About inverse problems related to deconvolution

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$$f \in C(X) \longmapsto \left(g \in G \mapsto \int_{gK_1} f d\mu \,, \dots , g \in G \mapsto \int_{gK_N} f d\mu \right) \in (C(G))^N$$

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For references, up to 1996...

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- An exhaustive bibliography by L. Zalcman
  [L. Zalcman, Approximation by solutions of PDE's, Kluwer, 1992, B. Fuglede ed.]
- An updated survey by C.A. Berenstein
  [C.A. Berenstein, *The Pompeïu problem, what's new ? in* Complex Analysis, Harmonic analysis and applications, Pitman Research Notes 347, 1996]

Shiffer's old (still open) question

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- $X = \mathbb{R}^n$, G : Euclidean motion group $M(n)$, $\mu = dx$;
- $N = 1$, $K_1 = \overline{\Omega}$, where Ω is an open bounded open set with Lipschitz boundary such that $\mathbb{R}^n \setminus K_1$ is connected.

Suppose the related Pompeïu transfom NON INJECTIVE ; **is K a disk ?**

A “reformulation” by A. Williams (1976) and a partial answer by C.A. Berenstein (1980)

Theorem (A. Williams, 1976)

The Pompeïu transform in the above setting is injective is and only if there is NO $\alpha > 0$ such that the overdetermined Neumann problem

$$\begin{aligned} \Delta u + \alpha u &= 0 \text{in } \Omega \\ u = 1\,,\ \partial u/\partial n_{\text{ext}} &= 0 \text{ on } \partial\Omega \end{aligned}$$

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Assuming ∂K $C^{2+\epsilon}$, if the Neumann problem admits solutions for an infinite number of real values α, then K is a disk ([C.A. Berenstein, 1980])

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- Yes (when $n = 2$) if Ω is conformally equivalent to the unit disk trough a rational (even in some cases algebraic) map : YES when Ω is a true ellipse, NO when it is a disk ! [P. Ehbenfelt, 1993]

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- Yes (when $n = 2$) if Ω is conformally equivalent to the unit disk trough a rational (even in some cases algebraic) map : YES when Ω is a true ellipse, NO when it is a disk ! [P. Ehbenfelt, 1993]
- Several attacks, still when $n = 2$, in particular through its natural companion (the holomorphy test of Morera with $K = \partial\Omega$, assuming $\partial\Omega$ is a piecewice Jordan curve and consider the path integral), mainly by L. Zalcman and V.V. Volchkov (1990-2000)

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- A companion problem : J. Delsarte's two radii theorem [J. Delsarte, Lectures at Tata Institute, 1961] :

$$f \in C(\mathbb{R}^n)\,,\ f(x) = \int f(x+y)d\sigma_{r_j}\ \forall x \in \mathbb{R}^n \Longrightarrow \Delta f \equiv 0$$

(r_1/r_2 outside some exceptional countable set).

The natural extension to irreducible symmetric spaces with rank 1

The reason : the crucial relation of these questions with Spectral Synthesis Problem for radially symmetric functions in $\mathbb{R}^n$! [L. Brown, B.M. Schreiber, B.A. Taylor, 1973]

- see [C.A. Berenstein-L. Zalcman, 1980], [C.A. Berenstein-M. Shahshahani, 1983], [C.A. Berenstein, D. Pasquas, 1994], [Molzon, 1991]

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- see the extensive work of M. Agranovsky, A. Semanov, V. Vochkov, C.A. Berenstein, D. Chen Chang, L. Zalcman (1990-1995)
- see also last chapter in : C.A. Berenstein, D.C. Chang, T. Tie's book on Laguerre calculus (International Press, 2001).

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A necessity : tools should come from multivariate complex analysis.

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  Versions in the symmetric spaces of rank 1 setting by A. Volchkov (injectivity), M. El Harchaoui (inversion) (around 1995).

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- This leads *via* identification of Fourier transforms to the following :

$$\begin{aligned} H_p(x)\sigma_r(\|x\|) &= \frac{(-1)^p}{2^{p-1}(p-1)!}\frac{r^{2-n}}{}\mathrm{vol}(S^n) \\ &\quad \times H\left(\frac{\partial}{\partial y}\right)\left[(r^2-\|y\|^2)^{p-1}\chi_{B_n(0,r)}(y)\right]_{|y=x} \end{aligned}$$

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$$\Phi(z)\chi_U(z) = \frac{F(z)}{2i\pi} \int_\Gamma \frac{\Phi(\zeta)d\zeta}{F(\zeta)(\zeta - z)} + \sum_{j=1}^{m} \sum_{\{\alpha \in U\,;\, f_j(\alpha)=0\}} \Big(\prod_{l \neq j} f_l(z)\Big) \mathrm{Res}_{\zeta=\alpha}\Big[\frac{\Phi(\zeta)(f_j(z) - f_j(\zeta))\, d\zeta}{(z-\zeta)F(\zeta)}\Big].$$

Inversion of the local two discs transformation *via* recovering the spherical decomposition (euclidean radial context) ;

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Let f be a C^∞ function in the open euclidean ball n-dimensional $B(0,R)$ (regularization of a continuous function)

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Theorem (C.A. Berenstein, R. Gay, A. Yger, 1990)

There are absolute constants c, γ, C, *a strictly increasing sequence* $R_0 = 0 < R_1 < R_2 < \ldots$ *with* $\lim_k(R_k) = R$ *such that for any* $k \geq 1$, *for any* $r \in [R_{k-1}, R_k[$, *for any spherical harmonic* $S_m = H_m\sigma_r$ *with degree* m, *one can construct two* *explicit* *sequences of "deconvolvers"* $(U_{r,l})_{l\geq 1}$ $(B(0, R - r_1)$ *supported) and* $(V_{r,l})_{l\geq 1}$ $(B(0, R - r_2)$ *supported) such that*

$$l \geq cm^2 \implies \left|\langle f\,,\,S_m\rangle - \langle U_{r,l}\,,\,\chi_{B(0,r_1)} * f\rangle - \langle V_{r,l}\,,\,\chi_{B(0,r_2)} * f\rangle\right| \leq \frac{\gamma}{l}(R-r)^{-N}\max_{|\alpha|\leq N}\|\partial^\alpha f\|_{B(0,R_{k+1})}\,.$$

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- For the hyperbolic quaternionic space [M. El Harchaoui, Thèse de Doctorat, Oujda, 2000] ;

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- For the hyperbolic quaternionic space [M. El Harchaoui, Thèse de Doctorat, Oujda, 2000] ;
- For the octonionic hyperbolic plane [M. El Harchaoui, Thèse de Doctorat, Oujda, 2000] .

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  (think for example, respect to potential applications, each ϕ is either a $\varphi_{k,j}$ or a $\psi_{k,j}$ from a multi-resolution analysis in $]-R, R[$).

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Theorem (C.A. Berenstein, A. Yger (1988), E. Maghras (1995))

There is an explicit procedure to recover

$$\langle f \, , \, \varphi_1(x_1) \otimes \cdots \otimes \varphi_n(x_n) \rangle$$

from the knowledge of each $\chi_{[-r_k,r_k]^n} * f$ *on the hypercube* $(]-R+r_k, R-r_k[)^n$, $k=1,...,n$.

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$$\mathrm{Supp}\, D^k_{\varphi,\psi} \subset (]-R+r_k, R-r_k[)^2\,,\ \ k=1,2,3\,.$$

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- Auxiliary tool in the Gerschberg-Papoulis extrapolation algorithm of signals with band-limited spectrum ?

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$$K(z,\zeta) := \sum_{\kappa_0+\kappa_1=n-1} \binom{n}{\kappa_1} [b(z,\zeta)]^{n-\kappa_1} \frac{\Big[S \wedge [\overline{\partial} S]^{\kappa_0} \wedge [\overline{\partial} a]^{\kappa_1} \Big](z,\zeta)}{\|\zeta - z\|^{2(\kappa_0+1)}}$$

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$$+ \sum_{\kappa_0+\kappa_1=n-1} \int_{\partial U} \binom{n}{\kappa_1} \frac{\left[\Phi\, b^{N-\kappa_1}\, S \wedge [\overline{\partial} S]^{\kappa_0} \wedge [\overline{\partial} a]^{\kappa_1}\right](z,\zeta)}{\|\zeta - z\|^{2(\kappa_0+1)}}$$

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  Find the $g_{j,k}$ is done either through divided differences or Taylor integral formula, so that convex enveloppes of supports are preserved both in ζ and z after inverse Paley-Wiener transform and the antecedents of the $g_{j,k}$ *via* Paley-Wiener are explicit in terms of the convolvers $h_1, ..., h_n$).

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(modulo a corrective boundary term expected to vanish at infinity with l when $U = U_l$ belongs to an exhaustive sequence $(U_l)_{l\geq 1}$ of $\mathbb{C}^n$).

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$$S(\omega,\zeta) = \sum_{j=1}^{n} (\overline{\zeta_j} - \overline{\omega_j})\, d\zeta_j$$

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$$\frac{1}{(2i\pi)^n}\Bigg(\int_{\partial U}\widehat{T}\sum_{p+q=n-1}\binom{n}{n-q}\frac{[b^{n-q}\,B\,S\wedge(\overline{\partial}S)^p\wedge(\overline{\partial}a)^p](\omega,\zeta)}{\|\zeta-\omega\|^{2(p+1)}}$$

$$+\int_{\partial U}\widehat{T}\sum_{p+q=n-2}\binom{n}{n-q}\frac{[b^{n-q}\,S\wedge(\overline{\partial}S)^p\wedge(\overline{\partial}a)^p\wedge\overline{\partial}A](\omega,\zeta)}{\|\zeta-\omega\|^{2(p+1)}}$$

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"For any $(\rho_1, ..., \rho_N)$ sufficiently close to $(\underline{1})$ in $(S^1)^N$, the set

$$\{\zeta \in \mathbb{C}^n\,;\ P_j(\zeta_1, ..., \zeta_n, \rho_1 e^{i\langle\gamma_1,\zeta\rangle}, ..., \rho_N e^{i\langle\gamma_N,\zeta\rangle})\,, j = 1, ..., M\}$$

remains discrete."

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Trick : Control the "growth" of the distribution $\| \ \|^{-2}$ *via* "fictive" integrations by parts.

$$\mathcal{Q}_j(\underline{\lambda}\,, X, e^{\langle \gamma, X\rangle})\Big[F_j \prod_{k=1}^{M} F_k^{\lambda_k}\Big] = b(\underline{\lambda}, [\ ]) \prod_{k=1}^{M} F_k^{\lambda_k}\,,\ \ k = 1, ..., M$$

(Bernstein-Sato type relations)

Some results (and the intrusion or arithmetics)

Two cases could be studied that way ([C.A. Berenstein, A.Y., 1995]) :

$$
\begin{aligned}
F_j &= P_j(\zeta_1, \dots, \zeta_n, e^{i\zeta_1})\,,\ j = 1, \dots, M,\ ,\ P_j \in \mathbb{C}[X_1, \dots, X_{n+1}] \\
F_j &= P_j(\zeta_1, \dots, \zeta_n, e^{i\zeta_1}, e^{i\omega\zeta_1})\,,\ j = 1, \dots, M\,,\ P_j \in \overline{\mathbb{Q}}[X_1, \dots, X_n]\,,\ \omega \in \overline{\mathbb{Q}}\,.
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$$

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As an example, related to J. Ritt's theorem : if an irreducible polynomial divides (as an entire function)

$$\zeta \longmapsto \sum_j A_j(\omega) e^{i\langle \gamma_j, \omega\rangle}\,,$$

either it divides all A_j, either it is an affine polynomial

$$P(\omega) = \langle \gamma_j - \gamma_l\,,\,\omega\rangle - \mathrm{Cst}\,.$$

Arithmetic constraints imply more rigidity.

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- A question after listening to the lectures ; is there any hope to state some theorem of the Delsarte type (probably with $n+1$ radii) to caracterize the harmonicity respect to the Dunkl Laplacian ?
- Can the machinery involved in toric geometry or in studying by indirect approaches problems where exponential polynomials (the "classical" exponential) are involved be of any help ?