Numerical simulation of BSDEs with drivers of quadratic growth

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1 Introduction
   - (Markovian) BSDEs
   - Simulation
   - Quadratic BSDEs

2 Different ideas for simulation

3 A new scheme
   - A time-dependent estimate of \( Z \)
   - Convergence of the scheme
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in $\mathbb{R}^d$, $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, $T$ be a nonnegative real number. We consider an SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

with standard assumptions on $b$ and $\sigma$, and a Markovian BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \((W_t)_{t \in \mathbb{R}^+}\) be a Brownian motion in \(\mathbb{R}^d\), \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) be his augmented natural filtration, \(T\) be a nonnegative real number. We consider an SDE

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\]

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\]

\[
\begin{align*}
1 & (Y, Z) \text{ is a predicable process with values in } \mathbb{R} \times \mathbb{R}^{1 \times d}, \\
2 & \mathbb{P} - a.s. \ t \mapsto Y_t \text{ is continuous and } \\
& \int_0^T |f(r, X_r, Y_r, Z_r)| + \|Z_r\|^2 dr < \infty
\end{align*}
\]
Theorem (Pardoux-Peng 1990)

Let us assume that $f$ is a Lipschitz function with respect to $y$ and $z$ and
\[ \mathbb{E} \left[ |g(X_T)|^2 + \int_0^T |f(r, X_r, 0, 0)|^2 dr \right] < \infty. \]
Then the previous equation has a unique solution $(Y, Z)$ such that

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty, \quad \mathbb{E} \left[ \int_0^T |Z_t|^2 dt \right] < \infty. \]
We consider a time discretization of the BSDE. We denote the
time step by $h = T/n$ and $(t_k = kh)_{0 \leq k \leq n}$ stands for the
discretization times. For $X$ we take the Euler scheme :

\[
\begin{align*}
X^n_0 &= x \\
X^n_{t_{k+1}} &= X^n_{t_k} + hb(t_k, X^n_{t_k}) + \sigma(t_k, X^n_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad 0 \leq k \leq n.
\end{align*}
\]

For $(Y, Z)$ we use the classical dynamic programming equation

\[
\begin{align*}
Y^n_{t_n} &= g(X^n_{t_n}) \\
Z^n_{t_k} &= \frac{1}{h} E^{t_k}[Y^n_{t_{k+1}} (W_{t_{k+1}} - W_{t_k})], \quad 0 \leq k \leq n - 1, \\
Y^n_{t_k} &= E^{t_k}[Y^n_{t_{k+1}}] + h E^{t_k}[f(t_k, X^n_{t_k}, Y^n_{t_{k+1}}, Z^n_{t_k})], \quad 0 \leq k \leq n - 1,
\end{align*}
\]

where $E^{t_k}$ stands for the conditional expectation given $\mathcal{F}_{t_k}$.
Remarks on simulation

- The dynamic programming equation is obtained by minimizing the difference

\[ \mathbb{E} \left[ \left( Y^n_{t_{k+1}} + h \mathbb{E}_{t_k} f(t_k, X^n_{t_k}, Y^n_{t_{k+1}}, Z) - Y - Z(W_{t_{k+1}} - W_{t_k}) \right)^2 \right] \]

over \( \mathcal{F}_{t_k} \)-measurable squared integrable random variables \((Y, Z)\).

- After time discretization, we need to use a spatial discretization in order to compute conditional expectation.

- We suppose that \( g \) and \( f \) are Lipschitz functions with respect to \( x, y, z \) and \( t \). If we define the error

\[
e(n) = \sup_{0 \leq k \leq n} \mathbb{E} \left| Y^n_{t_k} - Y_{t_k} \right|^2 + \mathbb{E} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left| Z^n_{t_k} - Z_t \right|^2 dt
\]

then \( e(n) = O(1/n) \).
References for simulation

See, for exemple:
- B. Bouchard, N. Touzi [2004],
- J. Zhang [2005],
- E. Gobet, J.P. Lemor, X. Warin [2005],
- F. Delarue, S. Menozzi [2006].
Quadratic BSDEs

What happened if $f$ has a quadratic growth with respect to $z$?

- when $g$ is bounded: existence and uniqueness results have been proved by M. Kobylanski [2000].
- when $g$ is unbounded: an existence result has been proved by P. Briand and Y. Hu [2006], partial uniqueness results has been proved by P. Briand and Y. Hu [2008], F. Delbaen, Y. Hu and A. R. [2010].

Such BSDEs have applications in finance: this class arises, for example, in the context of utility optimization problems with exponential utility functions (see e.g. Y. Hu, P. Imkeller and M. Müller [2005]).
BMO tool

Definition

For a brownian martingale $\Phi_t = \int_0^t \phi_s dW_s$, $t \in [0, T]$, we say that $\Phi$ is a BMO martingale if

$$\|\Phi\|_{BMO} = \sup_{\tau \in [0, T]} \mathbb{E} \left[ \int_\tau^T \phi_s^2 ds \bigg| \mathcal{F}_\tau \right]^{1/2} < +\infty,$$

where the supremum is taken over all stopping times in $[0, T]$. 
the very important feature of BMO martingales is the following lemma:

Lemma

Let $\Phi$ be a BMO martingale. Then we have:

1. The stochastic exponential

$$E(\Phi)_t = E_t = \exp \left( \int_0^t \phi_s dW_s - \frac{1}{2} \int_0^t |\phi_s|^2 \, ds \right), \quad 0 \leq t \leq T,$$

is a uniformly integrable martingale.

2. Thanks to the reverse Hölder inequality, there exists $p > 1$ such that $E_T \in L^p$. The maximal $p$ with this property can be expressed in terms of the BMO norm of $\Phi$. 

Adrien Richou Numerical simulation of quadratic BSDEs
Introduction

Different ideas for simulation

A new scheme

(Markovian) BSDEs

Simulation

Quadratic BSDEs

Theorem (Briand, Confortola (2008), Ankirchner and al. (2007))

We suppose that

\[ |f(t, x, y, z)| \leq M_f(1 + |y| + |z|^2), \]
\[ |f(t, x, y, z) - f(t, x', y', z')| \leq K_{f,x} |x - x'| + K_{f,y} |y - y'| \]
\[ + (K_{f,z} + L_{f,z}(|z| + |z'|)) |z - z'|, \]
\[ |g(x)| \leq M_g. \]

The SDE-BSDE system has a unique solution \((X, Y, Z)\) such that \(\mathbb{E}[\sup_{t \in [0, T]} |X|^2] < +\infty\), \(Y\) is a bounded measurable process and \(\mathbb{E}[\int_0^T |Z_s|^2 \, ds] < +\infty\). The martingale \(Z \ast W\) belongs to the space of BMO martingales and \(\|Z \ast W\|_{BMO}\) only depends on \(T\), \(M_g\) and \(M_f\). Moreover, there exists \(r > 1\) such that \(\mathcal{E}(Z \ast W) \in L^r\).
Proposition (Briand, Confortola (2008), Ankirchner and al. (2007))

If we denote \((Y^i, Z^i)\) the solution of a BSDE with a terminal condition \(g_i\) and a driver \(f_i\), then we have

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |Y^1_t - Y^2_t|^2 \right] + \mathbb{E}\left[ \int_0^T |Z^1_s - Z^2_s|^2 \, ds \right] \\
\leq \mathbb{E}\left[ |g_1(X_T) - g_2(X_T)|^{2q} + \left( \int_0^T |(f_1 - f_2)(s, X_s, Y^2_s, Z^2_s)| \, ds \right)^{2q} \right]^{1/q}.
\]

where \(1/r + 1/q = 1\).
The aim of our work is to give a time discretization scheme for quadratic BSDEs, and to obtain a “good” convergence rate for this scheme.
The exponential transformation

When the generator has the specific form

\[ f(t, x, y, z) = l(t, x, y) + a(t, z) + \frac{\gamma}{2} |z|^2, \]

with \(a\) and \(l\) Lipschitz functions and \(a\) homogeneous with respect to \(z\), it is possible to use an exponential transform (also known as the Cole-Hopf transformation) : \((e^{\gamma Y}, e^{\gamma Y} Z)\) is the solution of a BSDE with a driver of linear growth. See P. Imkeller, G. dos Reis and J. Zhang [2010].
If $g$ is a Lipschitz function with a Lipschitz constant $K_g$ and $\sigma$ does not depend on $x$, then, $\forall t \in [0, T]$, 

$$|Z_t| \leq C(1 + K_g).$$

In this situation the driver becomes a Lipschitz function with respect to $z$, and so we are allowed to use the classical discrete time approximation.
If $g$ is $\alpha$-Hölder, we have an explicit uniform Lipschitz approximation $g_N$ of $g$ with $K_{g_N} = N$. Then we do an approximation of $(Y, Z)$ by the solution $(Y^N, Z^N)$ to the BSDE

$$Y_t^N = g_N(X_T) + \int_t^T f(s, X_s, Y_s^N, Z_s^N)ds - \int_t^T Z_s^N dW_s.$$ 

- Thanks to BMO tools we have an error estimate for this approximation: $CN^{-\frac{\alpha}{1-\alpha}}$.
- We also need to have the error estimate for the time approximation of our BSDE with linear growth: $Ce^{CN^2}n^{-1}$.

Finally, if we take $N = \left(\frac{C}{\varepsilon} \log n\right)^{1/2}$ with $\varepsilon$ small, then the global error bound becomes

$$C_\varepsilon (\log n)^{-\frac{\alpha}{2(1-\alpha)}}.$$
An other idea is to do an approximation of \((Y, Z)\) by the solution \((Y^N, Z^N)\) to the truncated BSDE

\[
Y^N_t = g(X_T) + \int_t^T f(s, X_s, Y^N_s, h_N(Z^N_s))ds - \int_t^T Z^N_s dW_s,
\]

where \(h_N : \mathbb{R}^{1 \times d} \to \mathbb{R}^{1 \times d}\) is a smooth modification of the projection on the open Euclidean ball of radius \(N\) about 0. An error estimate is obtain by P. Imkeller and G. dos Reis [2009], but the same drawback appears.
Theorem (Delbaen, Hu, Bao (2010), R. (2010))

We suppose that $b$ is differentiable with respect to $x$ and $\sigma$ is differentiable with respect to $t$. There exists $\lambda \in \mathbb{R}^+$ such that

$$\forall \eta \in \mathbb{R}^d$$

$$\left| t \eta \sigma(s) \left[ t \sigma(s)^T \nabla b(s, x) - t \sigma'(s) \right] \eta \right| \leq \lambda \left| t \eta \sigma(s) \right|^2.$$  

Moreover, suppose that $g$ is lower (or upper) semi-continuous. Then, $\forall t \in [0, T[$,

$$|Z_t| \leq C_Z + C'_Z (T - t)^{-1/2}.$$
We suppose that

- $f$ does not depend on $x$ and $y$,
- $g$ is $C^1$ with respect to $x$ and $f$ is $C^1$ with respect to $z$.

Then $Y$ and $Z$ are differentiable with respect to $x$ the initial condition of $X$, and

\[ \nabla Y_t = \nabla g(X_T) \nabla X_T + \int_t^T \nabla_z f \nabla Z_s ds - \int_t^T \nabla Z_s d\tilde{W}_s \]

\[ = \nabla g(X_T) \nabla X_T - \int_t^T \nabla Z_s d\tilde{W}_s. \]

That is to say $\nabla Y$ is a $\mathbb{Q}$-martingale.
Thanks to the Malliavin calculus we have:

\[ Z_t = \nabla Y_t (\nabla X_t)^{-1} \sigma(t). \]

By applying the Itô formula to the process \(|e^{\lambda t} \nabla Y_t (\nabla X_t)^{-1} \sigma(t)|^2\), we show that \(|e^{\lambda t} Z_t|^2\) is a \(\mathbb{Q}\)-submartingale. Finally

\[
e^{2\lambda t} |Z_t|^2 (T - t) \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{2\lambda s} |Z_s|^2 \, ds \bigg| \mathcal{F}_t \right] \leq e^{2\lambda T} \|Z\|_{BMO(\mathbb{Q})}.
\]
This type of estimation is well known in the case of drivers with linear growth as a consequence of the Bismut-Elworthy formula. In our case, $\sigma$ does not depend on $x$ but we do not need to suppose that $\sigma$ is invertible.
How can we use this time-dependent estimate of $Z$?

In the Lipschitz case, to obtain a bound for the error

$$
\sup_{0 \leq k \leq n} \mathbb{E} \left[ \left| Y^n_{t_k} - Y^{t_k}_{t_k} \right|^2 \right]
$$

we show such an estimate:

$$
\mathbb{E} \left[ \left| Y^n_{t_k} - Y^{t_k}_{t_k} \right|^2 \right] \leq (1 + Ch + K^2_{f,z} h) \mathbb{E} \left[ \left| Y^n_{t_{k+1}} - Y^{t_{k+1}}_{t_{k+1}} \right|^2 \right] + h^2
$$

and then we use the Gronwall’s lemma.
In our case we have

\[
\mathbb{E} \left[ \left| Y_{t_k}^n - Y_{t_k} \right|^2 \right] \leq \left( 1 + C(t_k - t_{k+1}) + K \frac{t_{k+1} - t_k}{T - t_{k+1}} \right) \mathbb{E} \left[ \left| Y_{t_{k+1}}^n - Y_{t_{k+1}} \right|^2 \right] + h^2
\]

So, the idea is to find a new time net such that \( \frac{t_{k+1} - t_k}{T - t_{k+1}} \) is a constant: We define the \( n \) first discretization times by

\[
t_k = T \left( 1 - \left( \frac{\varepsilon}{T} \right)^{k/(n-1)} \right).
\]

\( \varepsilon \) is a parameter: \( t_{n-1} = T - \varepsilon \). We will set \( \varepsilon := T / n^a \) with \( a \) a parameter.
How can we use this time-dependent estimate of $Z$?

Lemma

$$\prod_{i=0}^{n-2} \left( 1 + C(t_{i+1} - t_i) + K \frac{t_{i+1} - t_i}{T - t_{i+1}} \right) \leq C n^{ak}.$$
Due to technical reason, we have to approximate our BSDE by an other one. Let \((Y_t^{N,\varepsilon}, Z_t^{N,\varepsilon})\) the solution of the BSDE

\[
Y_t^{N,\varepsilon} = g_N(X_T) + \int_t^T f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon}) ds - \int_t^T Z_s^{N,\varepsilon} dW_s.
\]

(3.2)

with

\[
f^\varepsilon(s, x, y, z) := \mathbb{1}_{s < T - \varepsilon} f(s, x, y, z) + \mathbb{1}_{s \geq T - \varepsilon} f(s, x, y, 0),
\]

and \(g_N\) a \(N\)-Lipschitz approximation of \(g\).
Our algorithm (2/2)

We denote $\rho_s : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^{1 \times d}$ the projection on the ball

$$B \left( 0, C_z + \frac{C'_z}{(T - s)^{1/2}} \right).$$

Finally we denote $(Y_{N,\varepsilon,n}, Z_{N,\varepsilon,n})$ our time approximation of $(Y_{N,\varepsilon}, Z_{N,\varepsilon})$. This couple is obtained by a slight modification of the classical dynamic programming equation:

$$Y_{t_{n}}^{N,\varepsilon,n} = g_{N}(X_{t_{n}}^{n})$$

$$Z_{t_{k}}^{N,\varepsilon,n} = \rho_{t_{k+1}} \left( \frac{1}{h_k} \mathbb{E}_{t_k} \left[ Y_{t_{k+1}}^{N,\varepsilon,n}(W_{t_{k+1}} - W_{t_k}) \right] \right), \quad 0 \leq k \leq n - 1,$$

$$Y_{t_{k}}^{N,\varepsilon,n} = \mathbb{E}_{t_{k}} \left[ Y_{t_{k+1}}^{N,\varepsilon,n} \right] + h_k \mathbb{E}_{t_{k}} \left[ f(t_k, X_{t_{k}}^{n}, Y_{t_{k+1}}^{N,\varepsilon,n}, Z_{t_{k}}^{N,\varepsilon,n}) \right], \quad 0 \leq k \leq n - 1.$$
A first speed of convergence

**Theorem**

Let us recall that $\varepsilon = T/n^a$. We set $N = n^b$. We assume that $g$ is $\alpha$-Hölder. Then we can set $a$ and $b$ such that for all $\eta > 0$, there exists a constant $C_\eta > 0$ that verifies

$$
\sup_{0 \leq k \leq n} \mathbb{E} \left[ \left| Y_{t_k}^{N,\varepsilon,n} - Y_{t_k} \right|^2 \right] + \sum_{k=0}^{n-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| Z_{t_k}^{N,\varepsilon,n} - Z_t \right|^2 dt \right] \leq \frac{C_\eta}{n^p},
$$

where

$$p = \frac{2\alpha}{(2 - \alpha)(2 + K(1 + \eta)) - 2 + 2\alpha}.
$$

$K$ is an explicit constant. It depends on constants that appear in assumptions on $g$ and $f$. 

Adrien Richou
Numerical simulation of quadratic BSDEs
A better speed of convergence

**Theorem**

*If, moreover, \( b \) is bounded, then we can take \( K \) as small as we want:*

\[
\sup_{0 \leq k \leq n} \mathbb{E} \left[ \left| Y_{t_k}^{N, \varepsilon, n} - Y_{t_k} \right|^2 \right] + \sum_{k=0}^{n-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left| Z_{t_k}^{N, \varepsilon, n} - Z_t \right|^2 dt \right] \leq \frac{C_n}{n^{\alpha-n}},
\]


A. R.,
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