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Lignes directrices

Introduction

Let f an integral indefinite binary quadratic form. Three important invariants of f are the discriminant $\mathrm{Disc}(f)$, the minimum $\min(f)$ of f over $\mathbb{Z}^2 \setminus \{(0,0)\}$ and the ratio $\mathrm{Spec}(f) = \mathrm{Disc}(f)/\min(f)^2$.

Markoff triples

An ordered Markoff triple is a triple $0 < m_1 \leq m_2 \leq m$ of integers that satisfy the equation

$$m_1^2 + m_2^2 + m^2 = 3m_1 m_2 m \quad . \tag{1}$$

The first Markoff triples are
$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34)$
The components of a Markoff triples are called Markoff numbers : $1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610$

Markoff forms

To such triple is associated k and l such that $k = \pm m1/m2 \pmod{m}$, $0 < k \leq m/2$ and $k^2 + 1 = lm$. and a quadratic form

$$f_m(X, Y) = mX^2 + (3m - 2k)XY + (l - 3k)Y^2 \quad . \tag{2}$$

Such form is called a Markoff form, has discriminant $\mathrm{Disc}(f) = 9m^2 - 4$ and minimium $\min(f) = m$ so $\mathrm{Disc}(f) = 9\min(f)^2 - 4$.

Mirror forms

Varnavides introduced two family of forms which have the property that

$$\mathrm{Disc}(f) = 9\min(f)^2 + 4 \tag{3}$$

and are linked to the equation

$$x^2 + y^2 = 3xyz + z^2 \tag{4}$$

Such forms also appear in Perrine in the $(2, 0, 1)$-Markoff theory.
In this talk we show that they are special cases of a more general construction.

We introduce a family of polynomials $U_n(X)$ defined by induction :

$$U_{-1} = 0 \tag{5}$$

$$U_0 = X \tag{6}$$

$$U_{n+2} = 3XU_{n+1} - U_n \tag{7}$$

For all $n \geq -1$, the triple (U_n, U_{n+1}, X) satisfies Varnavides equation

$$U_n^2 + U_{n+1}^2 = 3XU_nU_{n+1} + X^2$$

and furthermore $U_n(0) = 0$.

If m is a Markoff number, Equation (4) has a unique fundamental solution $(0, m, m)$, and the triples $(U_n(m), U_{n+1}(m), m)$ give all the solutions up to ordering and sign.

Remark

This not true if m is not a Markoff number, for example when $m = 10$*,* $(1, 33, 10)$ *and* $(0, 10, 10)$ *are two fundamental solutions of (4).*

Let (m_1, m_2, m) be a Markoff triple and (k, l) be as above. Set

$$\begin{aligned} u &= U_n(m) && (8)\\ v &= U_{n-1}(m) && (9)\\ A_n &= mu && (10)\\ B_n &= (3m-2k)u-2v && (11)\\ C_n &= (l-3k)u-2u/m+2kv/m && (12) \end{aligned}$$

and we define the mirror form $g_{m,n}$ by

$$g_{m,n}(X,Y) = A_nX^2 + B_nXY + C_nY^2 \ , \qquad (13)$$

the condition $U_n(0) = 0$ ensuring the integrality of C_n.

From the identity $k^2 + 1 = lm$ and Varnavides equation $u^2 + v^2 = 3muv + m^2$, it follows that

$$\mathrm{Disc}\, g_{m,n} = (9U_n(m)^2 + 4)m^2 \tag{14}$$

and

$$lA_n + kB_n + mC_n = 0 \ . \tag{15}$$

The goal of this talk is to establish the following result :

Theorem

The minimum of $g_{m,n}$ is equal to $U_n(m)m$, so

$$\mathrm{Disc}\, g_{m,n} = 9\min(g_{m,n})^2 + 4m^2 \quad .$$

Lignes directrices

Some examples

The “antisymmetric Markoff forms” of Varnavides and the $(2, 0, 1)$-Markoff theory of Perrine are the mirror forms associated to the Markoff triple $(1, 1, 1)$ and $(1, 1, 2)$. Varnavides paper establishes Theorem 1 for theses triples. In the sequel, we shall assume $m \leq 5$ to avoid this two cases. For each Markoff form f_m, we remark that the first term of our family is

$$g_{m,0}(X, Y) = mf_m(X, Y) - 2Y^2$$

which has discriminant $(9m^2 + 4)m^2$ and minimum m^2.

Continued fraction expansion

It is classical to associate a periodic continued fraction expansion to Markoff forms, or indeed any integral indefinite binary quadratic form.
In this section we give a formula for the period of the mirror forms in term of the period of the Markoff form.
We recall that the period of a form associated to a non-singular Markoff triple can always be written as $[2, a_1, a_2, \ldots, a_n, 1, 1, 2]$ with $a_i \in \{1, 2\}$. We shall see that mirror forms share this property.

Continued fraction expansion

Given two sequences $(a_i)_{i=1}^{p}$ and $(b_i)_{i=1}^{q}$ we define the sequence

$$a \wedge b = [a_1, a_2, \ldots, a_p, 2, 2, 1, 1, b_q, b_{q-1}, \ldots, b_1] \tag{16}$$

which is a sequence of length $p + q + 4$.
Let (m_1, m_2, m) a Markoff triple, and $[2, a_1, a_2, \ldots, a_p, 1, 1, 2]$ the period of the form f_m. We denote by $h_{m,n}$ the primitive form whose period is given by the sequence

$$S_{m,n} = [2, c_{n,1}, c_{n,2} \ldots, c_{n,r_n}, 1, 1, 2] \tag{17}$$

where c_n and r_n are defined inductively by :

$$c_0 = a \wedge a \tag{18}$$

$$r_0 = 2p + 4 \tag{19}$$

$$c_{i+1} = c_0 \wedge c_i \tag{20}$$

$$r_{i+1} = r_0 + r_i + 4 \tag{21}$$

Theorem

The period of the form $g_{m,n}$ is given by the sequence $S_{m,n}$, i.e, $g_{m,n}$ and $h_{m,n}$ are proportional.

This will be a consequence of Lemma 5 in the next section.

Matrices

Continued fraction expansion are associated to matrix factorization in the group $SL_2(\mathbb{Z})$. We denote by V_i the matrices $V_i = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$ and by T the matrix $T = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$. We recall the following lemma :

Lemma

Let $[p_1, \ldots, p_r]$ *be the period of an reduced integral indefinite binary quadratic form* f*. We denote by* M_f *the matrix* $M_f = \prod_{k=1}^{r} V_{p_i}$*. There exists some rational number* λ *such that*

$$f(X, Y) = \lambda(-Y, X)M(X, Y) \tag{22}$$

In particular, for the Markoff form f_m, we have the identity $M_{f_m} = \begin{pmatrix} l & k \\ k & m \end{pmatrix} T$.

Matrices

Lemma

Let (m_1, m_2, m) be a Markoff triple. The matrices $M_{m,n}$ of the forms $h_{m,n}$ statisfy

$$M_{m,0} = M_{f_m} T^{-1} M_{f_m}^t T \tag{23}$$

$$M_{m,n+1} = M_{m,0} T^{-1} M_{m,n}^t T \tag{24}$$

Démonstration.

This follows directly from the definitions by noting that the matrices V_i are symmetrical and that $T = V_2^{-1} V_1^2 V_2$. □

Lemma

Let $m_1 \leq m_2 \leq m$ be a non singular Markoff triple, and set

$$u = U_n(m) \tag{25}$$

$$v = U_{n-1}(m) \tag{26}$$

$$D_n = \frac{3u}{m}(ku + v) \tag{27}$$

then the matrix $M_{m,n}$ of the form $h_{m,n}$ satisfy the equation

$$M_{m,n} = \begin{pmatrix} 1 + D_n & -\frac{3u}{m} C_n \\ \frac{3u}{m} A_n & 9u^2 + 1 - D_n \end{pmatrix} \tag{28}$$

Démonstration.

This follows from Lemma 3 and the properties of U by a direct but extremly tedious computation best left to PARI/GP. It is easy to prove the lemma for $n = 0$. To prove the induction, replace A_n, B_n, C_n and D_n by their expression as rational functions of u, v, m, k, l, then compare the product $M_{m,0} T^{-1} M^t_{m,n} T$ with the first expression where (u, v) is substitued by $(3mu - v, u)$, and finally reduce using the equations $k^2 + 1 = lm$ and $u^2 + v^2 = 3muv + m^2$. □

Lemma

Let (m_1, m_2, m) be a Markoff triple. The matrix $M_{m,n}$ of the form $h_{m,n}$ statisfies the equation

$$\frac{3}{m}U_n g_{m,n}(X,Y) = (-Y,X)M_{m,n}(X,Y) \tag{29}$$

Démonstration.

This follows from Lemma 4 and the equality

$$D_n = \frac{9}{2}u^2 - \frac{3u}{2m}B_n \tag{30}$$

□

Theorem

The minimum of $g_{m,n}$ is equal to $U_{n+1}(m)m$, so

$$\mathrm{Disc}\, g_{m,n} = 9 \min(g_{m,n})^2 + 4m^2 \quad .$$

Démonstration.

This should follow from Theorem 2 and Dickson lemma. □