UNIVERSITÉ BORDEAUX I Sciences Technologies — Mathématiques et Interactions — MASTER

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UE : MHT734

Épreuve : Devoir surveillé d'Analyse Complexe

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Tous Documents Interdits

Exercise I

Let $\mathbb{D} = \{z \in \mathbb{C} \text{ tels que } |z| < 1\}$ be the unit disk of the complex plane.Let f be a injective holomorphic function in $\mathbb{D}$.

1. State the so called Open Mapping Theorem (for holomorphic functions). Prove that f is a diffeomorphism from $\mathbb{D}$ to $f(\mathbb{D})$ (i.e. f' does not vanish).
2. Let $\sum_{n=0}^{\infty} c_n z^n$ be the development of f. Prove that the area of $f(\mathbb{D})$ (i.e. $\int_{f(\mathbb{D})} d\lambda$, λ being the Lebesgue measure) is equal to $\pi \sum_{n=1}^{\infty} n|c_n|^2$ (use a change of coordinates).

Exercise II

Let f and g be two entire functions (i.e. holomorphic in $\mathbb{C}$).

1. We suppose that there exists a constant $C > 0$ such that, for $z \in \mathbb{C}$, $|f(z)| \leq C|g(z)|$.
   (a) Prove that the quotient f/g is a well defined entire function.
   (b) Prove that there exists a constant λ, $|\lambda| \leq C$, such that $f = \lambda g$.
2. We suppose now that there exist two constants A and B and an integer $k \geq 1$ such that, for $z \in \mathbb{C}$, $|f(z)| \leq A + B|z|^k$. Prove that f is a polynomial.

Exercise III

Let $\mathbb{D} = \{z \in \mathbb{C} \text{ tels que } |z| < 1\}$ be the unit disk of the complex plane and $\mathbb{T}$ it's boundary. For $w \in \mathbb{D}$ we consider the function

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

1. Prove that φ_w is holomorphic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and that $\varphi_w(\mathbb{D}) \subset \mathbb{D}$ (note first that for $z \in \mathbb{T}$, $|\varphi_w(z)| = 1$).
2. Verify that $\varphi_w(w) = 0$, $\varphi_w(0) = w$ and $(\varphi_w \circ \varphi_w)'(0) = 1$ and conclude that $\varphi_w \circ \varphi_w$ is the identity (make a direct calculus or apply Schwarz Lemma).
3. Let f be a holomorphic function from $\mathbb{D}$ into $\mathbb{D}$. Prove that, for $z, w \in \mathbb{D}$,

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(w)} f(z)} \right| \leq \left| \frac{w - z}{1 - \overline{w}z} \right|$$

Hint. Apply Schwarz Lemma to the function $\varphi_{f(w)} \circ f \circ \varphi_w$.

T.S.V.P.

Exercise IV

Let Ω be an open set of the complex plane and f a holomorphic function from Ω into itself. Let us write $f^{[1]} = f$, and, for any integer $n \geq 2$, $f^{[n]} = f \circ f^{[n-1]}$. Moreover we denote $\Omega_n = f^{[n]}(\Omega)$, $n \in \mathbb{N}$ and we suppose Ω_1 relatively compact in Ω.

1. Prove, by induction over n, that Ω_n is an open set which is relatively compact in Ω_{n-1} (show first $\overline{\Omega_n} \subset \Omega_{n-1}$) and conclude that the intersection K of the Ω_n is closed and then that it is compact.
2. Deduce that there exists a strictly increasing sequence of integers $(n_k)_k$ such that the sequence $\left(f^{[n_k]}\right)_k$ converges uniformly on every compact of Ω to a holomorphic function $g : \Omega \to \Omega$ such that $g(\Omega) \subset K$.
3. Let $z \in K$. For each integer k let ξ_k be a point of Ω_1 such that $z = f^{[n_k]}(\xi_k)$.
   (a) Show that there exists a convergent subsequence $\left(\xi_{k_p}\right)_p$ of the sequence $(\xi_k)_k$ and let $\xi \in \overline{\Omega_1}$ be it's limit.
   (b) Prove that $\lim_{p\to+\infty} f^{\left[n_{k_p}\right]}\left(\xi_{k_p}\right) = g(\xi)$ (note that the sequence of the derivatives of the functions $f^{\left[n_{k_p}\right]}$ converges to g' uniformly on $\overline{\Omega_1}$) and conclude that $g(\Omega) = K$.
4. Prove that if g is not constant then $K = \Omega$.
5. Conclude that K is reduced to a single point.

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