Exercise 1. Let X be a random variable with an exponential distribution of parameter λ.

1. Calculate the mean and the variance of X.
2. Prove that $\lambda X \sim \mathcal{E}(1)$.
3. Calculate the mean of $\exp(-tX)$ for all $t \geqslant 0$.

Exercise 2. Let $X \sim \mathcal{E}(\lambda)$ and $Y \sim \mathcal{E}(\mu)$ two independent random variables.

1. Give the law of $Z = \min\{X, Y\}$.
2. Calculate $\mathbb{P}(X \leqslant Y)$.
3. Show that events $(X < Y)$ and $(Z \geqslant t)$ are independent,$\forall t \geqslant 0$.

Exercise 3. The distribution Gamma with parameters (n, λ), $(n \geqslant 1, \lambda > 0)$ has a density function given by

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} \exp(-\lambda x) \mathbb{1}_{x \geqslant 0}.$$

1. Let X and Y two independent variables with distribution Gamma with parameters (n, λ) and (m, λ). Show that $X + Y$ has a distribution Gamma with parameters$(n + m, \lambda)$.
2. Let $X_1, ..., X_n$ i.i.d. random variables with law $\mathcal{E}(\lambda)$. Show that $X_1 + ... + X_n$ has a distribution Gamma with parameters (n, λ).

Exercise 4. Show that the filtration of a Poisson process is right continuous, i.e.

$$\mathcal{F}_t^N = \bigcap_{s>t} \mathcal{F}_s^N.$$

Deduce that τ is a stopping time if and only if $(\tau < t) \in \mathcal{F}_t^N$, $\forall t \geqslant 0$.

Exercise 5. Let S and T two stopping times.

1. Show that $\max\{S, T\}$, $\min\{S, T\}$ and $S + T$ are stopping times.
2. If $S \leqslant T$, is $T - S$ a stopping time ?
3. Show that $\mathcal{F}_T$ is a σ-algebra.
4. Let $R = \min\{S, T\}$. Show that $\mathcal{F}_R = \mathcal{F}_S \cap \mathcal{F}_T$.
5. Show that $(S \leqslant T) \in \mathcal{F}_T$.

Exercise 6. Let $s > 0$. Show that T_{N_s+1} is a stopping times, but not T_{N_s}.

Exercise 7. Let $s > 0$. Show that $T_{N_s+1} - s$ has the distribution $\mathcal{E}(\lambda)$.

Exercise 8.

1. We assume that we observe N_t for on $et > 0$. Calculate the maximum likelihood estimator of λ and study is asymptotic behavior.
2. We assume now that we observe T_n. Calculate the maximum likelihood estimator of λ and study is asymptotic behavior.

Exercise 9.

- Let N_t) a Poisson process and T_1 is first jump time. Calculate the cumulative distribution function of T_1 conditionally upon $N_T = 1$.
- Let I an interval of $[0, t]$ and n an integer. Conditionally upon $N_t = n$, calculate the probability such that (N_t) has exactly k jumps in I.

Exercise 10. Let $(X_t)_{t\geqslant 0}$ a Poisson process with parameter λ and $(Y_t)_{t\geqslant 0}$ a Poisson process with parameter μ independent with $(X_t)_{t\geqslant 0}$. Show that $(N_t = X_t + Y_t)$ is a Poisson process with intensity $\lambda + \mu$.

Exercise 11. Let (N_t) a Poisson process with intensity λ. We assume that jumps are of two types A and B: every jump is of type A with probability p and types of jumps are mutually independent and independent with the process (N_t). Let (N_t^A) and (N_t^B) the random counting functions of jumps A and jumps B.

1. For a given t, give the distribution of (N_t^A, N_t^B).
2. Deduce, for all t, the law of N_t^A.
3. Show that, for all t, random variables N_t^A and N_t^B are independent.
4. Show that (N_t^A) and (N_t^B) are two independent Poisson processes with intensities $p\lambda$ and $(1-p)\lambda$.

Exercise 12. At the bus stop Peixotto, there are six buses of line 10 and two buses of line 20 every hours. We assume that time arrivals of buses of these two lines are independent Poisson processes.

1. Give intensities of the two Poisson processes.
2. What is the probability such that there is exactly 8 buses in one hour at this bus stop.
3. During strikes, the half part of buses drive. What is the probability such that there is no bus at all during half an hour at the bus stop ?

Exercise 13. (part of a 2012 exam) We assume that goals scored by "les girondins de Bordeaux" are a Poisson process (N_t) with intensity λ. In this exercise we only consider football matches ended with two goals scored by "les girondins" (we do not care about the number of goals scored by the opponent). The duration of game is a constant denoted T and goals times are random variables denoted T_1 and T_2.

1. Explain why, for $i \in \{1, 2\}$ and $t \geqslant 0$, we have $(T_i \leqslant t) = (N_t \geqslant i)$.
2. By using previous question, calculate$\mathbb{P}(T_2 \leqslant T/2 | N_T = 2)$ the probability such that "les girondins" scored two goals during the first half.
3. Calculate the probability such that there is exactly one goal during the first half conditionally upon the fact that "les girondins" scored two goals during the football match.
4. Show that for all $0 < s < t < T$,
$$\mathbb{P}(T_1 \leqslant s, T_2 \leqslant t | N_T = 2) = \mathbb{P}(N_s = 1, N_t = 2 | N_T = 2) + \mathbb{P}(N_s = 2, N_t = 2 | N_T = 2).$$
5. By using stationary and independent increments property of Poisson process, show that , for all $0 < s < t < T$,
$$\mathbb{P}(T_1 \leqslant s, T_2 \leqslant t | N_T = 2) = \frac{s(2t - s)}{T^2}.$$
6. Deduce cumulative distribution functions and densities of T_1 and T_2 conditionally upon $N_T = 2$.
7. What is the mean time for scoring the first goal and the second goal ?

Exercise 14. (adapted from a 2013 exam) Let (N_t) a Poisson process with intensity λ. We define
$$G_t = t - T_{N_t}, \quad \text{and} \quad D_t = T_{N_t+1} - t.$$

1. Draw a trajectory of the Poisson process and point out on the draw times t, T_{N_t}, T_{N_t+1} and terms G_t and D_t.
2. For a given t and $0 < x \leqslant t$, $y \geqslant 0$, show that $(G_t < x, D_t \leqslant y) = (N_{t-x} < N_t < N_{t+y})$. Deduce that $\mathbb{P}(G_t < x, D_t \leqslant y)$.
3. For a given t and $y \geqslant 0$, show that $(G_t = t, D_t \leqslant y) = (N_t = 0, N_{t+y} > 0)$. Deduce that $\mathbb{P}(G_t = t, D_t \leqslant y)$.
4. For a given t and $y \geqslant 0$, calculate $(D_t \leqslant y)$ and deduce the distribution of D_t.
5. Calculate the cumulative distribution function of G_t.
6. Calculate $\mathbb{P}(\min(T_1, t) > x)$ for all $x \in \mathbb{R}$. Deduce that G_t has the same law than $\min(T_1, T)$.
7. Show that G_t and D_t are independent.
8. Calculate $\mathbb{E}[G_t]$. Deduce that $\mathbb{E}[G_t + D_t]$. What do you think about this result ?