

Numerical simulation of BSDEs with drivers of quadratic growth

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(W_t)_{t \in \mathbb{R}^+}$ be a Brownian motion in \mathbb{R}^d , $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be his augmented natural filtration, T be a nonnegative real number. We consider an SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

with standard assumptions on b and σ , and a Markovian BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

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Definition

A solution to this BSDE is a pair of processes $(Y_t, Z_t)_{0 \leq t \leq T}$ such that :

- 1 (Y, Z) is a predictable process with values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$,
- 2 \mathbb{P} - a.s. $t \mapsto Y_t$ is continuous and $\int_0^T |f(r, X_r, Y_r, Z_r)| + \|Z_r\|^2 dr < \infty$

Theorem (Pardoux-Peng 1990)

Let us assume that f is a Lipschitz function with respect to y and z and $\mathbb{E}\left[|g(X_T)|^2 + \int_0^T |f(r, X_r, 0, 0)|^2 dr\right] < \infty$. Then the previous equation has a unique solution (Y, Z) such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t|^2\right] < \infty, \quad \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty.$$

Time discretization

We consider a time discretization of the BSDE. We denote the time step by $h = T/n$ and $(t_k = kh)_{0 \leq k \leq n}$ stands for the discretization times. For X we take the Euler scheme :

$$\begin{aligned} X_0^n &= x \\ X_{t_{k+1}}^n &= X_{t_k}^n + hb(t_k, X_{t_k}^n) + \sigma(t_k, X_{t_k}^n)(W_{t_{k+1}} - W_{t_k}), \quad 0 \leq k \leq n. \end{aligned}$$

For (Y, Z) we use the classical dynamic programming equation

$$\begin{aligned} Y_{t_n}^n &= g(X_{t_n}^n) \\ Z_{t_k}^n &= \frac{1}{h} \mathbb{E}_{t_k} [Y_{t_{k+1}}^n (W_{t_{k+1}} - W_{t_k})], \quad 0 \leq k \leq n-1, \\ Y_{t_k}^n &= \mathbb{E}_{t_k} [Y_{t_{k+1}}^n] + h \mathbb{E}_{t_k} [f(t_k, X_{t_k}^n, Y_{t_{k+1}}^n, Z_{t_k}^n)], \quad 0 \leq k \leq n-1, \end{aligned}$$

where \mathbb{E}_{t_k} stands for the conditional expectation given \mathcal{F}_{t_k} .

Remarks on simulation

- the dynamic programming equation is obtained by minimizing the difference

$$\mathbb{E} \left[(Y_{t_{k+1}}^n + h\mathbb{E}_{t_k} f(t_k, X_{t_k}^n, Y_{t_{k+1}}^n, Z) - Y - Z(W_{t_{k+1}} - W_{t_k}))^2 \right]$$

over \mathcal{F}_{t_k} -measurable squared integrable random variables (Y, Z) .

- After time discretization, we need to use a spatial discretization in order to compute conditional expectation.
- We suppose that g and f are Lipschitz functions with respect to x, y, z and t . If we define the error

$$e(n) = \sup_{0 \leq k \leq n} \mathbb{E} |Y_{t_k}^n - Y_{t_k}|^2 + \mathbb{E} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |Z_{t_k}^n - Z_t|^2 dt$$

then $e(n) = O(1/n)$.

References for simulation

See, for exemple :

- B. Bouchard, N. Touzi [2004],
- J. Zhang [2005],
- E. Gobet, J.P. Lemor, X. Warin [2005],
- F. Delarue, S. Menozzi [2006].

Quadratic BSDEs

What happened if f has a quadratic growth with respect to z ?

- when g is bounded : existence and uniqueness results have been proved by M. Kobylanski [2000].
- when g is unbounded : an existence result has been proved by P. Briand and Y. Hu [2006], partial uniqueness results has been proved by P. Briand and Y. Hu [2008], F. Delbaen, Y. Hu and A. R. [2010].

Such BSDEs have applications in finance : this class arises, for example, in the context of utility optimization problems with exponential utility functions (see e.g. Y. Hu, P. Imkeller and M. Müller [2005]).

BMO tool

Definition

For a brownian martingale $\Phi_t = \int_0^t \phi_s dW_s$, $t \in [0, T]$, we say that Φ is a BMO martingale if

$$\|\Phi\|_{BMO} = \sup_{\tau \in [0, T]} \mathbb{E} \left[\int_{\tau}^T \phi_s^2 ds \middle| \mathcal{F}_{\tau} \right]^{1/2} < +\infty,$$

where the supremum is taken over all stopping times in $[0, T]$.

BMO tool

the very important feature of BMO martingales is the following lemma :

Lemma

Let Φ be a BMO martingale. Then we have :

- 1 The stochastic exponential

$$\mathcal{E}(\Phi)_t = \mathcal{E}_t = \exp \left(\int_0^t \phi_s dW_s - \frac{1}{2} \int_0^t |\phi_s|^2 ds \right), \quad 0 \leq t \leq T,$$

is a uniformly integrable martingale.

- 2 Thanks to the reverse Hölder inequality, there exists $p > 1$ such that $\mathcal{E}_T \in L^p$. The maximal p with this property can be expressed in terms of the BMO norm of Φ .

Theorem (Briand, Confortola (2008), Ankirchner and al. (2007))

We suppose that

$$\begin{aligned}
 |f(t, x, y, z)| &\leq M_f(1 + |y| + |z|^2), \\
 |f(t, x, y, z) - f(t, x', y', z')| &\leq K_{f,x} |x - x'| + K_{f,y} |y - y'| \\
 &\quad + (K_{f,z} + L_{f,z}(|z| + |z'|)) |z - z'|, \\
 |g(x)| &\leq M_g.
 \end{aligned}$$

*The SDE-BSDE system has a unique solution (X, Y, Z) such that $\mathbb{E}[\sup_{t \in [0, T]} |X|^2] < +\infty$, Y is a bounded measurable process and $\mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty$. The martingale $Z * W$ belongs to the space of BMO martingales and $\|Z * W\|_{BMO}$ only depends on T, M_g and M_f . Moreover, there exists $r > 1$ such that $\mathcal{E}(Z * W) \in L^r$.*

Proposition (Briand, Confortola (2008), Ankirchner and al. (2007))

If we denote (Y^i, Z^i) the solution of a BSDE with a terminal condition g_i and a driver f_i , then we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t^1 - Y_t^2|^2 \right] + \mathbb{E} \left[\int_0^T |Z_s^1 - Z_s^2|^2 ds \right] \\ & \leq \mathbb{E} \left[|g_1(X_T) - g_2(X_T)|^{2q} + \left(\int_0^T |(f_1 - f_2)(s, X_s, Y_s^2, Z_s^2)| ds \right)^{2q} \right]^{1/q}. \end{aligned}$$

where $1/r + 1/q = 1$.

Goal

The aim of our work is to give a time discretization scheme for quadratic BSDEs, and to obtain a “good” convergence rate for this scheme.

The exponential transformation

When the generator has the specific form

$$f(t, x, y, z) = l(t, x, y) + a(t, z) + \frac{\gamma}{2} |z|^2,$$

with a and l Lipschitz functions and a homogeneous with respect to z , it is possible to use an exponential transform (also known as the Cole-Hopf transformation) : $(e^{\gamma Y}, \gamma e^{\gamma Y} Z)$ is the solution of a BSDE with a driver of linear growth. See P. Imkeller, G. dos Reis and J. Zhang [2010].

g Lipschitz

Proposition

If g is a Lipschitz function with a Lipschitz constant K_g and σ does not depend on x , then, $\forall t \in [0, T]$,

$$|Z_t| \leq C(1 + K_g).$$

In this situation the driver becomes a Lipschitz function with respect to z , and so we are allowed to use the classical discrete time approximation.

If g is α -Hölder, we have an explicit uniform Lipschitz approximation g_N of g with $K_{g_N} = N$. Then we do an approximation of (Y, Z) by the solution (Y^N, Z^N) to the BSDE

$$Y_t^N = g_N(X_T) + \int_t^T f(s, X_s, Y_s^N, Z_s^N) ds - \int_t^T Z_s^N dW_s.$$

- Thanks to BMO tools we have an error estimate for this approximation : $CN^{\frac{-\alpha}{1-\alpha}}$.
- We also need to have the error estimate for the time approximation of our BSDE with linear growth : $Ce^{CN^2} n^{-1}$.

Finally, if we take $N = (\frac{C}{\varepsilon} \log n)^{1/2}$ with ε small, then the global error bound becomes

$$C_\varepsilon (\log n)^{\frac{-\alpha}{2(1-\alpha)}}.$$

Truncated BSDE

An other idea is to do an approximation of (Y, Z) by the solution (Y^N, Z^N) to the truncated BSDE

$$Y_t^N = g(X_T) + \int_t^T f(s, X_s, Y_s^N, h_N(Z_s^N)) ds - \int_t^T Z_s^N dW_s,$$

where $h_N : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^{1 \times d}$ is a smooth modification of the projection on the open Euclidean ball of radius N about 0. An error estimate is obtain by P. Imkeller and G. dos Reis [2009], but the same drawback appears.

A time-dependent estimate of Z

Theorem (Delbaen, Hu, Bao (2010), R. (2010))

We suppose that b is differentiable with respect to x and σ is differentiable with respect to t . There exists $\lambda \in \mathbb{R}^+$ such that
 $\forall \eta \in \mathbb{R}^d$

$$\left| {}^t\eta\sigma(s)[{}^t\sigma(s){}^t\nabla b(s, x) - {}^t\sigma'(s)]\eta \right| \leq \lambda |{}^t\eta\sigma(s)|^2. \quad (3.1)$$

Moreover, suppose that g is lower (or upper) semi-continuous. Then, $\forall t \in [0, T[$,

$$|Z_t| \leq C_Z + C'_Z(T - t)^{-1/2}.$$

Sketch of the proof (1/2)

We suppose that

- f does not depends on x and y ,
- g is C^1 with respect to x and f is C^1 with respect to z .

Then Y and Z are differentiable with respect to x the initial condition of X , and

$$\begin{aligned}\nabla Y_t &= \nabla g(X_T)\nabla X_T + \int_t^T \nabla_z f \nabla Z_s ds - \int_t^T \nabla Z_s dW_s \\ &= \nabla g(X_T)\nabla X_T - \int_t^T \nabla Z_s d\tilde{W}_s.\end{aligned}$$

That is to say ∇Y is a \mathbb{Q} -martingale.

Sketch of the proof (2/2)

Thanks to the Malliavin calculus we have :

$Z_t = \nabla Y_t (\nabla X_t)^{-1} \sigma(t)$. By applying the Itô formula to the process $|e^{\lambda t} \nabla Y_t (\nabla X_t)^{-1} \sigma(t)|^2$, we show that $|e^{\lambda t} Z_t|^2$ is a \mathbb{Q} -submartingale. Finally

$$e^{2\lambda t} |Z_t|^2 (T-t) \leq \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{2\lambda s} |Z_s|^2 ds \middle| \mathcal{F}_t \right] \leq e^{2\lambda T} \|Z\|_{BMO(\mathbb{Q})}.$$

Remark

This type of estimation is well known in the case of drivers with linear growth as a consequence of the Bismut-Elworthy formula. In our case, σ does not depend on x but we do not need to suppose that σ is invertible.

How can we use this time-dependent estimate of Z ?

In the Lipschitz case, to obtain a bound for the error

$$\sup_{0 \leq k \leq n} \mathbb{E} \left[|Y_{t_k}^n - Y_{t_k}|^2 \right]$$

we show such an estimate :

$$\mathbb{E} \left[|Y_{t_k}^n - Y_{t_k}|^2 \right] \leq (1 + Ch + K_{f,Z}^2 h) \mathbb{E} \left[|Y_{t_{k+1}}^n - Y_{t_{k+1}}|^2 \right] + h^2$$

and then we use the Gronwall's lemma.

How can we use this time-dependent estimate of Z ?

In our case we have

$$\mathbb{E} \left[|Y_{t_k}^n - Y_{t_k}|^2 \right] \leq (1 + C(t_k - t_{k+1})) + K \frac{t_{k+1} - t_k}{T - t_{k+1}} \mathbb{E} \left[|Y_{t_{k+1}}^n - Y_{t_{k+1}}|^2 \right] + h^2$$

So, the idea is to find a new time net such that $\frac{t_{k+1} - t_k}{T - t_{k+1}}$ is a constant : We define the n first discretization times by

$$t_k = T \left(1 - \left(\frac{\varepsilon}{T} \right)^{k/(n-1)} \right).$$

ε is a parameter : $t_{n-1} = T - \varepsilon$. We will set $\varepsilon := T/n^a$ with a a parameter.

How can we use this time-dependent estimate of Z ?

Lemma

$$\prod_{i=0}^{n-2} \left(1 + C(t_{i+1} - t_i) + K \frac{t_{i+1} - t_i}{T - t_{i+1}} \right) \leq Cn^{aK}.$$

Our algorithm (1/2)

Due to technical reason, we have to approximate our BSDE by an other one. Let $(Y_t^{N,\varepsilon}, Z_t^{N,\varepsilon})$ the solution of the BSDE

$$Y_t^{N,\varepsilon} = g_N(X_T) + \int_t^T f^\varepsilon(s, X_s, Y_s^{N,\varepsilon}, Z_s^{N,\varepsilon}) ds - \int_t^T Z_s^{N,\varepsilon} dW_s. \quad (3.2)$$

with

$$f^\varepsilon(s, x, y, z) := \mathbb{1}_{s < T-\varepsilon} f(s, x, y, z) + \mathbb{1}_{s \geq T-\varepsilon} f(s, x, y, 0),$$

and g_N a N -Lipschitz approximation of g .

Our algorithm (2/2)

We denote $\rho_s : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^{1 \times d}$ the projection on the ball

$$B\left(0, C_z + \frac{C'_z}{(T-s)^{1/2}}\right).$$

Finally we denote $(Y^{N,\varepsilon,n}, Z^{N,\varepsilon,n})$ our time approximation of $(Y^{N,\varepsilon}, Z^{N,\varepsilon})$. This couple is obtained by a slight modification of the classical dynamic programming equation :

$$\begin{aligned} Y_{t_n}^{N,\varepsilon,n} &= g_N(X_{t_n}^n) \\ Z_{t_k}^{N,\varepsilon,n} &= \rho_{t_{k+1}} \left(\frac{1}{h_k} \mathbb{E}_{t_k} [Y_{t_{k+1}}^{N,\varepsilon,n} (W_{t_{k+1}} - W_{t_k})] \right), \quad 0 \leq k \leq n-1, \\ Y_{t_k}^{N,\varepsilon,n} &= \mathbb{E}_{t_k} [Y_{t_{k+1}}^{N,\varepsilon,n}] + h_k \mathbb{E}_{t_k} [f(t_k, X_{t_k}^n, Y_{t_{k+1}}^{N,\varepsilon,n}, Z_{t_k}^{N,\varepsilon,n})], \quad 0 \leq k \leq n-1 \end{aligned}$$

A first speed of convergence

Theorem

Let us recall that $\varepsilon = T/n^a$. We set $N = n^b$. We assume that g is α -Hölder. Then we can set a and b such that for all $\eta > 0$, there exists a constant $C_\eta > 0$ that verifies

$$\sup_{0 \leq k \leq n} \mathbb{E} \left[\left| Y_{t_k}^{N,\varepsilon,n} - Y_{t_k} \right|^2 \right] + \sum_{k=0}^{n-1} \mathbb{E} \left[\int_{t_k}^{t_{k+1}} \left| Z_{t_k}^{N,\varepsilon,n} - Z_t \right|^2 dt \right] \leq \frac{C_\eta}{n^p},$$

where

$$p = \frac{2\alpha}{(2 - \alpha)(2 + K(1 + \eta)) - 2 + 2\alpha}.$$

K is an explicit constant. It depends on constants that appear in assumptions on g and f .




A better speed of convergence

Theorem

If, moreover, b is bounded, then we can take K as small as we want :

$$\sup_{0 \leq k \leq n} \mathbb{E} \left[\left| Y_{t_k}^{N,\varepsilon,n} - Y_{t_k} \right|^2 \right] + \sum_{k=0}^{n-1} \mathbb{E} \left[\int_{t_k}^{t_{k+1}} \left| Z_{t_k}^{N,\varepsilon,n} - Z_t \right|^2 dt \right] \leq \frac{C_\eta}{n^{\alpha-\eta}},$$

References (1/2)

-  P. Briand and F. Confortola.
BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces.
Stochastic Process. Appl., 118(5) :818–838, 2008.
-  P. Imkeller and G. dos Reis.
Path regularity and explicit convergence rate for bsde with truncated quadratic growth.
Stochastic Process. Appl., In Press, Corrected Proof, 2009.
-  F. Delbaen, Y. Hu, and X. Bao.
Backward SDEs with superquadratic growth.
To appear in *Probab. Theory Related Fields*, 2009.

References (2/2)



A. R.,

Numerical simulation of BSDEs with drivers of quadratic growth.

[arXiv :1001.0401v2](https://arxiv.org/abs/1001.0401v2)