements exist in class I. In joint work with B. de Mathan, we have proved the following theorem [LdM1].

Theorem 8.2. *Let K be a field of positive characteristic. Let α be an element of $K((T^{-1}))$, algebraic over $K(T)$, of degree $n > 1$. Assume that α is not an element of class* I. *Then for every positive real number ϵ we have*

$$|\alpha - P/Q| \geq |Q|^{-([n/2]+1+\epsilon)}$$

for all pairs $(P, Q) \in K[T] \times K[T]$ with $|Q|$ sufficiently large.

The conclusion of this theorem is equivalent to $\nu(\alpha) \leq [n/2] + 1$. This theorem highlights the specificity of the algebraic elements of class I. We may see that this theorem is nearly optimal. Indeed let α be the element given as *example 2* in section **4**. For p large, let us consider the element $\beta = \alpha^2$. We can show that β is algebraic of degree p, not of class I, and with $\nu(\beta) = p/2$. This remark is due to Voloch (see [V2] p. 321).

We indicate here the main steps of the proof. First we show that, for each integer $s \geq 1$, there are two polynomials U_s and V_s in $K[T][X]$ such that

$$U_s(\alpha^{p^s}) - \alpha V_s(\alpha^{p^s}) = 0$$

with

$$\mu_s = \max\ (\deg_X U_s, \deg_X V_s) \leq [n/2]$$

and

$$\max(\deg_T U_s, \deg_T V_s) \leq C^{p^s}$$

where C is a positive constant depending upon α. Here since α is not of class I, we have $\mu_s \geq 2$. Then the proof is obtained from the following intermediary result:

Theorem 8.3. *Suppose α is as in Theorem 8.2. With the above notation, if μ is a real number such that $\mu_s \leq \mu$ for all $s \geq 1$, then for all positive real ϵ, and for all pairs (P, Q) in $K[T] \times K[T]$ with* $\deg Q$ *sufficiently large, we have*

$$|P - \alpha Q| \geq |Q|^{-(\mu+\epsilon)}.$$

The proof is obtained by contradiction. We assume that the inequality $|P - \alpha Q| < |Q|^{-(\mu+\epsilon)}$ has solutions with $\deg Q$ arbitrarily large. Then we show that there are two pairs (P_1, Q_1) and (P_2, Q_2) satisfying this inequality and also an integer s such that $p^s \deg Q_1/\deg Q_2$ is close to 1. At last we obtain a contradiction using the elements of $K[T]$, defined by

$$A_s = Q_1^{\mu_s p^s} U_s((P_1/Q_1)^{p^s}) \quad \text{and} \quad B_s = Q_1^{\mu_s p^s} V_s((P_1/Q_1)^{p^s}).$$

We have to observe that, in some cases, because of Theorem 8.3, it is possible to get a better conclusion in Theorem 8.2. For instance, for elements satisfying an equation such as the one mentioned in Theorem 7.4 it is possible to show that the approximation exponent must be smaller or equal to $p + 1$ (see [LdM1], p 203-204). In the next paragraph, we will return to this example and prove a better upper bound for its approximation exponent.

As Theorem 8.2 is in some sense an analogue of Theorem 8.1, it is natural to expect to improve it in the direction of an analogue of Roth theorem. Evidently this can only be obtained by throwing away some other exceptional cases of too well aproximable numbers. We have to mention here a recent work in that direction, using tools of algebraic geometry, due to M. Kim, D. Thakur and J.-F. Voloch [KTV].

9. The differential method.

The use of differential algebra in rational approximation to formal power series was initiated by Kolchin [K], developed by Osgood [O1] [O2], and further by Schmidt [S1].

We have a formal derivation by differentiating the series term by term. If K has characteristic 0, then the field of constants is K, and if K has characteristic $p > 0$, then the field of constants is $K((T^{-p}))$. If $\alpha \in K((T^{-1}))$ is algebraic over $K(T)$, then it satisfies an algebraico-differential equation. Indeed, let $F \in K[T, X]$ be the minimal polynomial of α, with $\deg_X F = n$. We have $F(T, \alpha) = 0$, and differentiating this equality we get $F'_T(T, \alpha) + \alpha' F'_X(T, \alpha) = 0$. This shows that $\alpha' \in K(T, \alpha)$. Thus there is $G \in K(T)[X]$ such that $\alpha' = G(\alpha)$ with $0 \leq \deg_X G \leq n-1$.

If we have $\deg_X G \leq 2$, we will say that α satisfies a Riccati differential equation. In 1974, C. Osgood [O2] established the following theorem.

Theorem 9.1. *Let K be a field. Let $\alpha \in K((T^{-1}))$ be an algebraic element over $K(T)$ of degree $n > 1$. Assume that α does not satisfy a Riccati differential equation. Then there exists a positive real constant C depending only upon α such that*

$$|\alpha - P/Q| \geq C|Q|^{-([n/2]+1)}$$

for all $(P, Q) \in K[T] \times K[T]$ with $Q \neq 0$.

The conclusion of this theorem is equivalent to $B(\alpha, [n/2]+1) > 0$. We give here a sketch of the proof. First we show that there are two polynomials U and V of $K[T][X]$ such that

$$H(\alpha) = U(\alpha) - \alpha' V(\alpha) = 0$$

with

$$Q^{[n/2]+1} H(P/Q) \in K[T] \quad \text{for } P, Q \in K[T] \text{ and } Q \neq 0.$$

Now assume that $H(P/Q) \neq 0$. Then we can easily prove that P and Q satisfy the inequality in Theorem 9.1. We use the following argument. Given $x_0 \in K((T^{-1}))$, there exist two positive real numbers η and C_1 such that for $x \in K((T^{-1}))$ with $|x - x_0| < \eta$ we have $|H(x) - H(x_0)| \leq C_1|x - x_0|$. Consequently we obtain

$$|\alpha - P/Q| \geq C_1^{-1}|H(\alpha) - H(P/Q)| = C_1^{-1}|H(P/Q)| \geq C_1^{-1}|Q|^{-([n/2]+1)}.$$

The end of the proof is based on the following technical lemma which has been exposed by Schmidt (see [S1] p. 762).

Lemma 9.2. *Let R and S be two coprime polynomials in $K[T, X]$. We consider the differential equation*

$$(1) \qquad X' R(X) = S(X)$$

Assume that we do not simultaneously have $\deg_X R = 0$ and $\deg_X S \leq 2$, i.e. that the differential equation is not Riccati. There is a positive constant C depending on R and S, such that if $P, Q \in K[T]$ and P/Q is a solution of (1), then $|Q| \leq C$.

Now with the above notations, if $H(P/Q) = 0$ this lemma implies that $|\alpha - P/Q| \geq C_2$ for a certain positive constant. Therefore the inequality in Theorem 9.1 holds for all P and $Q \neq 0$ with $C = \min(C_1^{-1}, C_2)$.

We will now state a result which is easily obtained using the same arguments as above in Theorem 9.1 and Lemma 9.2. The basic idea is the one introduced by Liouville but using a differential equation instead of the minimal polynomial. Because of this analogy the following result can be seen as a differential version of Mahler's theorem.

Theorem 9.3. *Let K be a field. Let $\alpha \in K((T^{-1}))$ be an algebraic element over $K(T)$ of degree $n > 1$. Then we can write $\alpha' = G(\alpha)$ with $G \in K(T)[X]$and $\deg_X\ G < n$. Assume that $k = \deg_X\ G > 2$. Then there exists a positive real constant C depending only upon α such that*

$$|\alpha - P/Q| \geq C|Q|^{-k}$$

for all $(P, Q) \in K[T] \times K[T]$ with $Q \neq 0$.

Let us apply this theorem to an element α satisfying the equation introduced in Theorem 7.4. We have

$$\alpha = T^{-1} + \alpha^p + T^{-2}\alpha^{p^2}$$

and this implies

$$\alpha' = T^{-2} - 2T^{-1}\alpha + 2T^{-1}\alpha^p.$$

Therefore, by Theorem 9.3, we have $\nu(\alpha) \leq p$. Observe that for the element α_0 of Theorem 7.4 the approximation exponent which is explicitly computed tends to this upper bound when p tends to infinity.

10. Elements of class I and Riccati differential equation.

In this paragraph we suppose that the base field K has positive characteristic p. The likeness between Theorem 8.2 and Theorem 9.1 leads to a natural question . What is the link between elements of class I and Riccati differential equations ?

It is easy to show that an element of class I satisfies a Riccati differential equation and that this equation has a rational solution. On the other hand if α satisfies a Riccati differential equation which has a rational solution then there is some $\beta \in K((T^{-1}))$ such that $\alpha = f(\beta^p)$, where f is a linear fractional transformation with coefficients in $K[T]$.

Using these remarks and Theorem 9.1, in joint work with B. de Mathan we have obtained a theorem similar to Theorem 8.2 [LdM2]. We need a stronger hypothesis but get a stronger conclusion.

Theorem 10.1. *Let K be a finite field. Let $\alpha \in K((T^{-1}))$ be an algebraic element over $K(T)$ of degree $n > 1$. Assume that α is not an element of class* I. *Then there exists a positive real constant C depending only upon α such that*

$$|\alpha - P/Q| \geq C|Q|^{-([n/2]+1)}$$

for all $(P, Q) \in K[T] \times K[T]$ with $Q \neq 0$.

The proof is obtained by contradiction. Thus we assume that $B(\alpha, [n/2] + 1) = 0$. Then Theorem 9.1 implies that α satisfies a Riccati differential equation with a rational solution (see below in section **11**). It follows that there is an α_1 and a linear fractional transformation f_1 with polynomial coefficients such that $\alpha = f_1(\alpha_1^p)$. We show that α and α_1 have the same degree over $K(T)$ and that $B(\alpha_1, [n/2]+1) = 0$. By iteration, for all $n \geq 1$ there exists an α_n and a linear fractional transformation f_n with polynomial coefficients such that $\alpha = f_n(\alpha_n^{p^n})$. Then we consider the cross-ratio of α and of three of its conjugates and show that it belongs to L, a finite extension of K. Finally we prove that α is of class I and that the power of p involved in the equation satisfied by α is $q =$card(L).

When we started our investigations we had in mind Theorem 8.1 and Theorem 9.1. In a first step towards Theorem 8.2, we could prove, by elementary methods and using Theorem 9.1, the statement of Theorem 10.1 for $K = \mathbb{F}_2$ and $n = 4$. This confirmed the intuition that the subset of elements of class I was the right set of exceptional cases. Then we turned to the proof of Theorem 8.2, following Thue's ideas. The proof of Theorem 9.1 in the general case was later made possible by using the argument of the cross-ratio. This argument was introduced by J-F. Voloch (see [V2] p 324).

11. Badly approximable elements and Riccati differential equation.

Let us consider a Riccati differential equation

$$(R) \quad x' = ax^2 + bx + c \quad \text{with} \quad a, b, c \in K(T).$$

We set $H(x) = x' - ax^2 - bx - c$. Then it is easy to see that there exist $D \in K[T]$ such that, for all $P, Q \in K[T]$ with $Q \neq 0$, we have $DQ^2H(P/Q) \in K[T]$.

Let $\alpha \in K((T^{-1}))$ be an irrational element. Suppose that α satisfies (R). Let $P, Q \in K[T]$, with $Q \neq 0$, and assume that $H(P/Q) \neq 0$. Consequently we have $|DQ^2H(P/Q)| \geq 1$. Then we use the argument which has been exposed for the proof of Theorem 9.1. There is a positive real number C such that

$$|\alpha - P/Q| \geq C^{-1}|H(\alpha) - H(P/Q)| = C^{-1}|H(P/Q)| \geq C^{-1}|D|^{-1}|Q|^{-2}.$$

Now we suppose first that K has characteristic 0. In this case C. Osgood [O1] has remarked that if $u, v \in K((T^{-1}))$ with $u \neq v$ are such that $H(u) = 0$ and $H(v) = 0$, then there is a positive real constant C' such that $|u - v| \geq C'$. Therefore, in this case, if $H(P/Q) = 0$ we have $|\alpha - P/Q| \geq C'$.

Observe that this is false if K has positive characteristic. Indeed consider *example 2* of section **4**. We have $\alpha = \sum_{k\geq 0} T^{-q^k}$. Set $\alpha_n = \sum_{0\leq k\leq n} T^{-q^k}$. We see that α and α_n satisfy the same Riccati differential equation $x' = -1/T^2$. But $|\alpha - \alpha_n|$ tends to 0, when n tends to ∞.

In conclusion we have the following theorem.

Theorem 11.1. *Let K be a field. Let $\alpha \in K((T^{-1}))$ be an irrational solution of (R). Assume that either K has characteristic 0, or K has* *positive characteristic and* (R) *has no rational solution. Then there exists a positive real constant* C *such that*

$$|\alpha - P/Q| \geq C|Q|^{-2}$$

for all $(P,Q) \in K[T] \times K[T]$ *with* $Q \neq 0$.

Observe that the conclusion of this theorem is equivalent to saying that the partial quotients in the continued fraction for α are bounded, i.e. α is badly approximable. The constant C in the inequality can easily be made explicit and depends only on the coefficients of the Riccati differential equation satisfied by α.

In the case of characteristic 0, for instance every algebraic element of degree 3 satisfies a Riccati differential equation and is therefore badly approximable. This case has been considered by Osgood [O1] and latter by W. Schmidt [S1], obtaining explicit values for the constant C in the above inequality.

Osgood has also considered the n^{th} root of a rational function with rational coefficients (see [O1], p. 7). Such an element satisfies a Riccati differential equation. As an illustration, let us consider for $n \geq 2$ the element $\alpha \in \mathbb{Q}((T^{-1}))$ defined by $\alpha^n = 1 + 1/T$. Then we have $\alpha' = -(nT(T+1))^{-1}\alpha$ and, by applying Theorem 11.1, it can be proved that $|\alpha - P/Q| > |T|^{-2}|Q|^{-2}$. In other words, the degree of the partial quotients in the continued fraction for α are all equal to 1. It is natural to ask what these partial quotients really are. In fact this question can be answered and the continued fraction can be explicitly described. We have

$$(1+1/T)^{1/n} = [1, nT - (n-1)/2, u_2(T+1/2), \ldots.., u_k(T+1/2), \ldots.]$$

with for even k,

$$u_k = -4(2k-1)n \prod_{1 \leq i \leq (k-2)/2} (4i^2n^2 - 1) / \prod_{1 \leq i \leq k/2} ((2i-1)^2n^2 - 1),$$

and for odd k,

$$u_k = (2k-1)n \prod_{1 \leq i \leq (k-1)/2} ((2i-1)^2n^2 - 1) / \prod_{1 \leq i \leq (k-1)/2} (4i^2n^2 - 1).$$

For $n = 2$ these formulas show that the continued fraction expansion is periodic. Indeed we get $(1+1/T)^{1/2} = [1, 2T-1/2, (-8T-4, 2T+1)]$. The above formulas can be obtained by an adaptation of some other formulas which apparently go back to Euler.

The second case where K has positive characteristic p is interesting too. In joint work with B. de Mathan, we have obtained the following result [LdM2].

Theorem 11.2. *Let $\beta \in \mathbb{F}_2((T^{-1}))$, such that $|\beta| \leq 1$. There exists a unique $\alpha \in \mathbb{F}_2((T^{-1}))$ such that*

$$\alpha^2 + T\alpha + 1 = (T+1)\beta^2.$$

Then α is solution of a Riccati differential equation which has no rational solution. Consequently we have, for all $P, Q \in \mathbb{F}_2[T]$ with $Q \neq 0$,

$$|\alpha - P/Q| \geq |T|^{-1}|Q|^{-2}.$$

This result was first proved by Baum and Sweet [BS2], without any use of differential methods. Observe that if β is algebraic then α is also algebraic. Moreover if α is algebraic and non-quadratic then α is not an element of class I, since an element of class I satisfies a Riccati differential equation which has a rational solution and this equation is unique if α is not quadratic. In connection to Theorem 11.2, we must mention recent work by A. Lauder [La] who has followed the original ideas introduced by Baum and Sweet [BS2] and could extend their results.

When the characteristic of the base field K is $p > 2$, we have used elementary methods to prove the following result (see [LdM2] p. 5).
If K has characteristic $p = 3$ and if there is some $\alpha \in K((T^{-1}))$, neither rational nor quadratic, satisfying a Riccati differential equation, then this equation has infinitely many rational solutions.
M. van der Put [vdP] has studied such differential equations and it follows from his work that the same holds for all $p \geq 3$.

12. A normally approximable algebraic element.

Mills and Robbins [MR] have considered the following algebraic equation

$$x^4 + x^2 - Tx + 1 = 0.$$

They observed that, if the base field is $\mathbb{F}_3$, this equation has a unique root in $\mathbb{F}_3((T^{-1}))$ and that this root appears to have a continued fraction with a very peculiar pattern. Later Buck and Robbins [BR] proved the following theorem.

Theorem 12.1. *Let $\Omega_0 = \emptyset$, $\Omega_1 = T$, and for $n \geq 2$, let Ω_n be the finite sequence of polynomials defined by*

$$\Omega_n = \Omega_{n-1}, 2T, \Omega_{n-2}^{(3)}, 2T, \Omega_{n-1}$$

where commas indicate juxtapposition of sequences, and $\Omega_k^{(3)}$ is obtained by cubing every element of Ω_k. Let us denote by Ω_∞ the sequence begining by Ω_n for all n, and consider in $\mathbb{F}_3((T^{-1}))$ the element α defined by the continued fraction expansion $\alpha = [0, \Omega_\infty]$. Then α is the unique root in $\mathbb{F}_3((T^{-1}))$ of $x^4 + x^2 - Tx + 1 = 0$.

Their proof is obtained by considering some subsequences of convergents of the above continued fraction, say $p_{1,n}/q_{1,n}, p_{2,n}/q_{2,n}, ..., p_{k,n}/q_{k,n}$.

Then they prove an equality, say $F(p_{1,n}/q_{1,n}, p_{2,n}/q_{2,n}, ..., p_{k,n}/q_{k,n}) = 0$, satisfied for all $n \geq 0$. Finally by letting n go to infinity this implies that α satisfies the desired algebraic equation $f(\alpha) = 0$. The original proof can be made shorter (see [S2] section 10). The basic idea can be used for other algebraic elements (see the proofs of Theorem 4.1 and Theorem 6.2 for instance).

We have observed that the solution of the above quartic equation is directly connected to *example 1* in section **4** when $p = q = 3$. This has allowed us to consider the corresponding element for a general q. In this context we have obtained another proof of Theorem 12.1 [L1].

Let p be an odd prime and q a positive power of p. We consider $\alpha_q \in \mathbb{F}_p((T^{-1}))$, defined by $\alpha_q = [0, T, T^q, ..., T^{q^n}, ...]$. This element is the unique root, in $\mathbb{F}_p((T^{-1}))$, of the equation $x = (1/T)(1 - x^{q+1})$. We set $r = (q+1)/2$ and $\theta_q = \alpha_q^r$. Then we see that θ_q is the unique root in $\mathbb{F}_p((T^{-1}))$ of the equation

$$x = (1/T^r)(1 - x^2)^r.$$

Now if $p = q = 3$ then $r = 2$ and this equation becomes $x^4 + x^2 - T^2x + 1 = 0$. Thus the theorem of Buck and Robbins can be expressed by the following formula in $\mathbb{F}_3((T^{-1}))$:

$$[0, T, T^3, ..., T^{3^n}, ...]^2 = [0, \Omega_\infty(2)],$$

where $\Omega_\infty(2)$ denotes the sequence obtained from Ω_∞ by changing T into T^2.

We have studied the continued fraction defined in $\mathbb{F}_p((T^{-1}))$ by

$$\theta_q = [0, T, T^q, ..., T^{q^n}, ...]^r.$$

If $q = 3$, we can prove that the above formula holds. In the general case $q > 3$, we can only describe partially the pattern of the continued fraction expansion for θ_q. If we put $\alpha_{n,q} = [0, T, T^q, ..., T^{q^{n-1}}]$, we show that $\alpha_{n,q}^r$ is a convergent to θ_q. Thus we define, for $n \geq 0$, a sequence of polynomials in $\mathbb{F}_p[T]$, $\Omega_{1,n}$, by $\alpha_{n,q}^r = [0, \Omega_{1,n}]$. We have $\Omega_{1,0} = \emptyset$ and $\Omega_{1,1} = T^r$. We give here a description of $\Omega_{1,n}$ which is only partially proved.

Conjecture 12.2. *There exist $r-2$ sequences of finite sequences of polynomials in $\mathbb{F}_p[T]$, denoted $(\Omega_{i,n})_{n\geq 1}$ for $2 \leq i \leq r-1$, such that, for $n \geq 1$, we have*

$$\Omega_{1,n+1} = \Omega_{1,n}, 2T^r, \Omega_{2,n}, 2T^r, ..., \Omega_{1,n-1}^{(q)}, ..., 2T^r, \Omega_{2,n}', 2T^r, \Omega_{1,n}'.$$

In this formula, if $\Omega = \omega_1, \omega_2, ..., \omega_k$, we denote $\Omega' = \omega_k, \omega_{k-1}, ..., \omega_1$, and $\Omega_{1,n-1}^{(q)}$ is obtained by raising every element of $\Omega_{1,n-1}$ to the q^{th} power.

Observe that $\Omega_{1,n+1}$ is split in q blocks with $2T^r$ between two blocks. In the general case, it is not a full conjecture since we are not able to describe the intermediary blocks $\Omega_{i,n}$ for $2 \leq i \leq r-1$. For $p = q = 3$, we have $r - 2 = 0$ and there are no intermediary blocks, thus the previous formula contains only 3 blocks and in this case it is possible to get a proof of the conjecture.

Now we come back to the rational approximation to θ_q. Since θ_q is the unique root in $\mathbb{F}_p((T^{-1}))$ of the equation $x = (1/T^r)(1-x^2)^r$, we see that $\theta_q \in \mathbb{F}_p((T^{-r}))$. Thus we can define θ_q^* by $\theta_q^*(T^r) = \theta_q(T)$. So θ_3^* is the element introduced by Mills and Robbins. We recall that the approximation exponent of α_q is $q+1$. In the remark following Theorem 7.5, we have seen that $\nu(\alpha_q^k) = (q+1)/k$, if k is prime to p and small enough. Observe that for $k = (q+1)/2$ this formula would give $\nu(\theta_q) = 2$. Besides it is clear that θ_q and θ_q^* have the same rational approximation properties. In fact we proved the following theorem [L1].

Theorem 12.3. *Let θ_3^* be the unique root in $\mathbb{F}_3((T^{-1}))$ of the equation*

$$x^4 + x^2 - Tx + 1 = 0.$$

We have $\nu(\theta_3^*) = 2$ and $B(\theta_3^*, 2) = 0$. More precisely, there exist two explicit real constants λ_1 and λ_2 such that there is a sequence of rationals P_n/Q_n with $|Q_n|$ tending to infinity for which we have

$$|\theta_3^* - P_n/Q_n| \leq |Q_n|^{-(2+\lambda_1/\sqrt{\deg Q_n}\ )}$$

and for all rationals P/Q with $|Q|$ sufficiently large we have

$$|\theta_3^* - P/Q| \geq |Q|^{-(2+\lambda_2/\sqrt{\deg Q}\ )}$$

We can choose $\lambda_1 = 2/\sqrt{3}$ and $\lambda_2 > 2/\sqrt{3}$.

Observe, with our definitions, that θ_3^* is normally approximable and therefore is not an element of class I. The proof of Theorem 12.3 is obtained by means of the continued fraction for θ_3^*. Thus, because of the partial description for the continued fraction for θ_q^* exposed in Conjecture 12.2, we make a final conjecture

Conjecture 12.4. *Let $p \neq 2$ be a prime number and q a positive power of p. Let θ_q^* be the unique solution in $\mathbb{F}_p((T^{-1}))$ of the equation*

$$x = (1/T)(1-x^2)^{(q+1)/2}.$$

Then the same theorem as Theorem 12.3 holds for θ_q^*, but with $2/\sqrt{3}$ replaced by $\sqrt{2(q-1)/q}$.

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Institut de Mathématiques de Bordeaux
Université de Bordeaux I
351, Cours de la Libération
F33405 Talence Cedex - FRANCE
E-mail:lasjauni@math.u-bordeaux.fr

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