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Thesis report

TRANSFORMÉES DE RIESZ ASSOCIÈES AUX OPÉRATEURS DE SCHRÖDINGER AVEC DES POTENTIELS NÉGATIFS

by

Joyce Assaad

The thesis of Joyce Assaad deals with the boundedness on the Lebesgue spaces of the Riesz transforms associated to Schrödinger operators with potentials having a nontrivial negative part.

Riesz transforms are an important topic of investigation in harmonic analysis. In classical harmonic analysis they arise as generalizations to higher dimension of the Hilbert transform and play a key role in the study of the regularity of the solutions of partial differential equations. The boundedness of the Riesz transforms associated to the Laplacian on the Euclidean space follows from the classical theory of singular integrals, developed by Calderòn-Zygmund and the mathematicians of their school in the fifties. In more recent years this is still a very active research topic, which has attracted the attention of some of the leading researchers in the fields of harmonic analysis, partial differential equations, probability. Reaserchers have endeavoured to extend the theory to more general settings, replacing the Laplacian on the Euclidean space by other operators, such as second order operators associated to orthogonal expansions (Hermite, Laguerre, Jacobi expansions), invariant sub-Laplacians on Lie groups, Laplace-Beltrami operators on Riemannian manifolds, the De-Rham operator on differential forms, non self-adjoint operators with variable coefficients.

The analysis of the Riesz transforms associated to Schrödinger operators $-\Delta + V$ is fairly more recent, since it has been initiated only ten-odd years ago. All the results so far obtained are essentially concerned with Schrödinger operators with nonnegative potential V.

The main purpose of the thesis of Joyce Assaad is to address the study of the L^p boundedness of the first order Riesz transforms associated to Schrödinger operators with potentials that have a nontrivial negative part. The thesis is composed by five chapters, beginning by an introduction and revision of the state-of-the-art in the field. The third chapter deals with the problem on Euclidean space. In the fourth chapter she considers Schrödinger operators on doubling Riemannian manifolds. In the last chapter she proves estimates on weighted Euclidean space and investigates also the boundedness on weighted Lebesgue spaces of negative powers of the Schrödinger operator and the bounded holomorphic functional calculus.

First, she obtains results under the sole assumption that the potential V be "strongly subcritical". This means that $-\Delta + V \geq \epsilon V$, for some $\epsilon > 0$, in the sense of quadratic forms. This is a quite natural assumption, because it is equivalent to the boundedness of the Riesz transform on L^2. Then she proves that the range of p for which the Riesz transform is bounded on L^p can be improved, if the potential satisfies further regularity assumptions (V in a suitable Kato class or in a weak Lebesgue space). On doubling Riemannian manifolds she proves similar results. Here too she proves that the range of p for which the Riesz transform is bounded can be improved, if one makes further assumptions on the potential V or on the manifold (the Poincaré inequality holds, M has polynomial volume). It is noteworthy that, by providing counterexamples, the author proves also that the range of p for which she obtains the boundedness of the Riesz transforms is optimal.

It is important to remark that, when the potential has a nontrivial negative part, the Riesz transforms do not have a kernel, in the classical sense of the Calderòn-Zygmund theory. Therefore, the classical methods based on kernel estimates are not applicable. Thus, to achieve her goal, J. Assaad must adapt to her situation some new techniques developed recently by various authors, notably Blunck and Kunstmann, Auscher and Martell, Auscher, Coulhon, Duong and Hofmann, to deal with singular operators without integral kernel.

In my opinion this thesis is a fairly exhaustive treatment of the L^p boundedness of the Riesz transforms for Schrodinger operators with non positive potential in a variety of settings. The results are very interesting and will lead to at least three original papers that deserve to be published in good quality journals. The author has been able to master some recent and sophisticated techniques and to adapt them to the problem considered in the thesis. The proofs are correct, to the best of my knowledge, and the exposition is in general clear, but it could improved in a some points which I shall detail in a separate file.

In conclusion, it is my sincere opinion that Ms. Joyce Assaad fully deserves the degree of doctor by the University of Bordeaux I.