On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions

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Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in $\mathbb{R}^d$, $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, T be a nonnegative real number, ξ a real $\mathcal{F}_T$-measurable random variable, $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \longrightarrow \mathbb{R}$.

$$Y_t = \xi - \int_t^T g(r, Y_r, Z_r) dr + \int_t^T Z_r dW_r, \quad 0 \leqslant t \leqslant T. \qquad (1.1)$$

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Definition

A solution to (1.1) is a pair of processes $(Y_t, Z_t)_{0\leqslant t\leqslant T}$ such that :

1. (Y, Z) is a predicable process with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$,
2. $\mathbb{P} - a.s.\ t \mapsto Y_t$ is continuous and $\int_0^T |g(r, Y_r, Z_r)| + \|Z_r\|^2\, dr < \infty$
3. (Y, Z) verifies (1.1).

Simple example

We take $g = 0$.

1. A natural idea is to consider $Y_t := \mathbb{E}[\xi|\mathcal{F}_t]$.
2. The martingale representation theorem gives us $Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s$.
3. By a simple calculus we obtain

$$Y_t = \xi - \int_t^T Z_s dW_s.$$

Existence and unicity results

- Existence and uniqueness of BSDEs when g is Lipschitz with respect to y and z : E. Pardoux et S. Peng (1990).

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- Existence of solutions to quadratic BSDEs with ξ unbounded : P. Briand et Y. Hu (2006).
- Uniqueness of solutions to quadratic BSDEs with g a convex function with respect to z and ξ unbounded : P. Briand et Y. Hu (2008).

framework

Assumptions : There exist three constants $\bar{\beta} \geqslant 0$, $\bar{\gamma} > 0$ and $r \geqslant 0$ together with two progressively measurable nonnegative stochastic processes $(\bar{\alpha}_t)_{0 \leqslant t \leqslant T}$ and $(\underline{\alpha}_t)_{0 \leqslant t \leqslant T}$ such that, $\mathbb{P}$-a.s.,

1. $z \mapsto g(t, y, z)$ is a convex function $\forall (t, y) \in [0, T] \times \mathbb{R}$;
2. $\forall (t, z) \in [0, T] \times \mathbb{R}^{1 \times d}$,
$$\left|g(t, y, z) - g(t, y', z)\right| \leqslant \bar{\beta} \left|y - y'\right|, \quad \forall (y, y') \in \mathbb{R}^2;$$
3. growth condition : $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d}$,
$$-\underline{\alpha}_t - r(|y| + |z|) \leqslant g(t, y, z) \leqslant \bar{\alpha}_t + \bar{\beta} |y| + \frac{\bar{\gamma}}{2} |z|^2 .$$

Existence result

Theorem

If there exists $p > 1$ such that

$$\mathbb{E}\left[\exp\left(\gamma e^{\beta T}\left(\xi^- + \int_0^T \bar{\alpha}_t dt\right)\right) + (\xi^+)^p + \left(\int_0^T \underline{\alpha}_t dt\right)^p\right] < +\infty$$

then the BSDE (1.1) has a solution (Y, Z).

Fenchel-Legendre transform

Since $g(t, y, .)$ is a convex function, we can define the Fenchel-Legendre transform of g :

$$f(t, y, q) = \sup_z (zq - g(t, y, z)), \quad \forall t \in [0, T], q \in \mathbb{R}^d, y \in \mathbb{R}.$$

f is a function with value in $\mathbb{R} \cup \{+\infty\}$.

Proposition

- $\forall (t, y, y', q) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ *such that* $f(t, y, q) < +\infty$,

$$f(t, y', q) < +\infty \; et \; \left| f(t, y, z) - f(t, y', z) \right| \leqslant \bar{\beta} \left| y - y' \right|.$$

- *f is a convex function with respect to q,*
- *The Fenchel-Legendre transform of f is g.*

sketch of the proof for the uniqueness result

What happened when g does not depend on y ?

$$Y_t = \xi - \int_t^T g(s, Z_s)ds + \int_t^T Z_s dW_s.$$

We have $g(s, Z_s) = \sup_{q_s}(Z_s q_s - f(s, q_s)) = Z_s q_s^* - f(s, q_s^*)$.

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$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, q_s^*)ds + \int_t^T Z_s(dW_s - q_s^* ds) && (2.1) \\ &\leqslant \xi + \int_t^T f(s, q_s)ds + \int_t^T Z_s(dW_s - q_s ds). && (2.2) \end{aligned}$$

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Finally,

$$Y_t = \operatorname*{ess\,inf}_{q \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}} \left[\xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right].$$

Questions

- Are we allowed to apply Girsanov ?
- Which admissible control set $\mathcal{A}$ could we choice ?
- What happened when f depends on y ?

Construction of the control problem (1/4)

$$\mathcal{A} := \Bigg\{ (q_s)_{s\in[0,T]}, \quad \int_0^T |q_s|^2\, ds < +\infty \ \ \mathbb{P}-a.s., \\ (M_t)_{t\in[0,T]} \text{ is a martingale}, \\ \mathbb{E}^{\mathbb{Q}}\left[|\xi| + \int_0^T |f(s,0,q_s)|\, ds \right] < +\infty, \\ \text{with } M_t := \exp\left(\int_0^t q_s dW_s - \frac{1}{2}\int_0^t |q_s|^2\, ds \right) \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}} := M_T \Bigg\}$$

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$$\mathbb{E}^{\mathbb{Q}}\left[|\xi| + \int_0^T |f(s,0,q_s)| \, ds \right] < +\infty,$$
$$\left. \text{with } M_t := \exp\left(\int_0^t q_s dW_s - \frac{1}{2}\int_0^t |q_s|^2 \, ds \right) \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}} := M_T \right\}$$

There exists a solution to the BSDE

$$Y_t^q = \xi + \int_t^T f(s, Y_s^q, q_s) ds + \int_t^T Z_s^q dW_s^q, \quad 0 \leqslant t \leqslant T.$$

with $dW_t^q := dW_t - q_t dt$.

Construction of the control problem (2/4)

We have $Y \leqslant \operatorname{ess\,inf}_{q \in \mathbb{A}} Y^q$. We must show that $q^* \in \mathcal{A}$. If T is small enough and if there exist some given exponential moments for $\sup_{0 \leqslant t \leqslant T} Y_t^- + \int_0^t \bar{\alpha}_s ds$ and $\sup_{0 \leqslant t \leqslant T} Y_t^+$ then we are able to show that M_T^* is a martingale.
In the general case we divide $[0, T]$ into subsets. Thus, for N big enough we define $t_i := \frac{iT}{N}$ with $i \in \{0, ..., N\}$.

Construction of the control problem (3/4)

$$\mathcal{A}_{t_i,t_{i+1}}(\eta) := \Bigg\{ (q_s)_{s\in[t_i,t_{i+1}]}, \quad \int_{t_i}^{t_{i+1}} |q_s|^2\, ds < +\infty \ \ \mathbb{P}-a.s.,$$

$$(M_t^i)_{t\in[t_i,t_{i+1}]} \text{ is a martingale}, \quad \mathbb{E}^{\mathbb{Q}^i}\left[|\eta| + \int_{t_i}^{t_{i+1}} |f(s,0,q_s)|\, ds \right] < +\infty,$$

$$\text{with } M_t^i := \exp\left(\int_{t_i}^{t} q_s dW_s - \frac{1}{2}\int_{t_i}^{t} |q_s|^2\, ds \right) \text{ and } \frac{d\mathbb{Q}^i}{d\mathbb{P}} := M_{t_{i+1}}^i \Bigg\}.$$

There exists a solution to the BSDE

$$Y_t^{\eta,q} = \eta + \int_t^{t_{i+1}} f(s, Y_s^{\eta,q}, q_s)ds + \int_t^{t_{i+1}} Z_s^{\eta,q} dW_s^q, \quad t_i \leqslant t \leqslant t_{i+1}.$$

Construction of the control problem (4/4)

$$\mathcal{A} := \Big\{ (q_s)_{s\in[0,T]}, \quad q_{|[t_{N-1},T]} \in \mathcal{A}_{t_{N-1},T}(\xi), \\ \forall i \in \{N-2,\ldots,0\}, q_{|[t_i,t_{i+1}]} \in \mathcal{A}_{t_i,t_{i+1}}\left(Y^q_{t_{i+1}}\right) \\ \text{with } Y^q_{t_{i+1}} := Y^{Y^q_{t_{i+2}}, q_{|[t_{i+1},t_{i+2}]}}_{t_{i+1}} \text{ and } Y^q_T := \xi \Big\}.$$

We can define our cost functional

$$\forall i \in \{N-1,\ldots,0\}, \forall t \in [t_i, t_{i+1}], \quad Y^q_t := Y_t^{Y^q_{t_{i+1}}, q_{|[t_i,t_{i+1}]}}.$$

Result

Theorem

We suppose that there exists a solution (Y, Z) of the BSDE (1.1) verifying $\exists p > \bar{\gamma}, \exists \varepsilon > 0$,

$$\mathbb{E}\left[\exp\left(p \sup_{0 \leqslant t \leqslant T}\left(Y_t^- + \int_0^t \bar{\alpha}_s ds\right)\right) + \exp\left(\varepsilon \sup_{0 \leqslant t \leqslant T} Y_t^+\right)\right] < +\infty,$$

Then we have $Y = \text{ess inf}_{q \in \mathcal{A}} Y^q$, and there exists $q^* \in \mathcal{A}$ such that $Y = Y^{q^*}$. Moreover, this implies that the solution (Y, Z) is unique among solutions verifying such assumption.

Remarks :

1. To obtain the existence of a solution (Y, Z) that verifies such assumption it is sufficient to suppose that $\xi^- + \int_0^T \bar{\alpha}_t dt$ have an exponential moment of order $qe^{\bar{\beta}}$ avec $q > \bar{\gamma}$ and $\xi^+ + \int_0^T \underline{\alpha}_t dt$ have an exponential moment of order $\varepsilon > 0$.
2. When g does not depend on y, we do not have to divide $[0, T]$. We have

$$Y_t = \underset{q \in \mathcal{A}_{0,T}(\xi)}{\text{ess inf}} \, \mathbb{E}^{\mathbb{Q}} \left[\xi + \int_t^T f(s, q_s) ds \middle| \mathcal{F}_t \right], \quad \forall t \in [0, T].$$

Link with PDEs

Let us consider the following semi-linear PDE

$$\partial_t u(t,x) + \mathcal{L}u(t,x) - g(t,x,u(t,x),-\sigma^*\nabla_x u(t,x)) = 0, \quad u(T,.) = h, \tag{3.1}$$

with $\mathcal{L}$ is the infinitesimal generator of the diffusion

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r)dW_r, \quad t \leqslant s \leqslant T, \text{ and } X_s^{t,x} = x, \quad s \leqslant t, \tag{3.2}$$

and the BSDE

$$Y_t^{t,x} = h(X_T^{t,x}) - \int_t^T g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - \int_t^T Z_s^{t,x} dW_s, \quad 0 \leqslant t \leqslant T, \tag{3.3}$$

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The nonlinear Feynman-Kac formula consists in proving that the function defined by the formula

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad u(t,x) := Y_t^{t,x} \tag{3.4}$$

is a viscosity solution to the PDE (3.1).

Exponential Moments

We suppose σ continuous and b K-Lipschitz in x. We have

Lemma

$$\forall \lambda \in \left[0, \frac{1}{2e^{2KT}\left\|\sigma\right\|_{\infty}^{2} T}\right[, \exists C \geqslant 0, \quad \mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} e^{\lambda\left|X_t^{t_0,x_0}\right|^2}\right] \leqslant Ce^{C|x_0|^2}.$$

Assumptions

We suppose that $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ are continuous and that there exist five constants $r \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\alpha \geqslant 0$ et $\alpha' \geqslant 0$ such that :

1. $|g(t,x,y,z) - g(t,x,y',z)| \leqslant \beta |y - y'|$;
2. $z \mapsto g(t,x,y,z)$ is a convex function on $\mathbb{R}^{1\times d}$;
3. $-r(1 + |x|^2 + |y| + |z|) \leqslant g(t,x,y,z) \leqslant r + \alpha |x|^2 + \beta |y| + \frac{\gamma}{2} |z|^2$,
$$-r - \alpha' |x|^2 \leqslant h(x) \leqslant r(1 + |x|^2);$$
4. $|g(t,x,y,z) - g(t,x',y,z)| \leqslant r(1 + |x| + |x'|) |x - x'|$,
$$|h(x) - h(x')| \leqslant r(1 + |x| + |x'|) |x - x'|;$$
5. $$\alpha' + T\alpha < \frac{1}{2\gamma e^{3\beta T} \|\sigma\|_\infty^2 T}.$$

Results

Proposition

The function u defined by (3.4) is continuous on $[0, T] \times \mathbb{R}^d$ *and satisfies*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leqslant C(1 + |x|^2).$$

Moreover, u is a viscosity solution to the PDE (3.1).

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