Thesis report for Joyce ASSAAD. by Stefanie Petermichl

Transformées de Riesz associées aux opérateurs de Schroedinger avec des potentiels négatifs

In her thesis, the candidate studies boundedness of Riesz transforms associated to Schroedinger operators in a variety of settings, such as weighted and unweighted Euclidean Lebesgue spaces as well as Lebesgue spaces over manifolds. The main new part of the project is to allow for the potential to have negative parts, which introduces substantial additional difficulties, so that not much is known to date. Such operators are not necessarily represented by a kernel, methods using Gaussian estimates, Calderon Zygmund or singular integral theory do not apply or at least not in a direct way. In this light, the subject is daring for a thesis. A very broad background in modern harmonic analysis is required to be able to tackle such problems. The candidate demonstrates a great overview of both her area of interest as well as the origin and use of the tools needed for the proofs. Even more exotic results with very non standard proofs are known in detail to the candidate.

In the first chapter ASSAAD gives a very thorough introduction to her area of interest. The candidate first gives a summary of known results.

First, she describes various results of Ouhabez, Duoung, Yan ($1 < p \leq 2$ with very weak conditions on the potential) as well as Auscher and Ben Ali (a range of $p > 2$ with harder restrictions on the potential) concerning the boundedness of Riesz transforms in euclidean space associated to Schroedinger operators with non-negative potential. A result by Dragicevic and Volberg on the Hermite operator that covers all p is also mentioned.

Then, the candidate covers the known results on Riesz transforms on homogeneous Riemannian varieties, again with non-negative potential. Results of Coulhon, Duong and also of Ouhabez for the range $1 < p \leq 2$ are mentioned as well as a group of results considering a range of $p > 2$ by Carron and collaborators as well as Badr and Ben Ali.

It follows an illustration of results in weighted euclidean space. A summary is given on results of the classical Riesz transform as well as the fact that the Riesz transform associated to a non negative potential belonging to certain reverse Holder classes is, a Calderon Zygmund operator and thus automatically bounded.

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Next, the candidate describes the results of Auscher and Martell of boundedness of classical Riesz transforms on weighted Lebesgue spaces over homogeneous Riemannian varieties whose heat kernel has gaussian estimates.

In the last subsection of this introductory chapter, ASSAAD reviews what little is known if allowing negative potentials and outlines her own contributions.

In chapter 2, the candidate gives a careful review of operators and their associated semi groups as well as a motivation and definition of Riesz transforms associated to Schroedinger operators. A collection of criterions to show boundedness of Riesz transforms on certain manifolds is also given. A result of singular integral type is stated as well as a result for non-kernel operators with off diagonal estimates.

Chapter 3 consists of a separate paper with title "Riesz transforms associated to Schroedinger operators with negative potentials". ASSAAD considers for $N \geq 3$ Riesz transforms $\nabla A^{-1/2}$ associated to $A = -\Delta - V$ where V is non negative and strongly subcritical as well as operator $V^1/2A^{-1/2}$. A critical exponent p_0 that depends upon the dimension N comes into play that arises on the level of study of the associated semigroup. For exponents in (p_0', p_0) one has uniform boundedness while as the semigroup does not even act on L^p for other p. In her result, the candidate manages to prove boundedness of the Riesz transforms for the resulting natural range of $(p_0', 2]$ and gives a negative result for $(1, p_0') \cup (p_0N/(N + p_0), \infty)$. Better ranges of p are obtained for V in a Kato subclass. The first result is then extended to Lebesgue spaces on manifold M of homogeneous type with Sobolev inequality. Additional hypotheses are needed for the second result. Ingredients of the proof are delicate off diagonal estimates typical to non kernel operators and interpolation.

Chapter 4 considers the manifold setting. Riesz transforms arising from Schroedinger operators with signed potential on complete Riemannian manifolds of homogeneous type are considered. For L^p spaces with $p < 2$ the candidate obtains a positive result just with minimal conditions on the potential. The full range for large enough dimension, for very small dimension, the range is an interval $(p_0, 2]$ where p_0 in turn depends on a parameter in the definition of subcritical potentials. The case $p > 2$ is also considered, where additional assumptions both on the manifold and the potential are needed. This case exceeds the area of my expertise and I prefer to leave the evaluation of this chapter to other reporters.

Chapter 5 contains a thorough review of Singular integral/sublinear op-

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erator theory on weighted spaces. Reverse Hoelder classes and A_p classes with extrapolation theorems are reviewed. Both the classical case of kernel operators and the much newer considerations of Auscher and Martell on non-kernel operators are discussed. Then, the question of boundedness of the Riesz transform associated to Schroedinger operator with negative potential in weighted Lebesgue spaces is discussed. Due to extrapolation and counterexamples in the non weighted case for some p one cannot hope to cover all p with just the weakest condition of 'strongly subcritical' on the potential. The candidate manages to show for dimension $N \geq 3$ boundedness for all $p \in (N', N)$ as long as the potential belongs to all L^q with $q = N/2-\epsilon, N/2+\epsilon$ and as long as the weight belongs to $A_{p/N} \cap RH_{(N/p)'}$. The proof is delicate and considers the difference to the classical Riesz transform without potential. Various relatively new criterions are used to control these operators. Boundedness of $A^{-a/2}$ is also discussed.

This is an excellent thesis from all angles. The results are very strong and the candidate investigates a very new direction where previous results are scarce due to the lack of availability of certain tools in this negative potential situation. The thesis is extremely well written, the exposition of the proofs is clear and the reader enjoys to be provided with a meaningful summary of the history and development of the boundedness of Riesz transforms associated to Schroedinger operators.

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