Research statement

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1 Introduction

My field of research is enumeration of number fields by increasing discriminant. Let $n \geq 2$ be an integer and let $\mathscr{C}_n$ denote the set of isomorphism classes of number fields of degree n. For any positive real number X, let $\mathscr{C}_n(X)$ be the set of elements of $\mathscr{C}_n$ whose discriminant has absolute value at most X, and let $N_n(X)$ be its cardinality. From a theoretical point of view, we are interested in understanding the asymptotic behaviour of $N_n(X)$ when $X \to \infty$; from a practical point of view, we look for efficient algorithms to describe all elements of $\mathscr{C}_n(X)$.

These two problems are easy when $n = 2$, that is, in the case of quadratic fields. Indeed, there is a very simple classification of quadratic fields: they are the fields of the form $\mathbb{Q}(\sqrt{d})$ for $d \neq 1$ a squarefree integer. Moreover, the discriminant of $\mathbb{Q}(\sqrt{d})$ is equal to d if d is congruent to 1 modulo 4, and to $4d$ otherwise. From this it is possible to prove that $N_2(X) \sim \frac{1}{\zeta(2)}X$, and to enumerate $\mathscr{C}_2(X)$ in optimal time $O(X^{1+\varepsilon})$.

A classical conjecture predicts that for all $n \geq 2$, there should exist a constant $c_n > 0$ such that $N_n(X) \sim c_n X$. This conjecture remains open for $n \geq 6$. Ellenberg and Venkatesh [16] proved that $N_n(X) \gg X^{\frac{1}{2}+\frac{1}{n^2}}$, and the best upper bound known in arbitrary degree, due to Lemke Oliver and Thorne [23], is of the form $N_n(X) \ll X^{O(\log^2 n)}$.

From an algorithmic point of view, it is natural to conjecture that there should exist a way to enumerate the set $\mathscr{C}_n(X)$ using $O(X^{1+\varepsilon})$ elementary operations, which would match the expected size of the output and would therefore be optimal. The best general algorithm currently known is that of Hunter-Pohst-Martinet [20, 26, 24], which runs in time $O\big(X^{\frac{n+2}{4}+\varepsilon}\big)$.

The above questions have been solved in a satisfactory way for $n = 3$, and considerable progress has been made for $n = 4$ and $n = 5$. I will first describe the case of cubic fields, which is classical, then I will show how the various pieces generalise to quartic and quintic fields. Along the way I will present my thesis work. Finally I will describe new research directions.

2 Cubic fields

Enumeration of cubic fields by increasing discriminant is now a well-understood subject. The starting point is a remarkable bijection, discovered by Delone and Faddeev [14], between the set $\mathscr{C}_3$ of isomorphism classes of cubic fields and certain classes of binary cubic forms.

More precisely, let $V_{\mathbb{Z}}$ be the set of binary cubic forms with integer coefficients. The group $G_{\mathbb{Z}} = GL_2(\mathbb{Z})$ acts on $V_{\mathbb{Z}}$ by composition. Given any element of $V_{\mathbb{Z}}$, its irreducibility and its discriminant only depend on its class under this action, and there is a canonical, discriminant-preserving bijection between $\mathscr{C}_3$ and the set of equivalence classes of irreducible forms satisfying certain congruence relations.

To take advantage of this result, it is convenient to have at hand a fundamental domain $\mathscr{D}$ for the action of $G_{\mathbb{Z}}$ on the space $V_{\mathbb{R}}$ of binary cubic forms with real coefficients, or at least on the open subset $V_{\mathbb{R}}^* \subset V_{\mathbb{R}}$ of forms of nonzero discriminant. Such a fundamental domain can be constructed by combining the reduction theories of Hermite [17, 19] for positive discriminants and Mathews-Berwick [25] for negative discriminants.

Thanks to this fundamental domain, one obtains a bijection between $\mathscr{C}_3$ and certain irreducible integral points of $\mathscr{D}$, and, for all $X \geq 0$, a bijection between $\mathscr{C}_3(X)$ and certain irreducible integral points of $\mathscr{D}(X)$, where $\mathscr{D}(X)$ denotes the set of points of $\mathscr{D}$ whose discriminant has absolute value at most X. It is then natural to try to estimate $N_3(X)$ from the volume of $\mathscr{D}(X)$. In this way, Davenport and Heilbronn [12] could show, building on previous work of Davenport [10, 11], that $N_3(X) \sim \frac{1}{3\zeta(3)}X$.

From an algorithmic point of view, enumerating $\mathscr{C}_3(X)$ amounts to enumerating all irreducible integral points of $\mathscr{D}(X)$. Using this approach, Belabas [1] implemented an enumeration algorithm for cubic fields which runs in optimal time $O(X^{1+\varepsilon})$, and built a table of all cubic fields having discriminant at most 10^{11} in absolute value.

The set $\mathscr{D}(X)$, which plays an essential role in the study of cubic fields, is defined by complicated polynomial inequalities. It is useful to approximate it by a slightly larger domain described by simpler inequalities. Thus Davenport [10, 11] showed that every element $f = ax^3 + bx^2y + cxy^2 + dy^3$ of $\mathscr{D}(X)$ satisfies a *monomial* system of the form:

$$\begin{aligned} |a| &\ll X^{\frac{1}{4}}, & |b| &\ll X^{\frac{1}{4}}, \\ |ad| &\ll X^{\frac{1}{2}}, & |bc| &\ll X^{\frac{1}{2}}, \\ |ac^3| &\ll X, & |b^3d| &\ll X. \end{aligned} \tag{1}$$

Every irreducible integral point of $\mathscr{D}(X)$ corresponds to a integral solution of (1) such that $a \neq 0$, and it is not difficult to see that the number of such solutions is $O(X\log(X))$. Moreover, since the system (1) is monomial, its integral solutions can be enumerated in optimal time. This provides an algorithm to enumerate $\mathscr{C}_3(X)$ in time $O(X^{1+\varepsilon})$.

Besides the inequalities (1), Davenport also proved, under the same hypotheses, an additional, non-monomial inequality, which depends on the sign of disc(f):

$$\begin{aligned} c^2|bc - 9ad| &\ll X \qquad \text{if } \operatorname{disc}(f) > 0, \\ c^2|bc - ad| &\ll X \qquad \text{if } \operatorname{disc}(f) < 0. \end{aligned} \tag{2}$$

The number of integral solutions of (1) and (2) such that $a \neq 0$ is now $O(X)$. This can be used to improve the running time of the above algorithm. Let me stress that further improvements are possible, and that the algorithm of Belabas cannot be reduced to the ideas presented here.

3 Bhargava's bijections

In his thesis [2], Bhargava gave a list of parametrisations of arithmetic objects by integral orbits of algebraic representations. Two items of this list, which were extensively studied in later articles [4, 6], are concerned with quartic and quintic fields and are entirely analogous to the bijection of Delone and Faddeev for cubic fields.

Thus, in the case of quartic fields, $V_\mathbb{Z}$ is the set of pairs of ternary quadratic forms with integer coefficients. Such a pair can be seen as a map from $\mathbb{Z}^3$ to $\mathbb{Z}^2$, so that $V_\mathbb{Z}$ is naturally endowed with an action of $G_\mathbb{Z} = GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$. There is a notion of irreducibility on $V_\mathbb{Z}$, as well as a polynomial, called the discriminant, both of which are invariant under the action of $G_\mathbb{Z}$. In this setting, Bhargava constructed a canonical, discriminant-preserving bijection between $\mathscr{C}_4$ and the set of irreducible equivalence classes of $V_\mathbb{Z}$ satisfying certain congruence relations.

The case of quintic fields is similar; $V_\mathbb{Z}$ is now the set of quadruples of quinary alternating 2-forms with integer coefficients, on which $G_\mathbb{Z} = GL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$ acts in the natural way.

Using these bijections, Bhargava succeeded in proving the conjecture $N_n(X) \sim c_n X$ for $n = 4$ [5] and $n = 5$ [7], and gave formulas for the constants involved. The proof is based on the same geometry-of-numbers principle as that of Davenport and Heilbronn, but is technically more difficult and among other things involves a continuous family of fundamental domains.

Let us mention that similar parametrisations and counting techniques later enabled Bhargava to obtain other spectacular results in arithmetic statistics, especially bounds on the average rank of elliptic curves over $\mathbb{Q}$ [8, 9]. These achievements earned him the Fields medal in 2014.

In a short article [3], Bhargava emphasised the potential algorithmic applications of his bijections. In particular, they should allow to enumerate quartic and quintic fields in time $O(X^{1+\varepsilon})$. As will be apparent from the following sections, I was able to partially realise this goal for $n = 4$, through a strategy very analogous to that described above for cubic fields.

4 Reduction theories

To make algorithmic use of Bhargava's bijections, it is helpful to have at hand a good reduction theory for the action of $G_\mathbb{Z}$ on the open subset $V_\mathbb{R}^* \subset V_\mathbb{R}$ of elements of nonzero discriminant. In particular, one has to be able to decide efficiently whether a given element is reduced. The fundamental domains used by Bhargava do not fulfill this condition and are therefore not very practical.

The idea behind the reduction theories of Hermite and Mathews-Berwick used in the cubic case is to somehow reduce to Gauss's reduction of positive definite binary quadratic forms. Let f be a binary cubic form with real coefficients such that $\mathrm{disc}(f) > 0$ (respectively $\mathrm{disc}(f) < 0$). One attaches to f a positive definite quadratic form $c(f) = \alpha x^2 + \beta xy + \gamma y^2$, called its *covariant*, which depends on f equivariantly with respect to the action of $GL_2(\mathbb{Z})$. Then one defines f to be reduced if $c(f)$ belongs to the fundamental domain of Gauss, which is described by the inequalities $0 \le \beta \le \alpha \le \gamma$.

Hermite's covariant turns out to be more natural than Mathews-Berwick's, and lends itself to various generalisations and reinterpretations. First it can be extended to all binary forms f of degree d with complex coefficients which are *stable*, that is, whose zeros in $\mathbb{P}^1(\mathbb{C})$ have multiplicity less than $\frac{d}{2}$. In this case, $c(f)$ is a positive definite hermitian form, which depends on f equivariantly with respect to the actions of $\mathrm{GL}_2(\mathbb{C})$ and of complex conjugation. This construction results from work of Hermite [17, 18], Julia [21], and more recently Stoll and Cremona [29], who gave a remarkable geometric interpretation of it.

If f has real coefficients, equivariance with respect to complex conjugation shows that its covariant $c(f)$ also has real coefficients. By defining f to be reduced if $c(f)$ is reduced in the sense of Gauss, one obtains a reduction theory for the action of $\mathrm{GL}_2(\mathbb{Z})$ on stable binary forms with real coefficients. When restricted to cubic forms of negative discriminant, this reduction theory is different from that of Mathews-Berwick, over which it has certain advantages.

A nonzero binary form of degree d with complex coefficients, modulo scaling, can be identified with a formal sum of d points in $\mathbb{P}^1(\mathbb{C})$, that is, with a positive zero-cycle of degree d. Thus the covariant studied by Stoll and Cremona [29] can be seen as an $\mathrm{SL}_2(\mathbb{C})$-equivariant map from the set of stable positive zero-cycles of $\mathbb{P}^1(\mathbb{C})$ to the set of positive definite hermitian forms of determinant 1.

Now it turns out that this covariant extends to an $\mathrm{SL}_n(\mathbb{C})$-equivariant map from the set of positive zero-cycles of $\mathbb{P}^{n-1}(\mathbb{C})$ satisfying a certain stability property to the set of positive definite hermitian forms of determinant 1 in n variables, for all $n \geq 2$. This extension is due to Stoll [28]. Unfortunately, his approach is purely formal and lacks the geometric intuition of Stoll and Cremona [29].

Stoll's work provides a reduction theory for the action of $\mathrm{SL}_n(\mathbb{Z})$ on those stable positive zero-cycles of $\mathbb{P}^{n-1}(\mathbb{C})$ which are invariant under complex conjugation. Indeed, the covariant of such a zero-cycle has real coefficients; but there exist generalisations of Gauss's reduction for the action of $\mathrm{SL}_n(\mathbb{Z})$ on positive definite quadratic forms in n variables: two such generalisations are the reduction theory of Hermite-Korkine-Zolotarev and that of Minkowski.

The above yields a reduction theory for the action of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$ on the set of pairs of ternary quadratic forms with real coefficients and nonzero discriminant, which appears in Bhargava's bijection for quartic fields. Indeed, let $(\mathrm{F}_1, \mathrm{F}_2)$ be such a pair. On the one hand, the binary cubic form $f(x,y) = \det(x\mathrm{F}_1 + y\mathrm{F}_2)$ has nonzero discriminant; on the other hand, the formal sum Z of the four common zeros of F_1 and F_2 in $\mathbb{P}^2(\mathbb{C})$ is stable and invariant under complex conjugation. So one can define $(\mathrm{F}_1, \mathrm{F}_2)$ to be reduced if f and Z are so according to the reduction theories described previously.

It is very likely that the ideas of Stoll also allow to define a satisfactory reduction theory in the case of quintic fields, but I am not yet able to describe it in detail.

5 Reduction of positive zero-cycles in the grassmannian

As part of my thesis work, I extended Stoll's covariant to all positive zero-cycles of the Grassmannian $\mathrm{Gr}_n(\mathbb{C})$, that is, to all formal sums of linear subspaces of $\mathbb{C}^n$, provided that a certain stability condition is satisfied. I will briefly summarise my approach, which takes inspiration from the geometric ideas of Stoll and Cremona [29].

First of all, the set $\mathscr{H}_n$ of positive definite hermitian forms of determinant 1 in n variables can be identified with the quotient $\mathrm{SU}(n)\backslash \mathrm{SL}_n(\mathbb{C})$. It is a Hadamard manifold, that means a complete, simply connected, nonpositively curved Riemannian manifold. Its boundary at infinity in the sense of Eberlein and O'Neill [15] can be identified with the flag complex of $\mathbb{C}^n$, and thus contains all linear subspaces of $\mathbb{C}^n$ which are *proper*, that is neither 0 nor $\mathbb{C}^n$.

Given any proper subspace V of $\mathbb{C}^n$, seen as a point in the boundary at infinity of $\mathscr{H}_n$, one can introduce the vector field F_V on $\mathscr{H}_n$ whose norm is identically 1 and which points to V everywhere. It is enlightening to see F_V as a force field acting on a particle lying in $\mathscr{H}_n$ and attracted to the point V. Then it turns out that this force is *conservative*, meaning that it equals the negative of the gradient of a potential energy E_V, which is actually the Busemann function of V, well defined up to an additive constant.

To every positive zero-cycle Z of $\mathrm{Gr}_n(\mathbb{C})$ one can associate a vector field F_Z on $\mathscr{H}_n$, defined as a certain weighted sum of the fields F_V attached to the proper components of Z. Next, one shows that if Z is stable, then F_Z vanishes at a unique point, which is a global minimum of the corresponding potential energy E_Z. It is this point which is defined to be the covariant of Z.

I realised that the above is actually a special case of a very general construction due to Kempf and Ness [22]. Let G be a connected reductive algebraic group over $\mathbb{C}$, and let K be a maximal compact subgroup of G and V a finite-dimensional algebraic representation of G, together with a hermitian norm invariant under K. Given any nonzero vector $v \in V$, its Kempf-Ness function $\kappa_v : K\backslash G \to \mathbb{R}$ is defined by $\kappa_v(\overline{g}) = \log\|g \cdot v\|$.

One can show that v is stable, which means that the orbital map $g \mapsto g \cdot v$ is proper, if and only if κ_v has a unique minimum point. The function which maps any stable element v to the minimum point of κ_v is clearly G-equivariant. Moreover, the stability condition for v and the minimum point of κ_v, if it exists, only depend on the line $\mathbb{C}v$. Thus one obtains a covariant $\mathbb{P}(V) \to K\backslash G$, well defined on stable lines. This provides reduction theories in many contexts.

The connection with my construction is that a positive zero-cycle Z of $\mathrm{Gr}_n(\mathbb{C})$ can be seen, via the Plücker and the Segré embeddings, as a point in a certain projective space; then the associated Kempf-Ness function, well defined up to an additive constant, is none other than E_Z. Thus my covariant coincides with that of Kempf-Ness.

6 Monomial systems

Let us come back to Bhargava's bijection for quartic fields. We saw how to obtain a reduction theory for the action of $G_{\mathbb{Z}}$ on the open subset $V_{\mathbb{R}}^* \subset V_{\mathbb{R}}$ of elements of nonzero discriminant. Let $\mathscr{D}$ be the corresponding fundamental domain. As in the cubic case, enumerating $\mathscr{C}_4(X)$ amounts to enumerating all irreducible integral points of $\mathscr{D}(X)$.

There exists an over-approximation of $\mathscr{D}(X)$, defined by monomial inequalities, which is also valid for the fundamental domains used by Bhargava, except for the constants involved, and which appears in his article [5]. I showed that this over-approximation can be described by 178 inequalities analogous to those of the system (1).

It would obviously be very painful to estimate the number of solutions of these inequalities by hand. Fortunately, this can be done systematically using the next result, which follows from work of de la Bretèche [13].

Theorem 1. *Let* $(\alpha_{i,j})_{\substack{1\le i\le m\\ 1\le j\le n}}$ *be a family of nonnegative integers, and for any* $X \geq 0$*, let* $N(X)$ *be the number of* n*-tuples of positive integers satisfying the inequalities:*

$$x_1^{\alpha_{1,1}} \dots x_n^{\alpha_{1,n}} \leq X^{\alpha_{1,1}+\dots+\alpha_{1,n}},$$

$$\vdots$$

$$x_1^{\alpha_{m,1}} \dots x_n^{\alpha_{m,n}} \leq X^{\alpha_{m,1}+\dots+\alpha_{m,n}}.$$

Let C *be the convex cone of* $\mathbb{R}^n$ *generated by the* m *vectors* $(\alpha_{i,1}, \dots, \alpha_{i,n})$*, and:*

$$\mu = \min_{\substack{(t_1,\dots,t_n)\in C\\ t_1\geq 1,\dots,t_n\geq 1}} t_1 + \dots + t_n.$$

For all $\varepsilon > 0$*,* $N(X) = O(X^{\mu+\varepsilon})$*.*

Thanks to this result, I could show that the number of integral points in the monomial over-approximation of $\mathscr{D}(X)$ satisfying certain non-vanishing conditions which are necessary for irreducibility is $O\big(X^{\frac{5}{4}+\varepsilon}\big)$. This gives an enumeration algorithm for $\mathscr{C}_4(X)$ which is faster than the method of Hunter-Pohst-Martinet, whose running time is $O\big(X^{\frac{3}{2}+\varepsilon}\big)$ in degree 4. I implemented this algorithm, and I used it to build a table of all quartic fields having discriminant at most 10^9 in absolute value [30].

In the case of quintic fields, a serious difficulty arises. Indeed, applying theorem 1 to the monomial system implicit in Bhargava's work [7] suggests that the above strategy would yield an algorithm which would be less efficient than that of Hunter-Pohst-Martinet, whose running time is now $O\big(X^{\frac{7}{4}+\varepsilon}\big)$.

7 Prospects

My thesis work could be continued in several directions. First, in the case of quartic fields, there is a way to refine the over-approximation of $\mathscr{D}(X)$ with four additional, non-monomial inequalities, which depend on the signature and are analogous to the inequalities (2). This could make it possible to improve the running time of the enumeration algorithm, and maybe to reach the expected optimal complexity $O(X^{1+\varepsilon})$. Part of this strategy has already been completed in the totally real case [2, 27].

A difficulty of this project is that it will likely be necessary to prove a more general version of theorem 1 allowing for negative exponents $\alpha_{i,j}$. This will require checking that the arguments of de la Bretèche still work in this new setting.

Once the case of quartic fields is well understood, perhaps the quintic case will seem more within reach. I conjecture that in this case there will a refinement of the over-approximation of $\mathscr{D}(X)$ by 16 non-monomial inequalities.

On the other hand, questions of enumeration of number fields have a very natural generalisation, namely enumeration of extensions of a given number field, or even of an arbitrary global field. A sensible project would be to extend the methods described in this document to this setting.

I could also pursue my research in reduction theory. Indeed, it seems that the use of the work of Kempf and Ness [22] in this context is new, and it would be interesting to make explicit and to compare the different reduction theories which can be deduced from it, for instance for the action of $\mathrm{GL}_n(\mathbb{Z})$ on forms of degree d in n variables with real coefficients.

Finally, I am ready to work on new problems of enumeration of arithmetic objects, or more generally on new questions in algorithmic number theory.

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