Topics in the theory of Dirichlet series

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1 Introduction

The study of Dirichlet series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ has a long history beginning in the nineteenth century, and the interest was due mainly to the central role that such series play in analytic number theory. The general theory of Dirichlet series was developed by Hadamard, Landau, Hardy, Riesz, Schnee, and Bohr, to name a few. However, the main results were obtained before the central ideas of Functional Analysis became part of the toolbox of every analyst, and it would seem a good idea to insert this modern way of thinking into the study of Dirichlet series. Some effort has already been spent in this direction; we mention the papers by Helson [10, 11] and Kahane [12, 13]. However, the field did not seem to catch on. It is hoped that this paper can act as a catalyst by pointing at a number of natural open problems, as well as some recent advances. Fairly recently, in [7], Hedenmalm, Lindqvist, and Seip considered a natural Hilbert space $\mathcal{H}^2$ of Dirichlet series and began a systematic study thereof. The elements of $\mathcal{H}^2$ are analytic functions on the half-plane

$$\mathbb{C}_{\frac{1}{2}} = \left\{ s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2} \right\}$$

of the form

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \tag{1.1}$$

where the coefficients $a_1, a_2, a_3, \dots$ are complex numbers subject to the norm boundedness condition

$$\|f\|_{\mathcal{H}^2} = \left(\sum_{n=1}^{+\infty} |a_n|^2 \right)^{\frac{1}{2}} < +\infty.$$

In a natural sense, this is the analogue of the Hardy space H^2 for Dirichlet series. In [7], the pointwise multipliers of $\mathcal{H}$ were characterized, and the result was applied to a problem of Beurling concerning 2-periodic dilation bases in $L^2([0,1])$. The reader is referred to [8] for some historical comments on the topic. We need to introduce the right half plane

$$\mathbb{C}_+ = \left\{ s \in \mathbb{C} : \operatorname{Re} s > 0 \right\},$$

and the space $\mathcal{H}^\infty$ of bounded analytic functions on $\mathbb{C}_+$ which are given by a convergent Dirichlet series of the form (1.1) in some possibly remote half-plane $\operatorname{Re} s > \sigma_0$. By a theorem of Schnee [17], which was later improved by Bohr [1], the Dirichlet series for a function in $\mathcal{H}^\infty$ actually converges on $\mathbb{C}_+$.

2 Multipliers

We formulate the main result of [7]. We say that an analytic function on the half-plane $\mathbb{C}_{\frac{1}{2}}$ is a *multiplier* on $\mathcal{H}^2$ if $\varphi f \in \mathcal{H}^2$ whenever $f \in \mathcal{H}^2$.

THEOREM 2.1 *The collection of multipliers on $\mathcal{H}^2$ equals the space $\mathcal{H}^\infty$.*

The above theorem is analogous to the following well-known result for Hardy spaces: the (pointwise) multipliers of H^2 are the functions in H^∞. A noteworthy difference, however, is that the multipliers in the Dirichlet series case are defined as bounded and analytic on a bigger half-plane than the functions in the space. It should be mentioned that the proof of the above theorem in [7] is based on modelling $\mathcal{H}^2$ as the Hardy space on the infinite-dimensional polydisk $\mathbb{D}^\infty$, an idea which goes back to a 1913 paper of Bohr.

3 Convergence issues

The convergence and analyticity of $f \in \mathcal{H}^2$ given by the series (1.1) in the half-plane $\mathbb{C}_{\frac{1}{2}}$ is a simple consequence of the Cauchy-Schwarz inequality. A deeper fact is that the boundary values of f on the 'critical' line $\partial\mathbb{C}_{\frac{1}{2}} = \{s \in \mathbb{C} : \operatorname{Re} s = \frac{1}{2}\}$ are locally L^2-functions (see [15, formula (29), p. 140] or [7, Theorem 4.11]). It is well-known that functions in $\mathcal{H}^2$ need not have any analytic continuations beyond the half-plane $\mathbb{C}_{\frac{1}{2}}$, and so the Dirichlet series need not converge in any strictly larger open half-plane. The question, then, is what happens precisely on the boundary $\partial\mathbb{C}_{\frac{1}{2}}$. Here, we can compare with Carleson's theorem for Fourier series: given $f \in L^2$ on the unit circle, the corresponding Fourier series converges almost everywhere [2]. Recently, Hedenmalm and Saksman [9] established the validity of the counterpart for Dirichlet series of Carleson's convergence theorem ($\mathbb{R}$ is the set of all real numbers).

THEOREM 3.1 *Let $\sum_{n=1}^{+\infty} |a_n|^2 < +\infty$. Then the series*

$$\sum_{n=1}^{+\infty} a_n \, n^{-\frac{1}{2}+it}$$

converges for almost every $t \in \mathbb{R}$.

The proof uses an equivalent dual formulation of the strong L^2 maximal function estimate used to prove Carleson's theorem, in the form of a Strong Hilbert inequality.

PROBLEM 1 Suppose the function

$$f(s) = \sum_{n=1}^{+\infty} a_n \, n^{-s}$$

belongs to $\mathcal{H}^\infty$, so that the series converges on $\mathbb{C}_+$. Does the series then also converge almost everywhere on the imaginary axis?

We mention another type of convergence theorem. Given a function f of the form (1.1), we form the functions

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \, \chi(n) \, n^{-s}, \tag{3.1}$$

where $\chi(n)$ is a *character*, which means that $\chi(1) = 1$, $\chi(n) \in \mathbb{T}$ for all n, and $\chi(mn) = \chi(m)\,\chi(n)$ for all m and n. The functions f_χ are known as the *vertical limit functions* for f. The terminology is explained by the fact that $f_\chi(s)$ is obtained from f as a limit of a sequence of vertical translates $f(s - it)$, with $t \in \mathbb{R}$. Each character is determined uniquely by its values on the set of primes $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$, and the values at different primes may be chosen independently of each other. The set of all characters is denoted by Ξ, and we realize that it can be equated with the infinite-dimensional polycircle $\mathbb{T}^\infty$ by identifying each dimension with a prime number (see [7] for details). The polycircle $\mathbb{T}^\infty$

has a natural product probability measure defined on it, denoted $d\varpi$, the product of the normalized arc length measure $d\sigma$ in each dimension. The set of characters Ξ constitutes the dual group of the multiplicative group of positive rationals $\mathbb{Q}_+$, if the latter is given the discrete topology. The Haar probability measure on the compact group Ξ coincides with $d\varpi$. A natural question arises: given $f \in \mathcal{H}$, what is the almost sure convergence behavior of the series (3.1) for $f_\chi(s)$, where s is a point in the complex plane, and χ is a character? It is mentioned in [7] that for almost all χ, $f_\chi(s)$ extends to a holomorphic function on the right half plane $\operatorname{Re} s > 0$, and that this is best possible. The behavior of most of the vertical limit functions is thus in sharp constrast with that of individual functions! As a matter of fact, in [11] (see also [7], Theorem 4.4), Helson shows that for almost all χ, the Dirichlet series (3.1) actually converges in the half-plane $\operatorname{Re} s > 0$. By Theorem 4.1 of [7], the function $f_\chi(it)$ makes sense as a locally L^2 summable function on the real line, for almost all χ. This makes us suspect that we have convergence in (3.1) for almost all s on the imaginary line $\operatorname{Re} s = 0$ and almost all χ. In [9], the following theorem is obtained.

THEOREM 3.2 *Let $f \in \mathcal{H}$ be of the form (1.1), and let $f_\chi \in \mathcal{H}$ be defined by (3.1). Then the series*

$$f_\chi(it) = \sum_{n=1}^{\infty} a_n\, \chi(n)\, n^{-it}$$

converges for almost all characters χ and almost all reals t.

It is possible to use the above theorem to derive estimates of the almost sure growth behavior of partial sums of random characters. More precisely, we have, almost surely,

$$\sum_{n=1}^{N} \chi(n) = O\left(\sqrt{N \log N}\,\left(\log\log N\right)^{1/2+\varepsilon}\right), \qquad \text{as } N \to +\infty.$$

PROBLEM 2 *Find the best possible growth bound for the almost sure behavior of the above partial sums.*

This problem has an unmistakable Erdös-type flavor, in its combination of probability and number theory. And sure enough, in [3, pp. 251–252], Erdös states as a problem to determine the almost sure growth of the analoguous sums, where the $\chi(p)$ for prime indices p are replaced by independent random variables assuming the values ± 1 with equal probabilities $\frac{1}{2}$. Erdös looks to compare the growth of the partial sums with the classical law of the iterated logarithm (see [19]), where all the terms $\chi(n)$ are independent and take values ± 1 with equal probabilities $\frac{1}{2}$. In Erdös' problem, as in ours, the characters have the multiplicative property $\chi(mn) = \chi(m)\chi(n)$, which reduces the randomness and introduces a number-theoretic ingredient. A complete solution should thus shed light on the multiplicative structure of the integers. Some progress on Erdös' problem was obtained by Halász [5].

4 Composition operators

Let $f \in \mathcal{H}^2$ be of the form

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}, \qquad s \in \mathbb{C}_{\frac{1}{2}}.$$

Fix a $k = 1, 2, 3, \ldots$. Then

$$f_k(s) = f(ks) = \sum_{n=1}^{+\infty} a_n n^{-ks}, \qquad s \in \mathbb{C}_{\frac{1}{2}},$$

is another function in $\mathcal{H}^2$, of the same norm as f. In other words, if $\Phi(s) = ks$, and $\mathcal{C}_\Phi$ is the associated composition operator,

$$\mathcal{C}_\Phi f(s) = f \circ \Phi(s), \qquad s \in \mathbb{C}_{\frac{1}{2}},$$

then $\mathcal{C}_\Phi$ is an isometry on $\mathcal{H}^2$. One would tend to ask what other kinds of composition operators might be around. Recently, Gordon and Hedenmalm found a complete answer to this question. The space $\mathcal{D}$ consists of somewhere convergent Dirichlet series.

THEOREM 4.1 *An analytic function $\Phi\colon \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ generates a bounded composition operator $\mathcal{C}_\Phi : \mathcal{H} \to \mathcal{H}$ if and only if:*
(a) *it is of the form*

$$\Phi(s) = ks + \phi(s),$$

where $k \in \{0, 1, 2, 3, \ldots\}$ and $\varphi \in \mathcal{D}$; and
(b) *Φ has an analytic extension to $\mathbb{C}_+$, also denoted by Φ, such that*
  (i) *$\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$ if $k > 0$, and*
  (ii) *$\Phi(\mathbb{C}_+) \subset \mathbb{C}_{\frac{1}{2}}$ if $k = 0$.*

This constitutes a genuine Dirichlet series analogue of Littlewood's subordination principle [14]. Indeed, in case Φ fixes the point $+\infty$, which happens precisely when $k > 0$, the composition operator $\mathcal{C}_\Phi$ is a contraction on $\mathcal{H}^2$.

Note that we again have this dichotomy that sometimes the half-plane $\mathbb{C}_{\frac{1}{2}}$ is relevant, and sometimes we need the whole right half plane $\mathbb{C}_+$ instead.

PROBLEM 3 Suppose $\alpha = \Phi(+\infty) \in \mathbb{C}_{\frac{1}{2}}$. Find the optimal estimate of the norm $\|\mathcal{C}_\Phi\|$ in terms of α. Note that it is clear that $\zeta(2\,\mathrm{Re}\,\alpha) \leq \|\mathcal{C}_\Phi\|^2$.

PROBLEM 4 Characterize the compact composition operators on $\mathcal{H}^2$. Compare with Shapiro's characterization [18] of the compact composition operators on H^2 in terms of the Nevanlinna counting function.

5 Integral means

It is well-known that the norm on $\mathcal{H}^2$ can be expressed in terms of integral means of the function itself, provided the function is "nice". Suppose

$$f(s) = \sum_{n=1}^{+\infty} a_n\, n^{-s},$$

where the sum is finite, that is, all but finitely many of the a_n's are 0. We might call such functions *Dirichlet polynomials*. Then

$$\frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt \to \sum_{n=1}^{+\infty} \frac{|a_n|^2}{n^{2\sigma}} \quad \text{as} \quad T \to +\infty, \tag{5.1}$$

for each real σ. We can think of this as a Plancherel formula. However, it is not really useful for calculating the norm of functions in $\mathcal{H}^2$, as such functions need not even be defined along the imaginary axis where the integral mean should then be computed. In fact, functions in $\mathcal{H}^2$ need only be defined in $\mathbb{C}_{\frac{1}{2}}$, which is quite far from the imaginary axis! We shall view (5.1) as a combination of two things:

- a Plancherel formula, and

• an ergodic theorem.

The "genuine" Plancherel formula involves the characters we met earlier:

$$\int_\Xi |f_\chi(\sigma)|^2 d\varpi(\chi) = \sum_{n=1}^{+\infty} \frac{|a_n|^2}{n^{2\sigma}},$$

where we recall the notation

$$f_\chi(s) = \sum_{n=1}^{+\infty} a_n\,\chi(n)\,n^{-s}$$

for the vertical limit function associated with the character χ. The characters of the form

$$\chi_t(n) = n^{-it}, \qquad t \in \mathbb{R},$$

constitute a dense "one-dimensional" subset of Ξ; moreover, we can think of them as the result of a motion in Ξ. To make the latter idea precise, just think of the transformation $T_t(\chi) = \chi_t\chi$ which moves the point χ along the time flow parametrized by t. This flow is *ergodic*, because there are not subsets of Ξ of intermediate mass (that is, not equal to 0 or 1) which are preserved by it. The general ergodic theorem then says that the time average along the flow of a continuous function equals the space average, that is, the integral. And the limit

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2 dt = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left|f_{\chi_t}(\sigma)\right|^2 dt$$

is exactly a time average, whereas

$$\int_\Xi |f_\chi(\sigma)|^2 d\varpi(\chi)$$

is the space average. Now, we see that (5.1) holds for more general Dirichlet series f; what is needed is that $f_\chi(\sigma)$ defines a continuous function of $\chi \in \Xi$. For instance, this is true for all σ with $0 < \sigma < +\infty$ if $f \in \mathcal{H}^\infty$.

PROBLEM 5 Suppose $f \in \mathcal{H}^\infty$, so that f has well-defined nontangential boundary values almost everywhere on the imaginary line. Is it true that

$$\frac{1}{2T}\int_{-T}^{T} |f(it)|^2 dt \to \sum_{n=1}^{+\infty} |a_n|^2 \quad \text{as } T \to +\infty?$$

6 Hardy spaces for Dirichlet series

Suppose f is a Dirichlet polynomial (which means that the Dirichlet series is finite). Fix a p, $1 < p < +\infty$. One can show that the limit

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} |f(it)|^p dt$$

exists with ergodic methods like in the previous section; it equals the p-th power of the $L^p(\Xi)$ norm of $\chi \mapsto f_\chi(0)$. As a consequence, we can use the above limit to define a norm on the Dirichlet polynomials, and then form the completion of the space with respect to it. The result is the space $\mathcal{H}^p$, the *Hardy space for Dirichlet series*. For each p, the elements of $\mathcal{H}^p$ are Dirichlet series that define analytic functions on $\mathbb{C}_{\frac{1}{2}}$, and generally speaking, not on any other bigger domain.

PROBLEM 6 Find another scale of spaces (perhaps of Orlicz type) which is able to resolve the jump from finite p when the functions are analytic on $\mathbb{C}_{\frac{1}{2}}$, to $p = +\infty$, when the functions are analytic on $\mathbb{C}_+$.

PROBLEM 7 Study the properties of the spaces $\mathcal{H}^p$ in more detail.

7 General Dirichlet series

The theorem of Schnee [17] (see also the book of Hardy and Riesz [6]) mentioned earlier says the following: if

$$f(s)=\sum_{n=1}^{+\infty} a_n n^{-s}$$

converges in some (possibly remote) half-plane $\operatorname{Re} s>\sigma_0$, and the function has an analytic continuation to the right half-plane $\mathbb{C}_+$, and satisfies the growth bound for each $\varepsilon>0$,

$$|f(s)|=O(|s|^{\varepsilon}),\quad \text{as}\quad |s|\to+\infty,$$

in every half-plane $\operatorname{Re} s>\delta$ with $\delta>0$, then the Dirichlet series for $f(s)$ converges on $\mathbb{C}_+$. Schnee's theorem also applies to more general Dirichlet series of the form

$$f(s)=\sum_{n=1}^{+\infty} a_n e^{-\lambda_n s},$$

where $\lambda_n\in\mathbb{R}$ for all n, and $\lambda_n\to+\infty$ as $n\to+\infty$; the classical case corresponds to having $\lambda_n=\log n$. Schnee's theorem has certain regularity assumptions on the λ_n's. So, for instance, it does not apply when this sequence of frequences "clumps together" too much.

PROBLEM 8 Is it possible to handle the case when we have "clumping together" of the frequencies by enforcing a stronger growth condition on the function?

PROBLEM 9 To what extent are the results mentioned in the previous sections peculiar to $\lambda_n=\log n$?

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