Hilbert spaces of Dirichlet series

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Overview

- Ordinary Dirichlet series.
- Regions of convergence.
- Definition and reproducing kernel.
- Local behaviour and the discrete Hilbert transform.
- Almost periodicity.
- Pointwise convergence.
- Bohr correspondence.
- L^p variants.
- Multipliers and applications.
- Composition operators.
- Other weights.
- Estimates of coefficients.
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Ordinary Dirichlet series

We will consider series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

where $s = \sigma + it \in \mathbb{C}$ is a complex variable.
To make things interesting we assume that the series converges at least in a point $s = s_0$.
Actually the region of convergence will be a half plane by the

Theorem (Abel-Dedekind-Dirichlet)

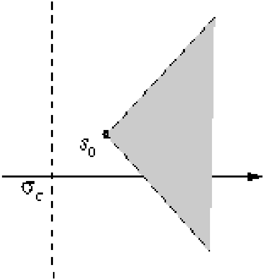
If the series $\sum_n a_n$ has bounded partial sums and the sequence b_n is of bounded variation and $\lim_n b_n = 0$ then the series $\sum_n a_n b_n$ converges.

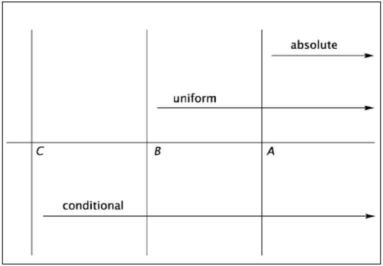
Replacing $f(s)$ by $g(s) = f(s + s_0)$ we may assume $s_0 = 0$ without loss of generality. Since $|n^{-s} - (n+1)^{-s}| = O(n^{-1-Re(s)})$, then n^{-s} is of bounded variation in $\mathbb{C}_0^+ = \{z \in \mathbb{C};\ \Re z > 0\}$.

Regions of convergence

Actually the convergence is uniform in a cone with vertex s_0 as in the picture. Thus $f(s)$ is an holomorphic function in the half plane to the right of s_0.

In contrast to the power series the regions of convergence differ when we consider pointwise convergence, uniform convergence or absolute convergence.





Regions of convergence

Given $f(s)$ we can define (at least) three abcissa $\sigma_C, \sigma_U, \sigma_A$.

- σ_C is the smallest σ such that $f(s)$ is convergent in $\mathbb{C}^+_{\sigma_C}$.
- σ_U is the smallest σ such that $f(s)$ converges uniformly in $\mathbb{C}^+_{\sigma_U+\varepsilon}$ for any $\varepsilon > 0$.
- σ_A is the smallest σ such that $f(s)$ converges absolutely in $\mathbb{C}^+_{\sigma_A}$.

Clearly $\sigma_C \leq \sigma_U \leq \sigma_A$. It is also easy that $\sigma_A - \sigma_C \leq 1$ (it can be anything in between 0 and 1).

Remark

The function $f(s)$ may continue analytically in a region bigger than $\mathbb{C}^+_{\sigma_C}$.

Keep in mind the following example:

$$g(s) = \sum_{n\geq 1}(-1)^n n^{-s}. \qquad g(s) = 2\zeta(s)(2^{-s} - 1/2)$$

This is an entire function such that $\sigma_C = 0$ and $\sigma_A = \sigma_U = 1$.

Regions of convergence

It remains to clarify what is the relation between σ_U and σ_A. Bohr studied another abscissa σ_B or abscissa of boundedness

- σ_B is the smallest σ such that $f(s)$ converges at some point s and it is bounded in $\mathbb{C}^+_{\sigma_B+\varepsilon}$ for any $\varepsilon > 0$.

And he proved

Theorem (Bohr)

$\sigma_B = \sigma_U$.

In particular this means that whenever we have an analytic function f that coincides with a Dirichlet series in a half plane $\mathbb{C}^+_a$ and it is holomorphic and bounded up to $\mathbb{C}^+_b$ with $b < a$, then the Dirichlet series converges to f up to $\mathbb{C}^+_b$.

Regions of convergence

Proof (Maurizi, Queffélec).

We only need to prove that $\sigma_U \leq \sigma_B$. Suppose that $f(s) = \sum a_n n^{-s}$ defines a bounded analytic function in $\mathbb{C}_0^+$ and let $\|f\|_\infty = \sup_{\mathbb{C}_0^+} |f(s)|$. Denote by $D_N(s) = D_N[f](s) = \sum_{n=1}^N a_n n^{-s}$. Then

Proposition (Balasubramanian, Calado, Queffélec)

$$\|D_N\|_\infty \leq C \log N \|f\|_\infty.$$

Now summing by parts:

$$\sum_{n=1}^{N} a_n n^{-(s+\varepsilon)} = \sum_{n=1}^{N-1} D_n(s)(n^{-\varepsilon} - (n+1)^{-\varepsilon}) + N^{-\varepsilon} D_N(s).$$

and the new series is dominated by $\frac{C\varepsilon \log n}{n^{\varepsilon+1}} \|f\|_\infty$.

$\square$

Regions of Convergence

Theorem

$$\sigma_A - \sigma_B \leq 1/2 \qquad \textit{(Bohr)}$$

and $1/2$ is sharp (Bohnenblust-Hille).

We will prove a refined version.

Theorem (DFOOS)

The series

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\left\{ c\sqrt{\log n \log\log n} \right\}$$

is convergent for every $\sum_n a_n n^{-s}$ bounded Dirichlet series in $\mathbb{C}_0^+$ if $c < 1/\sqrt{2}$ and it may be divergent if $c > 1/\sqrt{2}$.

Definition and reproducing kernel

Definition

The space $\mathcal{H}^2$ will be the Hilbert space of analytic functions defined by

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $\|f\|^2 := \sum |a_n|^2 < \infty$.

The functions in the space converge in $\sigma > 1/2$. In fact by Cauchy-Schwartz any function in $\mathcal{H}$ satisfies $\sigma_A \leq 1/2$.

There are function on $\mathcal{H}^2$ that are only defined in $\mathbb{C}^+_{1/2}$.

The values $f(s)$ determine the coefficients and of course the norm, but it will be interesting to express the norm directly in terms of the growth of $f(s)$.

Carlson's theorem

Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2$, then for any $\sigma > \sigma_U$ we have

$$\sum_n |a_n|^2 n^{-2\sigma} = \lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 \, dt.$$

In particular $\mathcal{H}^2$ is the closure of the Dirichlet polynomials $p(s) = \sum_{n=1}^{N} a_n n^{-s}$ under the norm

$$\|p\|^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T |p(it)|^2 \, dt.$$

This way of computing the norm is not valid for all $f \in \mathcal{H}^2$ because in general f is not defined in the imaginary axis.

The reproducing kernel

Definition

Since n^{-s} is an orthonormal basis and point evaluation is bounded for $\sigma > 1/2$, the function

$$\sum_{n=1}^{\infty} n^{-s} n^{-\bar{w}} = \zeta(s + \bar{w})$$

is the reproducing kernel for $\mathcal{H}^2$, i.e, for any $w \in \mathbb{C}^+_{1/2}$

$$f(w) = \langle f(s), \zeta(s + \bar{w}) \rangle$$

and the norm of the pointwise evaluation is $\sqrt{k(s,s)} = \sqrt{\zeta(2\sigma)}$ which blows up as $1/\sqrt{2\sigma}$ as we approach the boundary of $\mathbb{C}_{1/2}$.

Local behaviour

Theorem

For any $f \in \mathcal{H}$ the following holds:

$$\int_0^T |f(\sigma+it)|^2\,dt \leq C(T)\|f\|_{\mathcal{H}}^2 \quad \forall \sigma \geq 1/2.$$

Proof: Let $D(s) = \sum_{n=1}^N a_n n^{-s}$. The statement follows from:

$$\int_0^T |D(it)|^2\,dt = T\sum_{n=1}^N |a_n|^2 + O(\sum_{n=1}^N n|a_n|^2),$$

which trivially implies

$$\int_0^T |D(\sigma+it)|^2\,dt = T\sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}} + O(\sum_{n=1}^N \frac{n}{n^{2\sigma}}|a_n|^2).$$

Local behaviour

$$\boxed{\int_0^T |D(it)|^2\,dt = T\sum_{n=1}^N |a_n|^2 + O(\sum_{n=1}^N n|a_n|^2),}$$

$$\int_0^T |D(it)|^2\,dt = \sum_{n,m=1}^N a_n\bar{a}_m \int_0^T e^{i(\log n - \log m)t}\,dt =$$

$$= T\sum_n^N |a_n|^2 + i\sum_{n\neq m} \frac{a_n\bar{a}_m}{\log n - \log m} - i\sum_{n\neq m} \frac{a_n e^{i\log nT}\bar{a}_m e^{i\log mT}}{\log n - \log m}.$$

The last two terms are similar and can be controlled if we prove:

$$\left|\sum_{m\neq n} \frac{x_m y_n}{\lambda_m - \lambda_n}\right|^2 \leq C\sum_m \frac{|x_m|^2}{d_m}\sum_n \frac{|y_n|^2}{d_n},$$

where $\delta_n = \min_{m\neq n}|\lambda_m - \lambda_n|$. In our case $\lambda_n = \log n$ and $d_n = 1/n$

Local behaviour and the discrete Hilbert transform

In order to prove:

$$\left|\sum_{m\neq n} \frac{x_m y_n}{\lambda_m - \lambda_n}\right|^2 \leq C \sum_m \frac{|x_m|^2}{d_m} \sum_n \frac{|y_n|^2}{d_n},$$

consider the measure $\mu = \sum_n d_n \delta_{\lambda_n}$. The points $\lambda_n \in \mathbb{R}$ thus the curvature of $\mu = 0$ and by the definition of d_n it has linear growth (it is doubling).

Thus $C(f)(z) = \int \frac{f(w)}{z-w}\, d\mu(w)$ is bounded from $L^2(d\mu)$ to $L^2(d\mu)$.

Therefore there is a constant $C > 0$ such that for all a_n, b_m we have

$$\left|\sum_{m\neq n} \frac{a_m d_m}{\lambda_m - \lambda_n} \bar{b}_n d_n\right|^2 \leq C \sum_m |a_m|^2 d_m \sum_n |b_n|^2 d_n,$$

Finally taking $a_m = x_m/d_m$ and $b_n = \bar{y}_n/d_n$ we get the result. $\square$

Immediate consequences

This gives several immediate consequences:

Proposition

If $f \in \mathcal{H}^2$ then

- $g(s) = f(s)/s \in \mathbb{H}^2(\mathbb{C}^+_{1/2})$.
- *The zeros of f satisfy the Blaschke condition on the halfplane $\mathbb{C}^+_{1/2}$.*
- *For almost every $t \in \mathbb{R}$ there exists the non-tangential limit of f at the point $1/2 + it$*

And some not so immediate

Proposition

A compactly supported measure μ is a Carleson measure for $\mathcal{H}^2$ (i.e. $\int_{\mathbb{C}^+_{1/2}} |f|^2 \, d\mu \leq C\|f\|^2$ for all $f \in \mathcal{H}^2$) if and only if μ is a Carleson measure for the Hardy space $\mathbb{H}^2(\mathbb{C}^+_{1/2})$.

Almost periodicity

Let f be holomorphic in $\mathbb{C}_a^+$. Let $\varepsilon > 0$. A number τ is an ϵ translation number of f if

$$\sup_{s \in \mathbb{C}_a^+} |f(s + i\tau) - f(s)| \leq \varepsilon.$$

Definition

A function f is uniformly almost periodic in $\mathbb{C}_a^+$ if, for every $\varepsilon > 0$, there is an $M > 0$ such that every interval of length M contains at least one ϵ translation number of f.

One can prove that:

Theorem

Suppose that $f(s)$ is a Dirichlet series and $a > \sigma_U$, then f is uniformly almost periodic in $\mathbb{C}_a^+$.

Almost periodicity

This is a consequence of Kronecker's theorem that says

Theorem

Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the primes, then

$$\{(t \log p_1, t \log p_2, \ldots, t \log p_n) \subset \mathbb{T}^n, t \in \mathbb{R}\}$$

is dense in $\mathbb{T}^n$ (the flow is equidistributed).

Thus for any $\varepsilon > 0$ we can find a sequence $T_n \to \infty$ relatively dense such that $d(T_n \log p_j, 2\pi\mathbb{Z}) < \varepsilon$ for all $j = 1, \ldots, n$ and therefore $|d_n(s - iT_n) - d_n(s)| \leq C\varepsilon$ for a given Dirichlet polynomial of degree n.
By uniform approximation of the function f by Dirichlet polynomials in $\mathbb{C}_a^+$ we get the desired result.
In particular if $f \in \mathcal{H}^2$ and $f(a) = 0$ $\Re(a) > 1/2$, then in any strip $\Re(a) - \varepsilon < \Re f < \Re(a) + \varepsilon$ we can find infinitely many zeros.

Pointwise convergence

Theorem

For any $\sum a_n n^{-s} \in \mathcal{H}$*, the series* $\sum a_n n^{-s}$ *converges at* $s = 1/2 + it$ *for almost every* $t \in \mathbb{R}$*.*

Proof.

Let $f : [1, \infty) \to \mathbb{C}$ be defined by $f(x) = a_n$ if $n \leq x < n+1$ and $n = 1, 2, \ldots$. Now, define $g : [0, \infty) \to \mathbb{C}$ as $g(y) = f(e^y)e^{y/2}$. We have $g \in L^2(\mathbb{R}^+)$ and using Carleson's theorem for integrals we get that

$$\int_0^\infty g(y)e^{ity}\,dy = \lim_{A\to\infty}\int_0^A g(y)e^{ity}\,dy$$

exists a.e.in t.Thus

$$\lim_{N\to\infty}\int_1^N \frac{f(x)}{\sqrt{x}}e^{it\log x}\,dx.$$

exists a.e.in t and unwinding the definition of f we get the result. $\square$

Interpolating sequences

The techniques so far were from harmonic analysis. We can get more information in terms of the behaviour of the reproducing kernel. Observe that there is an entire function h such that

$$\zeta(s+\bar{w}) = \frac{1}{s+\bar{w}+1} + h(s+\bar{w}).$$

This is innocent looking, but $\frac{1}{s+\bar{w}+1}$ is the reproducing kernel of $\mathbb{H}^2(\mathbb{C}^+_{1/2})$ and $\zeta(s+\bar{w})$ is the reproducing kernel for $\mathcal{H}$.
From this relation Olsen and Seip proved:

Theorem

Let S be a bounded sequence in $\mathbb{C}^+_{1/2}$. Then S is an interpolating sequence for $\mathcal{H}$ if and only if S is an interpolating sequence for $\mathbb{H}^2(\mathbb{C}^+_{1/2})$.

A sequence S is interpolating if for any sequence of values $a_j/\|k_{a_j}\| \in \ell_2$ there is a function $f \in \mathcal{H}$ such that $f(s_j) = a_j$.
Similarly for $\mathbb{H}^2$

Interpolating sequences

The two key ingredients are the local embedding of $\mathcal{H}^2$ in $\mathbb{H}^2(\mathbb{C}_{1/2})$ and the dual formulation of interpolating sequence:

Lemma

A sequence S is interpolating if

$$\|\sum c_j \frac{k_{s_j}}{\|k_{s_j}\|}\|^2 \geq c(\sum |c_j|^2).$$

Then by a perturbative argument due to the nature of the reproducing kernels one can transfer the interpolating problem from one space to the other.

Local behaviour, again

The following theorem provides a precise description of the local behaviour of $\mathcal{H}^2$:

Theorem (Olsen and Saksman)

Let $f \in \mathbb{H}^2(\mathbb{C}^+_{1/2})$. Then there is an $F \in \mathcal{H}^2$ such that $f - F$ extends holomorphically through $I = [1/2 - i, 1/2 + i]$.

To prove this we will first prove that $R : \ell^2(\mathbb{Z} \setminus \{0\}) \to L^2[-1,1]$ is onto where

$$R(a_n) = \chi_{[-1,1]}(t) \sum_{n=1}^{\infty} \frac{a_n n^{-it} + a_{-n} n^{it}}{\sqrt{n}}$$

Then the theorem follows because for any $f \in \mathbb{H}^2(\mathbb{C}^+_{1/2})$ there are a_n such that $R(a_n) = \chi_{[-1,1]}(t)\Re f$. We may assume that $a_n \in \mathbb{R}$. We define $F(s) = 2\sum_{n\geq 1} a_n n^{-s}$ and it satisfies

$$(\Re F - \Re f)(1/2 - it) = 0, \qquad t \in [-1,1],$$

and the theorem follows.

Local behaviour, again

In order to check the exhaustivity of R we compute R^*

$$R^*(g) = \left(\ldots, \frac{\hat{g}(-\log 2)}{\sqrt{2}}, \hat{g}(0), \hat{g}(0), \frac{\hat{g}(-\log 2)}{\sqrt{2}}, \ldots\right)$$

Now R is onto if R^* is bounded below, i.e if for all $g \in L^2([-1,1])$ we have

$$\|g\|^2 \leq C \sum_{n\geq 1} \frac{|\hat{g}(\pm \log n)|^2}{n}.$$

In other words for all $f \in PW$ we should have

$$\|f\|^2 \leq C \sum_{n\geq 1} \frac{|f(\pm \log n)|^2}{n}.$$

This is the case because $\mu = \sum_{n\geq 1} \frac{\delta_{\pm \log n}}{n}$ is a sampling measure for PW.

Bohr correspondence

Another very important source of information on $\mathcal{H}^2$ is the Bohr correspondence. This is the identification of $\mathcal{H}^2$ with $\mathbb{H}^2(\mathbb{T}^\infty)$.

Definition

We endow $\mathbb{T}^\infty$ with the Haar measure μ which is the product measure. We denote by $\mathbb{H}^2(\mathbb{T}^\infty)$ the subspace generated by $z^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ where α are all finite multiindex with positive entries. i.e. $\alpha_1 \geq 0, \dots, \alpha_k \geq 0$.

The functions on $\mathbb{H}^2(\mathbb{T}^\infty)$ represent holomorphic functions of infinitely many variables. To any $f \in \mathbb{H}^2(\mathbb{T}^\infty)$ we associate a power series $f(z) = \sum_\alpha a_\alpha z^\alpha$, where $\sum |a_\alpha|^2 < \infty$. This series is not convergent (in general) when $z \in \mathbb{D}^\infty$. We need that $\sum |z|^{2\alpha} < +\infty$ and this is the case exactly when $z \in \ell_2$ because

$$\sum_\alpha |z|^{2\alpha} = \prod_j (1 - |z_j|^2)^{-1}.$$

Thus $f \in \mathbb{H}^2(\mathbb{T}^\infty)$ represent holomorphic functions in $\mathbb{D}^\infty \cap \ell_2$.

Infinitely many variables

To be precise, what is a function in $\mathbb{H}^2(\mathbb{D}^\infty)$? Given a formal series of infinitely many variables $f(z) = \sum a_\alpha z^\alpha$, and given $m \in N$, we define $f_m : \mathbb{D}^m \to \mathbb{C}$ as the power series where we replace

$$(z_1, \dots, z_m, z_{m+1}, \dots) \to (z_1, \dots, z_m, 0, 0, \dots).$$

Then $f \in \mathbb{H}^2(\mathbb{D}^\infty)$ if and only if $f_m \in \mathbb{H}^2(\mathbb{D}^m)$ with uniform control of the norms.

$$\|f_m\|^2_{\mathbb{H}^2(\mathbb{D}^m)} = \sup_{r<1} \int_{\mathbb{T}^m} |f(rz)|^2.$$

When $p = 2$ this is not really needed but this definition works for any $p \in [1, \infty]$

Bohr correspondence

The correspondence that Bohr established is the following: To any function $f \in \mathcal{H}^2$ we consider the function $F \in \mathbb{H}^2(\mathbb{T}^\infty)$ via the following relationship:

$$f(s) = \sum_n a_n n^{-s} \longleftrightarrow F(z) = \sum_\alpha b_\alpha z^\alpha,$$

where $b_\alpha = a_n$ when $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime number decomposition of n.There is an alternative way of looking at this identification: Consider the one complex dimensional curve γ in $\mathbb{D}^\infty$, $\gamma : \mathbb{C}^+_{1/2} \to \mathbb{D}^\infty$ defined as $\gamma(s) = (p_1^{-s}, p_2^{-s}, \ldots)$. Clearly for all $s \in \mathbb{C}^+_{1/2}$, $\gamma(s) \in \ell_2 \cap \mathbb{D}^\infty$ and $f \in \mathcal{H}$ if and only if $f(s) = F(\gamma(s))$ for some (unique) $F \in \mathbb{H}^2(\mathbb{D}^\infty)$.

Vertical convolution

A good smoothing approximation to functions on $\mathcal{H}^2$ and related spaces that are invariant under translations are vertical convolutions. Let $\phi \in L^1(\mathbb{R})$, $\phi \geq 0$ and with integral one. Let f be a Dirichlet series. Then if $s \geq \sigma_B$ we may define

$$f_\varepsilon(s) = \int_{-\infty}^{\infty} f(s+it) \frac{\phi(\frac{t}{\varepsilon})}{\varepsilon}\, dt.$$

One checks that if $f(s) = \sum a_n n^{-s}$, then

$$f_\varepsilon(s) = \sum_{n=1}^{\infty} a_n \hat{\phi}(\varepsilon \log n) n^{-s}$$

We will use as $\phi(t) = \frac{1}{2\pi}\left(\frac{\sin x}{x}\right)^2$. This ensures that f_ε is always a Dirichlet polynomial. Moreover $f_\varepsilon \to f$ in $\mathcal{H}^2$ and uniformly in $\sigma > \sigma_B + \delta$.

Multipliers

This is for instance a useful way to describe the pointwise multipliers in $\mathcal{H}^2$. We say that g is a multiplier in $\mathcal{H}^2$ whenever $g\mathcal{H}^2 \subset \mathcal{H}^2$. In particular, since $1 \in \mathcal{H}^2$ then any multiplier belongs to $\mathcal{H}^2$.Lindqvist, Hedenmalm and Seip described the multipliers. For this we need the following definition:

Definition

We say that $f \in \mathcal{H}^\infty$ if $f \in \mathcal{H}^2$ and moreover f extends analytically to a bounded function in $\mathbb{C}_0^+$.

Theorem

The multipliers of $\mathcal{H}^2$ are exactly $\mathcal{H}^\infty$ and the multiplier norm of g coincides with $\sup_{\mathbb{C}_0^+} |g|$.

Multipliers

This can be seen because the functions in $\mathcal{H}^\infty$ are in correspondence with functions in $\mathbb{H}^\infty(\mathbb{T}^\infty)$. These are functions $f \in L^\infty(\mathbb{T}^\infty)$ such that $\int_{\mathbb{T}^\infty} f(z)\bar{z}^\alpha d\mu(z) = 0$ for all positive multiindices. These functions define bounded holomorphic functions in $\mathbb{D}^\infty \cap c_0$.

Thus we are actually transferring the result that the multiplier of $\mathbb{H}^2(\mathbb{T}^\infty)$ is $\mathbb{H}^\infty(\mathbb{T}^\infty)$ and functions in $\mathbb{H}^\infty(\mathbb{T}^\infty)$ are holomorphic in the bigger set $\mathbb{D}^\infty \cap c_0$ and the curve $\gamma(\mathbb{C}_0^+) \subset \mathbb{D}^\infty \cap c_0$.

Multipliers

More precisely if g is a multiplier we may assume (taking vertical convolutions) that g is a Dirichlet polynomial and we want to proof that $\|g\|_{\mathcal{H}^\infty} \leq \|g\|_{\mathcal{M}}$.
Now we may identify g with an holomorphic polynomial in d variables that is a multiplier in $\mathcal{H}^2(\mathbb{T}^\infty)$, in particular in $\mathbb{H}^2(\mathbb{T}^d)$.
By iteration of the multiplier we know that

$$\|g^k 1\|_{L^2(\mathbb{T}^d)} \leq \|g\|_{\mathcal{M}}^k \|1\|_{L^2(\mathbb{T}^d)}$$

Thus

$$\int_{\mathbb{T}^d} |g|^{2k} \leq \|g\|_{\mathcal{M}}^{2k}.$$

Let $A_\varepsilon \subset \mathbb{T}^d$ be the set where $|g| \geq (1+\varepsilon)\|g\|_{\mathcal{M}}$, then

$$\mu(A_\varepsilon)^{1/2k}(1+\varepsilon)\|g\|_{\mathcal{M}} \leq \|g\|_{\mathcal{M}}$$

and if follows that $\|g\|_{L^\infty(\mathbb{T}^d)} \leq \|g\|_{\mathcal{M}}$.

Riesz basis of dilates

One application: Let $\phi \in L^2(0,1)$ and consider it extended as an odd periodic function of period 2. Beurling asked for which ϕ the system of dilates of ϕ

$$\phi(x), \phi(2x), \phi(3x), \ldots$$

is a Riesz basis in $L^2(0,1)$. The prototypical example is $\phi(x) = \sin(\pi x)$ when we have an orthonormal basis.For a general ϕ we express it as

$$\phi(x) = \sum_{n\geq 1} a_n \sin(n\pi x).$$

The solution is given in terms of a_n. More precisely, let $f(s) = \sum_{n\geq 1} a_n n^{-s} \in \mathcal{H}^2$.

Theorem (Hedenmalm, Lindqvist, Seip)

The system of dilates of ϕ is a Riesz basis for $L^2(0,1)$ if and only if $f \in \mathcal{H}^\infty$ and $0 < \varepsilon < |f(s)|$ for all $s \in \mathbb{C}_0^+$.

Riesz basis of dilates

The proof can be seen as follows. Define two operators: T, S as

$$T : L^2(0,1) \to L^2(0,1); \qquad T(sin(n\pi x)) = \phi(nx)$$

$$S : L^2(0,1) \to \mathcal{H}^2; \qquad S(sin(n\pi x)) = n^{-s}.$$

Clearly S is an invertible operator since it sends an orthonormal basis to another. On the other hand $\phi(nx)$ is a Riesz basis if and only if T is invertible.

Let us check that for any Dirichlet polynomial $p(s)$

$$STS^{-1}p = f(s)p(s)$$

Take $p(s) = n^{-s}$, then

$$STS^{-1}(p) = S(\phi(nx)) = S(\sum_k a_k sin(kn\pi x)) =$$

$$\sum a_k (kn)^{-s} = f(s)n^{-s} = f(s)p(s).$$

Vertical limits

We have seen that the multipliers “live” in a bigger half-plane $\mathbb{C}_0^+$. In fact, in a sense, many functions in $\mathcal{H}^2$ are defined there. Given a function $f(s) = \sum_n a_n n^{-s} \in \mathcal{H}^2$ if we consider its vertical translate $g(s) = f(s + it)$ it has coefficients:

$$g(s) = \sum_n a_n e^{-it \log n} n^{-s}.$$

We can generalize it. Consider a multiplicative character $\chi : \mathbb{N} \to \mathbb{S}^1$, $\chi(nm) = \chi(n)\chi(m)$ and $|\chi(n)| = 1$. Then given $f \in \mathcal{H}^2$ we can define

$$f_\chi(s) = \sum_n \chi(n) a_n n^{-s}.$$

The vertical translates are particular cases, but they are dense. That is for any given χ there is a sequence of vertical translates such that

$$f_\chi = \lim_n f(s + it_n).$$

uniformly on compacts of $\mathbb{C}_{1/2}^+$

Vertical translates

Why this is so?
This is again a consequence of Kronecker theorem. Any character χ can be identified with a point in $\mathbb{T}^\infty$. The characters are determined by its values on the primes $\chi(p_1), \chi(p_2), \ldots$ and these can be chosen independently, each one in $\mathbb{T}$.
So every character is given by the point in $\mathbb{T}^\infty$:

$$\chi(p_1), \chi(p_2), \ldots$$

The vertical translates correspond to the characters

$$\chi_t(p_j) = e^{it \log p_j}$$

and these are dense in $\mathbb{T}^\infty$.

Vertical translates

In the space of characters we can consider the probability measure m which is the Haar measure in $\mathbb{T}^\infty$ (the product measure). Then Helson proved:

Theorem (Helson)

Given $f \in \mathcal{H}^2$, for almost all characters χ the vertical limit function f_χ extends to $\mathbb{C}_0^{1/2}$.

Moreover there is pointwise convergence in the imaginary axis:

Theorem (Hedenmalm, Saksman)

Given $f \in \mathcal{H}^2$, for almost all characters χ and almost all $t \in \mathbb{R}$, the series $\sum a_n \chi(n) n^{-it}$ is convergent.

and we can compute the norm of f through the vertical translates:

Theorem (Hedenmalm, Lindqvist, Seip)

Let μ be an absolutely continuous probability measure on $\mathbb{R}$.

$$\|f\|_2^2 = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^2 d\mu(t)\, dm(\chi).$$

A curiosity on the Riemann hypothesis

This has the following curious corollary.

Proposition

If a_n is totally multiplicative and in ℓ_2 then $f_\chi(s) = \sum a_n \chi(n) n^{-s}$ converges for almost all characters to a zero free function in $\mathbb{C}_0^+$.

The proof: The function $f_\chi(s)$ converges to the half plane $\mathbb{C}_0^+$ as mentioned and moreover

$$1/f_\chi(s) = \sum_n \mu(n) a_n \chi(n) n^{-s}$$

where $\mu(n)$ is the Mobious function. Thus $1/f_\chi$ also extends to $\mathbb{C}_0^+$, therefore f_χ is zero free.

Corollary (Helson)

Almost all vertical translates of the Riemann ζ function

$$\zeta_\chi(s) = \sum \chi(n) n^{-s},$$

are zero free in $\mathbb{C}_{1/2}^+$.

L^p variants

Let $1 \leq p \leq +\infty$. We can try to extend the theory of Dirichlet series in $\mathcal{H}^2$ to $\mathcal{H}^p$ as proposed by Bayart. There are two possible equivalent definitions:

Definition

$\mathcal{H}^p$ is the closure of Dirichlet polynomials $d(s)$ with the norm

$$\|d\|_p^p := \lim_T \frac{1}{T} \int_0^T |d(it)|^p \, dt$$

or alternatively

Definition

$f \in \mathcal{H}^p$ if $f(s) = \sum a_n n^{-s}$ and there is a function $\tilde{f} \in L^p(\mathbb{T}^\infty)$ such that $\int_{\mathbb{T}^\infty} \tilde{f} \bar{z}^\alpha d\mu(z) = a_{n(\alpha)}$ for all multiindices α with all entries positive and $\int_{\mathbb{T}^\infty} \tilde{f} \bar{z}^\alpha d\mu(z) = 0$ otherwise.

L^p variants

The Bohr correspondence works well but the theory in $\mathcal{H}^p$ is much less developed. One of the most basic pieces of information lacking is the local embedding, i.e. is it true that

$$\int_0^1 |f(1/2+it)|^p \, dt \le C\|f\|^p.$$

for any Dirichlet polynomial f?The answer is true if $p = 2n$ but we cannot interpolate between the spaces $\mathcal{H}^p$.
The case $p = 1$ will follow if any function $f \in \mathcal{H}^1$ weakly factorizes as $f = \sum_j g_j h_j$, with $g_j, h_j \in \mathcal{H}^2$ and $\|f\|_1 \simeq \sum_j \|g_j\|_2 \|h_j\|_2$. This is known to be true for functions in $\mathbb{H}^1(\mathbb{T}^n)$.
Unfortunately this is not the case in $\mathbb{H}^1(\mathbb{T}^\infty)$.

Theorem

There are functions in $\mathcal{H}^1$ that cannot be weakly-factorized.

The Bourgain-Montgomery conjecture

The conjecture is not too unsimilar to a conjecture formulated by Montgomery and refined by Bourgain. Let $f(s) = \sum_{n \le N} a_n n^{-s}$ be a Dirichlet polynomial of degree N with coefficients $|a_n| \le 1$. Then

$$\int_0^T |f(it)|^q \, dt \le C(T + N^{q/2}) N^{q/2+\varepsilon}.$$

This is known if $q = 2n$ (it is the version of the local embedding that we proved!) but it is open for other values of q.

Composition operators (boundedness)

Gordon and Hedenmalm have studied the composition operators in $\mathcal{H}^2$. More precisely they studied the following question:

Question

For which analytic mappings $\phi : \mathbb{C}^+_{1/2} \to \mathbb{C}^+_{1/2}$ is the composition operator $C_\phi(f) = f \circ \phi$ a bounded linear operator on $\mathcal{H}^2$?

Their complete answer is the following:

Theorem

C_ϕ is a bounded composition operator if and only if:

- *$\phi(s) = c_0 s + \psi(s)$, where $c_0 \in \mathbb{N} \cup \{0\}$ and $\psi(s)$ is an ordinary Dirichlet series converging in some point.*
- *ϕ admits an analytic extension to $\mathbb{C}^+_0$ such that*
  - *$\phi(\mathbb{C}^+_0) \subset \mathbb{C}^+_0$ if $c_0 > 0$, and*
  - *$\phi(\mathbb{C}^+_0) \subset \mathbb{C}^+_{1/2}$ if $c_0 = 0$.*

Composition operators (compactness)

Several authors (Bayart, Finet, Lefèvre, Queffélec, Volberg, at least) have studied the compactness of the composition operators in $\mathcal{H}^2$. So far, the description is not complete. We give one of the results in this area:

Theorem (Bayart)

Let $\phi : \mathbb{C}_0^+ \to \mathbb{C}_0^+$ be of the form $\phi(s) = c_0 s + \psi(s)$ $c_0 \geq 1$. Suppose

- *$\Im\psi$ is bounded on $\mathbb{C}_0^+$*
- *$\mathcal{N}_\phi(s) = o(\sigma)$ as $\sigma \to 0$.*

Then C_ϕ is compact on $\mathcal{H}^2$.

In this statement

$$\mathcal{N}_\phi(s) = \begin{cases} \sum_{w\in\phi^{-1}(s)} \Re w & \text{if } s \in \phi(\mathbb{C}_0^+) \\ 0 & \text{otherwise.} \end{cases}$$

Different weights

McCarthy introduced analogous spaces of Dirichlet series that behave locally like the Bergman space and obtained similar properties. This theme has been further studied by Olsen. His setting is the following:

Definition

Let $\mathcal{H}^2_\omega$ be the space of ordinary Dirichlet series such that

$$f(s) = \sum_{n\geq 1} a_n n^{-s} : \ \sum \frac{|a_n|^2}{\omega_n} < +\infty.$$

The main results studied are the local behaviour of functions in the space $\mathcal{H}^2_\omega$ in terms of the weight. Suppose that the weight ω satisfies

$$A\frac{x}{(\log x)^\alpha} \leq \sum_{n\leq x} \omega_n \leq B\frac{x}{(\log x)^\alpha}$$

for some $\alpha \leq 1$.Then the main result is that under this hypothesis on the weight $\mathcal{H}^2_\omega$ behave locally as functions in $D_\alpha(\mathbb{C}^+_{1/2})$.

Different weights

The functions in $D_\alpha(\mathbb{C}^+_{1/2})$ correspond to weighted Bergman spaces for $\alpha < 0$:

$$\int_{\mathbb{C}^+_{1/2}} |f(s)|^2(\sigma - 1/2)^{-1-\alpha} dm(s) < +\infty.$$

When $\alpha = 0$, $D_\alpha(\mathbb{C}^+_{1/2})$ is the Hardy space $\mathbb{H}^2(\mathbb{C}^+_{1/2})$.
When $0 < \alpha \leq 1$, they are weighted Dirichlet spaces:

$$\int_{\mathbb{C}^+_{1/2}} |f'(s)|^2(\sigma - 1/2)^{+1-\alpha} dm(s) < +\infty.$$

By local behaviour we mean the local embedding, the description of compact Carleson measures, the bounded interpolation sequences and local extension as in the Hardy-Dirichlet space.

Estimates for the coefficients $\sum |a_n|$

Let D_N be a Dirichlet polynomial, then We have

$$\sum_{n=1}^{N} |a_n| \leq \begin{cases} (1+o(1))\sqrt{N \log N}\|D_N\|_1 \\ \sqrt{N}\|D_N\|_2 \\ \sqrt{N} e^{-(1/\sqrt{2}+o(1))\sqrt{log N \log\log N}} \|D_N\|_\infty, \end{cases}$$

Estimates for the coefficients $\sum |a_n|$

The first estimate on terms of $\|D_N\|_1$ follows from the isoperimetric type inequality due to Bayart and Helson:

$$\sum_{n=1}^{N} \frac{|a_n|^2}{d(n)} \leq \|D_N\|_1$$

where $d(n)$ is the number of divisor of n.
We may estimate then by Cauchy-Schwartz and using some known estimates for $\sum_{n=1}^{N} d(n)$ we get

$$\sum_{n=1}^{N} |a_n| \leq (1 + o(1))\sqrt{N \log N}\|D_N\|_1$$

The estimate in terms of $\|D_N\|_\infty$

We let $S(N)$ be the smallest constant C such that the inequality $\sum_{n=1} |a_n| \le C\|D_N\|_\infty$ holds for every nontrivial D_N.

Theorem (Konyagin–Queffélec 2001, de la Bretèche 2008, DFOOS 2011)

We have

$$S(N) = \sqrt{N} \exp\Big\{ \big(-\frac{1}{\sqrt{2}} + o(1)\big) \sqrt{\log N \operatorname{loglog} N} \Big\}$$

when $N \to \infty$.

A refined version of Bohr theorem

With the estimates for $S(N)$ and the estimate

$$\sup_{\sigma>0}\left|\sum_{n\le N} a_n n^{-s}\right| \le C\log N \sup_{\sigma>0}\left|\sum_{n\ge 1} a_n n^{-s}\right|$$

one can prove that:

Theorem

The supremum of the set of real numbers c such that

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\left\{c\sqrt{\log n \log\log n}\right\} < \infty$$

for every $\sum_{n=1}^{\infty} a_n n^{-s}$ *in* $\mathcal{H}^{\infty}$ *equals* $1/\sqrt{2}$*.*

This is really a refinement of a theorem of R. Balasubramanian, B. Calado, and H. Queffélec. The basic point is the conection of Dirichlet series with power series in the infinite polydisk.

Sidon sets

Definition

If G is an Abelian compact group and Γ its dual group, a subset of the characters $S \subset \Gamma$ is called a Sidon set if

$$\sum_{\gamma \in S} |\alpha_\gamma| \leq C \| \sum_{\gamma \in S} \alpha_\gamma \gamma \|_\infty$$

The smallest constant $C(S)$ is called the Sidon constant of S.

We estimate the Sidon constant for homogeneous polynomials

Definition

Then $S(m,n)$ is the smallest constant C such that the inequality $\sum_{|\alpha|=m} |a_\alpha| \leq C \|P\|_\infty$ holds for every m-homogeneous polynomial in n complex variables $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$.

The Sidon constant for homogeneous polynomials

Proposition

Let m and n be positive integers larger than 1. Then

$$S(m,n) \le e\sqrt{m}(\sqrt{2})^{m-1}\binom{n+m-1}{m}^{\frac{m-1}{2m}}.$$

Note that we also have the following trivial estimate:

$$S(m,n) \le \sqrt{\binom{n+m-1}{m}},$$

This inequality is essentially sharp. This is a corollary of the hypercontractivity of the Bohnenblust and Hille estimate for polynomials.

A multilinear inequality

In 1930, Littlewood proved that for every bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\left(\sum_{i,j} |B(e^i, e^j)|^{4/3} \right)^{3/4} \leq \sqrt{2} \sup_{z,w \in \mathbb{D}^n} |B(z,w)|\,,$$

This was extended by Bohnenblust and Hille who proved in 1931:

$$\left(\sum_{i_1,\dots,i_m} |B(e^{i_1}, \dots, e^{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \sqrt{2}^{m-1} \sup_{z^i \in \mathbb{D}^n} |B(z^1, \dots, z^m)|.$$

The exponent $\frac{2m}{m+1}$ cannot be improved if one wants constants independent of the number of variables.

A symmetric Bohnenblust-Hille inequality

Our main result is that the polynomial Bohnenblust–Hille inequality is in fact hypercontractive:

Theorem

Let m and n be positive integers larger than 1. Then we have

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq e\sqrt{m}(\sqrt{2})^{m-1} \sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|$$

for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$.

Polarization

There is an obvious relationship between both identities.If we restrict a symmetric multilinear form to the diagonal $P(z) = B(z, \dots, z)$, then we obtain a homogeneous polynomial. The converse is also true: Given a homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree m, by polarization, we may define the symmetric m-multilinear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ so that $B(z, \dots, z) = P(z)$.

$$B(z^1, \dots, z^m) = \frac{1}{2^m m!} \sum_{\substack{\varepsilon_i = \pm 1 \\ 1 \le i \le m}} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m P\Big(\sum_{i=1}^m \varepsilon_i z^i\Big)$$

Harri's lemma states that

$$|B(z^1, \dots, z^m)| \le \frac{m^m}{m!} \|P\|_\infty.$$

Two lemmas

Lemma (Blei)

For all sequences $(c_i)_i$ where $i = (i_1, \dots, i_m)$ and $i_k = 1, \dots, n$, we have

$$\Big(\sum_{i_1,\dots,i_m=1}^{n} |c_i|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq \prod_{1\leq k\leq m} \Big[\sum_{i_k=1}^{n} \Big(\sum_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_m} |c_i|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{1}{m}}.$$

Lemma (Bayart)

For any homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$:

$$\big(\sum_{|\alpha|=m} |a_\alpha|^2 \big)^{\frac{1}{2}} \leq (\sqrt{2})^m \Big\| \sum_{|\alpha|=m} a_\alpha z^\alpha \Big\|_{L^1(\mathbb{T}^n)}.$$

The proof 1/3

We denote the polynomial p as

$$p(z) = \sum_{i_1 \leq \cdots \leq i_m} c(i_1, \ldots, i_m) z_{i_1} \cdots z_{i_m}.$$

We have

$$\sum_{i_1 \leq \cdots \leq i_m} |c(i_1, \ldots, i_m)|^{2m/(m+1)} \leq \sum_{i_1, \ldots, i_m} \Big(\frac{|c(i_1, \ldots, i_m)|}{|i|^{1/2}} \Big)^{2m/(m+1)}$$

If we use now Blei's lemma, the last sum is bounded by

$$\prod_{k=1}^{m} \Big[\sum_{i_k=1}^{m} \Big(\sum_{i^k} \frac{|c(i_1, \ldots, i_m)|^2}{|i|} \Big)^{1/2} \Big]^{1/m} \leq \sqrt{m} \prod_{k=1}^{m} \Big[\sum_{i_k=1}^{m} \Big(\sum_{i^k} |i^k| \frac{|c(i_1, \ldots, i_m)|^2}{|i|^2} \Big)^{1/2} \Big]^{1/m}.$$

The proof 2/3

Now observe that we freeze the variable i_k and we group the terms to make a polynomial again:

$$\Big(\sum_{i^k} |i^k| \frac{|c(i_1,\ldots,i_m)|^2}{|i|^2}\Big)^{1/2} = \Big(\sum_{i^k} |i^k| |B(e^{i_1},\ldots,e^{i_m})|^2\Big)^{1/2} = \|P_k\|_2.$$

where $P_k(z)$ is the polynomial
$P_k(z) = B(z,\ldots,z,e^{i_k},z,\ldots,z)$.Now we use Bayart estimate and

$$\Big(\sum_{i^k} |i^k| \frac{|c(i_1,\ldots,i_m)|^2}{|i|^2}\Big)^{1/2} \leq \sqrt{2}^{m-1} \int_{\mathbb{T}^n} |B(z,\ldots,z,e^{i_k},z,\ldots,z)|.$$

The proof 3/3

We replace e^{i_k} by λe^{i_k} with $|\lambda| = 1$. We take $\tau_k(z) = \sum \lambda_k(z) e^{i_k}$ in such a way that

$$\begin{aligned}\sum_{i_k=1}^{n}\Big(\sum_{i^k}|i^k|\frac{|c(i_1,\dots,i_m)|}{|i|^2}\Big)^{1/2} &\leq \int_{\mathbb{T}^n} B(z,\dots,\tau_k(z),\dots,z)\\ &\leq e\sqrt{2}^{m-1}\|P\|_\infty.\end{aligned}$$

Finally we have

$$\sum_{i_1\leq\cdots\leq i_m}|c(i_1,\dots,i_m)|^{2m/(m+1)} \leq e\sqrt{2}^{m-1}\sqrt{m}\|P\|_\infty.$$

Open Problems

- Describe the bounded zero sequences of functions in $\mathcal{H}^2$.
- Prove the "baby corona" problem, i.e. if f_1 and f_2 are multipliers in $\mathcal{H}^2$, such that its associated functions $F_1, F_2 \in \mathbb{H}^\infty(\mathbb{D}^\infty)$ satisfy $|F_1(z)| + |F_2(z)| \geq \delta$ for all $z \in c_0 \cap \mathbb{D}^\infty$ and g is an arbitrary function in $\mathcal{H}^2$, are there functions $g_1, g_2 \in \mathcal{H}^2$ such that $f_1 g_1 + f_2 g_2 = g$?
- Disprove the embedding problem for $\mathcal{H}^1$,i.e. prove that the (local) Carleson measures of $\mathcal{H}^1$ are different from the (local) Carleson measures of $\mathcal{H}^2$.
- Describe the compact composition operators on $\mathcal{H}^2$.

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Caution!

The true corona problem, i.e. in $\mathcal{H}^\infty$ can be missleading. This is incorrectly formulated:

Wrong problem

Consider $f_1, f_2 \in \mathcal{H}^\infty$ such that $|f_1| + |f_2| \geq \delta > 0$ in $\mathbb{C}_0^+$. Is it true that there are functions $g_1, g_2 \in \mathcal{H}^\infty$ such that $f_1 g_1 + f_2 g_2 = 1$?

This is not right. Take $f_1(s) = 2^{-s} - 1/5$ and $f_2(s) = 3^{-s} - 1/5$. If there were solutions $g_1, g_2 \in \mathcal{H}^\infty$, then there will bounded holomorphic functions $G_1, G_2 \in \mathbb{H}^\infty(\mathbb{D}^2)$ such that $(z_1 - 1/5)G_1(z_1, z_2) + (z_2 - 1/5)G_2(z_1, z_2) = 1$!

Right problem (too hard)

Consider $f_1, f_2 \in \mathcal{H}^\infty$ such that its associated functions $F_1, F_2 \in \mathcal{H}^\infty(\mathbb{D}^\infty)$ satisfy $|F_1| + |F_2| \geq \delta > 0$ in $c_0 \cap \mathbb{D}^\infty$. Is it true that there are functions $g_1, g_2 \in \mathcal{H}^\infty$ such that $f_1 g_1 + f_2 g_2 = 1$?

This seems very hard because one needs to solve the Corona theorem in the polydisk.