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Matrices
Let $f$ an integral indefinite binary quadratic form. Three important invariants of $f$ are the discriminant $\text{Disc}(f)$, the minimum $\min(f)$ of $f$ over $\mathbb{Z}^2 \setminus \{(0, 0)\}$ and the ratio $\text{Spec}(f) = \text{Disc}(f) / \min(f)^2$. 
Markoff triples

An ordered Markoff triple is a triple $0 < m_1 \leq m_2 \leq m$ of integers that satisfy the equation

$$m_1^2 + m_2^2 + m^2 = 3m_1m_2m.$$  \hspace{1cm} (1)

The first Markoff triples are

$$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34)$$

The components of a Markoff triples are called Markoff numbers: $1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610$
To such triple is associated $k$ and $l$ such that $k = \pm \frac{m_1}{m_2} \pmod{m}$, $0 < k \leq \frac{m}{2}$ and $k^2 + 1 = lm$. and a quadratic form

$$f_m(X, Y) = mX^2 + (3m - 2k)XY + (l - 3k)Y^2.$$  \hspace{1cm} (2)

Such form is called a Markoff form, has discriminant $\text{Disc}(f) = 9m^2 - 4$ and minimum $\text{min}(f) = m$ so $\text{Disc}(f) = 9 \text{min}(f)^2 - 4$. 

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Markoff forms

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Mirror forms

Varnavides introduced two family of forms which have the property that

$$\text{Disc}(f) = 9 \min(f)^2 + 4$$  \hspace{1cm} (3)

and are linked to the equation

$$x^2 + y^2 = 3xyz + z^2$$  \hspace{1cm} (4)

Such forms also appear in Perrine in the $(2, 0, 1)$-Markoff theory.
In this talk we show that they are special cases of a more general construction.
We introduce a family of polynomials $U_n(X)$ defined by induction:

\begin{align*}
U_{-1} &= 0 \\
U_0 &= X \\
U_{n+2} &= 3XU_{n+1} - U_n
\end{align*}

For all $n \geq -1$, the triple $(U_n, U_{n+1}, X)$ satisfies Varnavides equation

$$U_n^2 + U_{n+1}^2 = 3XU_nU_{n+1} + X^2$$

and furthermore $U_n(0) = 0$. 
If $m$ is a Markoff number, Equation (4) has a unique fundamental solution $(0, m, m)$, and the triples $(U_n(m), U_{n+1}(m), m)$ give all the solutions up to ordering and sign.

**Remark**

*This not true if $m$ is not a Markoff number, for example when $m = 10$, $(1, 33, 10)$ and $(0, 10, 10)$ are two fundamental solutions of (4).*
Let \((m_1, m_2, m)\) be a Markoff triple and \((k, l)\) be as above. Set

\[
\begin{align*}
u & = U_n(m) \quad (8) \\
v & = U_{n-1}(m) \quad (9) \\
A_n & = mu \quad (10) \\
B_n & = (3m - 2k)u - 2v \quad (11) \\
C_n & = (l - 3k)u - 2u/m + 2kv/m \quad (12)
\end{align*}
\]

and we define the mirror form \(g_{m,n}\) by

\[
g_{m,n}(X, Y) = A_nX^2 + B_nXY + C_nY^2 ,
\]

the condition \(U_n(0) = 0\) ensuring the integrality of \(C_n\).
From the identity $k^2 + 1 = lm$ and Varnavides equation $u^2 + v^2 = 3muv + m^2$, it follows that

$$\text{Disc} g_{m,n} = (9U_n(m)^2 + 4)m^2$$

(14)

and

$$lA_n + kB_n + mC_n = 0.$$  

(15)
The goal of this talk is to establish the following result:

**Theorem**

*The minimum of $g_{m,n}$ is equal to $U_n(m)m$, so*

\[
\text{Disc} g_{m,n} = 9 \min(g_{m,n})^2 + 4m^2.
\]
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The “antisymmetric Markoff forms” of Varnavides and the $(2, 0, 1)$-Markoff theory of Perrine are the mirror forms associated to the Markoff triple $(1, 1, 1)$ and $(1, 1, 2)$. Varnavides paper establishes Theorem 1 for theses triples. In the sequel, we shall assume $m \leq 5$ to avoid this two cases. For each Markoff form $f_m$, we remark that the first term of our family is

$$g_{m,0}(X, Y) = mf_m(X, Y) - 2Y^2$$

which has discriminant $(9m^2 + 4)m^2$ and minimum $m^2$. 
Continued fraction expansion

It is classical to associate a periodic continued fraction expansion to Markoff forms, or indeed any integral indefinite binary quadratic form. In this section we give a formula for the period of the mirror forms in term of the period of the Markoff form. We recall that the period of a form associated to a non-singular Markoff triple can always be written as \([2, a_1, a_2, \ldots, a_n, 1, 1, 2]\) with \(a_i \in \{1, 2\}\). We shall see that mirror forms share this property.
Continued fraction expansion
Given two sequences \((a_i)_{i=1}^p\) and \((b_i)_{i=1}^q\) we define the sequence

\[
a \wedge b = [a_1, a_2, \ldots, a_p, 2, 2, 1, 1, b_q, b_{q-1}, \ldots, b_1]
\]  

which is a sequence of length \(p + q + 4\).

Let \((m_1, m_2, m)\) a Markoff triple, and \([2, a_1, a_2, \ldots, a_p, 1, 1, 2]\) the period of the form \(f_m\). We denote by \(h_{m,n}\) the primitive form whose period is given by the sequence

\[
S_{m,n} = [2, c_{n,1}, c_{n,2} \ldots, c_{n,r_n}, 1, 1, 2]
\]  

where \(c_n\) and \(r_n\) are defined inductively by:

\[
\begin{align*}
c_0 &= a \wedge a \\
r_0 &= 2p + 4 \\
c_{i+1} &= c_0 \wedge c_i \\
r_{i+1} &= r_0 + r_i + 4
\end{align*}
\]
Theorem

The period of the form $g_{m,n}$ is given by the sequence $S_{m,n}$, i.e., $g_{m,n}$ and $h_{m,n}$ are proportional.

This will be a consequence of Lemma 5 in the next section.
Continued fraction expansion are associated to matrix factorization in the group $SL_2(\mathbb{Z})$. We denote by $V_i$ the matrices $V_i = \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$ and by $T$ the matrix $T = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$. We recall the following lemma:

**Lemma**

Let $[p_1, \ldots, p_r]$ be the period of an reduced integral indefinite binary quadratic form $f$. We denote by $M_f$ the matrix $M_f = \prod_{k=1}^{r} V_{p_i}$. There exists some rational number $\lambda$ such that

$$f(X, Y) = \lambda(-Y, X)M(X, Y)$$

In particular, for the Markoff form $f_m$, we have the identity

$$M_{f_m} = \begin{pmatrix} l & k \\ k & m \end{pmatrix}T.$$
Lemma

Let \((m_1, m_2, m)\) be a Markoff triple. The matrices \(M_{m,n}\) of the forms \(h_{m,n}\) satisfy

\[
M_{m,0} = M_{f,m} T^{-1} M_{f,m}^t T \quad (23)
\]
\[
M_{m,n+1} = M_{m,0} T^{-1} M_{m,n}^t T \quad (24)
\]

Démonstration.

This follows directly from the definitions by noting that the matrices \(V_i\) are symmetrical and that \(T = V_2^{-1} V_1^2 V_2\). \qed
Lemma

Let $m_1 \leq m_2 \leq m$ be a non singular Markoff triple, and set

$$u = U_n(m)$$  \hspace{1cm} (25)$$
$$v = U_{n-1}(m)$$  \hspace{1cm} (26)$$
$$D_n = \frac{3u}{m}(ku + v)$$  \hspace{1cm} (27)$$

then the matrix $M_{m,n}$ of the form $h_{m,n}$ satisfy the equation

$$M_{m,n} = \begin{pmatrix}
1 + D_n & -\frac{3u}{m} C_n \\
\frac{3u}{m} A_n & 9u^2 + 1 - D_n
\end{pmatrix}$$  \hspace{1cm} (28)$$
Démonstration.
This follows from Lemma 3 and the properties of $U$ by a direct but extremely tedious computation best left to PARI/GP. It is easy to prove the lemma for $n = 0$. To prove the induction, replace $A_n$, $B_n$, $C_n$ and $D_n$ by their expression as rational functions of $u$, $v$, $m$, $k$, $l$, then compare the product $M_{m,0} T^{-1} M_{m,n}^t T$ with the first expression where $(u, v)$ is substituted by $(3mu - v, u)$, and finally reduce using the equations $k^2 + 1 = lm$ and $u^2 + v^2 = 3muv + m^2$. □
Lemma

Let \((m_1, m_2, m)\) be a Markoff triple. The matrix \(M_{m,n}\) of the form \(h_{m,n}\) satisfies the equation

\[
\frac{3}{m} U_{n,g_{m,n}}(X, Y) = (-Y, X) M_{m,n}(X, Y)
\]  

(29)

Démonstration.

This follows from Lemma 4 and the equality

\[
D_n = \frac{9}{2} u^2 - \frac{3u}{2m} B_n
\]  

(30)
Theorem

The minimum of $g_{m,n}$ is equal to $U_{n+1}(m)m$, so

$$\text{Disc } g_{m,n} = 9 \min(g_{m,n})^2 + 4m^2.$$ 

Démonstration.

This should follow from Theorem 2 and Dickson lemma. 

□